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E. E. LEVI CONVEXITY AND THE HANS LEWY PROBLEM. PART I: REDUCTION TO VANISHING THEOREMS

By Aldo Andreotti and C. Denson Hill (*)

Let S be a portion of a smooth real $(2n-1)$-dimensional hypersurface in an n dimensional complex analytic manifold M. On M there is the Cauchy-Riemann operator $\bar{\partial}$, and on S there is the tangential Cauchy-Riemann operator $\bar{\partial}_S$. The purpose of this paper is to make a study of the relationship between $\bar{\partial}$ and $\bar{\partial}_S$.

Motivation for this work stemmed originally from a paper of Hans Lewy [12] There Lewy dealt with a three-dimensional S in $\mathbb{C}^2$: He considered the problem of extending a smooth function on S, which satisfies the tangential Cauchy-Riemann equations, to a smooth function that is holomorphic in a (possibly one-sided) neighborhood of S. Lewy showed that, locally, whether or not such an extension is always possible is dependent on the convexity of S in the sense of E. E. Levi [11].

Let U^- be an appropriate one-sided neighborhood of S. Lewy's extension problem is a homogeneous Cauchy problem in U^- for the operator $\bar{\partial}$ acting on functions; the initial data on S have to satisfy homogeneous compatibility conditions determined by $\bar{\partial}_S$.

We generalize the above Cauchy problem by letting $\bar{\partial}$ and $\bar{\partial}_S$ act on differential forms of type (p, q) and are led to a formulation of the Cauchy problem in terms of cohomology classes: On U^- we define a certain Cauchy cohomology group $H^{pq}(U^-)$; on S we define a boundary cohomology group

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$H^{pq}(S)$. Given a boundary cohomology class $\xi_0 \in H^{pq}(S)$, the problem consists in finding a class $\xi \in H^{pq}(U^-)$ such that ξ_0 is the restriction of ξ to S.

Let U^+ be a similar one-sided neighborhood on the side of S opposite from U^-. Given a $\xi_0 \in H^{pq}(S)$ we also consider the problem of finding a $\xi^+ \in H^{pq}(U^+)$ and a $\xi^- \in H^{pq}(U^-)$ such that ξ_0 is the jump between ξ^+ and ξ^- across S. This is the additive Riemann-Hilbert problem for cohomology classes.

For these problems we pose the usual questions about existence and uniqueness. The central theme of Part I of this work is to reduce all such questions to corresponding questions about the vanishing of certain cohomology groups.

In the additive Riemann-Hilbert problem the Levi convexity of S does not enter into the picture. What is important there is the vanishing of the standard Dolbeault cohomology for the two-sided neighborhood $U = U^+ \cup U^-$; hence a complete treatment (Theorem 2) is obtained within the context of well-known results. There the main point is an analytical version of the Mayer-Vietoris sequence (Theorem 1). This has an analogue (Theorem 3) for cohomology with compact supports; it leads to a generalization (Theorem 4) of a result of Bochner-Fichera-Martinelli about the holomorphic extension of functions to the interior of a compact region in $\mathbb{C}^n$ $(n > 1)$, from its smooth connected boundary S. Using Serre duality we obtain, in particular, the analogue of Bochner's result for a Stein manifold. Again this is independent of the Levi convexity of S.

For the Cauchy problem, however, the Levi convexity of S is important. We show (Theorem 5, 6, 8) that existence and uniqueness depend on the vanishing of certain cohomology groups that are not standard they involve behavior at the boundary. In Part I we are concerned primarily with that reduction to vanishing theorems; hence we refrain here from using any vanishing theorems except the simplest ones, namely the vanishing of the Dolbeault cohomology for a Stein manifold. Part II of this work will be devoted to proving the necessary vanishing theorems, under suitable hypotheses on the Levi convexity of S. The case of functions $(q = 0)$, however, is special. There the role of the vanishing theorem is played by another question which, essentially, boils down to a question about envelopes of holomorphy (Section 4.5) Therefore, interesting results for functions can be obtained by using wellknown results about envelopes of holomorphy. In particular, one can see a clear distinction between the local Cauchy problem and the global Cauchy problem. We also obtain some useful isomorphisms (Theorems 7, 9), and make un application to the unique continuation property for $\bar{\partial}_S$ (Section 4.6).

There is a connection between this work and a recent paper of the authors. It was shown in [3] that, whenever sufficiently many characteristic coordinates are available, an essentially arbitrary first-order linear system of partial differential equations in one unknown function has a local canonical form in which the non trivial portion of the principal part of the system is the tangential Cauchy-Riemann operator to a real submanifold S of $\mathbb{C}^q$, for some q. This paper treats the case where S has real codimension one. As an illustration of these ideas we discuss, at the end of the paper, the original example of Lewy of a linear partial differential equation without solutions. Lewy's operator-even though it is not locally solvable represents the compatibility conditions for two different, perfectly good, Cauchy problems.

In the first two sections we formulate the necessary notation and definitions, and prove several preliminary propositions. In particular, we give an intrinsic definition of $\bar{\partial}_S$ that does not involve the introduction of any Hermitian metric on M. It should be noted that the boundary cohomology groups $H^{pq}(S)$ we define here are essentially the same as those introduced in a different way by Kohn [9] and Kohn and Rossi [10].

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The results of this work have been announced in [1].

§ 1. **Preliminaries.**

1.1. In this section we introduce some notation and terminology that will be used throughout the remainder of the paper. Consider a connected complex analytic manifold M of complex dimension n, and an open connected subset U of M (1). Let S be a closed C^∞-differentiable submanifold of U of real dimension $2n$-1. We assume that *S has two sides in U.* This latter condition means the following:

(i) S is oriented. A certain side of S, which we call the « — » side, is determined locally by the requirement that, from that side, the orientation of S is induced from the orientation of M. The other side of S is called the « + » side.

(1) We assume M and U are paracompact.

(ii) $U - S = \overset{\circ}{U}{}^{+} \cup \overset{\circ}{U}{}^{-}$, where $\overset{\circ}{U}{}^{+}$, $\overset{\circ}{U}{}^{-}$ are disjoint non-void open subsets of M.

(iii) Every point $p \in S$ has a neighborhood ω in U such that $\omega \cap \overset{\circ}{U}{}^{+}$ is on the « + » side of S and $\omega \cap \overset{\circ}{U}{}^{-}$ is on the « — » side of S(2).

Set $U^{+} = \overset{\circ}{U}{}^{+} \cup S$ and $U^{-} = \overset{\circ}{U}{}^{-} \cup S$. Then $U^{+} \cup U^{-} = U$, and $U^{\pm}$ is a C^{∞} manifold (which in general is partly with and partly without boundary) of real dimension $2n$, whose interior $\overset{\circ}{U}{}^{\pm}$ is a complex manifold of complex dimension n.

These general considerations are motivated by the important special case where $M = \mathbb{C}^{n}$. Essentially there are two situations of principal interest: 1^{0}. (motivated by the *Cauchy problem*) where S is not compact, but is a connected portion of a hypersurfaee which divides U into two connected pieces U^{+} and U^{-}. 2^{0}. (motivated by the *Dirichlet problem*) where S is compact and forms the boundary of the compact connected region U^{-}, whose complement in U is $\overset{\circ}{U}{}^{+}$. In the first case « initial values » will be prescribed on S; in the second case « boundary values » will be prescribed on S.

1.2. *Global definition of S.* Under the assumptions listed above we have

PROPOSITION 1.1. *There exists a real-valued function $\varrho \in C^{\infty}(U)$ such that $S = \{z \in U \mid \varrho(z) = 0\}$, $U^{-} = \{z \in U \mid \varrho(z) \leq 0\}$, $U^{+} = \{z \in U \mid \varrho(z) \geq 0\}$ and $d\varrho|_{S} \neq 0$.*

PROOF. Let $\{V_i ; i \in I\}$ be a locally finite open covering of S in U such that in each V_i there is a $\varrho_i \in C^{\infty}(V_i)$ with $d\varrho_i \neq 0$ on $S \cap V_i$ and $S \cap V_i = \{z \in V_i \mid \varrho_i(z) = 0\}$. By changing the sign of ϱ_i, if necessary, we may assume that $\varrho_i > 0$ on the « + » side of S and $\varrho_i < 0$ on the « — » side of S. On any non-void intersection $V_i \cap V_j$ it follows that $\varrho_i/\varrho_j > 0$. Adjoin the open sets $\overset{\circ}{U}{}^{+}$ and $\overset{\circ}{U}{}^{-}$ to the covering $\{V_i ; i \in I\}$ and define corresponding functions $\varrho_{+} \equiv +1$ in $\overset{\circ}{U}{}^{+}$ and $\varrho_{-} \equiv -1$ in $\overset{\circ}{U}{}^{-}$. To simplify notation we denote the new covering and enlarged system of functions again by $\{V_i, \varrho_i, i \in I\}$. Since $\varrho_i/\varrho_j > 0$ in each non-vold intersection $V_i \cap V_j$, we can set $h_{ij} = \log \varrho_i/\varrho_j$. Then $h_{ij} + h_{ji} = 0$ in $V_i \cap V_j$ and $h_{ij} + h_{jk} + h_{ki} = 0$ in each non-void $V_i \cap V_j \cap V_k$. This assignment is a 1-cocycle h which

(2) I.e., on $\omega \cap S$ the positive normal to S points into $\omega \cap \overset{\circ}{U}{}^{+}$.

represents a class in $\check{H}^1(U, \{V_i\}, \underline{C}^\infty)$, where $\underline{C}^\infty$ is the sheaf of germs of C^∞ functions in U. But $\check{H}^1(U, \{V_i\}, \underline{C}^\infty) = 0$ because $\underline{C}^\infty$ is a fine sheaf; hence h is a coboundary, and there exist functions $h_i \in C^\infty(V_i)$ such that $h_i - h_j = h_{ij}$ on each non-void $V_i \cap V_j$. It is easily verified that the function ϱ, defined by $\varrho = \varrho_i e^{-h_i}$ in each V_i, is well-defined as a C^∞ function on U and has all the properties required by the proposition.

1.3. $C^\infty(U)$, $C^\infty(U^\pm)$ and $C^\infty(S)$ denote the complex-valued infinitely differentiable functions defined on U, $U^\pm$, and S, respectively. We shall use $C_0^\infty(U)$, $C_0^\infty(U^\pm)$, and $C_0^\infty(S)$ to denote the corresponding classes of functions which have compact support. A function in $C^\infty(U^+)$, for example, is infinitely differentiable up to that part of the boundary of U^+ formed by S; if in addition it is in $C_0^\infty(U^+)$, then its support may meet S but it vanishes identically in a neighborhood of the boundary of U.

Next we consider forms on M and place various restrictions on their coefficients: If C is any space of functions defined on a subset of M, $C_{(p,q)}$ shall denote the space of forms on M of type (p, q) whose coefficients are in C. Thus for (p, q)-forms we have the corresponding spaces $C^\infty_{(p,q)}(U)$, $C^\infty_{(p,q)}(U^\pm), \dots, C^\infty_{0(p,q)}(S)$.

Since $U^\pm$ and S are relatively closed in U, it follows from the Whitney extension theorem [19] by using a partition of unity that any element of $C^\infty_{(p,q)}(U^\pm)\,(C^\infty_{0(p,q)}(U^\pm))$ or $C^\infty_{(p,q)}(S)\,(C^\infty_{0(p,q)}(S))$ can be extended to an element of $C^\infty_{(p,q)}(U)\,(C^\infty_{0(p,q)}(U))$. Hence whenever it is convenient to do so we may regard a form in $C^\infty_{(p,q)}(U^\pm)$ or $C^\infty_{(p,q)}(S)$, etc., as an equivalence class of forms in $C^\infty_{(p,q)}(U)$, etc.

In terms of local holomorphic coordinates on M, a form u of type (p, q) is given by

$$u = \sum_{|I|=p,\,|J|=q}' u_{I,J}\, dz^I \wedge d\bar{z}^J ,$$

where $I = (i_1, i_2, \dots, i_p)$ and $J = (j_1, j_2, \dots, j_q)$ are multi-indices of integers between 1 and n. Here

$$dz^I \wedge d\bar{z}^J = dz_{i_1} \wedge dz_{i_2} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \wedge \dots \wedge d\bar{z}_{j_q} ,$$

and the notation Σ' means that the summation extends only over strictly increasing multi-indices. The functions $u_{I,J}$ are assumed to be antisymmetric in both I and J.

The exterior differentiation operator d decomposes into the sum $d = \partial + \bar{\partial}$ such that $\partial : C^\infty_{(p,q)}(U) \to C^\infty_{(p+1,q)}(U)$, $\bar{\partial} : C^\infty_{(p,q)}(U) \to C^\infty_{(p,q+1)}(U)$, and

$\partial^2 = \bar{\partial}^2 = 0$, where

$$\partial u = \sum_k \sum'_{|I|=p, |J|=q} \frac{\partial u_{I,J}}{\partial z_k} d z_k \wedge d z^I \wedge d \bar{z}^J,$$

(1.3.1)

$$\bar{\partial} u = \sum_k \sum'_{|I|=p\ |J|=q} \frac{\partial u_{I,J}}{\partial \bar{z}_k} d \bar{z}_k \wedge d z^I \wedge d \bar{z}^J.$$

The operators ∂ and $\bar{\partial}$ can be defined on $C^\infty_{(p,q)}(U^\pm)$, etc., by (1.3.1) in the interior $\mathring{U}^\pm$, and by continuity up to S. We also have the relations

$$\partial (u \wedge v) = (\partial u) \wedge v + (-1)^{p+q} u \wedge \partial v,$$

(1.3.2)

$$\bar{\partial} (u \wedge v) = (\bar{\partial} u) \wedge v + (-1)^{p+q} u \wedge \bar{\partial} v,$$

which hold when u is a form of type (p, q) and when v is a form of any type.

1.4. *The ideal generated by ϱ and $\bar{\partial}\varrho$.* Let $\mathcal{J}_{(p,q)}(U)$ denote the «differential» ideal (cf. Kähler [8], page 8) in $C^\infty_{(p,q)}(U)$ generated by ϱ and $\bar{\partial}\varrho$; that is, $u \in \mathcal{J}_{(p,q)}(U)$ means that $u = \varrho\alpha + \bar{\partial}\varrho \wedge \beta$ where $\alpha \in C^\infty_{(p,q-1)}(U)$ and $\beta \in C^\infty_{(p,q-1)}(U)$. In the special case $q = 0$, any $u \in \mathcal{J}_{(p,0)}(U)$ can ben written more simply as $u = \varrho\alpha$ — it vanishes on S. Observe that $\mathcal{J}_{(p,q)}(U)$ is actually independent of the function ϱ used to define S globally. Indeed if ϱ' is another global dofining function having all the properties listed in Proposition 1.1, then $\varrho' = \varrho h$ for some $h \in C^\infty(U)$ with $h > 0$ on U. It follows that a form $u \in C^\infty_{(p,q)}(U)$ can be written as $u = \varrho\alpha + \bar{\partial}\varrho \wedge \beta$ if and only if it can be written as $u = \varrho' \alpha' + \bar{\partial}\varrho' \wedge \beta'$.

In the same fashion we define the analogous ideal $\mathcal{J}_{(p,q)}(U^\pm)$, $\mathcal{J}_{0(p,q)}(U)$, or $\mathcal{J}_{0(p,q)}(U^\pm)$ by requiring that α belong to $C^\infty_{(p,q)}(U^\pm)$, $C^\infty_{0(p,q)}(U)$, or $C^\infty_{0(p,q)}(U^\pm)$ and that β belong to $C^\infty_{(p,q-1)}(U^\pm)$, $C^\infty_{0(p,q-1)}(U)$, or $C^\infty_{0(p,q-1)}(U^\pm)$, respectively. As above, any form in $\mathcal{J}_{(p,q)}(U^\pm)$ ($\mathcal{J}_{0(p,q)}(U^\pm)$) can always be extended to a form in $\mathcal{J}_{(p,q)}(U)$ ($\mathcal{J}_{0(p,q)}(U)$). When there is no danger of confusion we shall sometimes write $\mathcal{J}_{(p,q)}$ to stand for any one of the above ideals, and use $C^\infty_{(p,q)}$ in a corresponding notation. When $p = q = 0$, we shall occasionally write $\mathcal{J}$ and C^∞ instead of $\mathcal{J}_{(0,0)}$ and $C^\infty_{(0,0)}$.

It follows from (1.3.2) that

(1.4.1) $$\bar{\partial} : \mathcal{J}_{(p,q)} \rightarrow \mathcal{J}_{(p,q+1)}.$$

The following proposition gives an alternate characterization of $\mathcal{J}_{(p,q)}$.

PROPOSITION 1.2. *Let* $f \in C^{\infty}_{(p,q)}$. *Then* $f \in \mathcal{I}_{(p,q)}$ *if and only if*

$$\bar{\partial} \varrho \wedge f|_S = 0 \ (^3). \tag{1.4.2}$$

PROOF. If f has the form $f = \varrho\alpha + \bar{\partial}\varrho \wedge \beta$, then (1.4.2) clearly holds. On the other hand, (1.4.2) implies that $f|_S = \bar{\partial}\varrho \wedge \beta_0$ for some $\beta_0 \in C^{\infty}_{(p,q-1)}(S)$. Let $\beta \in C^{\infty}_{(p,q-1)}$ be a C^{∞} extension of β_0. Then $f - \bar{\partial}\varrho \wedge \beta = \varrho\alpha$ for some $\alpha \in C^{\infty}_{(p,q)}$.

1.5. *The tangential Cauchy-Riemann operator ou S.* Consider two forms $u^{\pm} \in C^{\infty}_{(p,q)}(U^{\pm})$ with $\bar{\partial}u^{+} = 0$ on U^{+}, $\bar{\partial}u^{-} = 0$ on U^{-}, and denote their jump across S by $u_0 = u^{+}|_S - u^{-}|_S$. Let $\tilde{u}_0 \in C^{\infty}_{(p,q)}(U)$ be any extension of u_0 and assume that the $u^{\pm}$ have also been extended to $C^{\infty}_{(p,q)}(U)$. Since $\tilde{u}_0 - (u^{+} - u^{-})$ vanishes on S we have $\tilde{u}_0 - (u^{+} - u^{-}) = \varrho h$ for some $h \in C^{\infty}_{(p,q)}(U)$. Similarly $\bar{\partial}(u^{+} - u^{-}) = \varrho h_1$, for some $h_1 \in C^{\infty}_{(p,q+1)}(U)$, because $\bar{\partial}(u^{+} - u^{-})$ vanishes on S. It follows that $\bar{\partial}\tilde{u}_0 = \varrho(\bar{\partial}h + h_1) + \bar{\partial}\varrho \wedge h$. Thus a necessary condition for u_0 to represent the jump across S of two $\bar{\partial}$ closed forms, defined on either side of S, is that $\bar{\partial}\tilde{u}_0 \in \mathcal{I}_{(p,q+1)}$ for any C^{∞} extension $\tilde{u}_0$. According to (1.4.2) this is equivalent to $\bar{\partial}\varrho \wedge \bar{\partial}\tilde{u}_0|_S = 0$.

On the other hand, let $u_1, u_2 \in C_{(p,q)}(U)$ be two forms such that $u_1 - u_2 \in \mathcal{I}_{(p,q)}$. Then it follows from (1.4.1) and (1.4.2) that $\bar{\partial}\varrho \wedge \bar{\partial}u_{1}|_S = \bar{\partial}\varrho \wedge \bar{\partial}u_{2}|_S$. In particular, if $\tilde{\tilde{u}}_0$ and $\tilde{u}_0$ are two different C^{∞} extensions of some $u_0 \in C^{\infty}_{(p,q)}(S)$, then $\bar{\partial}\varrho \wedge \bar{\partial}\tilde{\tilde{u}}_{0}|_S = \bar{\partial}\varrho \wedge \bar{\partial}\tilde{u}_{0}|_S$. Since, in general, the the operator $\bar{\partial}\varrho \wedge \bar{\partial}|_S$ does not distinguish between u_1 and u_2 if their difference is in the ideal, it is convenient to define the tangential Cauchy-Riemann operator $\bar{\partial}_S$ directly on equivalence classes of such forms u_1 and u_2.

Therefore we introduce the quotient

$$Q_{(p,q)}(U) = C^{\infty}_{(p,q)}(U)/\mathcal{I}_{(p,q)}(U)$$

and use $\{u\}$ to denote the equivalence class in $Q_{(p,q)}(U)$ represented by $u \in C^{\infty}_{(p,q)}(U)$ *Then*

$$\bar{\partial}_S : \ Q_{(p,q)}(U) \to Q_{(p,q+1)}(U)$$

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(3) Restriction here means only restriction of the coefficients; i. e., (1.4.2) means that $\bar{\partial}\varrho \wedge f$ has coefficients which vanish at each point of S.

is defined by $\bar\partial_S\{u\} = \{\bar\partial u\}$. It follows that $\bar\partial_S\{u\} = 0$ if and only if $\bar\partial u \in \mathcal{J}_{(p,q+1)}(U)$, and $\bar\partial_S\{u\} = \{f\}$ means that $\bar\partial u - f \in \mathcal{J}_{(p,q+1)}(U)$. Note that $\bar\partial_S^2 = 0$. Let $H^{p0}(U)$ be the kernel of $\bar\partial$ in $C^\infty_{(p,0)}(U)$ and $H^{p0}(S)$ be the kernel of $\bar\partial_S$ in $Q_{(p,0)}(U)$. Then $H^{p0}(U)$ consists of the holomorphic p-forms on U and $H^{p0}(S)$ is isomorphic to the space of p-forms whose coefficients f are C^∞ on S and satisfy the tangential Cauchy-Rieman equuatinn $\bar\partial\varrho\wedge\bar\partial f|_S = 0$ there.

The previous paragraph can be paraphrased by saying that $\bar\partial_S$ is defined so as to make the following diagram commutative:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & H^{p0}(U) & & H^{p0}(S) & & \\
 & & & & \downarrow i & & \downarrow i & & \\
0 & \longrightarrow & \mathcal{J}_{(p,0)}(U) & \xrightarrow{i} & C^\infty_{(p,0)}(U) & \xrightarrow{j} & Q_{(p,0)}(U) & \longrightarrow & 0 \\
 & & \downarrow \bar\partial & & \downarrow \bar\partial & & \downarrow \bar\partial_S & & \\
0 & \longrightarrow & \mathcal{J}_{(p,1)}(U) & \xrightarrow{i} & C^\infty_{(p,1)}(U) & \xrightarrow{j} & Q_{(p,1)}(U) & \longrightarrow & 0 \\
 & & \downarrow \bar\partial & & \downarrow \bar\partial & & \downarrow \bar\partial_S & & \\
0 & \longrightarrow & \mathcal{J}_{(p,2)}(U) & \xrightarrow{i} & C^\infty_{(p,2)}(U) & \xrightarrow{j} & Q_{(p,2)}(U) & \longrightarrow & 0 \\
 & & \downarrow \bar\partial & & \downarrow \bar\partial & & \downarrow \bar\partial_S & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Here the maps labeled i represent inclusion and those labeled j represent projection onto the quotient; all the rows are exact.

The operator $\bar\partial_S$ does not depend on the global defining function ϱ because the ideals $\mathcal{J}_{(p,q)}(U)$ are independent of ϱ.

The above considerations can be restricted to either of the sets $U^\pm$. Then one obtains a commutative diagram exactly like the one above, except that U must be replaced everywhere by $U^\pm$. In that case the kernel $H^{p0}(U^\pm)$ consists of the p-forms which are holomorphic in the interior

$\overset{\circ}{U}\pm$, and whose coefficients are smooth up to S; but the kernel $H^{p0}(S)$ of $\bar\partial_S$ in $Q_{(p,0)}(U\pm)$ is isomorphic to the previous one based on $Q_{(p,0)}(U)$.

The above considerations can also be carried out with $\mathcal{I}_{(p,q)}(U)$, $C^\infty_{(p,q)}(U)$, and $Q_{(p,q)}(U)$ replaced by $\mathcal{I}_{0(p,q)}(U)$, $C^\infty_{0(p,q)}(U)$, and $Q_{0(p,q)}(U)$ (using an obvious notation). Similarly they can also be carried out for the case $\mathcal{I}_{0(p,q)}(U\pm)$, $C^\infty_{0(p,q)}(U\pm)$, and $Q_{0(pq)}(U\pm)$. It what follows we shall have occasion to refer to all of these cases.

§ 2. Cohomology for the Cauchy Problem.

2.1. In this section, we introduce certain cohomology groups that are important in the study of the Cauchy problem (as well as in other problems) for the $\bar\partial$ operator. They arise quite naturally. However, *with the exception of part (a) below, we emphasize that these cohomology groups are not the usual ones; they should not be confused with the usual Dolbeault cohomology or with any sheaf cohomology on U.*

(*a*) The usual Dolbeault cohomology in U we shall denote by $H^{pq}(U)$; that is,

$$H^{p0}(U) = \ker\{\bar\partial : C^\infty_{(p,0)}(U) \to C^\infty_{(p,1)}(U)\},$$

and

$$H^{pq}(U) = \frac{\ker\{\bar\partial : C^\infty_{(p,q)}(U) \to C^\infty_{(p,q+1)}(U)\}}{\operatorname{im}\{\bar\partial : C^\infty_{(p,q-1)}(U) \to C^\infty_{(p,q)}(U)\}}, \quad q \geq 1.$$

According to the Dolbeault isomorphism, $H^{pq}(U) \cong \check{H}^q(U, \mathcal{O}_{(p)})$, where $\check{H}^q(U, \mathcal{O}_{(p)})$ is the *q-th* cohomology group of U with coefficients in the sheaf $\mathcal{O}_{(p)}$ of germs of holomorphic p-forms on U.

By $H^{pq}_k(U)$, we shall mean the analogous cohomology with compact supports in U; that is,

$$H^{p0}_k(U) = \ker\{\bar\partial : C^\infty_{0(p,0)}(U) \to C^\infty_{0(p,1)}(U)\},$$

and

$$H^{pq}_k(U) = \frac{\ker\{\bar\partial : C^\infty_{0(p,q)}(U) \to C^\infty_{0(p,q+1)}(U)\}}{\operatorname{im}\{\bar\partial : C^\infty_{0(p,q-1)}(U) \to C^\infty_{0(p,q)}(U)\}}, \quad q \geq 1.$$

Since U is open, we have $H^{p0}_k(U) = 0$ for all p, except for the special case where M is compact and $U = M$.

(*b*) *Cohomology for the Cauchy problem.* On either of the two sets $U^{\pm}$, we define the cohomology groups $H^{pq}(U^{\pm})$ by

$$H^{p0}(U^{\pm}) = \ker\{\bar\partial : C^{\infty}_{(p,0)}(U^{\pm}) \to C^{\infty}_{(p,1)}(U^{\pm})\},$$

and

$$H^{pq}(U^{\pm}) = \frac{\ker\{\bar\partial : C^{\infty}_{(p,q)}(U^{\pm}) \to C^{\infty}_{(p,q+1)}(U^{\pm})\}}{\operatorname{im}\{\bar\partial : C^{\infty}_{(p,q-1)}(U^{\pm}) \to C^{\infty}_{(p,q)}(U^{\pm})\}},\ q \geq 1.$$

Remark. In the above, the operator $\bar\partial$ is defined in the usual way in the interior $\overset{\circ}{U}{}^{\pm}$, and is defined by continuity up to S. Thus an element of $H^{00}(U^{+})$, for example, is a function which is holomorphic in the interior of U^{+} and C^{∞} up to that part of the boundary of U^{+} formed by S.

The analogous cohomology $H^{pq}_{k}(U^{\pm})$ with compact supports in $U^{\pm}$ is defined by

$$H^{p0}_{k}(U^{\pm}) = \ker\{\bar\partial : C^{\infty}_{0\,(p,0)}(U^{\pm}) \to C^{\infty}_{0\,(p,1)}(U^{\pm})\},$$

and

$$H^{pq}_{k}(U^{\pm}) = \frac{\ker\{\bar\partial : C^{\infty}_{0\,(p,q)}(U^{\pm}) \to C^{\infty}_{0\,(p,q+1)}(U^{\pm})\}}{\operatorname{im}\{\bar\partial : C^{\infty}_{0\,(p,q-1)}(U^{+}) \to C^{\infty}_{0\,(p,q)}(U^{\pm})\}},\ q \geq 1.$$

Remark. If $U^{\pm}$ is compact, then $H^{pq}_{k}(U^{\pm}) = H^{pq}(U^{\pm})$ for all p and q. If $U^{\pm}$ is not compact, then $H^{p0}_{k}(U^{\pm}) = 0$ for all p on each noncompact component of $U^{\pm}$.

(*c*) *Cohomology in the ideal.* In U, define the groups

$$H^{pq}(U, \mathcal{J}) = \frac{\ker\{\bar\partial : \mathcal{J}_{(p,q)}(U) \to \mathcal{J}_{(p,q+1)}(U)\}}{\operatorname{im}\{\bar\partial : \mathcal{J}_{(p,q-1)}(U) \to \mathcal{J}_{(p,q)}(U)\}},\ q \geq 1.$$

There is no point in defining these groups for the case $q = 0$ because $H^{p0}(U, \mathcal{J}) = H^{p0}(U^{\pm}, \mathcal{J}) = 0$ for all p. This is a consequence of the following.

Remark. Let Ω be a connected component of U^{+} (or U^{-}), and let σ be an open set in S such that $\sigma \subset \partial\Omega$. Suppose $f \in C^{1}(\Omega)$ is a function which is holomorphic in the interior of Ω and vanishes on σ. Then $f \equiv 0$ in Ω. To see this, let $z_0 \in \sigma$, and consider a sufficiently small connected neighborhood ω of z_0 in M. Then if f is extended to be identically zero in $\omega \cap \complement\Omega$, the hypotheses on f imply that $\bar\partial f = 0$ in ω in the sense of distributions. By interior regnlarity and the uniqueness of analytic continuation, we conclude that $f \equiv 0$ in ω; hence $f \equiv 0$ in Ω.

We shall always assume that $S \neq \emptyset$. Then, according to the remark we can add the exact row

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & H^{p0}(U) & \xrightarrow{\text{restriction}} & H^{p0}(S) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
\end{array}$$

to the upper left of the diagram on page 8.

After a fashion by now familiar to the reader, we also define, for $q \geq 1$, the cohomology groups $H_k^{pq}(U, \mathcal{J})$ and $H_k^{pq}(U^{\pm}, \mathcal{J})$ in the ideal with compact supports.

(*d*) *The boundary cohomology.*

PROPOSITION 2.1. *For all p and q, we have the isomorphisms*

$$\text{(i)} \qquad Q_{(p,q)}(U) \cong Q_{(p,q)}(U^+) \cong Q_{(p,q)}(U^-),$$

and

$$\text{(ii)} \qquad Q_{0(p,q)}(U) \cong Q_{0(p,q)}(U^+) \cong Q_{0(p,q)}(U^-).$$

PROOF. We shall show $Q_{(p,q)}(U^+) \cong Q_{(p,q)}(U)$, the other cases being similar. Consider a class $\{u^+\} \in Q_{(p,q)}(U^+)$ represented by $u^+ \in C^\infty_{(p,q)}(U^+)$, and let $\tilde{u}^+ \in C^\infty_{(p,q)}(U)$ be an extension of u^+. Then $\tilde{u}^+$ represents a class $\{\tilde{u}^+\} \in Q_{(p,q)}(U)$. The class $\{\tilde{u}^+\}$ is independent of the particular C^∞ extension chosen; the difference between any two extensions vanishes on U^+ and hence belongs to $\mathcal{J}_{(p,q)}(U)$. Since any element of $\mathcal{J}_{(p,q)}(U^+)$ can be extended to an element of $\mathcal{J}_{(p,q)}(U)$, it follows that the class of $\{\tilde{u}^+\}$ is independent of the parlicular representative u^+. Thus we have a homomorphism $h: Q_{(p,q)}(U^+) \longrightarrow Q_{(p,q)}(U)$. By taking restrictions from U to U^+, one sees that h is one-to-one and onto, hence an isomorphism.

The boundary cohomology $H^{pq}(S)$ of S is defined by

$$\begin{aligned}
H^{p0}(S) &= \ker\{\bar{\partial}_S : Q_{(p,0)}(U) \longrightarrow Q_{(p,1)}(U)\} \\
&\cong \ker\{\bar{\partial}_S : Q_{(p,0)}(U^+) \longrightarrow Q_{(p,1)}(U^+)\} \\
&\cong \ker\{\bar{\partial}_S : Q_{(p,0)}(U^-) \longrightarrow Q_{(p,1)}(U^-)\},
\end{aligned}$$

and

$$H^{pq}(S) = \frac{\ker\{\bar\partial_S : Q_{(p,q)}(U) \to Q_{(p,q+1)}(U)\}}{\operatorname{im}\{\bar\partial_S : Q_{(p,q-1)}(U) \to Q_{(p,q)}(U)\}}$$

$$\cong \frac{\ker\{\bar\partial_S : Q_{(p,q)}(U^+) \to Q_{(p,q+1)}(U^+)\}}{\operatorname{im}\{\bar\partial_S : Q_{(p,q-1)}(U^+) \to Q_{(p,q)}(U^+)\}}$$

$$\cong \frac{\ker\{\bar\partial_S : Q_{(p,q)}(U^-) \to Q_{(p,q+1)}(U^-)\}}{\operatorname{im}\{\bar\partial_S : Q_{(p,q-1)}(U^-) \to Q_{(p,q)}(U^-)\}}, \quad q \geq 1.$$

Similarly, we define the boundary cohomology $H_k^{pq}(S)$ with compact support. Note that $H_k^{pq}(S) \cong H^{pq}(S)$ if S is compact.

It will be useful in what follows to observe that the above characterizations of $H^{pq}(S)$ (or $H_k^{pq}(S)$) can be rewritten in the slightly different form

$$H^{p0}(S) \cong \frac{\{u \in C^\infty_{(p,0)}(U) \mid \bar\partial u \in \mathcal{J}_{(p,1)}(U)\}}{\{u \in C^\infty_{(p,0)}(U) \mid u = 0 \text{ on } S\}}$$

$$\cong \frac{\{u \in C^\infty_{(p,0)}(U^+) \mid \bar\partial u \in \mathcal{J}_{(p,1)}(U^+)\}}{\{u \in C^\infty_{(p,0)}(U^+) \mid u = 0 \text{ on } S\}}$$

$$\cong \frac{\{u \in C^\infty_{(p,0)}(U^-) \mid \bar\partial u \in \mathcal{J}_{(p,1)}(U^-)\}}{\{u \in C^\infty_{(p,0)}(U^-) \mid u = 0 \text{ on } S\}},$$

and, for $q \geq 1$,

$$H^{pq}(S) \cong \frac{\{u \in C^\infty_{(p,q)}(U) \mid \bar\partial u \in \mathcal{J}_{(p,q+1)}(U)\}}{\{u \in C^\infty_{(p,q)}(U) \mid u - \bar\partial v \in \mathcal{J}_{(p,q)}(U) \text{ for some } v \in C^\infty_{(p,q-1)}(U)\}}$$

$$\cong \frac{\{u \in C^\infty_{(p,q)}(U^+) \mid \bar\partial u \in \mathcal{J}_{(p,q+1)}(U^+)\}}{\{u \in C^\infty_{(p,q)}(U^+) \mid u - \bar\partial v \in \mathcal{J}_{(p,q)}(U^+) \text{ for some } v \in C^\infty_{(p,q-1)}(U^+)\}} \cong$$

$$\cong \frac{\{u \in C^\infty_{(p,q)}(U^-) \mid \bar\partial u \in \mathcal{J}_{(p,q+1)}(U^-)\}}{\{u \in C^\infty_{(p,q)}(U^-) \mid u - \bar\partial v \in \mathcal{J}_{(p,q)}(U^-) \text{ for some } v \in C^\infty_{(p,q-1)}(U^-)\}}.$$

2.2. *Distinguished representatives.* In this section, we consider $H^{pq}(S)$, $H_k^{pq}(S)$, $H^{pq}(U, \mathcal{J})$, $H_k^{pq}(U, \mathcal{J})$, $H^{pq}(U^\pm, \mathcal{J})$ and $H_k^{pq}(U^\pm, \mathcal{J})$, and show that each cohomology class from one of these groups has certain distinguished representatives. These distinguished representatives are important because they exhibit very nice behavior upon approach to S.

PROPOSITION 2.2. *Let $h^{(k)} \in C^{\infty}_{(p,q)}(S)$ $(k = 0, 1, 2, \ldots)$ be any sequence of forms. Then there is a form $f \in C^{\infty}_{(p,q)}(U)$ such that*

$$\frac{\partial}{\partial \varrho^k} f|_S = h^{(k)}$$

for $k = 0, 1, 2, \ldots$.

PROOF. Let $\{V_i;\ i \in I\}$ be a locally finite covering of S in U such that, in each V_i, there is a local coordinate system of the form $(x_1, x_2, \ldots, x_{2n})$ with $x_1 = \varrho$. Then on each $S \cap V_i$, we have $h^{(k)} = h^{(k)}(x_2, x_3, \ldots, x_{2n})$. By applying the Whitney extension theorem $\binom{n}{p}\binom{n}{q}$ timen to the corresponding sequences of coefficients, we can construct a form $f_i \in C^{\infty}_{(p,q)}(V_i)$ such that

$$\frac{\partial^k f_i}{\partial x_1^k}(0, x_2, x_3, \ldots, x_{2n}) = h^{(k)}(x_2, x_3, \ldots, x_{2n})$$

on $S \cap V_i$ for $k = 0, 1, 2, \ldots$. Adjoin the sets $\overset{\circ}{U}{}^{\pm}$ with corresponding forms $f_{\pm} \equiv 0$ to the system constructed above. For simplicity, denote the enlarged collection of sets and forms again by $\{V_i, f_i;\ i \in I\}$. Let $\{\varphi_i;\ i \in I\}$ be a partition of unity subordinate to the covering $\{V_i;\ i \in I\}$ of U, and define $f \in C^{\infty}_{(p,q)}(U)$ by $f = \sum_i \varphi_i f_i$. Consider this form in a neighborhood of S:

$$\frac{\partial^k}{\partial \varrho^k} f = \sum_{j=0}^{k} \sum_i \binom{k}{j} \frac{\partial^j \varphi_i}{\partial \varrho^j} \frac{\partial^{k-j} f_i}{\partial \varrho^{k-j}} = \sum_j \left(\sum_i \frac{\partial^j}{\partial \varrho^j} \varphi_i\right) \binom{k}{j} h^{(k-j)} = \sum_i \varphi_i \frac{\partial^k f_i}{\partial \varrho^k},$$

because $\sum_i \frac{\partial^j \varphi_i}{\partial \varrho^j} \equiv 0$ for $j > 0$, since $\sum_i \varphi_i \equiv 1$. Restricting to S, we obtain

$$\frac{\partial^k}{\partial \varrho^k} f|_S = \sum_i \varphi_i h^{(k)} = h^{(k)} \qquad (k = 0, 1, 2, \ldots).$$

For the case of compact supports, Proposition 2.2 must be modified to

PROPOSITION 2.2'. *Let $h^{(k)} \in C^{\infty}_{0(p,q)}(S)$ $(k = 0, 1, 2, \ldots)$ be any sequence of forms, all of whose supports are contained in a fixed compact set K of S. Then there is a form $f \in C^{\infty}_{0(p,q)}(U)$ such that*

$$\frac{\partial^k}{\partial \varrho^k} f|_S = h^{(k)}$$

for $k = 0, 1, 2, \ldots$.

PROOF. Same as for Proposition 2.2 except that, at the end, f must be multiplied by a cutoff function $\zeta \in C_0^\infty(U)$ with $\zeta \equiv 1$ in a neighborhoodof K.

DEFINITION. If f is a C^∞ form in U (or $U^\pm$), *we shall write* $f|_S = 0^\infty$ *to mean that* f *vanishes to infinite order on* S; that is, for any integer m, we have $f = 0(\varrho^m)$ as $\varrho \to 0$, and the 0 is uniform on compact subsets of U (or $U^\pm$).

In the following lemma, $\mathcal{J}_{(p,q)}$ stands for any one of the differential ideals $\mathcal{J}_{(p,q)}(U)$, $\mathcal{J}_{(p,q)}(U^\pm)$, $\mathcal{J}_{0(p,q)}(U)$ or $\mathcal{J}_{0(p,q)}(U^\pm)$. In the proof, we shall write $C^\infty_{(p,q)}$ for the corresponding $C^\infty_{(p,q)}(U)$, $C^\infty_{(p,q)}(U^\pm)$, $C^\infty_{0(p,q)}(U)$ or $C^\infty_{0(p,q)}(U^\pm)$.

LEMMA 2.1. (i) *Let* $u \in \mathcal{J}_{(p,0)}$ *be such that* $\bar\partial u|_S = 0^\infty$. *Then* $u|_S = 0^\infty$.

(ii) *Let* $g \geq 1$ *and* $u \in \mathcal{J}_{(p,g)}$ *be such that* $\bar\partial u|_S = 0^\infty$. *Then there exists a* $v \in \mathcal{J}_{(p,g-1)}$ *such that* $(u - \bar\partial v)|_S = 0^\infty$.

PROOF. The hypothesis of part (i) means that $u = \varrho\alpha_1$ where $\alpha_1 \in C^\infty_{(p,0)}$. Then $\bar\partial u = \bar\partial\varrho \wedge \alpha_1 + \varrho\bar\partial\alpha_1$, and we obtain $\bar\partial\varrho \wedge \alpha_1|_S = 0$ because $\bar\partial u$ vanishes on S. Since α_1 is of type $(p, 0)$, this means that α_1 vanishes on S; hence $\alpha_1 = \varrho\alpha_2$ and $u = \varrho^2\alpha_2$ for some $\alpha_2 \in C^\infty_{(p,0)}$. Arguing by induction, suppose $u = \varrho^m\alpha_m$ for some $\alpha_m \in C^\infty_{(p,0)}$. Then $\bar\partial u = m\varrho^{m-1}\bar\partial\varrho \wedge \alpha_m + \varrho^m\bar\partial\alpha_m$, and we obtain $\bar\partial\varrho \wedge \alpha_m|_S = 0$ because $\bar\partial u$ vanishes to order $0(\varrho^m)$. But this means that $\alpha_m = \varrho\alpha_{m+1}$ and $u = \varrho^{m+1}\alpha_{m+1}$ for some $\alpha_{m+1} \in C^\infty_{(p,0)}$. It follows that $u = 0(\varrho^m)$ for any m, and the proof of part (i) is complete.

As preparation for the proof of part (ii), consider any $f \in \mathcal{J}_{(p,q)}$, $q \geq 1$. Then $f = \varrho\alpha + \bar\partial\varrho \wedge \beta$ where $\alpha \in C^\infty_{(p,q)}$ and $\beta \in C^\infty_{(p,q-1)}$. Using Proposition 2.2 (or Proposition 2.2′ in the case of compact supports), we can find a form $\varrho_1 \in C^\infty_{p,q-1)}$ such that

$$\beta_1|_S = \beta|_S,$$

and

$$\left.\frac{\partial^k\beta_1}{\partial\varrho^k}\right|_S = 0 \qquad (k = 1, 2, 3, \ldots).$$

This new form β_1 has the advantage that, near S, $\beta_1 - \beta_1|_S$ vanishes to infinite order. Since $f - \bar\partial\varrho \wedge \beta_1 = \varrho\alpha + \bar\partial\varrho \wedge (\beta - \beta_1)$ vanishes on S, it must be equal to $\varrho\alpha_1$ for some $\alpha_1 \in C^\infty_{(p,q)}$. Thus f can always be written as $f = \varrho\alpha_1 + \bar\partial\varrho \wedge \beta_1$ with $\beta_1 - \beta_1|_S$ vanishing to infinite order.

Let u be the form given in part (ii), and write it in the above fashion as $u = \varrho\alpha_1 + \bar\partial\varrho \wedge \beta_1$. Since $\bar\partial(\varrho\beta_1) = \bar\partial\varrho \wedge \beta_1 + \varrho\bar\partial\beta_1$, we can write $u - \bar\partial(\varrho\beta_1) = \varrho(\alpha_1 - \bar\partial\beta_1)$. Set $\alpha_1 - \bar\partial\beta_1 = \gamma_1$. Then $\bar\partial u = \bar\partial\varrho \wedge \gamma_1 + \varrho\bar\partial\gamma_1$, and we obtain $\bar{\partial}\varrho \wedge \gamma_1 |_S = 0$. This means that $\gamma_1 \in \mathcal{J}_{(p,q)}$, and hence γ_1 can be written as $\gamma_1 = \varrho\alpha_2 + \bar{\partial}\varrho \wedge \beta_2$ where $\alpha_2 \in C^\infty_{(p,q)}$, $\beta_2 \in C^\infty_{(p,q-1)}$, and where $\beta_2 - \beta_2|_S$ vanishes to infinite order. Since $\bar{\partial}\left(\frac{1}{2}\varrho^2\beta_2\right) = \varrho\bar{\partial}\varrho \wedge \beta_2 + \frac{1}{2}\varrho^2\bar{\partial}\beta_2$, we can therefore write $u - \bar{\partial}\left(\varrho\beta_1 + \frac{1}{2}\varrho^2\beta_2\right) = \varrho^2\left(\alpha_2 - \frac{1}{2}\bar{\partial}\beta_2\right)$. Proceeding by induction, suppose that

$$u - \bar{\partial}\left(\varrho\beta_1 + \frac{1}{2}\varrho^2\beta_2 + \dots + \frac{1}{m}\varrho^m\beta_m\right) = \varrho^m\gamma_m$$

with $\gamma_m = \alpha_m - \frac{1}{m}\bar{\partial}\beta_m$, and where each $\beta_k - \beta_k|_S$ vanishes to infinite order. Then

$$\bar{\partial}u = m\varrho^{m-1}\bar{\partial}\varrho \wedge \gamma_m + \varrho^m\bar{\partial}\gamma_m ,$$

which implies that $\bar{\partial}\varrho \wedge \gamma_m |_S = 0$ because $\bar{\partial}u$ vanishes to order $0(\varrho^m)$. Thus $\gamma_m = \varrho\alpha_{m+1} + \bar{\partial}\varrho \wedge \beta_{m+1}$ for some $\alpha_{m+1} \in C^\infty_{(p,q)}$ and some $\beta_{m+1} \in C^\infty_{(p,q-1)}$ such that $\beta_{m+1} - \beta_{m+1}|_S$ vanishes to infinite order. Since

$$\bar{\partial}\left(\frac{1}{m+1}\varrho^{m+1}\beta_{m+1}\right) = \varrho^m\bar{\partial}\varrho \wedge \beta_{m+1} + \frac{1}{m+1}\varrho^{m+1}\bar{\partial}\beta_{m+1} ,$$

we obtain

$$u - \bar{\partial}\left(\varrho\beta_1 + \frac{1}{2}\varrho^2\beta_2 + \dots + \frac{1}{m+1}\varrho^{m+1}\beta_{m+1}\right) = \varrho^{m+1}\gamma_{m+1}$$

with $\gamma_{m+1} = \alpha_{m+1} - \frac{1}{m+1}\bar{\partial}\beta_{m+1}$. This completes the induction step.

For $k = 0, 1, 2, \dots$, set

$$h^{(k)} = \frac{k!}{k+1}\beta_{k+1}|_S ,$$

and note that $h^{(k)} \in C^\infty_{(p,q-1)}(S)$. According to Proposition 2.2, there exists a form $f \in C^\infty_{(p,q-1)}$ such that

$$\text{(2.2.1)} \qquad \frac{\partial^k}{\partial\varrho^k} f|_S = h^{(k)} \qquad (k = 0, 1, 2, \dots).$$

(In the case of compact supports, it is necessary to use Proposition 2.2′, and to observe that, from the above construction, it follows that the β_k, and hence the $h^{(k)}$, can all be chosen to have their supports contained in some fixed compact set of S). Let $v = \varrho f$. Then $v \in \mathcal{I}_{(p, q-1)}$, and near S, we have

$$v - \sum_{k=0}^{m-1} \frac{1}{k+1} \varrho^{k+1} \beta_{k+1}|_S = \varrho \left(f - \sum_{k=0}^{m-1} \frac{1}{k!} \varrho^k \frac{\partial^k}{\partial \varrho^k} f|_S \right) = 0\,(\varrho^{m+1}).$$

Therefore

$$u - \bar\partial v = \varrho^m \gamma_m - \bar\partial \left(v - \sum_{k=0}^{m-1} \frac{1}{k+1} \varrho^{k+1} \beta_{k+1}|_S \right)$$

$$+ \bar\partial \left(\sum_{k=0}^{m-1} \frac{1}{k+1} \varrho^{k+1} \beta_{k+1} - \sum_{k=0}^{m-1} \frac{1}{k+1} \varrho^{k+1} \beta_{k+1}|_S \right)$$

$$= 0\,(\varrho^m),$$

as $\varrho \to 0$, where the last term on the right is $0\,(\varrho^m)$ because each $\beta_{k+1} - \beta_{k+1}|_S$ vanishes to infinite order on S. As this holds for any m, we have $(u - \bar\partial v)|_S = 0^\infty$, and the proof of the lemma is complete.

DEFINITION. (*a*) A form $u \in C^\infty_{(p,q)}(U)$ is said to be a *distinguished representative* of a class $[u] \in H^{pq}(S)$ if $\bar\partial u|_S = 0^\infty$.

(*b*) A form $u \in \mathcal{I}_{(p,q)}(U)$ (or $u \in \mathcal{I}_{(p,q)}(U^\pm)$) is said to be a *distinguished representative* of a class $[u] \in H^{pq}(U, \mathcal{I})$ (or $[u] \in H^{pq}(U^\pm, \mathcal{I})$) if $u|_S = 0^\infty$.

Replacing $C, \mathcal{I}, H$ in the above definition by $C_0, \mathcal{I}_0, H_k$, we obtain the corresponding definition in the case of compact supports.

LEMMA 2.2. *Every class in $H^{pq}(S)$ has a distinguished representative.*

PROOF. Consider any class $[u] \in H^{pq}(S)$ represented by some $u \in C^\infty_{(p,q)}(U)$. Then $\bar\partial u \in \mathcal{I}_{(p,q+1)}(U)$ and $\bar\partial(\bar\partial u) \equiv 0$. Since $q + 1 \geq 1$, we can apply Lemma 2.1 to $\bar\partial u$. Hence there is a $v \in \mathcal{I}_{(p,q)}(U)$ such that $(\bar\partial u - \bar\partial v)|_S = 0^\infty$. Let $u' = u - v$. Then $[u'] = [u]$ in $H^{pq}(S)$, and u' is the distinguished representative required by the lemma.

LEMMA 2.3. *Every class in $H^{pq}(U, \mathcal{I})$ (or $H^{pq}(U^\pm, \mathcal{I})$) has a distinguished representative.*

PROOF. Let $[u] \in H^{pq}(U, \mathcal{I})$ be a class represented by $u \in \mathcal{I}_{(p,q)}(U)$. Then $\bar\partial u \equiv 0$, and, since we need only consider the case $q \geq 1$, Lemma 2.1 is again applicable. Hence there exists a $v \in \mathcal{I}_{(p,q-1)}(U)$ such that $(u - \bar\partial v)|_S = 0^\infty$. Let $u' = u - \bar\partial v$. Then $[u'] = [u]$ in $H^{pq}(U, \mathcal{I})$, and u' is a distinguished representative. The proof for $H^{pq}(U^\pm, \mathcal{I})$ is identical.

REMARK. Lemmas 2.2 and 2.3 also hold for the case of compact supports, with the same proof.

§ 3. The Riemann-Hilbert Problem.

3.1. *A Mayer-Vietoris sequence.* The purpose of this section is to derive an exact sequence that relates the cohomology of $U^+ \cup U^- = U$ and the cohomology of $U^+ \cap U^- = S$ to that of U^+ and U^-, and which resolves the additive Riemann-Hilbert problem in U whenever U is Stein. This is the analogue of the Mayer-Vietoris sequence for cohomology; however, we must give a proof from first principles since our cohomology groups are not the standard ones. We also wish to make explicit all of the maps involved. First, we define these maps and then prove a lemma showing that they are well-defined.

DEFINITION. For any p and for $q = 0, 1, \dots$, the maps α_q, β_q and γ_q are defined as follows:

(*a*) $\alpha_q : H^{pq}(U) \to H^{pq}(U^+) \oplus H^{pq}(U^-)$ is *the restriction from* U *to* U^+ *and* U^-. Consider a class $[u] \in H^{pq}(U)$ represented by u. Set $u^{\pm} = u|_{U^{\pm}}$, and let $[u^{\pm}]$ be the class in $H^{pq}(U^{\pm})$ represented by $u^{\pm}$. Then

$$\alpha_q([u]) = [u^+] \oplus [u^-] \in H^{pq}(U^+) \oplus H^{pq}(U^-).$$

(*b*) $\beta_q : H^{pq}(U^+) \oplus H^{pq}(U^-) \to H^{pq}(S)$ is *the jump across* S. Consider a class $[u^+] \oplus [u^-] \in H^{pq}(U^+) \oplus H^{pq}(U^-)$ represented by u^+ and u^-. Set $u_0 = u^+|_S - u^-|_S$, and let $[\tilde{u}_0]$ be the class in $H^{pq}(S)$ represented by $\tilde{u}_0$, where $\tilde{u}_0$ is any C^∞ extension of u_0. Then

$$\beta_q([u^+] \oplus [u^-]) = [\tilde{u}_0] \in H^{pq}(S).$$

(*c*) $\gamma_q : H^{pq}(S) \to H^{p,q+1}(U)$ *is defined as follows*: Consider a class $[u] \in H^{pq}(S)$ and for it a *distinguished* representative u. Set

$$u_1 = \begin{cases} -\bar{\partial} u, & U^+ \\ +\bar{\partial} u, & U^-, \end{cases}$$

and let $[u_1]$ be the class in $H^{p,q+1}(U)$ represented by u_1. Then

$$\gamma_q([u]) = [u_1] \in H^{p,q+1}(U).$$

LEMMA 3.1. *The maps* α_q , β_q *and* γ_q *are well defined homomorphisms.*

PROOF. (*a*) Since α_q is defined by restriction, it is clearly a map of classes.

(*b*) Consider β_q . At the beginning of Section 1.5, it was shown that $\bar\partial \tilde u_0 \in \mathcal{J}_{(p,q+1)}(U)$, so $\tilde u_0$ does represent a class in $H^{pq}(S)$. That class is independent of which C^∞ extension of u_0 is taken because the difference between any two C^∞ extensions of the same u_0 belongs to $\mathcal{J}_{(p,q)}(U)$. To see that β_q is a map of classes, consider any other representatives w^+ and w^- such that $w^\pm - u^\pm = \bar\partial v^\pm$ in $U^\pm$, and set $w_0 = w^+|_S - w^-|_S$. According to the above, we are free to choose convenient extensions $\tilde u_0$ and $\tilde w_0$ in order to prove that $[\tilde u_0] = [\tilde w_0]$ in $H^{pq}(S)$. Choose C^∞ extensions $\tilde v^\pm$ of $v^\pm$ and $\tilde u^\pm$ of $u^\pm$, and define $\tilde w^\pm$ by $\tilde w^\pm - \tilde u^\pm = \bar\partial \tilde v^\pm$. Letting $\tilde u_0 = \tilde u^+ - \tilde u^-$ and $\tilde w_0 = \tilde w^+ - \tilde w^-$ in U, we have $\tilde w_0 - \tilde u_0 - \bar\partial(\tilde v^+ - \tilde v^-) \equiv 0$ in U. Hence $[\tilde u_0] = [\tilde w_0]$.

(*c*) Finally, consider γ_q . Since $\bar\partial u|_S = 0^\infty$ for a distinguished representative, it follows that $u_1 \in C^\infty_{(p,q+1)}(U)$ and $\bar\partial u_1 = 0$ in U; hence u_1 does represent a class in $H^{p,q+1}(U)$. First, take the case $q = 0$. Let w be any other distinguished representative which represents the same class as u, and set

$$w_1 = \begin{cases} -\bar\partial w, & U^+ \\ +\bar\partial w, & U^-. \end{cases}$$

Since we have $w - u \in \mathcal{J}_{(p,0)}(U)$ and $\bar\partial(w-u)|_S = 0^\infty$, it follows from part (i) of Lemma 2.1 that $(w-u)|_S = 0^\infty$. Let

$$\psi = \begin{cases} -(w-u), & U^+ \\ +(w-u), & U^-. \end{cases}$$

Then $\psi \in C^\infty_{(p,0)}(U)$, and we obtain $w_1 - u_1 = \bar\partial\psi$ in U. This shows that $[u_1] = [w_1]$ in $H^{p1}(U)$, and completes the proof for the case $q = 0$. Now take the case $q \geq 1$. Again, let w be any other distinguished representative of the same class as u, and define w_1 as above. This time, we have $w - u - \bar\partial\varphi \in \mathcal{J}_{(p,q)}(U)$ for some $\varphi \in C^\infty_{(p,q-1)}(U)$ and $\bar\partial(w - u - \bar\partial\varphi)|_S = 0^\infty$. Hence by part (ii) of Lemma 2.1, there is a $v \in \mathcal{J}_{(p,q-1)}(U)$ such that $(w - u - \bar\partial\varphi - \bar\partial v)|_S = 0^\infty$. Define

$$\psi = \begin{cases} -(w - u - \bar\partial\varphi - \bar\partial v), & U^+ \\ +(w - u - \bar\partial\varphi - \bar\partial v), & U^-. \end{cases}$$

Then $\psi \in C^{\infty}_{(p,q)}(U)$, and we obtain $w_1 - u_1 = \bar{\partial}\psi$ in U. Thus $[u_1] = [w_1]$. γ_q is a map of classes, and the proof of the lemma is complete.

THEOREM 1. *The Mayer-Vietoris sequence*

$$0 \longrightarrow H^{p0}(U) \xrightarrow{\alpha_0} H^{p0}(U^+) \oplus H^{p0}(U^-) \xrightarrow{\beta_0} H^{p0}(S) \xrightarrow{\gamma_0} H^{p1}(U) \xrightarrow{\alpha_1}$$

$$\cdots \xrightarrow{\gamma_{q-1}} H^{pq}(U) \xrightarrow{\alpha_q} H^{pq}(U^+) \oplus H^{pq}(U^-) \xrightarrow{\beta_q} H^{pq}(S) \xrightarrow{\gamma_q} H^{p,q+1}(U) \xrightarrow{\alpha_{q+1}} \cdots$$

is exact.

PROOF. *Exactness at $H^{pq}(U)$.* When $q = 0$, this is trivial because α_0 is obviously injective. Therefore, consider the case $q \geq 1$.

(*a*) $\alpha_q \circ \gamma_{q-1} = 0$: Let $[u] \in H^{p,q-1}(S)$ and $[u_1] = \gamma_{q-1}([u])$ be as in the definition of γ_{q-1}. Set $u^{\pm} = u\,|_{U^{\pm}}$ and $u_1^{\pm} = u_1\,|_{U^{\pm}}$. Then $[u_1^+] \oplus [u_1^-] = \alpha_q \circ \gamma_{q-1}([u])$. But $u_1^+ = \bar{\partial}(-u^+)$ on U^+, and $u_1^- = \bar{\partial}(u^-)$ on U^-; therefore, $[u_1^+] \oplus [u_1^-]$ is the zero class. (*b*) $\alpha_q([u_1]) = 0 \Longrightarrow [u_1] = \gamma_{q-1}([u])$ for some $[u] \in H^{p,q-1}(S)$: $\alpha_q([u_1]) = 0$ means that there is a ψ^+ such that $u_1 = \bar{\partial}\psi^+$ on U^+ and that there is a ψ^- such that $u_1 = \bar{\partial}\psi^-$ on U^-. We may assume that ψ^+ and ψ^- have been extended and belong to $C^{\infty}_{(p,q-1)}(U)$. Define $u = \frac{1}{2}\psi^- - \frac{1}{2}\psi^+$ and $\varphi = \frac{1}{2}\psi^+ + \frac{1}{2}\psi^-$. Observe that $u_1 - \bar{\partial}\psi^- \equiv 0$ on U^- and $u_1 - \bar{\partial}\psi^+ \equiv 0$ on U^+; hence their difference vanishes to infinite order on S, which means that $\bar{\partial}u\,|_S = 0^{\infty}$. So u is the distinguished representative of a certain class $[u]$ in $H^{p,q-1}(S)$. But $\varphi \in C^{\infty}_{(p,q-1)}(U)$, and an easy calculation shows that

$$u_1 - \bar{\partial}\varphi = \begin{cases} -\bar{\partial}u, & U^+ \\ +\bar{\partial}u, & U^-. \end{cases}$$

Therefore, $[u_1] = \gamma_{q-1}([u])$.

Exactness at $H^{pq}(U^+) \oplus H^{pq}(U^-)$. When $q = 0$, this is trivial: A holomorphic function in U has no jump across S; two holomorphic functions, one in $\mathring{U}^+$ and one in $\mathring{U}^-$, which are smooth up to S and agree there, form together a holomorphic function in U. Therefore, consider the case $q \geq 1$. (*a*) $\beta_q \circ \alpha_q = 0$. Again, this is trivial since the jump $u_0 = u^+|_S - u^-|_S$ is zero whenever u^+ and u^- are the restrictions of a $u \in C^{\infty}_{(p,q)}(U)$; hence the class $[\tilde{u}_0]$ is zero in $H^{pq}(S)$. (*b*) $\beta_q([u^+] \oplus [u^-]) = 0 \Longrightarrow [u^+] \oplus [u^-] = \alpha_q([u])$ for some $[u] \in H^{pq}(U)$: We may assume that u^+ and u^- have been extended to be in $C^{\infty}_{(p,q)}(U)$, $\tilde{u}_0 = u^+ - u^-$. Then the hypothesis $\beta_q([u^+] \oplus [u^-]) = 0$ means that $\tilde{u}_0 - \bar{\partial}\varphi \in \mathcal{J}_{(p,q)}(U)$ for some $\varphi \in C^{\infty}_{(p,q-1)}(U)$. We have $\bar{\partial}(\tilde{u}_0 - \bar{\partial}\varphi)|_S = 0^{\infty}$, so by Lemma 2.1, there is a $v \in \mathcal{J}_{(p,q-1)}(U)$ such that $(\tilde{u}_0 - \bar{\partial}\varphi - \bar{\partial}v)|_S = 0^{\infty}$. Define $u'^- = u^- + \bar{\partial}\varphi + \bar{\partial}v$ and

$$u = \begin{cases} u^+ , & U^+ \\ u'^- , & U^- . \end{cases}$$

Since $(u^+ - u'^-)|_S = 0^{\infty}$, we observe that $u \in C^{\infty}_{(p,q)}(U)$; hence u represents a class $[u]$ in $H^{pq}(U)$. But $[u'^-] = [u^-]$ in $H^{pq}(U^-)$, and consequently, $[u^+] \oplus [u^-] = \alpha_q([u])$.

Exactness at $H^{pq}(S)$. (*a*) $\gamma_q \circ \beta_q = 0$: Let $[\tilde{u}_0] = \beta_q([u^+] \oplus [u^-])$. As above, we can assume that $\tilde{u}_0 = u^+ - u^-$ where u^+ and u^- have been extended to $C^{\infty}_{(p,q)}(U)$. Then $\bar{\partial}\tilde{u}_0|_S = 0^{\infty}$, so $\tilde{u}_0$ is a distinguished representative. We have $[u_1] = \gamma_q([\tilde{u}_0])$ where

$$u_1 = \begin{cases} -\bar{\partial}\tilde{u}_0 , & U^+ \\ +\bar{\partial}\tilde{u}_0 , & U^- . \end{cases}$$

But $[u_1] = 0$ in $H^{p,q+1}(U)$ because $u_1 = \bar{\partial}(u^+ + u^-)$ in U.

(*b*) $\gamma_q([u]) = 0 \Longrightarrow [u] = \beta_q([u^+] \oplus [u^-])$ for some $[u^+] \oplus [u^-] \in H^{pq}(U^+) \oplus H^{pq}(U^-)$: With u a distinguished representative of $[u]$ and u_1 as in the definition of γ_q, the hypothesis $\gamma_q([u]) = 0$ means that $u_1 = \bar{\partial}\psi$ for some $\psi \in C^{\infty}_{(p,q)}(U)$. Define $u^+ = \frac{1}{2}(u + \psi)$ and $u^- = \frac{1}{2}(\psi - u)$. Then $\bar{\partial}u^+ = 0$ on U^+, $\bar{\partial}u^- = 0$ on U^-, and $u^+ - u^- = u$. Thus u^+ represents a class $[u^+]$ in $H^{pq}(U^+)$, u^- represents a class $[u^-]$ in $H^{pq}(U^-)$, and $u = \beta_q([u^+] \oplus [u^-])$.

The proof of Theorem 1 is complete.

3.2. *The additive Riemann-Hilbert problem.* When $p = q = 0$, the additive Riemann-Hilbert problem is the problem of finding a function u^+ holomorphic in $\mathring{U}^+$ and a function u^- holomorphic in $\mathring{U}^-$, with u^+ and u^- smooth up to S, such that their jump across S is equal to a prescribed function u_0.

More generally, consider the same problem for arbitrary p and q: Suppose a form $u_0 \in C^{\infty}_{(p,q)}(S)$ is given. The problem is to find forms $u^{\pm} \in C^{\infty}_{(p,q)}(U^{\pm})$

with $\bar{\partial} u^{\pm} = 0$ on $U^{\pm}$ such that $u^{+}|_S - u^{-}|_S = u_0$. At the beginning of Section 1.5, we derived the necessary condition that $\bar{\partial} \tilde{u}_0 \in \mathcal{J}_{(p,q+1)}(U)$ for any C^∞ extension $\tilde{u}_0$, and observed that this condition is independent of which C^∞ extension is taken. Thus any such form u_0 for which the problem is possible determines a unique boundary cohomology class $[\tilde{u}_0] \in H^{pq}(S)$ represented by an arbitrary extension $\tilde{u}_0 \in C^\infty_{(p,q)}(U)$. Suppose $[\tilde{u}_0]$ is the image, under the map β_q from the Mayer-Vietoris sequence, of a class $[u^+] \oplus [u^-] \in H^{pq}(U^+) \oplus H^{pq}(U^-)$ with representatives u^+ and u^- as indicated. Then $[\tilde{u}_0] = [\tilde{u}'_0]$ where $\tilde{u}'_0$ is some extension of $u'_0 = u^+|_S - u^-|_S$. This means that there is a $v \in C^\infty_{(p,q-1)}(U)$ such that $w = \tilde{u}_0 - \tilde{u}'_0 - \bar{\partial} v \in \mathcal{J}_{(p,q)}(U)$ (the special case $q = 0$ is even easier because there is no v). Writing $w = \varrho\alpha + \bar{\partial}\varrho \wedge \beta$ and $v' = v + \varrho\beta$, we observe that $\tilde{u}_0 - \tilde{u}'_0 - \bar{\partial} v' = \varrho(\alpha - \bar{\partial}\beta)$; hence $u_0 = u'_0 + \bar{\partial} v'|_S$. Replace u^+ by $u'^+ = u^+ + \bar{\partial} v'|_{U^+}$. Then $[u^+] = [u'^+]$, and we have $u'^+|_S - u^-|_S = u_0$ with $\bar{\partial} u'^+ = 0$ on U^+ and $\bar{\partial} u^- = 0$ on U^-.

We shall formulate the additive Riemann-Hilbert problem directly in terms of cohomology classes; it follows from the preceding discussion that nothing is lost at the level of forms with such a formulation. Thus the problem is: *Given a boundary cohomology class $\xi_0 \in H^{pq}(S)$, it is required to find Cauchy cohomology classes $\xi^{\pm} \in H^{pq}(U^{\pm})$ such that $\beta_q(\xi^+ \oplus \xi^-) = \xi_0$.* We say the problem has a unique solution if ξ^+ and ξ^- are uniquely determined by ξ_0.

The following theorem states that the additive Riemann-Hilbert problem for cohomology classes is always solvable in U if U is a Stein manifold, and that the solution is unique for $q \geq 1$.

THEOREM 2. *Let U be a Stein manifold. Then*

(i) *the sequence*

$$0 \longrightarrow H^{p0}(U) \xrightarrow{\alpha_0} H^{p0}(U^+) \oplus H^{p0}(U^-) \xrightarrow{\beta_0} H^{p0}(S) \longrightarrow 0$$

is exact, and for $q \geq 1$, we have the isomorphisms

$$\text{(ii)} \qquad H^{pq}(U^+) \oplus H^{pq}(U^-) \xrightarrow[\sim]{\beta_q} H^{pq}(S).$$

PROOF. If U is a Stein manifold, we have $H^{pq}(U) = 0$ for $q \geq 1$; the result follows as a corollary of Theorem 1.

REMARK. In an arbitrary U, the Mayer-Vietoris sequence also gives a precise result: For a given $\xi_0 \in H^{pq}(S)$, the Riemann Hilbert problem is solvable if and only if the image of ξ_0 under the map γ_q is the zero class in $H^{p,q+1}(U)$; the solution is unique modulo the image of $H^{pq}(U)$ by α_q. In particular, part (ii) above holds if only $H^{pq}(U) = H^{p,q+1}(U) = 0$.

3.3. *Solution of Riemann-Hilbert barrier relations.* In one complex variable, the classical Riemann-Hilbert problem is a multiplicative problem that consists of solving certain homogeneous or inhomogeneous barrier relations. However, in the standard theory (see for example [7]), the multiplicative problem is always reduced to the additive one by taking logarithms. The same reduction can be used together with the results of the previous section to solve analogous barrier relations in several complex variables. Here we indicate how this is done in the case of functions ($p = q = 0$) by treating a simple situation.

Consider functions $G, g \in C^\infty(S)$ with $G \neq 0$ everywhere on S, and assume that G and g satisfy the tangential Cauchy-Riemann equations on S. This can be expressed by $G, g \in H^{00}(S)$ if we regard G and g as equivalence classes of C^∞ extensions to U. The *homogeneous barrier problem* is the problem of finding functions $h^\pm$ holomorphic in $\overset{\circ}{U}{}^\pm$, smooth up to S, with $h^\pm \neq 0$ on $U^\pm$, and such that

$$h^+|_S = G \cdot h^-|_S. \tag{3.2.1}$$

The *inhomogeneous barrier problem* asks for functions $f^\pm$ holomorphic in $\overset{\circ}{U}{}^\pm$, smooth up to S, and such that

$$f^+|_S = G \cdot f^-|_S + g. \tag{3.2.2}$$

We assume that $H^{01}(U) = 0$, so that part (i) of Theorem 2 is valid with $p = 0$. Moreover, we assume the situation is such that G has a well-defined single-valued logarithm. Then (3.2.1) and (3.2.2) can be solved in U: Let $u_0 = \log G$, and note that u_0 also satisfies the tangential Cauchy-Riemann equations on S. According to part (i) of Theorem 2, the map β_0 is onto; hence there are functions $u^\pm \in H^{00}(U^\pm)$ such that $u^+|_S - u^-|_S = u_0$. The required solution of (3.2.1) is then given by $h^\pm = e^{u^\pm}$. Next, let $v_0 = \dfrac{g}{h^+|_S}$, and observe that v_0 again satisfies the tangential Cauchy-Riemann equations on S. Therefore, we can find functions $v^\pm \in H^{00}(U^\pm)$ such that $v^+|_S - v^-|_S = v_0$. It is easily verified that the functions $f^\pm = h^\pm v^\pm$ provide a solution of (3.2.2).

3.4. *Mayer-Vietoris sequence for compact supports.* The considerations of Section 3.1, including the proof of exactness of the Mayer-Vietoris sequence, carry over without change to the case of compact supports.

THEOREM 3. *The sequence*

$$0 \longrightarrow H_k^{p0}(U) \xrightarrow{\alpha_0} H_k^{p0}(U^+) \oplus H_k^{p0}(U^-) \xrightarrow{\beta_0} H_k^{p0}(S) \xrightarrow{\gamma_0} H_k^{p1}(U) \xrightarrow{\alpha_1}$$

$$\dots \xrightarrow{\gamma_{q-1}} H_k^{pq}(U) \xrightarrow{\alpha_q} H_k^{pq}(U^+) \oplus H_k^{pq}(U^-) \xrightarrow{\beta_q} H_k^{pq}(S) \xrightarrow{\gamma_q} H_k^{p,q+1}(U) \xrightarrow{\alpha_{q+1}} \dots$$

is exact.

REMARK 1. In the above we have $H_k^{p0}(U) = 0$, except for the special case where M is compact and $U = M$.

REMARK 2. If U is q-complete, then $H_k^{rs}(U) = 0$ for all r and $s < n - q$ (see [2]). Here U is said to be q-complete if there is a real function $\varphi \in C^\infty(U)$ such that

(*a*) $$B_c = \{z \in U \mid \varphi(z) < c\} \subset\subset U \quad \text{for all } c,$$

and

(*b*) at each point $z_0 \in U$ the Levi form

$$\mathcal{L}(\varphi)(z_0) = \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \, \partial \bar{z}_k}(z_0)\, w_j \bar{w}_k$$

has at least $n - q$ positive eigenvalues.

In particular $H_k^{rs}(U) = 0$ for $s < n$ if $n > 1$ and U is Stein.

REMARK 3. Suppose $H^{p1}(U) = 0$ and let S be such that neither U^+ nor U^- has any compact components. Then from Theorem 3 we obtain that $H_k^{p0}(S) = 0$.

It is straightforward to write down the precise analogue of Theorem 2. However, in the case of compact supports we obtain a more interesting analogue of the additive Riemann-Hilbert problem by looking at the following situation:

THEOREM 4. *Assume S and U^- are compact.*

(i) *If $H_k^{p1}(U) = 0$ and U^+ has no compact components, then we have an isomorphism* $H^{p0}(U^-) \xrightarrow[\sim]{r_0} H^{p0}(S)$.

(ii) *If* $H_k^{pq}(U) = H_k^{p,q+1}(U) = 0$ *and* $H_k^{pq}(U^+) = 0$, *then we have an isomorphism* $H^{pq}(U^-) \xrightarrow[\sim]{r_q} H^{pq}(S)$.

The maps r_0, r_q *represent restriction from* U^- *to* S.

PROOF. The maps r_0, r_q differ from β_0, β_q only by a minus sign. Part (ii) is immediate from Theorem 3; $H_k^{pq}(S) = H^{pq}(S)$ and $H_k^{pq}(U^-) = H^{pq}(U^-)$ because S and U^- are compact. For Part (i) it is sufficient to observe that $H_k^{p0}(U^+) = 0$ and $H_k^{p0}(U) = 0$. The latter follows from the fact that if M were compact and $U = M$, then U^+ would have to be closed in M and hence compact.

Part (i) above provides a generalization to manifolds of a well known theorem in $\mathbb{C}^n$ that is substantially due to Bochner [4] (see also Martinelli [15], [16] and Fichera [6]). The problem of Bochner is similar to the Dirichlet problem: Let U^- be a *compact* region on a complex manifold M with smooth boundary ∂U^-. Given a smooth function on ∂U^- which satisfies the tangential Cauchy-Riemann equations on ∂U^-, the problem is to extend it to a smooth function on U^- which is holomorphic in $\overset{\circ}{U}{}^-$. We say the Bochner problem is solvable if such an extension is always possible. Set $U = M$, $S = \partial U^-$, $U^+ = M - \overset{\circ}{U}{}^-$ and $p = 0$ in Part (i) above. Since r_0 is the restriction from U^- to U^-, we obtain:

COROLLARY 4.1. *The Bochner problem is solvable if* $H_k^{01}(M) = 0$ *and* U^+ *has no compact components.*

The classical result in $\mathbb{C}^n$ follows easily: by using the Cauchy integral formula it is easy to show that $H_k^{01}(\mathbb{C}^n) = 0$ if $n > 1$.

According to Remark 2 above, $H_k^{01}(M) = 0$ if M is $(n-2)$-complete or, less generally, if M is Stein and $n > 1$. For M Stein one can also obtain $H_k^{01}(M) = 0$ by Serre duality [17]: $H_k^{01}(M)$ is dual to $H^{n,n-1}(M) = 0$.

COROLLARY 4.2. *On an* $(n-2)$*-complete manifold* M *(in particular if* $n > 1$ *and* M *is Stein) the Bochner problem is always solvable provided* U^+ *has no compact components.*

The interesting thing about the above results is that they involve only global hypotheses on M and U^-, and do not depend on the « shape » of U^-; i. e., the local Levi convexity of ∂U^- does not enter. In [10] Kohn and Rossi remarked that the generalization of Bochner's theorem to complex manifolds presents some difficulties: They gave the example $M = \mathbb{C}^1 \times \mathbb{P}^1$, $U^- = \{|z_1| \leq 1\} \times \mathbb{P}^1$. Any function independent of the second factor satisfies the tangential Cauchy-Riemann equations on ∂U^-; but if such a function is not identically zero, and it vanishes on an open set in ∂U^-, then it cannot be extended. Note that for this example $H_k^{01}(M) \neq 0$ because $H_k^{01}(\mathbb{C}^1) \neq 0$ (in fact, they are infinite-dimensional).

However, by imposing a convexity condition on ∂U^-, Kohn and Rossi [10], proved the following extension theorem for an arbitrary M:

COROLLARY 4.3. *Let U^- be compact and $S = \partial U^-$ be connected. At every point of S assume that the Levi form $\mathcal{L}(\varrho)|_{HT}$ (the Levi form restricted to the holomorphic tangent space — see (4.5.1) for a definition) has at least one positive eigenvalue. Then the Bochner problem is solvable in U^-.*

PROOF. By using Theorem 3 and imitating part of the argument of Kohn-Rossi we obtain a simpler proof of their result: The hypothesis about the Levi convexity of S means that there is a sufficiently small open neighhorhood U of U^- which is $(n-2)$-convex (4). From the $(n-2)$ convexity of U it follows (see [2]) that

$$\dim_{\mathbb{C}} H_k^{01}(U) < \infty.$$

Since U^+ has no compact components, we obtain from Theorem 3 an exact sequence

$$0 \longrightarrow H^{00}(U^-) \xrightarrow{r_0} H^{00}(S) \xrightarrow{\gamma_0} H_k^{01}(U).$$

It remains to show that the finite dimensional space $H_k^{01}(U)$ does not obstruct the surjectivity of r_0.

If every element of $H^{00}(S)$ is constant, then the corollary is trivial. Otherwise let $[u] \in H^{00}(S)$ be such that u is not constant on S, and $\gamma_0([u]) = [u_1] \neq 0$, where u and u_1 are as in the definition of γ_0. Let $P(u)$ be any polynomial with complex coefficients. Then $P(u)$ represents a class in $H^{00}(S)$ and $\gamma_0([P(u)]) = [P'(u)\, u_1] \in H_k^{01}(U)$. If $\dim_{\mathbb{C}} H_k^{01}(U) = m$, then $\{\gamma_0([1]), \gamma_0([u]), \gamma_0([u^2]), \dots, \gamma_0([u^m])\}$ are linearly dependent. From $\sum_{j=1}^{m} a_j \gamma_0([u^j]) = 0$ it follows that $\gamma_0([P(u)])$ is the zero class in $H_k^{01}(U)$, where P is the polynomial $P(u) = \sum_{j=1}^{m} a_j u^j$. We can assume $a_m \neq 0$. By exactness it follows that there is a holomorphic function $H \in H^{00}(U^-)$ such that $H|_S = P(u)|_S$.

---

(4) U is said to be q-convex if there exists a function $\varphi \in C^\infty(U)$ such that part (a) under Remark 2 holds, and part (b) holds in $U - K$, where K is some compact subset of U. Here it suffices to choose φ of the form $\varphi = \mu(e^{\lambda\varrho} - 1)$ where $\lambda > 0$ is sufficiently large and $\mu(t)$ is a strictly increasing convex function.

H is not constant because $u|_S$ is not constant. Each $H^j u$ represents a class in $H^{00}(S)$ and $\gamma_0([P_1(H)u]) = [P_1(H)u_1]$ for any polynomial P_1. Applying the same arguments as above to $\{\gamma_0([u]), \gamma_0([Hu]), \gamma_0([H^2u]), \dots, \gamma_0([H^m u])\}$, we conclude that there is a polynomial $P_1(H) = \sum_{j=1}^{m} b_j H^j$ and a holomorphic function $F \in H^{00}(U^-)$ such that $F|_S = \{P_1(H)u\}|_S$. Set $G = P_1(H)$. Then $G^m H$ and $G^m P(F/G)$ must coincide because they belong to $H^{00}(U^-)$ and have the same boundary values. Hence $P(F/G) \equiv H$ on $U^- - \{G = 0\}$ and it follows that F/G is holomorphic and locally bounded there. Since $G \not\equiv 0$ because H is not constant, the Riemann extension theorem implies that $F/G \in H^{00}(U^-)$ is the desired extension of u. The proof of the corollary is complete.

In the next section we treat the case of general U^- and S; so the Bochner problem is also included. However, the situation we wish to emphasize there is the case where U^- and S are not compact.

§ 4. The Cauchy Problem.

4.1. *The inhomogeneous Cauchy problem.* Consider the general Cauchy problem for $\bar\partial$ in U^-: Given $f \in C^\infty_{(p,q)}(U^-)$ and $u_0 \in C^\infty_{(p,q)}(S)$, the problem is to find $u \in C^\infty_{(p,q)}(U^-)$ such that

$$(4.1.1) \qquad \begin{aligned} \bar\partial u &= f \quad \text{in} \quad U^-, \\ u|_S &= u_0 . \end{aligned}$$

Clearly some compatibility conditions are necessary: we must have $\bar\partial f = 0$ in U^- and $f - \bar\partial \tilde u_0 \in \mathcal{I}_{(p,q+1)}(U^-)$ for any C^∞ extension $\tilde u_0$ of u_0. Problem (4.1.1) can be written equivalently as

$$(4.1.2) \qquad \begin{aligned} \bar\partial u' &= f' \quad \text{in} \quad U^-, \\ u'|_S &= 0 , \end{aligned}$$

where $u' = u - \tilde u_0$ and $f' = f - \bar\partial \tilde u_0$. Suppose a u'' can be found such that

$$(4.1.3) \qquad \begin{aligned} \bar\partial u'' &= f' \quad \text{in} \quad U^-, \\ u'' &\in \mathcal{I}_{(p,q)}(U^-). \end{aligned}$$

If $q = 0$, then (4.1.2) and (4.1.3) are the same. If $q \geq 1$, we have $u'' = \varrho\alpha + \bar{\partial}\varrho \wedge \beta$ and $u' = u'' - \bar{\partial}(\varrho\beta) = \varrho(\alpha - \bar{\partial}\beta)$ will be a solution of (4.1.2). The existence of such a u'' means that f' represents the zero cohomology class in $H^{p,q+1}(U^-, \mathcal{I})$. Note that the class $[f'] \in H^{p,q+1}(U^-, \mathcal{I})$ represented by f' does not depend on which extension $\tilde{u}_0$ of u_0 is used. Therefore any prescription of compatible Cauchy data f and u_0 in (4.1.1) determines a unique cohomology class $[f'] \in H^{p,q+1}(U^-, \mathcal{I})$; a solution u of (4.1.1) exists if and only if $[f'] = 0$. In particular, we have proved

PROPOSITION 4.1. *The existence of a solution $u \in C^\infty_{(p,q)}(U^-)$ to (4.1.1) for all compatible data $f \in C^\infty_{(p,q+1)}(U^-)$ and $u_0 \in C^\infty_{(p,q)}(S)$ is equivalent to* $H^{p,q+1}(U^-, \mathcal{I}) = 0$.

When $q = 0$ solutions of (4.1.1) are unique. Suppose $q \geq 1$ and let $w = u_2 - u_1$, where u_1 and u_2 are two solutions of (4.1.1). Then

$$\bar{\partial} w = 0 \quad \text{in} \quad U^-, \qquad w|_S = 0, \tag{4.1.4}$$

so w represents a class in $H^{pq}(U^-, \mathcal{I})$. Conversely any cohomology class in $H^{pq}(U^-, \mathcal{I})$ has a representative w that satisfies (4.1.4). This suggests we should identify two solutions u_1 and u_2 if their difference is cohomologous to zero in $H^{pq}(U^-, \mathcal{I})$. Then the set of solutions to (4.1.1) is partitioned into equivalence classes, and the uniqueness question is shifted to the question of how many distinct equivalence classes of solutions there are. Thus we arrive at

PROPOSITION 4.2. *The solution of (4.1.1) is unique, in the sense that there is only one equivalence class of solutions, if and only if $H^{pq}(U^-, \mathcal{I}) = 0$.*

Obviously U^- can be replaced everywhere by U^+ in the above discussion. We can also replace U^- everywhere by U and consider a two-sided Cauchy problem in which Cauchy data are prescribed on S and a solution is sought in U. With these replacements Propositions 4.1 and 4.2 remain valid.

4.2. *The homogeneous Cauchy problem for cohomology classes.* Consider the homogeneous version of (4.1.1): Given $u_0 \in C^\infty_{(p,q)}(S)$ a solution $u \in C^\infty_{(p,q)}(U^-)$ is sought to

$$\bar{\partial} u = 0 \quad \text{in} \quad U^-, \qquad u|_S = u_0. \tag{4.2.1}$$

As we have seen the problem is possible only if $\bar{\partial}\tilde{u}_0 \in \mathcal{J}_{(p,q+1)}(U^-)$ for any C^∞ extension u_0 of u_0. Therefore any allowable assignment of Cauchy data determines a unique boundary cohomology class $[\tilde{u}_0] \in H^{pq}(S)$. Likewise any solution u determines a unique Cauchy cohomology class $[u] \in H^{pq}(U^-)$.

Definition. For any p and $q = 0, 1, 2, \ldots$ the map $r_q : H^{pq}(U^-) \to H^{pq}(S)$ is defined by *restriction from* U^- *to* S: Given any class $[u] \in H^{pq}(U^-)$ represented by u, $r_q([u]) = [\tilde{u}_0] \in H^{pq}(S)$ where $\tilde{u}_0$ is any C^∞ extension of $u_0 = u|_S$. It follows as in part (*b*) of the proof of Lemma 3.1 that r_q is a well-defined homomorphism. In the obvious way we also define the restriction maps $r_q : H^{pq}(U) \to H^{pq}(U^+)$ and $r_q : H^{pq}(U) \to H^{pq}(U^-)$.

Now we can formulate the Cauchy problem in U^- for cohomology classes: *Given a boundary cohomology class* $\xi_0 \in H^{pq}(S)$, *the problem is to find a Cauchy cohomology class* $\xi \in H^{pq}(U^-)$ *such that* $r_q(\xi) = \xi_0$.

Just as with the additive Riemann-Hilbert problem, nothing is lost at the level of forms by such a formulation in terms of cohomology classes.

Of course U^- can be replaced by U^+ and the same problem can be considered in U^+; in either case we have a one-sided Cauchy problem. Likewise U^- could be replaced by U and a two-sided Cauchy problem could be considered.

4.3. *The one-sided Cauchy problem.* The existence or the uniqueness of solutions to the Cauchy problem for cohomology classes in U^- is equivalent to the surjectivity or the injectivity, respectively, of the homomorphism

$$H^{pq}(U^-) \xrightarrow{r_q} H^{pq}(S). \tag{4.3.1}$$

These questions will be reduced to corresponding questions about the vanishing of certain cohomology groups.

Proposition 4.3. *We have the commutative diagram*

$$\begin{array}{ccccccccccc}
0 \to & H^{p0}(U) & \xrightarrow{r_0} & H^{p0}(U^+) & \xrightarrow{\bar{\partial}'_0} & H^{p1}(U^-, \mathcal{J}) & \xrightarrow{i'_1} & H^{p1}(U) & \xrightarrow{r_1} & \ldots \\
 & \downarrow r_0 & & \downarrow r_0 & & \downarrow & & \downarrow r_1 & & \\
0 \to & H^{p0}(U^-) & \xrightarrow{r_0} & H^{p0}(S) & \xrightarrow{\bar{\partial}_0} & H^{p1}(U^-, \mathcal{J}) & \xrightarrow{t_1} & H^{p1}(U^-) & \xrightarrow{r_1} & \ldots
\end{array}$$

$$
\begin{array}{ccccccccccc}
\cdots \to H^{pq}(U^-,\mathcal{I}) & \xrightarrow{i'_q} & H^{pq}(U) & \xrightarrow{r_q} & H^{pq}(U^+) & \xrightarrow{\bar\partial'_q} & H^{p,q+1}(U^-,\mathcal{I}) & \xrightarrow{i'_{q+1}} & H^{p,q+1}(U) \to \cdots \\
\downarrow & & \downarrow{\scriptstyle r_q} & & \downarrow{\scriptstyle r_q} & & \downarrow & & \downarrow{\scriptstyle r_{q+1}} \\
\cdots \to H^{pq}(U^-,\mathcal{I}) & \xrightarrow{i_q} & H^{pq}(U^-) & \xrightarrow{r_q} & H^{pq}(S) & \xrightarrow{\bar\partial_q} & H^{p,q+1}(U^-,\mathcal{I}) & \xrightarrow{i_{q+1}} & H^{p,q+1}(U^-) \to \cdots
\end{array}
$$

in which the rows are exact.

PROOF. The unlabeled vertical arrows represent the obvious isomorphisms. The homomorphisms i_q and $\bar\partial_q$ are induced by inclusion and by $\bar\partial$, respectively. The bottom row is the standard exact cohomology sequence in U^- associated with the short exact sequence

$$0 \to \bigoplus_q \mathcal{I}_{(p,q)}(U^-) \xrightarrow{i} \bigoplus_q C^\infty_{(p,q)}(U^-) \xrightarrow{j} \bigoplus_q Q_{(p,q)}(U^-) \to 0$$

of graded groups and allowable homomorphisms. The top row is motivated by an analogous standard exact cohomology sequence in U; that is, the one obtained by replacing $\mathcal{I}_{(p,q)}(U^-)$ by the ideal of forms in $C^\infty_{(p,q)}(U)$ which vanish identically on U^+. The maps i'_q and $\bar\partial'_q$ are defined as follows:

$i'_q : H^{pq}(U^-,\mathcal{I}) \to H^{pq}(U)$ is the *extension by zero of a distinguished representative* (here $q \geq 1$). Consider a class $[u^-] \in H^{pq}(U^-,\mathcal{I})$ with *distinguished* representative u^-. Then

$$i'_q([u^-]) = [u] \in H^{pq}(U),$$

where $u \in C^\infty_{(p,q)}(U)$ is defined by

$$u = \begin{cases} 0, & U^+ \\ u^-, & U^-. \end{cases}$$

$\bar\partial'_q : H^{pq}(U^+) \to H^{p,q+1}(U^-,\mathcal{I})$ is *the $\bar\partial$ of an extension to U^-*. Consider $[u^+] \in H^{pq}(U^+)$ represented by u^+ and let $\tilde{u}^+ \in C^\infty_{(p,q)}(U)$ be any extension of u^+. Then

$$\bar\partial'_q([u^+]) = [u^-] \in H^{p,q+1}(U^-,\mathcal{I}).$$

where $u^- = \bar\partial\,\tilde{u}^+|_{U^-}$.

These maps i'_q and $\bar\partial'_q$ are well-defined homomorphisms. We omit the proof because it is easy and is very similar to the proof of Lemma 3.1.

As for the exactness of the top row, we do the case $q > 0$ and leave the obvious changes in the proof for $q = 0$ to the reader :

Exactness at $H^{pq}(U^-, \mathcal{J})$. (*a*) $i'_q \circ \bar\partial'_{q-1} = 0$: Let $[u^-] \in H^{pq}(U^-, \mathcal{J})$ be such that $[u^-] = \bar\partial'_{q-1}([u^+])$ where $u^- = \bar\partial\, \tilde u^+|_{U^-}$, as in the definition of $\bar\partial'_{q-1}$. Since $\bar\partial\, \tilde u^+ = \bar\partial\, u^+ = 0$ on U^+, u^- is a distinguished representative of its class in $H^{pq}(U^-, \mathcal{J})$ and u^- can be extended by zero to U^+ without destroying the relation $u^- = \bar\partial\, \tilde u^+$. This means that $i'_q([u^-])$ is the zero class in $H^{pq}(U)$. (*b*) $i'_q([u^-]) = 0 \Longrightarrow [u^-] = \bar\partial'_{q-1}([u^+])$ for some $[u^+] \in H^{p,q-1}(U^+)$: $i'_q([u^-]) = 0$ implies that there is a $w \in C^\infty_{(p,q-1)}(U)$ such that $u^- = \bar\partial\, w$ in U where u^- is a distinguished representative that has been extended by zero to U. Then $u^+ = w|_{U^+}$ represents a class $[u^+] \in H^{p,q-1}(U^+)$ and $\bar\partial'_{q-1}([u^+]) = [u^-]$.

Exactness at $H^{pq}(U)$. (*a*) $r_q \circ i'_q = 0$: This is trivial because extension by zero to U^+ followed by restriction to U^+ gives zero. (*b*) $r_q([u]) = 0 \Longrightarrow [u] = i'_q([u^-])$ for some $[u^-] \in H^{pq}(U^-, \mathcal{J})$: If $r_q([u]) = 0$, then $u = \bar\partial\, w$ in U^+ for some $w \in C^\infty_{(p\ q-1)}(U^+)$. Set $u^- = u|_{U^-} - \bar\partial\, \tilde w|_{U^-}$ where $\tilde w \in C^\infty_{(p,q-1)}(U)$ is an extension of w. Then $\bar\partial\, u = 0$ in U^- and $u^-|_S = 0^\infty$. Hence u^- is a distinguished representative of some class $[u^-] \in H^{pq}(U^-, \mathcal{J})$, and the equation $u^- = u - \bar\partial\, \tilde w$ in U provides the extension of u^- by zero in U^+. Therefore $i'_q([u^-]) = [u]$.

Exactness at $H^{pq}(U^+)$. (*a*) $\bar\partial'_q \circ r_q = 0$: Consider a $[u] \in H^{pq}(U)$. First u is restricted to U^+ and then extended back to U^-; the extension can be taken to be u again. But $\bar\partial\, u = 0$ in U, so we obtain that $\bar\partial'_q \circ r_q([u])$ is the zero class in $H^{p,q+1}(U^-, \mathcal{J})$. (*b*) $\bar\partial'_q([u^+]) = 0 \Longrightarrow [u^+] = r_q([u])$ for some $[u] \in H^{pq}(U)$: With u^+ and $\tilde u^+$ as in the definition of $\bar\partial'_q$, the hypothesis $\bar\partial'_q([u^+]) = 0$ means that, in U^-, we have $\bar\partial\, \tilde u^+ = \bar\partial\, w$ for some $w \in \mathcal{J}_{(p,q)}(U^-)$. But $\bar\partial\, w|_S = 0^\infty$, so by Lemma 2.1, there is a $v \in \mathcal{J}_{(p,q-1)}(U^-)$ such that $(w - \bar\partial\, v)|_S = 0^\infty$. Define

$$w' = \begin{cases} 0, & U^+ \\ w - \bar\partial v, & U^-, \end{cases}$$

and observe that $w' \in C^\infty_{(p,q)}(U)$ and $\bar\partial\, \tilde u^+ = \bar\partial w'$ is valid in U. Let $u = \tilde u^+ - w'$. Then u represents a class $[u] \in H^{pq}(U)$, and we have $r_q([u]) = [u^+]$.

Finally, as the reader can see, the diagram is obviously commutative. This completes the proof.

THEOREM 5 (UNIQUENESS). *Either* $H^{pq}(U^-, \mathcal{J}) = 0$ (*which is automatic for* $q = 0$) *or* $H^{pq}(U) = 0$ *is a sufficient condition for the injectivity of* $H^{pq}(U^-) \xrightarrow{r_q} H^{pq}(S)$.

PROOF. From the commutative diagram in Proposition 4.3, one sees that either condition implies that the image of $H^{pq}(U^-, \mathcal{J})$ in $H^{pq}(U)$ by i_q is zero; therefore r_q is injective.

THEOREM 6 (EXISTENCE). *For the surjectivity of* $H^{pq}(U^-) \xrightarrow{r_q} H^{pq}(S)$,

(*a*) $H^{p,q+1}(U^-, \mathcal{J}) = 0$ *is sufficient*;

(*b*) $H^{p,q+1}(U^-, \mathcal{J}) = 0$ *is also necessary if* $H^{p,q+1}(U) = 0$;

(*c*) $H^{pq}(U^+) = 0$ *is sufficient if* $H^{p,q+1}(U) = 0$.

PROOF. (*a*) and (*c*) are obvious. For (*b*), it suffices to observe that the surjectivity of r_q implies that the image of $H^{pq}(U^+)$ in $H^{p,q+1}(U^-, \mathcal{J})$ by $\bar{\partial}'_q$ is zero; hence $H^{p,q+1}(U^-, \mathcal{J}) = 0$.

THEOREM 7. *Assume that* $H^{p,q+1}(U) = 0$. *Then*

(*a*) *if* $H^{pq}(U^-, \mathcal{J}) = 0$ (*which is automatie for* $q = 0$), *there is an isomorphism*

$$H^{p,q+1}(U^-, \mathcal{J}) \cong \frac{H^{pq}(U^+)}{H^{pq}(U)};$$

(*b*) *if* $H^{pq}(U) = 0$, *there is an isomorphism*

$$H^{p,q+1}(U^-, \mathcal{J}) \cong H^{pq}(U^+).$$

PROOF-follows immediately from Proposition 4.3.

As a corollary of the above results, we obtain:

THEOREM 8. *Let* U *be a connected Stein manifold of dimension* $n > 1$, *and consider the Cauchy problem* (4.3.1) *in* U^-. *Then*

(i) *for* $q = 0$, *it is uniquely solvable if and only if*

$$H^{p0}(U^+)/H^{p0}(U) = 0;$$

(ii) *for* $q > 0$, *it is uniquely solvable if and onlg if*

$$H^{pq}(U^+) = 0.$$

4.4. *The two sided Cauchy problem.* One can also consider the Cauchy problem for cohomology classes in U:

$$H^{pq}(U) \xrightarrow{r_q} H^{pq}(S). \tag{4.4.1}$$

This is less interesting than the one-sided problem because if U is Stein, for example, only $H^{p0}(U)$ is nontrivial. Also, the two-sided problem can be reduced to solving the separate one-sided problems on either side. Here we indicate briefly how this is done.

PROPOSITION 4.4. *There is an isomorphism*

$$H^{pq}(U, \mathcal{I}) \xrightarrow[\sim]{r} H^{pq}(U^+, \mathcal{I}) \oplus H^{pq}(U^-, \mathcal{I})$$

induced by restriction.

PROOF. r is defined by

$$r([u]) = [u^+] \oplus [u^-], \quad u^\pm = u\,|_{U^\pm}.$$

Since every class in $H^{pq}(U^\pm, \mathcal{I})$ has a distinguished representative $u^\pm$, u can be taken as $u = u^\pm$ on $U^\pm$. Hence r is surjective. If $r([u]) = 0$, then u can be assumed to be a distinguished representative; it follows from Lemma 2.1 that $u^\pm = \bar\partial v^\pm$ on $U^\pm$, where $v^\pm|_S = 0^\infty$. Therefore $u = \bar\partial v$ in U with v defined as $v = v^\pm$ on $U^\pm$. Hence r is injective, so r is an isomorphism.

THEOREM 9. *Assume that* $H^{p,q+1}(U) = 0$ *and* $H^{pq}(U^\pm, \mathcal{I}) = 0$ (*which is automatic for* $q = 0$). *Then there is an isomorphism*

$$\frac{H^{pq}(S)}{H^{pq}(U)} \simeq \frac{H^{pq}(U^+)}{H^{pq}(U)} \oplus \frac{H^{pq}(U^-)}{H^{pq}(U)}.$$

PROOF. From the short exact sequence

$$0 \to \bigoplus_q \mathcal{I}_{(p,q)}(U) \xrightarrow{i} \bigoplus_q C^\infty_{(p,q)}(U) \xrightarrow{j} \bigoplus_q Q_{(p,q)}(U) \to 0$$

in U, we have the exact cohomology sequence

$$0 \longrightarrow H^{p0}(U) \xrightarrow{r_0} H^{p0}(S) \xrightarrow{\bar{\partial}_0} H^{p1}(U, \mathcal{J}) \xrightarrow{i_1} \ldots$$

$$\ldots \longrightarrow H^{pq}(U, \mathcal{J}) \xrightarrow{i_q} H^{pq}(U) \xrightarrow{r_q} H^{pq}(S) \xrightarrow{\bar{\partial}_q} H^{p,q+1}(U, \mathcal{J}) \longrightarrow \ldots .$$

Using Proposition 4.4 and Theorem 7 we obtain a short exact sequence

$$0 \longrightarrow H^{pq}(U) \xrightarrow{r_q} H^{pq}(S) \longrightarrow \frac{H^{pq}(U+)}{H^{pq}(U)} \oplus \frac{H^{pq}(U-)}{H^{pq}(U)} \longrightarrow 0,$$

which gives the desired result.

It follows from Theorems 5-9 that, under the assumptions of Theorem 9, the two sided problem (4.4.1) is uniquely solvable if and only if both of the one-sided problems are uniquely solvable.

4.5. *Extension of holomorphic functions.* Suppose U and S are such that

(i) $H^{01}(U) = 0$,

(ii) $U^- \subset$ envelope of holomorphy of $\overset{\circ}{U}{}^+$.

Then $H^{00}(U^+)/H^{00}(U) = 0$, and it follows from Theorems 6-7 that $H^{00}(U^-) \xrightarrow[\sim]{r_0} H^{00}(S)$ is an isomorphism. This means that *every smooth function on S, which satisfies the tangential Cauchy-Riemann equations, has a smooth extension to U^- which is holomorphic in $\overset{\circ}{U}{}^-$.* If we also have that

(iii) $U^+ \subset$ envelope of holomorphy of $\overset{\circ}{U}{}^-$, then from Theorem 9 we obtain the isomorphism $H^{00}(U) \xrightarrow[\sim]{r_0} H^{00}(S)$; hence *in that case the extension is to all of U.*

Examples of U and S in $\mathbb{C}^n$ $(n > 1)$ satisfying either (i), (ii) or (i), (ii), (iii) are easy to construct. A general example can also be abtained by taking U and S such that U is equal to the envelope of holomorphy of $\overset{\circ}{U}{}^+$. Then (i), (ii) are both satisfied because U is Stein.

The problem considered above is a *global problem*; the functions to be extended from S are required to satisfy the tangential Cauchy-Riemann equations on all of S, and the holomorphic extensions are sought in a fixed global region U^- (or U^+ or U). One can also consider the *local problem* on S: Given an *arbitrarily small* neighborhood U_p of a point $p \in S$, one can consider smooth functions that satisfy the tangential Cauchy-Riemann equations only on $S_p = S \cap U_p$, and require that all such functions should be holomorphically extendable to some small (depending on U_p) region U_p^+ (or U_p^+ or U_p).

It is well-known that solvability of the local problem is determined by the local Levi convexity of S near p: The Levi form $\mathcal{L}(\varrho)|_{HT}(z_0)$ at $z_0 \in S$ (the Levi form at z_0 restricted to the holomorphic tangent space) is defined by

$$\mathcal{L}(\varrho)|_{HT}(z_0) = \sum_{j,k=1}^{n} \frac{\partial^2 \varrho}{\partial z_j \partial \bar{z}_k}(z_0) w_j \bar{w}_k ,$$

(4.5.1)

$$\sum_{j=1}^{n} \frac{\partial \varrho}{\partial z_j}(z_0) w_j = 0.$$

The signature of $\mathcal{L}(\varrho)|_{HT}(z_0)$ is independent of the function ϱ used to define S, and is invariant under holomorphic changes of coordinates. The well-known results are:

(*a*) The local problem at p is solvable (at least on one side) if $\mathcal{L}(\varrho)|_{HT}(p) \neq 0$.

(*b*) The local problem at p is not solvable (on either side) if $\mathcal{L}(\varrho)|_{HT}(z^0) \equiv 0$ for all z_0 in a neighborhood S_p of p on S.

By taking U_p to be an arbitrarily small Stein neighborhood of p, and by using elementary techniques of Levi convexity, one can easily deduce (*a*) and (*b*) from the above discussion involving (i), (ii).

What is more interesting is the fact that *the global problem on S may be solvable even when the local problem is not solvable on S*: It is easy to construct examples of U and S that satisfy (i), (ii) above, but such that the Levi form of S vanishes identically on a portion S_p of S. In fact S can be considerably deformed without destroying extendability from S to the corresponding U^- (it is only necessary that (ii) be maintained in the deformation).

4.6. *Unique continuation for $\bar{\partial}_S$*. The tangential Cauchy-Riemann operator $\bar{\partial}_S$ is said to have the *unique continuation property* on S if the following is true: If given any open connected set σ in S and any function $u_0 \in H^{00}(S)$ with $u_0 \equiv 0$ in σ, then necessarily $u_0 \equiv 0$ in the connected component of S that contains σ.

It is easy to see (for example, when the Levi form of S is identically zero) that $\bar{\partial}_S$ may not have the unique continuation property. However, $\bar{\partial}_S$ *has the unique continuation property on S if the followtng conditions are satisfied*:

1. $H^{01}(U) = 0$.

2. Either $U^- \subset$ envelope of holomorphy of $\overset{\circ}{U}{}^+$, or else $U^+ \subset$ envelope of holomorphy of $\overset{\circ}{U}{}^-$.

Indeed, the above conditions imply that $H^{00}(S)$ is isomorphic to either $H^{00}(U^-)$ or else $H^{00}(U^+)$. But if there is (say) some $u \in H^{00}(U^+)$ such that $u|_S = u_0$, and $u_0 = 0$ on σ, then it follows (see the remark in Section 2.1) that $u_0 \equiv 0$ in the connected component of S that contains σ.

The criterion given above can be generalized in an obvious fashion: It is only necessary to observe that $\bar{\partial}_S$ has the unique continuation property on S if every point of S has a neighborhood in S such that the restriction of $\bar{\partial}_S$ to that neighborhood has the unique continuation property. The latter remark follows because $F = \operatorname{supp} u_0$ is closed; but the hypotheses imply that F consists entirely of interior points, so F is also open. Since $S - F$ contains the open set σ, it follows that the intersection of F with the connected component of S that contains σ is void.

From (*a*) of the previous section we obtain, as a special case, that $\bar{\partial}_S$ has the unique continuation property on S if the Levi form $\mathcal{L}(\varrho)|_{HT}$ does not vanish at any point of S.

§ 5. The equation of Hans Lewy.

In [13] Lewy gave the first example,

$$L \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) - i(x+iy)\frac{\partial}{\partial \xi}, \tag{5.1}$$

of a linear partial differential operator L such that the inhomogeneous equation $Lu = f$ is not locally solvable at any point in $\mathbb{R}^3$. Lewy's equation is not locally solvable even in the sense of distributions or in the sense of hyperfunctions. For the definition of local solvability, and a discussion of recent results on solvability, see the survey [18]. In general there is a connection — through the use of complex characteristic coordinates — between operators such as (5.1) and tangential Cauchy-Riemann equations. That is explained in detail in [3]. Here we discuss the special case (5.1); we exhibit a particularly simple three-dimensional real hypersurface Σ in $\mathbb{C}^2$ (an imbedding of $\mathbb{R}^3$ in $\mathbb{C}^2$) such that Σ has Lewy's operator as its $\bar{\partial}_\Sigma$.

Let $\mathbb{R}^3$: (x, y, ξ), $\mathbb{C}^2 : (z, \zeta)$, $z = x + iy$ and $\zeta = \xi + i\eta$. Then Σ is given by

$$\Sigma = \{(z,\zeta) \in \mathbb{C}^2 \mid \varrho(x, y, \xi, \eta) = 0\},$$

where

$$\varrho(x, y, \xi, \eta) = \eta - (x^2 + y^2).$$

Note that Σ has the form of $\mathbb{R}^1 \times \{\text{a paraboloid}\}$. Alternately, in terms of the complex characteristic coordinates $x + iy$ and $\xi + i(x^2 + y^2)$, Σ is determined by the imbedding $\mathbb{R}^3 \to \mathbb{C}^2$ given by

$$z = x + iy, \ \zeta = \xi + i(x^2 + y^2).$$

Obviously $\varrho \in C^\infty$ and $d\varrho \neq 0$ on Σ. A trivial calculation shows that the Levi form $\mathcal{L}(\varrho)|_{HT}$, defined by (4.5.1), has one negative eigenvalue at every point of Σ. This implies, in particular, that Lewy's operator has the unique continuation property in any connected region in $\mathbb{R}^3$.

Let p be an arbitrary point on Σ and consider an open connected neighborhood U of p in $\mathbb{C}^2$. Set $S = \Sigma \cap U$, $U^+ = \{\varrho \geq 0\} \cap U$ and $U^- = \{\varrho \leq 0\} \cap U$. In what follows U will be assumed to be such that S, U^+, U^- are all connected

In order to interpret $\bar\partial_S$ on S, we write $Q_{(p,q)}$ for any of the isomorphic $Q_{(p,q)}$'s occuring in part (i) of Proposition 2.1. It is clear from the definition that $Q_{(0,0)} \cong C^\infty(S)$. In a sufficiently small neighborhood of any point on S, we may take $d\bar z$ and $\bar\partial \varrho$ as a basis for all form of type (0, 1). It follows that there is an isomorphism $Q_{(0,1)} \cong C^\infty(S) \wedge d\bar z$. Similarly $Q_{(0,2)} = 0$ because $d\bar z \wedge \bar\partial \varrho$ is a basis near S for all forms of type (0, 2). Therefore, by a slight abuse of our previous notation, $\bar\partial_S$ on S can be regarded as

$$C^\infty(S) \xrightarrow{\bar\partial_S} C^\infty(S) \wedge d\bar z \xrightarrow{\bar\partial_S} 0.$$

The first $\bar\partial_S$ above in given by $\bar\partial_S = (Lu) \wedge d\bar z$, where L is a linear map $L: C^\infty(S) \to C^\infty(S)$. This L is the operator (5.1) of Hans Lewy: Let $u \in C^\infty(S)$ and consider a C^∞ extension $\tilde u$ of u; it represents a class $\{\tilde u\}$ in $Q_{(0,0)}$. Since the class of $\bar\partial_S \{\tilde u\}$ in $Q_{(0,1)}$ is independent of the particular extension $\tilde u$ chosen, we may conveniently take $\tilde u$ such that

$$\left.\frac{\partial \tilde u}{\partial \eta}\right|_S = 0.$$

Near S we have

$$\bar\partial \tilde u = \frac{\partial \tilde u}{\partial \bar z} d\bar z + \frac{\partial \tilde u}{\partial \bar\zeta} d\bar\zeta = \left\{\frac{\partial \tilde u}{\partial \bar z} - 2iz \frac{\partial \tilde u}{\partial \bar\zeta}\right\} d\bar z - 2i \frac{\partial \tilde u}{\partial \bar\zeta} \bar\partial \varrho.$$

Restricting to S and using

$$\frac{\partial \tilde u}{\partial \bar\zeta} = \frac{1}{2}\left(\frac{\partial \tilde u}{\partial \xi} + i\, \frac{\partial \tilde u}{\partial \eta}\right),$$

we find that the class $\bar{\partial}_S \{\tilde{u}\} = \{\bar{\partial}\, \tilde{u}\}$ in $Q_{(0,1)}$ corresponds to

$$(Lu) \wedge d\bar{z} = \left\{\frac{\partial u}{\partial \bar{z}} - iz \frac{\partial u}{\partial \xi}\right\} d\bar{z},$$

as was claimed.

Therefore, in terms of Lewy's operator L we obtain the interpretations

$$H^{00}(S) \cong \ker \{L : C^{\infty}(S) \to C^{\infty}(S)\},$$

$$H^{01}(S) \cong \frac{C^{\infty}(S)}{\operatorname{im} \{L : C^{\infty}(S) \to C^{\infty}(S)\}}.$$

From the non-solvability of L it follows that $H^{01}(S) \neq 0$ and indeed it is infinite-dimensional. Note that this is true no matter how small the neighborhood U of p is taken-the boundary cohomology is not trivial even locally. Moreover the Poincaré lemma does not hold for the sequence of « sheaves »

$$Q^{00} \xrightarrow{\bar{\partial}_S} Q^{01} \longrightarrow 0.$$

Now take U to be Stein. This choice of U implies a special class of $S = \Sigma \cap U$.

First consider $H^{00}(S)$: Since the Levi form $\mathcal{L}(\varrho)|_{HT}$ of S has one negative eigenvalue everywhere on S, U may be readjusted to a Stein $\tilde{U} =$ the envelolope ef holomorphy of $\overset{\circ}{U}{}^{-}$. [For instance, given $(x_0, y_0, \xi_0, \eta_0) \in \Sigma$ and $\varepsilon > 0$ one can find $\delta = \delta(\epsilon) > 0$ so small that $W_\varepsilon \cap \{\eta < x^2 + y^2\}$ has $W_\varepsilon\}$ as its envelope of holomorphy, where W_ε is defined by

$$W_\varepsilon : \begin{cases} |\xi - \xi_0 - 2(x - x_0) y_0 + 2(y - y_0) x_0| < \delta(\varepsilon) \\ |\eta - \eta_0 - 2(x - x_0) x_0 + 2(y - y_0) y_0| < \delta(\varepsilon) \\ |x - x_0| < \varepsilon, \; |y - y_0| \, \varepsilon. \end{cases}$$

What is involved here is essentially the disc theorem.] This gives us a new $\tilde{U}^{+}$. Then according to Section 4.5 we have

$$H^{00}(\tilde{U}^{+}) \xrightarrow[\sim]{r_0} H^{00}(S).$$

Next consider $H^{01}(S)$: Return to the original Stein U. In Part II of this work [4] we prove: the fact that $\mathcal{L}(\varrho)|_{HT}$ has one negative eigenvalue everywhere on S implies that $H^{01}(U^+) = 0$. Hence from Theorem 8 we obtain

$$H^{01}(U^-) \xrightarrow[\sim]{r_1} H^{01}(S).$$

Thus there are two perfectly good Cauchy problems associated with Lewy's operator-one on each side of S. The first is Lewy's original extension problem [12]. The second provides an « explanation » for the non-solvability in Lewy's example.

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