ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Almost *r*-contact structures

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 26, nº 1 (1972), p. 97-115

<http://www.numdam.org/item?id=ASNSP_1972_3_26_1_97_0>

ALMOST r-CONTACT STRUCTURES (*)

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Introduction.

The almost r-contact structures introduced in the paper generalize the almost contact structures as defined in [1] by Sasaki. *Almost r-contact structure* is defined on a manifold M^{2n+r} of dimension $2n+r$ and consists of r differentiable vector fields $\xi_{(1)}, \dots, \xi_{(r)}$, r differentiable 1-forms $\eta^{(1)}, \dots, \eta^{(r)}$, and a differentiable tensor field Φ of type (1, 1) on M^{2n+r} such that

(0.1) $$\eta^{(i)}(\xi_{(j)}) = \delta^i_j \qquad i, j = 1, \dots, r$$

(0.2) $$\Phi\, \xi_{(i)} = 0, \ \eta^{(i)} \circ \Phi = 0 \qquad i = 1, \dots, r$$

(0.3) $$\Phi^2 = -I + \sum_{i=1}^{r} \xi_{(i)} \otimes \eta^{(i)}$$

Very often we call it simply $(\Phi, \xi_{(i)}, \eta^{(i)})$-*structure.* If moreover M^{2n+r} admits a positive definite Riemannian metric g such that

(0.4) $$\eta^{(i)}(X) = g(X, \xi_{(i)}) \ \ i = 1, \dots, r$$

(0.5) $$g(\Phi X, \Phi Y) = g(X, Y) - \sum_{i=1}^{r} \eta^{(i)}(X)\, \eta^{(i)}(Y)$$

we speak about *almost r-contact metric structure,* denoting it by $(\Phi, \xi_{(i)}, \eta^{(i)}, g)$. From this point of view almost-complex structures (almost hermitian structures) can be considered as almost 0-contact structures (almost 0-contact

Pervenuto alla Redazione il 10 Novembre 1970.

(*) During the preparation of this paper the author was supported by a fellowship at the Scuola Normale Superiore di Pisa.

metric structures) as well as almost contact structures (almost-contact metric structures) can be considered as almost 1-contact structures (almost 1 contact metric structures).

In paragraphs 1, 2, 3 we generalize the results of Sasaki and Hatakeyama in [1] and [2] to the almost r contact structures. As a new factor appears here the presence of r vector fields $\xi_{(1)}, \dots, \xi_{(r)}$ instead of the only one in the case of almost contact structures.

We denote by F the distribution generated by $\xi_{(1)}, \dots, \xi_{(r)}$ and according to its properties we distinguish special types of $(\Phi, \xi_{(i)}, \eta^{(i)})$ structures. In paragraph 4 we study some topological properties of almost r-contact structures, giving at the same time certain classes of examples.

All structures in the paper are supposed to be C^∞-differentiable. We denote by E the distribution on M^{2n+r} defined by $E = \{X \in T(M^{2n+r});\ \eta^{(1)}(X) = \dots = \eta^{(r)}(X) = 0\}$. If not otherwise specified the latin indices i, j, k take values $1, \dots, r$.

1. Admissible Riemannian metric.

We start with

LEMMA : Let $\xi_{(1)}, \dots, \xi_{(r)}$ and $\eta^{(1)}, \dots, \eta^{(r)}$ be r vector fields and r 1-forms on a manifold M^{2n+r}. Let us suppose that there is

$$\eta^{(i)}(\xi_{(j)}) = \delta^i_j$$

Then there exists a positive definite Riemannian metric g on M^{2n+r} such that

$$\eta^{(i)}(X) = g(X, \xi_{(i)})$$

for all $i = 1, \dots, r$. Clearly with this metric $\xi_{(1)}, \dots, \xi_{(r)}$ and $\eta^{(1)}, \dots, \eta^{(r)}$ are orthonormal vector fields and orthonormal 1-forms respectively.

PROOF : First let us take any Riemannian metric g' on M^{2n+r}. We can find an open covering $\{U_\alpha\}$ of M^{2n+r} such that on every U_α there exist orthonormal vector fields ${}_\alpha\xi_{(r+1)}, \dots, {}_\alpha\xi_{(2n+r)}$ which represent a basis of E on U_α. Thus on any U_α we can define a Riemannian metric ${}^\alpha g$ by setting

$${}^\alpha g^{ij} = \sum_{\varepsilon=1}^{2n+r} \xi^i_{(\varepsilon)}\, \xi^j_{(\varepsilon)}\,.$$

Now in the same way as in [1] we can see that the just defined Riemannian metrics coincide on all intersections $U_\alpha \cap U_\beta$ and that in this way constructed global Riemannian metric has the required properties.

PROPOSITION 1.: Let $(\Phi, \xi_{(i)}, \eta^{(i)})$ be an almost r contact structure on M^{2n+r}. Then M^{2n+r} admits a positive definite Riemannian metric g such that

$$\eta^{(i)}(X) = g(X, \xi_{(i)}) \tag{1.1}$$

$$g(\Phi X, \Phi Y) = g(X, Y) - \sum_{i=1}^{r} \eta^{(i)}(X)\, \eta^{(i)}(Y). \tag{1.2}$$

PROOF: According to the previous lemma we can find a metric g' such that $\eta^{(i)}(X) = g'(X, \xi_{(i)})$.

Then we define

$$g(X, Y) = \frac{1}{2}\left[g'(X, Y) + g'(\Phi X, \Phi Y) + \sum_{i=1}^{r} \eta^{(i)}(X)\, \eta^{(i)}(Y)\right].$$

The proof then proceeds as in [1].

REMARK: First we notice that the endomorphism Φ satisfies the equation $\Phi^3 + \Phi = 0$. It can be easily seen that the endomorphism $\Phi + \sum_{i=1}^{r} \eta^{(i)} \otimes \xi_{(i)}$ is an automorphism, namely we can verify that

$$\left(\Phi + \sum_{i=1}^{r} \eta^{(i)} \otimes \xi_{(i)}\right)\left(-\Phi + \sum_{i=1}^{r} \eta^{(i)} \otimes \xi_{(i)}\right) = I.$$

Moreover we can show that in case of $(\Phi, \xi_{(i)}, \eta^{(i)}, g)$ — structure the automorphism $\Phi + \sum_{i=1}^{r} \eta^{(i)} \otimes \xi_{(i)}$, and then naturally also the automorphism $-\Phi + \sum_{i=1}^{r} \eta^{(i)} \otimes \xi_{(i)}$, is orthogonal.

For a $(\Phi, \xi_{(i)}, \eta^{(i)}, g)$-structure the following skew symmetric bilinear form

$$\varphi(X, Y) = g(X, \Phi Y) \tag{1.3}$$

is important. It is called the *fundamental 2-form* of the almost r-contact metric structure. One can easily see that rank $\varphi = 2n$.

Taking into account that for a $(\Phi, \xi_{(i)}, \eta^{(i)}, g)$-structure the restriction of Φ to any E_x is a complex structure on E_x and that the restriction of g to any E_x is a hermitian metric on E_x with respect to the just mentioned complex structure on E_x, we can prove, following [1]

PROPOSITION 2.: On M^{2n+r} there is $1-1$ correspondence between almost r-contact metric structures and the reductions of the structural group of the tangent bundle of M^{2n+r} to the subgroup $\underbrace{1 \times \ldots \times 1}_{r\times} \times U(n)$.

Remark: For a $(\Phi, \xi_{(i)}, \eta^{(i)}, g)$-structure we can quite easily compare the volume element dV of the Riemannian metric g with the $(2n+r)$ form $\eta^{(1)} \wedge \ldots \wedge \eta^{(r)} \wedge \varphi^n$. Clearly we can find an orthogonal basis of $T_x(M^{2n+r})$ in the form $(\xi_{(1)}, \ldots, \xi_{(r)}, X_1, \ldots, X_n, \Phi X_1, \ldots, \Phi X_n)$. We denote by

$$(\eta^{(1)}, \ldots, \eta^{(r)}, \omega^{(1)}, \ldots, \omega^{(n)}, \bar{\omega}^{(1)}, \ldots, \bar{\omega}^{(n)})$$

its dual basis. We get

$$dV = \eta^{(1)} \wedge \ldots \wedge \eta^{(r)} \wedge \omega^{(1)} \wedge \ldots \wedge \omega^{(n)} \wedge \bar{\omega}^{(1)} \wedge \ldots \wedge \bar{\omega}^{(n)}$$

$$\varphi = -2 \sum_{\alpha=1}^{n} \omega^{(\alpha)} \wedge \bar{\omega}^{(\alpha)}.$$

Then calculating $\eta^{(1)} \wedge \ldots \wedge \eta^{(r)} \wedge \varphi^n$ we get

$$\eta^{(1)} \wedge \ldots \wedge \eta^{(r)} \wedge \varphi^n = (-1)^{\frac{n(n+1)}{2}} 2^n \, dV.$$

2. Normality of an almost r-contact structure.

Let us have a $(\Phi, \xi_{(i)}, \eta^{(i)})$-structure on a manifold M^{2n+r}. Now following [2] we introduce an almost complex structure J on $M^{2n+r} \times \mathbb{R}^r$.

We denote only by $(t_1, \ldots, t_r)$ the canonical coordinates on $\mathbb{R}^r$ and we can define

$$J\left(X, \sum_{i=1}^{r} f_i \frac{\partial}{\partial t_i}\right) = \left(\Phi X + \sum_{i=1}^{r} f_i \xi_{(i)}, -\sum_{i=1}^{r} \eta^{(i)}(X) \frac{\partial}{\partial t_i}\right)$$

where X is a vector field on M^{2n+r} and f_i are real functions on $M^{2n+r} \times \mathbb{R}^r$. It can be easily seen that J is an almost complex structure, i. e. $J^2 = -I$.

In the sequel we shall study the integrability conditions of this almost complex structure. First we calculate the components of the Nijenhuis torsion of J, namely the components of the $(1, 2)$-tensor

$$N(A, B) = 2\{[A, B] - [JA, JB] + J[A, JB] + J[JA, B]\}.$$

Here A and B are vector fields on $M^{2n+r} \times \mathbb{R}^r$. We denote by P and Q^i the projections of $M^{2n+r} \times \mathbb{R}^r$ on M^{2n+r} and on the i-th factor of $\mathbb{R}^r$ respectively, and we define the following four groups of tensors on M^{2n+r}:

$$(2.1)_1 \qquad N^{(1)}(X, Y) = P_* N(X, Y)$$

$$N^{(2)}_{(i)}(X, Y) = Q^i_* N(X, Y) \tag{2.1$_2$}$$

$$N_i^{(3)}(X) = P_* N\left(X, \frac{\partial}{\partial t_i}\right) \tag{2.3$_3$}$$

$$N^{(4)}_{(i,j)}(X) = Q^j_* N\left(X, \frac{\partial}{\partial t_i}\right) \tag{2.1$_4$}$$

Here X, Y are vector fields on M^{2n+r} and P_*, Q^i_* denote the differentials of P, Q^i respectively. It is not very difficult to show that

$$N^{(1)}(X, Y) = [\Phi, \Phi](X, Y) - 4 \sum_{i=1}^{r} d\eta^{(i)}(X, Y)\, \xi_{(i)} \tag{2.2$_1$}$$

$$N^{(2)}_{(i)}(X, Y) = 2\,(L_{\Phi X}\, \eta^{(i)})(Y) - 2\,(L_{\Phi Y}\, \eta^{(i)})(X) \tag{2.2$_2$}$$

$$N^{(3)}_{(i)}(X) = 2\,(L_{\xi_{(i)}} \Phi)(X) \tag{2.2$_3$}$$

$$N^{(4)}_{(i,j)}(X) = -2\,(L_{\xi_{(i)}}\, \eta^{(j)})(X) \tag{2.2$_4$}$$

where $[\Phi, \Phi]$ denotes the Nijenhuis torsion of Φ and L denotes the Lie derivative. Clearly the tensors of the four groups are of types (1.2), (0.2), (1,1) and (0,1) respectively. We notice here also that

$$\begin{aligned} P_* N\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}\right) &= -2\,[\xi_{(i)}, \xi_{(j)}] \\ Q^k_* N\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}\right) &= 0. \end{aligned} \tag{2.3}$$

Thus we can see that the tensor $N = [J, J]$ on $M^{2n+r} \times \mathbb{R}^r$ vanish if and only if all the four groups of tensors $N^{(1)}$, $N^{(2)}$, $N^{(3)}$, $N^{(4)}$ and all the brackets $[\xi_{(i)}, \xi_{(j)}]$ vanish. In the next we shall see that the vanishing of the only tensor $N^{(1)}$ from the first group together with the vanishing of all the brackets $[\xi_{(i)}, \xi_{(j)}]$ implies vanishing of all tensors from the remaining three groups $N^{(2)}$, $N^{(3)}$, $N^{(4)}$.

To prove the just announced result we use up the fact that N is a hybrid and pure tensor, what means that we have

$$N(A, JB) = -JN(A, B) \quad \text{and} \quad N(A, JB) = N(JA, B). \tag{2.4}$$

The first relation of (2.4) gives the following four identities

$(2.5)_1$ $$N(X, JY) = -JN(X, Y)$$

$(2.5)_2$ $$N\left(X, J\frac{\partial}{\partial t_i}\right) = -JN\left(X, \frac{\partial}{\partial t_i}\right)$$

$(2.5)_3$ $$N\left(\frac{\partial}{\partial t_i}, JX\right) = -JN\left(\frac{\partial}{\partial t_i}, X\right)$$

$(2.5)_4$ $$N\left(\frac{\partial}{\partial t_i}, J\frac{\partial}{\partial t_j}\right) = -JN\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}\right)$$

X and Y denote again vector fields on M^{2n+r}.

From these four identities we get applying projectors P_* and Q^i_* the next 8 identities

$(2.5)_{1P}$ $$\Phi N^{(1)}(X, Y) + N^{(1)}(X, \Phi Y) + \sum_{i=1}^{r} N^{(2)}_{(i)}(X, Y)\,\xi_{(i)} - \sum_{i=1}^{r} \eta^{(i)}(Y)\,N^{(3)}_{(i)}(X) = 0$$

$(2.5)_{1Q}$ $$\eta^{(i)}(N^{(1)}(X, Y)) - N^{(2)}_{(i)}(X, \Phi Y) + \sum_{j=1}^{r} \eta^{(j)}(Y)\,N^{(4)}_{(j,i)}(X) = 0$$

$(2.5)_{2P}$ $$N^{(1)}(X, \xi_{(i)}) + \Phi N^{(3)}_{(i)}(X) + \sum_{j=1}^{r} N^{(4)}_{(i,j)}(X)\,\xi_{(j)} = 0$$

$(2.5)_{2Q}$ $$\eta^{(j)}(N^{(3)}_{(i)}(X)) - N^{(2)}_{(j)}(X, \xi_{(i)}) = 0$$

$(2.5)_{3P}$ $$\Phi N^{(3)}_{(i)}(X) + N^{(3)}_{(i)}(\Phi X) + \sum_{j=1}^{r} N^{(4)}_{(i,j)}(X)\,\xi_{(j)} = 0$$

$(2.5)_{3Q}$ $$\eta^{(j)}(N^{(3)}_{(i)}(X)) - N^{(4)}_{(i,j)}(\Phi X) = 0$$

$(2.5)_{4P}$ $$N^{(3)}_{(i)}(\xi_{(j)}) - \Phi([\xi_{(i)}, \xi_{(j)}]) = 0$$

$(2.5)_{4Q}$ $$N^{(4)}_{(i,k)}(\xi_{(j)}) - \eta^{(k)}([\xi_{(i)}, \xi_{(j)}]) = 0.$$

Now we use up the second relation of (2.4) thus obtaining these four identities

$(2.6)_1$ $$N(X, JY) = N(JX, Y)$$

$$(2.6)_2 \qquad N\left(X, J\frac{\partial}{\partial t_i}\right) = N\left(JX, \frac{\partial}{\partial t_i}\right)$$

$$(2.6)_3 \qquad N\left(\frac{\partial}{\partial t_i}, JX\right) = N\left(J\frac{\partial}{\partial t_i}, X\right)$$

$$(2.6)_4 \qquad N\left(\frac{\partial}{\partial t_i}, J\frac{\partial}{\partial t_j}\right) = N\left(J\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}\right).$$

Before proceeding as above we notice that the identities $(2.6)_2$ and $(2.6)_3$ are equivalent. Thus we get this time only these 6 identities

$$(2.6)_{1P} \quad N^{(1)}(X, \Phi Y) - N^{(1)}(\Phi X, Y) - \sum_{i=1}^{r} \eta^{(i)}(Y) N^{(3)}_{(i)}(X) - \sum_{i=1}^{r} \eta^{(i)}(X) N^{(3)}_{(i)}(Y) = 0$$

$$(2.6)_{1Q} \quad N^{(2)}_{(i)}(X, \Phi Y) - N^{(2)}_{(i)}(\Phi X, Y) - \sum_{j=1}^{r} \eta^{(j)}(Y) N^{(4)}_{(j,i)}(X)$$

$$- \sum_{j=1}^{r} \eta^{(j)}(X) N^{(4)}_{(j,i)}(Y) = 0$$

$$(2.6)_{2P} \qquad N^{(1)}(X, \xi_{(i)}) - N^{(3)}_{(i)}(\Phi X) - \sum_{j=1}^{r} \eta^{(j)}(X)[\xi_{(i)}, \xi_{(j)}] = 0$$

$$(2.6)_{2Q} \qquad N^{(2)}_{(j)}(X, \xi_{(i)}) - N^{(4)}_{(i,j)}(\Phi X) = 0$$

$$(2.6)_{4P} \qquad N^{(3)}_{(i)}(\xi_{(j)}) + N^{(3)}_{(j)}(\xi_{(i)}) = 0$$

$$(2.6)_{4Q} \qquad N^{(4)}_{(i,k)}(\xi_{(j)}) + N^{(4)}_{(j,k)}(\xi_{(i)}) = 0.$$

Now we shall combine the just obtained two groups of identities $(2.5)_{1P}, \dots, (2.5)_{4Q}$ and $(2.6)_{1P}, \dots, (2.6)_{4Q}$:

$(2.6)_{2Q} + (2.5)_{4Q} \Longrightarrow$

$$(2.7)_1 \qquad N^{(4)}_{(i,j)}(X) = - N^{(2)}_{(j)}(\Phi X, \xi_{(i)}) + \sum_{k=1}^{r} \eta^{(k)}(X)\, \eta^{(j)}([\xi_{(i)}, \xi_{(k)}])$$

$(2.5)_{3Q} + (2.5)_{4Q} \Longrightarrow$

$$(2.7)_2 \qquad N^{(4)}_{(i,j)}(X) = \eta^{(j)}(N^{(3)}_{(i)}(\Phi X)) + \sum_{k=1}^{r} \eta^{(k)}(X)\, \eta^{(j)}([\xi_{(i)}, \xi_{(k)}])$$

$(2.5)_{2P} \Longrightarrow$

$$(2.7)_3 \qquad N^{(4)}_{(i,j)}(X) = -\eta^{(j)}(N^{(1)}(X, \xi_{(i)}))$$

$(2.5)_{1P} \Longrightarrow$

$$(2.7)_4 \qquad N^{(2)}_{(i)}(X, Y) = -\eta^{(i)}(N^{(1)}(X, \Phi Y)) + \sum_{j=1}^{r} \eta^{(j)}(Y)\, \eta^{(i)}(N^{(3)}_{(j)}(X))$$

$(2.5)_{1P} \Longrightarrow$

$$(2.7)_5 \qquad N^{(3)}_{(i)}(X) = \Phi N^{(1)}(X, \xi_{(i)}) + \sum_{j=1}^{r} N^{(2)}_{(j)}(X, \xi_{(i)})\, \xi_{(j)} .$$

Now we are in position to prove.

PROPOSITION 3: If $N^{(1)} = 0$ and all the brackets $[\xi_{(i)}, \xi_{(j)}]$ vanish, then all tensors from the three groups $N^{(2)}, N^{(3)}, N^{(4)}$ vanish.

PROOF.: Vanishing of all $N^{(4)}_{(i,j)}$ follows immediately from $(2.7)_3$. Further from $(2.5)_{1Q}$ and $(2.6)_{2Q}$ we get $N^{(2)}_{(i)}(X, \Phi Y) = 0$ and $N^{(2)}_{(i)}(X, \xi_{(j)}) = 0$ respectively, what implies immediately $N^{(2)}_{(i)} = 0$. Finally $(2.6)_{2P}$ and $(2.5)_{4P}$ gives $N^{(3)}_{(i)}(\Phi X) = 0$ and $N^{(3)}_{(i)}(\xi_{(j)}) = 0$ from which we get again $N^{(3)}_{(i)} = 0$.

DEFINITION 1. An almost r-contact structure $(\Phi, \xi_{(i)}, \eta^{(i)})$ for which the tensor $N^{(1)}$ and all the brackets $[\xi_{(i)}, \xi_{(j)}]$ vanish will be called a *normal almost r-contact structure.*

3. Special connections on an almost r-contact manifold.

As usual here we shall try to find connections with respect to which all tensors appearing in the definition of an almost r-contact structure are covariant constants.

PROPOSITION 4: Let ∇ be any connection on a manifold M^{2n+r} with a $(\Phi, \xi_{(i)}, \eta^{(i)})$-structure. We define a new connection $\widetilde{\nabla}$ on M^{2n+r} by

$$\widetilde{\nabla}_Y X = \nabla_Y X - \frac{1}{2} \Phi [(\nabla_Y \Phi)(X)] - \frac{1}{2} \sum_{i=1}^{r} \eta^{(i)}(X)\, \nabla_Y \xi_{(i)} + \sum_{i=1}^{r} (\nabla_Y \eta^{(i)})(X)\, \xi_{(i)} .$$

Then with respect to this new connection Φ is a covariant constant, i. e. $\widetilde{\nabla} \Phi = 0$.

Proof is the same as the proof of the corresponding proposition in [2] (Theorem 7).

PROPOSITION 5: Let M^{2n+r} be a manifold with a $(\Phi, \xi_{(i)}, \eta^{(i)}, g)$-structure. If the connection ∇ in proposition 5 is taken to be the Riemannian connection associated with g, then we have not only $\widetilde{\nabla}\,\Phi = 0$, but also $\widetilde{\nabla}\, g = 0$.

Proof is the same as the proof of the corresponding proposition in [2] (Theorem 10).

PROPOSITION 6; Let M^{2n+r} be a manifold with a $(\Phi, \xi_{(i)}, \eta^{(i)}, g)$-structure, and let ∇ be a connection on M^{2n+r} such that $\nabla\,\Phi = 0$ and $\nabla\, g = 0$. Then the connection $\widetilde{\nabla}$ defined by

$$\widetilde{\nabla}_Y X = \nabla_Y X + \sum_{i=1}^{r} (\nabla_Y\, \eta^{(i)})\,(X)\,\xi_{(i)}$$

leaves all the tensors Φ, g, $\xi_{(i)}$, $\eta^{(i)}$ covariant constant.

PROOF:

$$(\widetilde{\nabla}_Y\, \Phi)(X) = \sum_{i=1}^{r} (\nabla_Y\, \eta^{(i)})\,(\Phi\, X)\,\xi_{(i)} - \sum_{i=1}^{r} (\nabla_Y\, \eta^{(i)})\,(X)\,\xi_{(i)} =$$

$$= \sum_{i=1}^{r} Y\,\eta^{(i)}\,(\Phi X)\,\xi_{(i)} - \sum_{i=1}^{r} \eta^{(i)}\,(\nabla_Y\,(\Phi\, X))\,\xi_{(i)} =$$

$$= - \sum_{i=1}^{r} \eta^{(i)}\,((\nabla_Y\, \Phi)\,(X) + \Phi\,(\nabla_Y\, X))\,\xi_{(i)} = 0$$

$$(\widetilde{\nabla}_Y\, g)\,(X_1, X_2) = - \sum_{i=1}^{r} (\nabla_Y\, \eta^{(i)})\,(X_1)\; g\,(\xi_{(i)}, X_2) -$$

$$- \sum_{i=1}^{r} (\nabla_Y\, \eta^{(i)})(X_2)\; g\,(\xi_{(i)}, X_1) = - \sum_{i=1}^{r} \{(\nabla_Y \eta^{(i)})(X_1)\,\eta^{(i)}(X_2) + \eta^{(i)}\,(X_1)\,(\nabla_Y\, \eta^{(i)})(X_2)\} =$$

$$= - \sum_{i=1}^{r} (\nabla_Y(\eta^{(i)} \otimes \eta^{(i)}))\,(X_1 \otimes X_1) = (\nabla_Y(g \circ (\Phi \otimes \Phi) - g))\,(X_1 \otimes X_2) = 0.$$

Here we have used the identity (0.5)

$$\widetilde{\nabla}_Y\;\; \xi_{(j)} = \nabla_Y\, \xi_{(j)} + \sum_{i=1}^{r}\; Y\,\eta^{(i)}\,(\xi_{(j)})\,\xi_{(i)} - \sum_{i=1}^{r} \eta^{(i)}\,(\nabla_Y\, \xi_{(j)})\,\xi_{(i)} =$$

$$= \nabla_Y\, \xi_{(j)} - \nabla_Y\, \xi_{(j)} - \Phi^2\,(\nabla_Y\, \xi_{(j)}) = - \Phi[\nabla_Y\,(\Phi\, \xi_{(j)}) - (\nabla_Y\, \Phi)\,(\xi_{(j)})] = 0$$

$$(\widetilde{\nabla}_Y \eta^{(j)})(X) = (\nabla_Y \eta^{(j)})(X) - \eta^{(j)}\left(\sum_{i=1}^{r} (\nabla_Y \eta^{(i)})(X)\, \xi_{(i)}\right) =$$

$$= (\nabla_Y \eta^{(j)})(X) - (\nabla_Y \eta^{(j)})(X) = 0.$$

DEFINITION 2: A connection leaving all the tensors Φ, $\xi_{(i)}$, $\eta^{(i)}$ covariant costant will be called *almost r-contact connection* or simply $(\Phi, \xi_{(i)}, \eta^{(i)})$-connection. A connection leaving moreover g covariant constant will be called *almost r-contact metric connection* or simply $(\Phi, \xi_{(i)}, \eta^{(i)}, g)$-connection.

Having found a connection leaving all our tensors covariant constant we ask as usual for a symmetric connection having the same property. We start with

PROPOSITION 7: Le M be a manifold and let $\xi_{(1)}, \dots, \xi_{(r)}$ and $\eta^{(1)}, \dots, \eta^{(r)}$ be r vector fields and r 1-forms on M satisfying

$$\eta^{(i)}(\xi_{(j)}) = \delta^i_j, \quad [\xi_{(i)}, \xi_{(j)}] = 0$$

for all i, j. Then there exits a symmetric connection leaving all $\xi_{(i)}$ and $\eta^{(i)}$ covariant constant if and only if all 1-forms $\eta^{(i)}$ are closed.

PROOF: If there exists a symmetric connection leaving all $\xi_{(i)}$ and $\eta^{(i)}$ covariant constant we have $d\eta^{(i)} = 0$ by virtue of the formula $d\omega = A(\nabla\omega)$ which holds for any h-form under the assumption that the connection ∇ is symmetric. A denotes here the alternation.

On the other hand because of $[\xi_{(i)}, \xi_{(j)}] = 0$ we can according to Ishihara and Obata's theorem (see [3]) find a symmetric connection ∇ leaving all $\xi_{(i)}$ covariant constant. We define a new connection $\widetilde{\nabla}$ by

$$\widetilde{\nabla}_Y X = \nabla_Y X + \sum_{i=1}^{r} (\nabla_Y \eta^{(i)})(X)\, \xi_{(i)}.$$

First we notice that $\widetilde{\nabla}$ is again a symmetric connection because of

$$(\nabla_Y \eta^{(i)})(X) - (\nabla_X \eta^{(i)})(Y) = [A(\nabla\eta^{(i)})](X, Y) = d\eta^{(i)}(X, Y) = 0.$$

Moreover we have

$$\widetilde{\nabla}_Y \xi_{(j)} = \sum_{i=1}^{r} (\nabla_Y \eta^{(i)})(\xi_{(j)})\, \xi_{(i)} = -\sum_{i=1}^{r} \eta^{(i)}(\nabla_Y \xi_{(j)})\, \xi_{(i)} = 0$$

$$(\widetilde{\nabla}_Y \eta^{(j)})(X) = (\nabla_Y \eta^{(j)})(X) - \sum_{i=1}^{r} \eta^{(j)}(\xi_{(i)}) \cdot (\nabla_Y \eta^{(i)})(X) = 0$$

DEFINITION 3: Let $(\Phi, \xi_{(i)}, \eta^{(i)})$ be an almost r-contact structure on a manifold M^{2n+r}. If all the brackets $[\xi_{(i)}, \xi_{(j)}]$ vanish we shall call this structure a *commutative Lie almost r-contact structure.*

PROPOSITION 8: Let $(\Phi, \xi_{(i)}, \eta^{(i)})$ be a commutative Lie almost r contact structure on a manifold M^{2n+r}. If all the 1-forms $\eta^{(i)}$ are closed and all the tensors $N_{(i)}^{(3)}$ vanish, then we can find a $(\Phi, \xi_{(i)}, \eta^{(i)})$-connection whose torsion is equal to $-\frac{1}{8} N^{(1)}$.

PROOF: Because all $\eta^{(i)}$ are closed and the structure is commutative we can, by virtue of the preceding proposition, find a symmetric connection leaving all $\xi_{(i)}$ and $\eta^{(i)}$ covariant constant. Then, as a consequence of vanishing of $N_{(i)}^{(3)}$ and the symmetry of ∇, we get (see $(2.2)_3$)

$$0 = N_i^{(3)}(X) = (L_{\xi_{(i)}} \Phi)(X) = [\xi_{(i)}, \Phi X] - \Phi [\xi_{(i)}, X] \tag{3.1}$$

$$= \nabla_{\xi_{(i)}} \Phi X - \nabla_{\Phi X} \xi_{(i)} - \Phi (\nabla_{\xi_{(i)}} X - \nabla_X \xi_{(i)}) = (\nabla_{\xi_{(i)}} \Phi)(X).$$

Moreover we have the following obvious identities

$$(\nabla_Y \Phi)(\xi_{(i)}) = 0 \, , \, \eta^{(i)} \circ (\nabla_Y \Phi) = 0, \nabla_Y \Phi^2 = 0. \tag{3.2}$$

We define a new connection $\widetilde{\nabla}$ on M^{2n+r} by

$$\widetilde{\nabla}_Y X = \nabla_Y X - \frac{1}{4} \{(\nabla_{\Phi X} \Phi)(Y) - (\nabla_Y \Phi)(\Phi X) +$$

$$+ \Phi [(\nabla_X \Phi)(Y)] + \Phi [(\nabla_Y \Phi)(X)]\}$$

Now in the same way as in [2] (Theorem 12) we can prove that $\widetilde{\nabla} \Phi = 0$ and $\widetilde{\nabla} \xi_{(i)} = 0$. There is also $\widetilde{\nabla} \eta^{(i)} = 0$ which is an immediate consequence of the following lemma.

LEMMA: If $\widetilde{\nabla}$ is a connection on M^{2n+r} leaving Φ covariant constant, then there exist r^2 1-forms $\lambda_{(j)}^{(i)}$ on M^{2n+r} such that .

$$\widetilde{\nabla}_Y \xi_{(i)} = \sum_{=1}^{r} \lambda_{(i)}^{(j)}(Y) \xi_{(j)} \, , \widetilde{\nabla}_Y \eta^{(i)} = - \sum_{j=1}^{r} \lambda_{(j)}^{(i)} (Y) \eta^{(j)} \tag{3.3}$$

Proof of the lemma : Applying $\widetilde{\nabla}_Y$ on $\Phi\xi_{(i)} = 0$ we get $\Phi\widetilde{\nabla}_Y\xi_{(i)} = 0$ and from this we conclude that there exist r^2 differentiable functions $\lambda^{(j)}_{(i)}(Y)$ such that that

$$\widetilde{\nabla}_Y \xi_{(i)} = \sum_{j=1}^{r} \lambda^{(j)}_{(i)}(Y)\, \xi_{(j)}.$$

Similarly we can see that there are r^2 differentiable functions $\mu^{(i)}_{(j)}(Y)$ such

$$\widetilde{\nabla}_Y \eta^{(i)} = \sum_{j=1}^{r} \mu^{(i)}_{(j)}(Y)\, \eta^{(j)}$$

Evidently both $\lambda^{(i)}_{(j)}(Y)$ and $\mu^{(i)}_{(j)}(Y)$ are linear in Y. Finally applying $\widetilde{\nabla}_Y$ to $\eta^{(i)}(\xi_{(j)}) = 0$ we get

$$\mu^{(i)}_{(j)}(Y) + \lambda^{(i)}_{(j)}(Y) = 0.$$

Now to finish the proof of the proposition we must only calculate the torsion of $\widetilde{\nabla}$. We get easily.

$$\begin{aligned} 4(\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X, Y]) &= (\nabla_{\Phi X}\Phi)(Y) - (\nabla_Y \Phi)(\Phi X) - \\ &- (\nabla_{\Phi Y}\Phi)(X) + (\nabla_X \Phi)(\Phi Y) = \nabla_{\Phi X}(\Phi Y) - \Phi(\nabla_{\Phi X} Y) - \\ &- \nabla_Y(\Phi^2 Y) + \Phi(\nabla_Y(\Phi X)) - \nabla_{\Phi Y}(\Phi X) + \Phi(\nabla_{\Phi Y} X) + \\ &+ \nabla_X(\Phi^2 Y) - \Phi(\nabla_X(\Phi Y)) = [\Phi X, \Phi Y] - \Phi[\Phi X, Y] + \\ &+ \Phi[\Phi Y, X] - [X, Y] - \nabla_Y\left(\sum_{i=1}^{r} \eta^{(i)}(X)\,\xi_{(i)}\right) + \\ &+ \nabla_X\left(\sum_{i=1}^{r} \eta^{(i)}(Y)\,\xi_{(i)}\right) = [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \\ &- \Phi[X, \Phi Y] - [X, Y] + \sum_{i=1}^{r} \eta^{(i)}([X, Y])\,\xi_{(i)} = \\ &= -\frac{1}{2}[\Phi, \Phi](X, Y) = -\frac{1}{2} N^{(1)}(X, Y) \end{aligned}$$

where we used two times the fact that $d\eta^{(i)} = 0$. Now the proof is finished.

PROPOSITION 9 : On a manifold with a commutative $(\Phi, \xi_{(i)}, \eta^{(i)})$ — structure, there exists a symmetric $(\Phi, \xi_{(i)}, \eta^{(i)})$-connection if and only if the following two conditions are satisfied

(i) all $\eta^{(i)}$ are closed

(ii) the $(\Phi, \xi_{(i)}, \eta^{(i)})$-structure is normal

PROOF : If (i) and (ii) are satisfied then we can find a symmetric $(\Phi, \xi_{(i)}, \eta^{(i)})$-connection by virtue of the previous proposition. On the other hand, if there exists a symmetric $(\Phi, \xi_{(i)}, \eta^{(i)})$-connection ∇ then all 1-forms $\eta^{(i)}$ are closed according to proposition 7. Moreover starting from $(2.2)_1$ we get

$$N^{(1)}(X, Y) = [\Phi, \Phi](X, Y) = 2\,\{[X, Y] - [\Phi X, \Phi Y] +$$

$$+ \Phi[X, \Phi Y] + \Phi[\Phi X, Y] - \sum_{i=1}^{r} \eta^{(i)}([X, Y])\, \xi_{(i)}\} =$$

$$= 2\,\{\nabla_X Y - \nabla_Y X - \nabla_{\Phi X}(\Phi Y) + \nabla_{\Phi Y}(\Phi X) +$$

$$+ \Phi[\nabla_X(\Phi Y)] - \Phi[\nabla_{\Phi Y} X] + \Phi[\nabla_{\Phi X} Y] -$$

$$- \Phi[\nabla_Y(\Phi X)] - \sum_{i=1}^{r} \eta^{(i)}([X, Y])\, \xi_{(i)} =$$

$$= 2\,\{\nabla_X Y - \nabla_Y X - \Phi[\nabla_{\Phi X} Y] + \Phi[\nabla_{\Phi Y} X] - \nabla_X Y +$$

$$+ \sum_{i=1}^{r} \eta^{(i)}(\nabla_X Y)\, \xi_{(i)} - \Phi[\nabla_{\Phi Y} X] + \Phi[\nabla_{\Phi X} Y] + \nabla_Y X$$

$$- \sum_{i=1}^{r} \eta^{(i)}(\nabla_Y X)\, \xi_{(i)} - \sum_{i=1}^{r} \eta^{(i)}([X, Y])\, \xi_{(i)} = 0.$$

4. Some topological properties.

Let M^{2n+r} be a manifold with a $(\Phi, \xi_{(i)}, \eta^{(i)})$-structure. Considering the vector fields $\xi_{(i)}$ we can distinguish some special types of $(\Phi, \xi_{(i)}, \eta^{(i)})$-structures.

DEFINITION 4 : Let F be the r-dimensional distribution on M^{2n+r} spanned by the vector fields $\xi_{(i)}$.

We shall call this distribution *fundamental distribution* of the $(\Phi, \xi_{(i)}, \eta^{(i)})$-structure. If F is an involutive distribution we say that the $(\Phi, \xi_{(i)}, \eta^{(i)})$-structure is *foliated*. Further, if the vector fields $\xi_{(i)}$ form a basis of a finite dimensional subalgebra of the Lie algebra of all vector fields on M^{2n+r}, we speak about a *Lie* $(\Phi, \xi^{(i)}, \eta_{(i)})$-*structure* . Finally if all the brackets $[\xi_{(i)}, \xi_{(j)}]$ vanish we call the $(\Phi, \xi_{(i)}, \eta^{(i)})$-structure *commutative Lie* $(\Phi_1\, \xi_{(i)}, \eta^{(i)})$-*structure.*

In the next we shall be interested only in the topological properties of foliated $(\Phi, \xi_{(i)}, \eta^{(i)})$-structures. Here we shall use very often results and terminology of [4] and [5]. We start by giving an example of the almost r-contact structure.

Let X be a $2n$-dimensional manifold with an almost complex structure $\mathcal{J}$. Let (M^{2n+r}, p, X) be a fibered manifold over X with fibers of dimension r, and let $\xi_{(1)}, \dots, \xi_{(r)}$ be vector fields on M^{2n+r}, tangent to the fibers, and linearly independent at every point of M^{2n+r}. We denote by F the n-dimensional distribution on M^{2n+r} spanned by these vector fields. Moreover let E be a horizontal distribution on M^{2n+r}, i. e. such a distribution that at every point $x \in M^{2n+r}$ we have $T_x(M^{2n+r}) = F_x \oplus E_x$. We can define r 1-forms on M^{2n+1} as follows

$$\eta^{i)}(E) = 0, \ \eta^{(i)}(\xi_{(j)}) = \delta^i_j \, . \tag{4.1}$$

Clearly all $\eta^{(i)}$ are differentiable 1-forms. Finally we can define on M^{2n+r} a tensor field Φ of type (1,1) at every point $x \in M^{2n+r}$ maps the tangent space $T_x(M^{2n+r})$ into its subspace E_x. Namely we set

$$\Phi\xi_{(i)} = 0 \quad i = 1, \dots, r$$

$$\Phi X = p_*^{-1} \mathcal{J} p_* X \text{ for } X \in E \tag{4.2}$$

where p_* denotes the differential of p. For other tangent vectors is Φ defined by the linear extension.

The following proposition is trivial.

PROPOSITION 10 : Let (M^{2n+r}, p, X) be a fibered manifold over a $2n$-dimensional almost-complex manifold X with fibers of dimension r. Let $\xi_{(1)}, \dots, \xi_{(r)}$ be vector fields on M^{2n+r}, tangent to the fibers, and linearly independet at every point. Finally let E be a horizontal distribution on M^{2n+r}. Then the vector fields $\xi_{(i)}$ together with the tensors $\eta^{(i)}$ and Φ introduced above, define on M^{2n+r} a foliated $(\Phi, \xi_{(i)}, \eta^{(i)})$-structure. Moreover the involutive distribution F defines a foliation on M^{2n+r}.

The leaves of this foliation are connected components of the fibers of M^{2n+r} over X and are therefore closed. The foliation F defines on the restriction of the vector bundle E to any leaf a linear connection (see [4], p. 448), which we denote by ∇. The holonomy group of ∇ is everywhere trivial. Denoting again by Φ the restriction of Φ to E over a leaf we have $\nabla\Phi = 0$.

If there is on X also a hermitian metric, i. e. a Riemannian metric h satisfying $h(JX, JY) = h(X, Y)$, we can define a Riemannian metric g on M^{2n+r} as follows

$$g(\xi_{(i)}, \xi_{(j)}) = \delta_{ij} \qquad i, j = 1, \dots, r$$

$$g(\xi_{(i)}, X) = 0 \qquad X \in E, \quad i = 1, \dots, r$$

$$g(X, Y) = h(p_* X, p_* Y) \qquad X, Y \in E.$$

For other tangent vectors is g defined by the linear extension. It can be easily seen that g is a differentiable tensor field. We get

PROPOSITION 11.: The just defined Riemannian metric g is an admissible metric for the $(\Phi, \xi_{(i)}, \eta^{(i)})$-structure from proposition 10. Thus the tensors Φ, $\xi_{(i)}$, $\eta^{(i)}$, g define a $(\Phi, \xi_{(i)}, \eta^{(i)}, g)$ structure on M^{2n+r}. Denoting again by g the restriction of g to E over a leaf we have easily $\nabla g = 0$ i. e. the metric g is bundle-like.

Now we shall prove a kind of converse propositions to propositions 10 and 11. We have

PROPOSITION 10*: Let $(\Phi, \xi_{(i)}, \eta^{(i)})$ be a foliated almost r-contact structure on a manifold M^{2n+r}.

Let us suppose that all leaves of the foliation are closed and that there exists on M^{2n+r} a complete, bundle-like with respect to the foliation, Riemannian metric g (of course this metric need not be admissible for our structure). Denoting again by ∇ the linear connection defined by the foliation on the restriction of E to any leaf, we shall suppose that its holonomy group is everywhere trivial and that $\nabla\Phi = 0$. Then our $(\Phi, \xi_{(i)}, \eta^{(i)})$ — structure is the one constructed in proposition 10.

PROOF: By virtue of the theorem 4.4 from [4], there exist a manifold X and a projection $p: M^{2n+r} \rightarrow X$ of maximal rank such that the r-dimensional fibers of the fibered manifold (M^{2n+r}, p, X) are precisely the leaves of our foliation of M^{2n+r}. E obviously a horizontal distribution on the just constructed fibered manifold. Finally the vanishing of $\nabla\Phi$ allows us, using the vector-bundle projection $p_*: E \rightarrow T(X)$ to transfer Φ from E to $T(X)$, thus obtaining an almost-complex structure on X.

PROPOSITION 11*: We keep the assumptions and notations from the preceding proposition and we suppose more that there exists on M^{2n+r} a complete bundle-like Riemannian metric g which is admissible for the considered $(\Phi, \xi_{(i)}, \eta^{(i)}, g)$-structure. Then the so obtained $(\Phi, \xi_{(i)}, \eta^{(i)}, g)$-structure on M^{2n+r} is the one constructed in proposition 11.

PROOF: Using the results of proposition 10* we must show only that the metric g can be obtained from a hermitian metric on X. But it can be obtained from a Riemannian metric h on X because it is bundle-like, and this metric h is hermitian because g is admissible.

Now we shall treat the case of Lie almost r-contact structure. Let (P, p, X) be a fiber manifold with $\dim P = 2n + r$, $\dim X = 2n$. Let G be a r-dimensional Lie group operating on P in such a way that its classes of transitivity are precisely the fibers of P over X. Moreover we shall suppose that all isotropy subgroups of G are discrete. (As an example we can take a principal fiber bundle with the bundle space P, basis X, and the structural group G). We denote by $\mathcal{L}(G)$ the Lie algebra of G and we fix its basis $\Xi_{(1)}, \dots, \Xi_{(r)}$. Let $\xi_{(1)}, \dots, \xi_{(r)}$ be the vertical vector fields on P generated in a well-known way by the elements $\Xi_{(1)}, \dots, \Xi_{(r)}$. It can be easily seen that $\xi_{(1)}, \dots, \xi_{(r)}$ are linearly independent at every point of P. This is an immediate consequence of the fact that G operates transitively on the fibers and that its isotropy subgroups are discrete. Finally let M^{2n+r} be an open subset of P such that $p(M^{2n+r}) = X$, let E be a horizontal distribution on M^{2n+r} and let J be an automorphism of the vector bundle E such that $J^2 = -I$. Then we can construct a $(\Phi, \xi_{(i)}, \eta^{(i)})$-structure on M^{2n+r} as follows. We restrict $\xi_{(i)}$ to M^{2n+r}, and define 1-forms $\eta^{(i)}$ by (4.1). Endomorphism Φ is defined by

$$\Phi\xi_{(i)} = 0 \qquad i = 1, \dots, r$$

$$\Phi X = JX \qquad \text{for} \quad X \in E$$

and for other tangent vectors is defined by the linear extension. We get easily

PROPOSITION 12: The above constructed $(\Phi, \xi_{(i)}, \eta^{(i)})$-structure is a Lie almost r-contact structure. If moreover $M^{2n+r} = P$ is a principal fiber bundle with a structure group G, and if the horizontal distribution E is a connection on P, then its connection form

$$\omega = \sum_{i=1}^{r} \eta^{(i)} \otimes \Xi_{(i)} .$$

Now again we are going to prove a kind of converse proposition to proposition 12.

PROPOSITION 12*: Let $(M^{2n+r}\ p', X)$ be a fibered manifold with connected fibers and $\dim M^{2n+r} = 2n + r$, $\dim X = 2n$. Let $(\Phi, \xi_{(i)}, \eta^{(i)})$ be a Lie almost r-contact structure on M^{2n+r} such that the vector fields $\xi_{(i)}$ are tangent to the fibers of M^{2n+r} over X. Let G be a connected r dimensional Lie group, $\mathcal{L}(G)$ its Lie algebra and let $\Xi_{(1)}, \dots, \Xi_{(r)}$ be a basis of $\mathcal{L}(G)$ such that the linear extension Θ of the mapping $\Xi_{(i)} \longrightarrow \xi_{(i)}$ is a Lie algebra homomorphism of $\mathcal{L}(G)$ into the Lie algebra of all vector fields on M^{2n+r}. Moreover let the infinitesimal G-transformation group Θ (see [5]; Def. IV, p. 34) on M^{2n+r} be univalent (ibid., Def. VI, p. 62). Then the $(\Phi, \xi_{(i)}, \eta^{(i)})$-structure on M^{2n+r} is the one constructed in proposition 12.

PROOF: Let $x \in X$ be any point, and let M_x^{2n+r} denote a fiber of M^{2n+r} over x. It can be immediately seen from the definition of univalent infinitesimal G-transformation group that the restriction Θ_x of Θ to M_x^{2n+r} (i. e. the mapping $\Xi_{(i)} \longrightarrow$ restriction of $\xi_{(i)}$ to M_x^{2n+r}) is a univalent infinitesimal G transformation group on M_x^{2n+r}. Thus by virtue of the Principal Theorem (Theorem X, p. 75) from [5], we can find a universal globalization of Θ_x (see [5], Chap. III, § 1). In this way we get a manifold P_x of which M_x^{2n+r} is an open submanifold, and on which G operates in the way described in definition II of Chap. III in [5] (p. 59). Let $P = \bigcup_{x \in X} P_x$ and let us denote the operation of G on P_x and P by φ_x and φ respectively.

We shall provide P with a structure of differentiable manifold. We start with introducing a topology on P, which will be done using fundamental systems of neighborhoods of a point. M^{2n+r} is a subset of P and has its original topology. We keep this topology of M^{2n+r}, i. e. we define a fundamental system of neighborhoods of a point from M^{2n+r} to be a fundamental system of neighborhoods of the point under the original topology of M^{2n+r}. For a point $a \in P$, $a \notin M^{2n+r}$ we define a fundamental system of its neighborhoods as follows. Then is $a \in P_x$ for some $x \in X$, and we can find $b \in M_x^{2n+r}$ and $g \in G$ such that $a = bg$. If $\{\mathcal{U}_\alpha ; \alpha \in I\}$ is a fundamental system of neighborhoods of b, then a fundamental system of neighborhoods of a is defined to be $\{\mathcal{U}_\alpha g ; \alpha \in I\}$. Of course, now we must show that this way of introducing topology does not depend on the choice of b and g. Clearly, it is sufficient to show that having two points $b_1, b_2 \in M_x^{2n+r}$ and $g \in G$ such that $b_2 = b_1 g$, and being $\{\mathcal{U}_\alpha ; \alpha \in I\}$ a fundamental system of neighborhoods of b_1, then $\{\mathcal{U}_\alpha g ; \alpha \in I\}$ is a fundamental system of neighborhoods of b_2. But this can be proved in the following way. The infinitesimal G-transformation group Θ on M^{2n+r} is univalent and therefore, again by virtue of the Principal Theorem, admits a globalization. Hence we get a manifold Q of which M^{2n+r} is an open submanifold and on which G operates again in the way described in definition II of Chap. II in [5]. This operation of G on Q we denote by φ'. It generates on Q an involutive distribution of which M_x^{2n+r} is an integral submanifold. We denote by Q_x the maximal integral submanifold of this distribution containing M_x^{2n+r}.

Such a manifold clearly exists because M_x^{2n+r} is connected. Using the fact that Q_x is maximal and that the group G is connected we can find easily that Q_x is G-invariant and that the restriction φ'_x of φ' to Q_x is a globalization of Θ_x. Now using the universality of the globalization (P_x, φ_x) we get a homomorphism of (P_x, φ_x) into (Q_x, φ'_x). From existence of this homomorphism we can conclude immediately that the actions φ_x and φ'_x coincide on M_x^{2n+r} and therefore also the actions φ and φ' of G coincide on M^{2n+r}. But the action φ' of G on Q is differentiable and of course also continuous, and from this fact we can easily see that our topology is well defined. As the manifold structure of P is concerned, there is no more trouble. M^{2n+r} has its original manifold structure, and this one can be extended to P in the same way as the topology. Here we need only to know that the action φ of G on M^{2n+r} is differentiable. But this fact we have just proved above.

There is a natural projection $p: P \longrightarrow X$, namely for $a \in P_x$ there is $p(a) = x$. It is not difficult to see that with this projection (P, p, X) becomes a fibered manifold. The action φ of G on P is differentiable as a consequence of the fact that it is differentiable on M^{2n+r}, and the transitivity classes of this action are precisely the fibers P_x. Finally as the vector fields $\xi_{(1)}, \dots, \xi_{(r)}$ are linearly independent at every point, all isotropy subgroups of G an discrete. The proof is completed.

REMARK: We notice here that if M^{2n+r} is compact and G is taken to be simply connected, then every-infinetesimal G-transformation group acting on M^{2n+r} is even proper (see [5], Corollary 2, p. 82).

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