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Singular integrals on homogeneous spaces and some problems of classical analysis

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CONTENTS

Introduction.

In this paper we generalize some classical results of Calderón and Zygmund to the context of homogeneous spaces of locally compact groups and use these results to solve certain problems of classical type which can not be dealt with by the presently existing versions of the theory of singular integrals. Problems of this kind arise in studying the Cauchy-Szegö integral on the boundary of the complex unit ball and of the generalized halfplane in $\mathbf{C}^n$ holomorphically equivalent to the unit ball. In one variable this leads to the classical Hilbert transform. In [16] we sketched a theory

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which allowed us to prove L^p-continuity of the relevant operators for all $1 < p < \infty$. In the present paper we extend this theory somewhat further and bring more applications within its scope. We consider the case of vector-valued functions, which enables us to deal with the functions of Littlewood-Paley and Lusin on the generalized halfplane equivalent to the complex ball. Moreover by considering a class of operators slightly more general than ordinary convolutions (« twisted convolutions », cf. (1.1)) we get a theory which also applies to Riesz transforms defined on the (real) n-sphere.

In § 1 we describe the class of spaces (« homogeneous spaces with gauge ») and operators to which our theory applies. A homogeneous space with gauge is similar to an ergodic group in the sense of Calderón [2] but is somewhat more general, corresponding to the needs of our applications. The most important result in this section is Lemma 1.1, which is a variant of a result of Wiener.

The main result in § 2 is Theorem 2.2 which states that if a singular integral operator is continuous in L^2 then it is continuous in every L^p $(1 < p < \infty)$, provided that its kernel satisfies certain conditions analogous to those in [1]. In this section we follow [1] fairly closely; the only major change in the argument is that the covering lemma of Calderón and Zygmund [3, Lemma 1] has to be replaced by a slightly different one (Lemma 2.1), whose proof is based on our Lemma 1.1. Results very similar to those contained in this section have been announced for the case of ordinary convolutions by N. M. Rivière [20]; closely related results have also been obtained in unpublished work by A. P. Calderón and by R. R. Coifman and M. de Guzmán.

Theorem 2.2 is sufficient for three out of our four applications. In the case of the Riesz transforms, however, we have no way of directly proving L^2-continuity. In § 3 we prove a general theorem which under hypotheses on the kernel stronger than those of Theorem 2.2 guarantees L^2 continuity of the singular integral operator. This is a direct extension of a result of Knapp and Stein [13] which is, in turn, based on an idea of M. Cotlar [5], [6]. The main difficulty here is, of course, that Plancherel's theorem can not be used in the usual way because of the non commutativity of our groups.

In § 4 we prove a theorem about preservation of Lipschitz classes by an extension of the method of [4]. The conditions we seem to need here are slightly stronger than in Theorem 2.1 but not as strong as in Theorem 3.1.

In § 5 we deal with the important special case of homogeneous kernels on nilpotent Lie groups. In this case we obtain a very simple result (Theorem 5.1) on L^p-continuity; the special case $p = 2$ of this is due to Knapp and Stein [13].

In Part II. we discuss the applications listed at the beginning of this Introduction.

We wish to express our thanks to A. W. Knapp and E. M. Stein for showing us their results [13] before their publication, and to E. M. Stein for some useful conversations about the material in § 8.

The present paper was completed and preprints of it were distributed in February 1970. We also wish to thank those who by pointing out minor errors helped us to make some improvements on the text.

PART I. - GENERAL THEORY

§ 1. Definitions and basic facts.

Let G be a locally compact Hausdorff topological group, K a compact subgroup, $\pi: G \to G/K$ the canonical map. Let μ denote a left Haar measure on G, which we assume to be normalized in case G is compact.

DEFINITION 1.1. A gauge for (G, K) is a map $G \to [0, \infty)$, right-invariant under K and denoted $g \mapsto |g|$, such that

(i) the sets $B(r) = \{g \in G \mid |g| < r, \ (r > 0)\}$ are relatively compact and measurable; the sets $\pi B(r)$ form a basis of neighborhoods of $\pi(e)$ in G/K,

(ii) $|g^{-1}| = |g|$ for all $g \in G$,

(iii) $|gh| \leq \varkappa(|g| + |h|)$ for all $g, h \in G$ with some positive constant $\varkappa$.

(iv) $\mu(B(3\varkappa r)) \leq \bar{\varkappa}\, \mu(B(r))$ for all $r > 0$, with some constant $\bar{\varkappa}$ independent of r [1].

REMARK 1. In terms of the homogeneous space $X = G/K$ a gauge is equivalent with a function $\gamma: X \times X \to [0, \infty)$ such that $\gamma(x, y) = \gamma(y, x)$, $\gamma(gx, gy) = \gamma(x, y)$, $\gamma(x, z) \leq \varkappa(\gamma(x, y) + \gamma(y, z))$ for all $x, y, z \in X$, $g \in G$, and such that for some (and hence for all) $x \in X$ the sets $B_x(r) = \{y \in X \mid \gamma(x, y) < r\}$ $(r > 0)$ form a neighborhood basis at x, and $m(B_x(3\varkappa r)) \leq \bar{\varkappa}\, m(B_x(r))$ for

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(1) We have chosen this form of stating (iv) because it is the most convenient in our later applications. Of course, it is equivalent with saying that for some (and hence all) $c > 1$ there exists $\bar{c}$ such that $\mu(B(cr)) \leq \bar{c}\, \mu(B(r))$ for all $r > 0$.

all $r > 0$. This is clear by setting $|g| = \gamma(gp, p)$ where p denotes the identity coset.

2. G is necessarily unimodular; it is σ-compact, and totally σ-finite as a measure space. In fact (cf. [2]), let Δ be the modular function. Given $g \in G$, let $r > |g|$. Now $\Delta(g)\,\mu(B(r)) = \mu(B(r)\,g) \leq \mu(B(r)^2) \leq \bar{\varkappa}\,\mu(B(r))$, whence $\Delta(g) \leq \bar{\varkappa}$ for all g, and the first assertion follows. The others are obvious from (i).

3. By (i), (ii), every compact set $S \subset G$ is contained in some $B(r)$. (Note that, for every g, $gB(1) \subset B(\varkappa(|g| + 1))$ so g is in the interior of some $B(r)$). In particular, if G is compact then $G \subset B(r)$ for $r \geq r_0$ with some $r_0 > 0$.

4. There exist C, $a > 0$ such that $\mu(B(r)) \leq Cr^a$ for all $r > 0$. In fact, $\mu(B(2^n)) \leq \bar{2}^n \mu(B(1))$ for natural n; for any $x > 0$ it follows that $\mu(B(2^x)) \leq \bar{2}\cdot\bar{2}^x \mu(B(1))$. Writing $\bar{2} = 2^a$, $2^x = r$, we have $\mu(B(r)) \leq$ $\leq \bar{2}\mu(B(1))\, r^a$ (cf. [2]).

5. $|g| = 0$ if and only if $g \in K$. In fact, there exist g's with arbitrarily small gauge, otherwise $B(r)$ would be empty for small r, contradicting (i). Now $|e| \leq \varkappa(|g| + |g^{-1}|)$ implies $|e| = 0$, whence $|g| = 0$ for $g \in K$. Conversely, if $|g| = r > 0$ then $g \notin B(r/2)$, so $g \notin K$.

6. The gauge is measurable on G (by (i)) and continuous at e (also by (i)). The function $r \longmapsto \mu(B(r))$ is left continuous. If each $B(r)$ is open, the gauge is upper semi-continuous. These statements remain true for the function induced by the gauge on G/K.

7. The range of the gauge is discrete (in $\mathbf{R}$) if and only if G/K is discrete. In fact, if the range is discrete, K is open by the continuity of the gauge at e and by Remark 5. Conversely, if G/K is discrete, then K, being open, has positive measure. If $r \in \mathbf{R}$ were a limit point of the gauge, there would exist an infinite sequence of elements $g_n \in B(r + 1)$ with the $|g_n|$ all different, hence the sets $g_n K$ all disjoint. This would contradict the finiteness of $\mu(B(r + 1))$.

8. If $g \longmapsto |g|$ is a gauge for (G, K), then so is $g \longmapsto |g|^\alpha$ with any fixed $\alpha > 0$.

9. One could also consider gauges that are not defined everywhere on G, but satisfy (i)-(iv). One would assume then that if the right hand side

of (ii) or (iii) is meaningful, then so is the left. This means that the gauge would be defined on a subgroup $\widetilde{G}$ of G; by (i) and (iii) $\widetilde{G}$ would have to be open. This gives a generalization of the « ergodic groups » of Calderón [2]; ergodic groups correspond to the case $\varkappa = 1$.

The following lemma will play a fundamental role in § 2. For the case of $\mathbf{R}^n$ it is due to Wiener; more general variants of it, very close to the present one, can be found in [2] and [7].

LEMMA 1.1. Let $S \subset G$ and let $r : S \to (0, \infty)$ be a function such that $\mu(\bigcup_n g_n B(r(g_n))) < M$ whenever the sets $g_n B(r(g_n))$ are mutually disjoint. Then there exists a (finite or infinite) sequence $\{g_n\}$ in S such that

(i) the sets $g_n B(r(g_n))$ are mutually disjoint,

(ii) $\bigcup_n g_n B(3\varkappa\, r(g_n)) \supset S$.

PROOF. In case G is compact we may assume that $r(g) \leq r_0$ for all $g \in S$, where r_0 is the number in Remark 3. We pick $g_1 \in S$ such that

$$r(g_1) > \frac{1}{2} \sup_{g \in S} r(g).$$

The latter number is finite by our hypothesis even if G is non-compact, since in that case $\lim_{r \to \infty} \mu(B(r)) = \infty$ by Remark 3. By induction we pick $g_{n+1} \in S$ such that (writing r_j for $r(g_j)$)

$$(a) \quad g_{n+1} B(r_{n+1}) \cap g_j B(r_j) = \emptyset \qquad (1 \leq j \leq n),$$

$$(b) \quad r_{n+1} > \frac{1}{2} \sup \{ r(g) \mid gB(r(g)) \cap g_j B(r_j) = \emptyset \qquad (1 \leq j \leq n)\}.$$

Note that if this sequence does not end somewhere, then $\lim_{n \to \infty} r_n = 0$, since otherwise there would exist infinitely many disjoint $g_n B(r_n)$ with $r_n \geq \varepsilon > 0$, and their union would be of infinite measure.

We have to show that our sequence has property (ii). For this, let $g \in S$. Let l be the smallest number such that either $r_l < 1/2\, r(g)$ or that there is no g_l. (Note that if $S \neq \emptyset$, $l > 1$ by the choice of g_1). Now $gB(r(g))$ intersects some $g_j B(r_j)$ $(1 \leq j \leq l)$, or else g would have been picked instead of g_l in the construction of our sequence. Let h be an element in this intersection. Then $|g_j^{-1} g| = |g_j^{-1} h (g^{-1} h)^{-1}| \leq \varkappa (r_j + r(g)) \leq 3\varkappa r_j$. Hence $g \in g_j B(3\varkappa\, r_j)$, finishing the proof.

Corollary. Defining the « maximal function »

$$f^*(g) = \sup_{r>0} \frac{1}{\mu(B(r))} \int_{gB(r)} |f| \, d\mu$$

for every right K-invariant L^p-function f on G $(1 \leq p < \infty)$, the Hardy-Littlewood maximal theorem holds. Furthermore, for every right K-invariant locally integrable function f we have, for almost every $g \in G$,

$$f(g) = \lim_{r \to 0} \frac{1}{\mu(B(r))} \int_{gB(r)} f \, d\mu.$$

The proof is classical; for details see e. g. [7].

Definition 1.2. Let E be a Banach space. For $1 \leq p \leq \infty$, $L^p(G:K, E)$ is the subspace formed by (equivalence classes of) right K-invariant functions of the usual E-valued L^p-space on G. $L_0^\infty(G:K, E)$ is the set of (equivalence classes of) functions with compact support in $L^\infty(G:K, E)$. $M(G:K, E)$ is the space of strongly measurable right K-invariant functions $G \longrightarrow E$.

These spaces are of course just the E-valued L^p and other spaces on $X = G/K$, lifted to G. We will be interested in integral operators (and limits of such) on these spaces of the form

$$A\varphi(x) = \int_X S(x, y) \, \varphi(y) \, dy$$

where $S : X \times X \longrightarrow \mathcal{L}(E_1, E_2)$ such that $S(gx, gy) = \sigma_2(g) \, S(x, y) \, \sigma_1(g)^{-1}$ for all $g \in G$, with some uniformly bounded representations (2) σ_1, σ_2 of G on the Banach spaces E_1, E_2. These operators have the property $T_g A = \sigma_2(g) \, A T_g \, \sigma_1(g)^{-1}$ for all $g \in G$, where T_g denotes the action of G on X. If we lift all functions to G the operator A is of the form (3)

$$\text{(1.1)} \qquad Af(g) = \int_G \sigma_2(h) \, k(h^{-1}g) \, \sigma_1(h)^{-1} f(h) \, dh$$

(2) Throughout this paper by a « representation » we mean a strongly measurable representation.

(3) For reasons of convenience $\int f \, d\mu$ and $\int f(g) \, dg$ are used interchangably to denote the integral of f on G.

where $k(g) = S(gp, p)$, p denoting the identity coset; k satisfies the identity

$$k(hgh') = \sigma_2(h)\, k(g)\, \sigma_1(h)^{-1} \tag{1.2}$$

for all $h, h' \in K$. In case $E_1 = E_2 = \mathbf{C}$ and σ_1, σ_2 are trivial, (1.2) only means that k is left and right invariant under K, and (1.1) is an ordinary convolution $f * k$. In the general case it could be called a « twisted convolution » and denoted $f \underset{1,2}{*} k$ (one has to think of operators acting on E_1, E_2 on the right). The following three lemmas extend well known simple facts about convolutions to the twisted case.

LEMMA 1.2. Let σ_1, σ_2 be representations of G on the Banach spaces E_1, E_2 having uniform bound M_1, M_2, respectively.

Let $k \in L^1(G : K, \mathcal{L}(E_1, E_2))$ satisfy (1.2). Then, for all $1 \leq p \leq \infty$, A defined by (1.1) is a continuous linear transformation $L^p(G : K, E_1) \to L^p(G : K, E_2)$ with bound not greater than $M_1 M_2 \| k \|_1$.

PROOF. A change of variable in (1.1) gives

$$Af(g) = \int_G \sigma_2(gl)\, k(l^{-1})\, \sigma_1(gl)^{-1} f(gl)\, dl.$$

Now, by Minkowski's inequality,

$$\| Af \|_p = \left(\int_G | Af(g) |^p \, dg \right)^{1/p} \leq$$

$$\int_G \left(\int_G | \sigma_2(gl)\, k(l^{-1})\, \sigma_1(gl)^{-1} f(gl) |^p \, dg \right)^{1/p} dl \leq$$

$$M_1 M_2 \int_G | k(l^{-1}) | \left(\int_G | f(gl) |^p \, dg \right)^{1/p} dl = M_1 M_2 \| k \|_1 \| f \|_p ,$$

finishing the proof.

We denote by $\langle , \rangle$ the bilinear form connecting E and its dual E'. This may be a complex bilinear or a Hermitian form; the results that follow hold in either case. If E is a Hilbert space we always identify E with E' and $\langle , \rangle$ with the inner product on E.

In any case, the dual of $L^p(G : K, E)$ contains $L^{p'}(G : K, E')$ (here p' is the dual exponent to p, $p' = p/(p-1)$), and coincides with it if E is reflexive.

LEMMA 1.3. Let A be as in Lemma 1.2. Then, for any $\varphi \in L^{p'}(G: K, E_2')$, the adjoint A^* of A is given by

$$A^*\varphi(g) = \int \sigma_1(h)^{*-1} \check{k}(h^{-1}g)^* \sigma_2(h)^* \varphi(h)\, dh$$

where

$$\check{k}(l) = \sigma_2(l)\, k(l^{-1})\, \sigma_1(l)^{-1}.$$

PROOF. For every $f \in L^p(G: K, E_1)$,

$$\langle f, A^*\varphi \rangle = \langle Af, \varphi \rangle = \int \Big\langle \int \sigma_2(h)\, k(h^{-1}g)\, \sigma_1(h)^{-1} f(h)\, dh, \varphi(g) \Big\rangle\, dg =$$

$$\int \Big\langle f(h), \int \sigma_1(h)^{*-1}\, k(h^{-1}g)^*\, \sigma_2(h)^*\, \varphi(g)\, dg \Big\rangle\, dh =$$

$$\int \Big\langle f(h), \int \sigma_1(g)^{*-1}\, \check{k}(g^{-1}h)^*\, \sigma_2(g)^*\, \varphi(g)\, dg \Big\rangle\, dh$$

by Fubini's theorem. The assertion follows.

REMARK. If $k(g) = S(gp, p)$ with a kernel S as in the discussion preceding Lemma 1.2, then $\check{k}(g) = S(p, gp)$ for all $g \in G$.

LEMMA 1.4. Let σ_j be uniformly bounded representations of G on the Banach spaces E_j $(j = 1, 2, 3)$, let $k_{j,j+1} \in L^1(G: K, \mathcal{L}(E_j, E_{j+1}))$ $(j = 1, 2)$, and $A_{j,j+1} f(g) = \int \sigma_{j+1}(h)\, k_{j,j+1}(h^{-1}g)\, \sigma_j(h)^{-1} f(h)\, dh$. Then

$$A_{23} A_{12} f(g) = \int \sigma_3(h)\, k(h^{-1}g)\, \sigma_1(h)^{-1} f(h)\, dh$$

with

$$k(g) = \int \sigma_3(h)\, k_{2,3}(h^{-1}g)\, \sigma_2(h)^{-1}\, k_{1,2}(h)\, dh.$$

The proof is an easy computation left to the reader.

It may be remarked that, if we use the twisted convolutions mentioned before, this lemma expresses the associativity

$$(f_{1,2}^{*} k_{1,2})_{2,3}^{*} k_{2,3} = f_{1,2}^{*} (k_{1,2}{}^{*}_{2,3}\, k_{2,3}).$$

DEFINITION 1.3. Given a function k on G, for any $0 < \varepsilon < R$, we define $k^{\varepsilon,R}$ by

$$k^{\varepsilon,R}(g) = \begin{cases} k(g) & \text{if } \varepsilon < |g| < R \\ 0 & \text{otherwise} \end{cases}$$

and for any $0 < \varepsilon$,

$$k^{\varepsilon}(g) = \begin{cases} k(g) & \text{if } \varepsilon < |g| \\ 0 & \text{otherwise.} \end{cases}$$

DEFINITION 1.4. Let $\alpha > 0$ and let E be a Banach space. We denote by $\Lambda^{\alpha}(G : K, E)$ the class of right K-invariant functions $f : G \longrightarrow E$ such that, for some M, $\delta > 0$,

$$(1.3) \qquad |f(gh) - f(g)| \leq M |h|^{\alpha}$$

whenever $|h| < \delta$. We denote by $\Lambda_0^{\alpha}(G : K, E)$ the subclass made up of functions with compact support.

REMARK. Formulated for functions $\varphi : X \longrightarrow E$ (1.3) is equivalent with $|\varphi(x) - \varphi(y)| \leq M \gamma(x, y)^{\alpha}$ for $\gamma(x, y) < \delta$. Here γ is the same as in Remark 1 after Definition 1.1.

LEMMA 1.5. Let $f \in \Lambda^{\alpha}(G : K, E) \cap L^{p}(G : K, E)$ for some $\alpha > 0$, $1 \leq p < \infty$. Then f is bounded and tends to 0 at infinity.

PROOF. Clearly f is continuous, so it suffices to prove the second statement. Suppose it is false. Then there exists $\eta > 0$ such that outside of every ball there exists g with $|f(g)| \geq \eta$. By (1.3) there exists $\delta_1 > 0$ such that $|f(g)| \geq \eta$ and $|h| < \delta_1$ imply $|f(gh)| > \eta/2$. So we can find a sequence $\{g_n\}$ such that the balls $g_n B(\delta_1)$ are disjoint and $|f(g)| > \eta/2$ on each of them. This contradicts $f \in L^{p}$.

COROLLARY. If $f \in \Lambda^{\alpha}(G : K, E) \cap L^{p}(G : K, E)$ $(1 \leq p < \infty)$, then $|f(gh) - f(g)| \leq M' |h|^{\alpha}$ for all $g, h \in G$ with some $M' > 0$.

PROOF. The statement is true by (1.3) for $|h| < \delta$. If $|h| \geq \delta$, we have

$$|f(gh) - f(g)| \leq 2 \|f\|_{\infty} \leq \frac{2 \|f\|_{\infty}}{\delta^{\alpha}} |h|^{\alpha}.$$

DEFINITION 1.5. Let $\alpha, \beta > 0$. We say that the gauge satisfies $(L_{\alpha\beta})$ if there exists $\eta > 0$ such that

$$(1.4) \qquad \big|\, |gh|^{\beta} - |g|^{\beta} \,\big| \leq M |h|^{\alpha}$$

whenever $|g|$, $|h| < \eta$. We say that the gauge satisfies (L'_α) if

$$(1.5) \qquad ||gh| - |g|| \leq M |h|^\alpha |g|^{1-\alpha}$$

whenever $|h| \leq |g|$.

REMARK. If it is known that (1.5) holds whenever $N|h| \leq |g|$ with some $N > 1$, then it also holds automatically for all $|h| \leq |g|$ (possibly with a different M). In fact, if $|h| \leq |g| < N|h|$, we have

$$||gh| - |g|| \leq \varkappa(|g| + |h|) + |g| \leq (2\varkappa + 1)|g| \leq (2\varkappa + 1) N^\alpha |h|^\alpha |g|^{1-\alpha}.$$

LEMMA 1.6. (L'_α) implies $(L_{\alpha\alpha})$.

PROOF. If $|h| \leq |g|$, we have

$$||gh|^\alpha - |g|^\alpha| = ||gh| - |g|| \left| \frac{|gh|^\alpha - |g|^\alpha}{|gh| - |g|} \right| \leq M |h|^\alpha |g|^{1-\alpha} \left| \frac{\xi^\alpha - 1}{\xi - 1} \right|$$

with $\xi = |gh|/|g|$. By property (iii) of the gauge ξ is bounded, hence the last factor on the right hand side is bounded. If $|h| > |g|$, then

$$||gh|^\alpha - |g|^\alpha| \leq [\varkappa(|g| + |h|)]^\alpha + |g|^\alpha < (2^\alpha \varkappa^\alpha + 1)|h|^\alpha$$

which finishes the proof.

LEMMA 1.7. If the gauge satisfies $(L_{\alpha\beta})$ with some $\alpha, \beta > 0$, then $\Lambda_0^\alpha(G : K, E)$ is dense in every $L^p(G : K, E)$ $(1 \leq p < \infty)$.

PROOF. for $0 < r < \eta/2\varkappa$ we define ψ_r by

$$\psi_r(g) = \operatorname{Max}\left\{1 - \left(\frac{|g|}{r}\right)^\beta, 0\right\}.$$

Then $\psi_r \in \Lambda_0^\alpha(G : K, \mathbf{R})$. In fact, let $\delta = \eta/2\varkappa$. We show (1.3) for $|h| \leq \delta$ by distinguishing the four cases $|g|, |gh| \gtrless r$. If $|g| \geq r$, $|gh| < r$, we have $|g| \leq \varkappa(|gh| + |h^{-1}|) \leq \varkappa(r + \delta) < \eta$, and hence

$$|\psi_r(gh) - \psi_r(g)| = 1 - \left(\frac{|gh|}{r}\right)^\beta \leq \frac{|g|^\beta - |gh|^\beta}{r^\beta} \leq \frac{M}{r^\beta}|h|^\alpha.$$

The other three cases are obvious.

Let a_r be a scalar multiple of ψ_r such that $\int a_r \, d\mu = 1$. It is clear that for every $f \in L^1(G : K, E), f * a_r \in \Lambda^\alpha(G : K, E)$. We are going to show that every continuous right K- invariant $f \colon G \to E$ with compact support can be approximated to within any $\varepsilon > 0$ in the L^∞-norm by an $f * a_r$; this will immediately imply the lemma. We have

$$\|f * a_r - f\|_\infty = \sup_g \left| \int [f(gh) - f(g)] \, a_r(h^{-1}) \, dh \right| \leq \int \|R_h f - f\|_\infty \, a_r(h^{-1}) \, dh$$

where $R_h f$ denotes the right translate of f by h. By the uniform continuity of f we can find $r > 0$ such that $\|R_h f - f\|_\infty < \varepsilon$ for $|h| \leq r$. Since the support of a_r is the set $|h| \leq r$, the assertion follows.

The next lemma describes a case where the density of Λ_0^α can be proved under even less restrictive hypotheses about the gauge.

LEMMA 1.8. If G is a Lie group and there exists some $\alpha > 0$ and a local coordinate system $\{q_j\}$ on G/K at $\pi(e)$ such that $|q_j(\pi g)| \leq c|g|^\alpha$, then $\Lambda_0^\alpha(G : K, E)$ is dense in every $L^p(G : K, E)$ $(1 \leq p < \infty)$.

PROOF. It is known that the C^∞-functions f with compact support are dense in every L^p. But every such f is also in Λ_0^α. In fact by Taylor's formula,

$$|f(gh) - f(g)| = |\Sigma \, q_j(\pi h) f_j(g, h)|$$

with some smooth functions f_j, and on a compact neighborhood of $\pi(e)$ this is majorized by $M|h|^\alpha$ with some $M > 0$.

§ 2. The main result on L^p-continuity.

LEMMA 2.1. Let $f \geq 0$ be an integrable function on G. Then for every $\lambda > 0$ (resp. for every $\lambda \geq \|f\|_1$ if G is compact) there exists a sequence of mutually disjoint, right K-invariant measurable sets Q_n such that

(i) $g_n B(r_n) \subset Q_n \subset g_n B(3\varkappa r_n)$ with some $g_n \in G$, $r_n > 0$,

(ii) $\bar{\varkappa}^{-1} \lambda \leq \dfrac{1}{\mu(Q_n)} \displaystyle\int_{Q_n} f \, d\mu \leq \bar{\varkappa} \lambda$,

(iii) $f \leq \lambda$ a. e. outside of $\bigcup_n Q_n$.

PROOF. Denote by $(M_r f)(g)$ the mean of f on $gB(r)$, and let

$$E_\lambda = \{g \in G \mid \sup_{r>0} (M_r f)(g) > \lambda\}.$$

Then $f \leq \lambda$ outside of E_λ by the differentation theorem (Corollary to Lemma 1.1).

For each $g \in E_\lambda$ choose $r(g) > 0$ such that

$$(M_{r(g)} f)(g) > \lambda,$$

$$(M_{3\varkappa r(g)} f)(g) \leq \lambda.$$

This is possible since $\overline{\lim}_{r\to\infty} (M_r f)(g) \leq \lambda$ (trivially in the non-compact case; in the compact case because $(M_{r_0} f)(g) = \|f\|_1 \leq \lambda$).

The hypotheses of Lemma 1.1 are now clearly satisfied. Let $\{g_n\}$ be a sequence as in Lemma 1.1, write $r_n = r(g_n)$, and define, by induction,

$$Q_1 = g_1 B(3\varkappa r_1) - \bigcup_{j>1} g_j B(r_j)$$

$$Q_n = g_n B(3\varkappa r_n) - \bigcup_{j<n} Q_j - \bigcup_{j>n} g_j B(r_j).$$

Now (i) is obvious, and (iii) follows since $E_\lambda \subset \bigcup_n Q_n$ by Lemma 1.1. (ii) follows using property (iv) of the gauge by the following chain of inequalities:

$$\bar{\varkappa}^{-1}\lambda \leq \bar{\varkappa}^{-1}(M_{r_n} f)(g_n) \leq \frac{1}{\mu(B(3\varkappa r_n))} \int_{g_n B(r_n)} f\,d\mu \leq \frac{1}{\mu(Q_n)} \int_{Q_n} f\,d\mu \leq$$

$$\leq \frac{1}{\mu(B(r_n))} \int_{g_n B(3\varkappa r_n)} f\,d\mu \leq \bar{\varkappa}\,(M_{3\varkappa r_n} f)(g_n) \leq \bar{\varkappa}\,\lambda.$$

THEOREM 2.1. Let $A : L_0^\infty(G:K, E_1) \to M(G:K, E_2)$ be a linear map such that for some $r > 1$ and $c_1, c_2, c_3 > 0$,

(i) $\mu\{g \mid |Af(g)| > \lambda\} \leq \frac{c_1}{\lambda} \|f\|_r^r$ for all $\lambda > 0$ and all f,

(ii) $\int_{g_0 B(c_2\varrho)'} |Af|\,d\mu < c_3 \|f\|_1$ for all f supported on $g_0 B(\varrho)$ and such that $\int f\,d\mu = 0$ [4]. Then, for all $1 < p < r$, $Af \in L^p(G:K, E_2)$ and $\| Af \|_p \leq \leq c_p \| f \|_p$ with a constant c_p depending only on p, c_1, c_2, c_3.

PROOF. By the operator-valued extension of the Marcinkiewicz interpolation theorem [1, Lemma 1] it suffices to prove that

$$\mu \{ g \mid | Af(g) | > \lambda \} \leq \frac{c}{\lambda} \| f \|_1 \tag{2.1}$$

for all $\lambda > 0$ and all f, with some constant c.

Let therefore $\lambda > 0$ be given. If G is compact and $\lambda \leq \| f \|_1$, then $\mu \{ | Af | > \lambda \} \leq 1 \leq \lambda^{-1} \| f \|_1$, so (2.1) holds with $c = 1$. If $\lambda > \| f \|_1$, or if G is non-compact, we take the sets Q_n of Lemma 2.1 corresponding to $|f|$, aud define

$$\varphi(g) = \begin{cases} f(g) & \text{if } g \notin \bigcup_n Q_n \\ \dfrac{1}{\mu(Q_n)} \displaystyle\int_{Q_n} f\,d\mu & \text{if } g \in \bigcup_n Q_n \end{cases}$$

and $\psi = f - \varphi$. We have then $| \varphi(g) | \leq \bar{\varkappa} \lambda$ a. e.,

$$\| \varphi \|_1 \leq \| f \|_1, \quad \int_{Q_n} \varphi\,d\mu = 0, \quad \| \psi \|_1 \leq 2 \| f \|_1 .$$

If follows that

$$\mu \{ | A\varphi | > \lambda/2 \} \leq \frac{c_1 2^r}{\lambda^r} \| \varphi \|_r^r \leq \frac{c_1 2^r}{\lambda^r} \int (\bar{\varkappa} \lambda)^{r-1} | \varphi | \, d\mu \leq \frac{2^r c_1 \bar{\varkappa}^{r-1}}{\lambda} \| f \|_1 . \tag{2.2}$$

Now let $D = \bigcup_n g_n B(3\varkappa c_2 r_n)$ and let $\psi_n = \psi / _{Q_n}$. The support of ψ_n is in $g_n B(3\varkappa r_n)$, and $\int \psi_n\,d\mu = 0$. So hypothesis (ii) gives

$$\int_{D'} | A\psi | \, d\mu \leq \sum_n \int_{g_n B(3\varkappa c_2 r_n)'} | A\psi_n | \, d\mu \leq \sum_n c_3 \| \psi_n \|_1 \leq c_3 \| \psi \|_1 \leq 2 c_3 \| f \|_1 .$$

It follows that

$$\mu \{ | A\psi | > \lambda/2 \} \leq \mu(D) + \frac{2}{\lambda} \int_{D'} | A\psi | \, d\mu \leq \mu(D) + \frac{4c_3}{\lambda} \| f \|_1 .$$

---

(4) Here and elsewhere, for any $S \subset G$ we denote the complement of S by S'.

By Remark 4 after Definition 1.1 and by Lemma 2.1 we have $\mu(D) \leq \leq \sum_n \mu(B(3\varkappa c_2 r_n)) \leq \bar{c}_2 \sum_n \mu(B(r_n)) \leq \bar{c}_2 \sum_n \mu(Q_n)$, and again by Lemma 2.1, $\mu(Q_n) \leq (\bar{\varkappa}/\lambda) \int_{Q_n} |f|\, d\mu$. So, finally

$$(2.3) \qquad \mu\{|A\psi| > \lambda/2\} \leq \frac{\bar{c}_2\,\bar{\varkappa}}{\lambda} \sum_n \int_{Q_n} |f|\, d\mu + \frac{4c_3}{\lambda}\|f\|_1 \leq \frac{c_2\bar{\varkappa} + 4c_3}{\lambda}\|f\|_1$$

and (2.1) follows from (2.2) and (2.3).

LEMMA 2.2. Let σ_1, σ_2 be representations of G on the Banach spaces E_1, E_2, uniformly bounded by M_1, M_2, respectively. Assume that $k: G \to \to \mathcal{L}(E_1, E_2)$ satisfies (1.2) and is integrable on compact sets. If for all $h \in G$ and $u \in E_1$

$$\int_{|g| > c_2 |h|} |[\sigma_2(h)\, k(h^{-1}g)\, \sigma_1(h)^{-1} - k(g)]\, u|\, dg \leq c_3 |u|$$

with some constants c_2, c_3, then the operator A defined by

$$Af(g) = \int \sigma_2(h)\, k(h^{-1}g)\, \sigma_1(h)^{-1} f(h)\, dh$$

satisfies condition (ii) of Theorem 2.1.

PROOF. Suppose that $f \in L_0^\infty(G:K, E_1)$ with support contained in $g_0 B(\varrho)$ and $\int f\, d\mu = 0$. Making the variable change $h = g_0 l$ and then the change $g = g_0 m$ we find

$$\int_{g_0 B(c_2\varrho)'} |Af|\, d\mu = \int_{|g_0^{-1} g| > c_2\varrho} \left| \int \sigma_2(h)\, k(h^{-1}g)\, \sigma_1(h)^{-1} f(h)\, dh \right| dg =$$

$$\int_{|m| > c_2\varrho} \left|\sigma_2(g_0) \int_{|l| < \varrho} [\sigma_2(l)\, k(l^{-1}m)\, \sigma_1(l)^{-1} - k(m)]\, \sigma_1(g_0)^{-1} f(g_0 l)\, dl\right| dm.$$

In the last step we used $\int f \, d\mu = 0$. Using our hypothesis, the last expression is seen to be majorized by

$$M_2 \int\limits_{|l| < \varrho} c_3 \, | \, \sigma_1 (g_0)^{-1} f (g_0 \, l) \, | \, dl \leq M_1 \, M_2 \, c_3 \, \| f \|_1$$

finishing the proof.

LEMMA 2.3. Let σ_1, σ_2 be as in Lemma 2.2. Assume that $k : G \rightarrow \mathcal{L}(E_1, E_2)$ satisfies (1.2) and is integrable on all compact sets disjoint from K. If k satisfies the conditions

$$\text{(i)} \qquad \int\limits_{|g| < \varrho} | \, g \, |^\beta \, | \, k(g) \, u \, | \, dg \leq c_1 \, \varrho^\beta \, | \, u \, |$$

for all $\varrho > 0$, all $u \in E_1$ and some (hence all) fixed $\beta > 0$,

$$\text{(ii)} \qquad \int\limits_{|g| > 2\varkappa |h|} | \, [\sigma_2 (h) \, k (h^{-1} \, g) \, \sigma_1 (h)^{-1} - k(g)] \, u \, | \, dg \leq c_3 \, |u|$$

for all $u \in E_1$, then $k^{\varepsilon, R}$ satisfies them too, with a constant c_3' possibly different from c_3 but independent of ε and R.

PROOF. It is immediate that (i) for any $\beta > 0$ is equivalent with the condition

$$\int\limits_{a\varrho < |g| < b\varrho} | \, k (g) \, u \, | \, dg < c' \, | \, u \, |$$

for some (hence every) fixed $0 < a < b$ and all $\varrho > 0$, $u \in E_1$.

Now observe that if $\varrho > 4\varkappa^2 \, | \, h \, |$ then

$$\int\limits_{\varrho/2\varkappa < |g| < 2\varkappa\varrho} | \, \sigma_2 (h) \, k (h^{-1} \, g) \, \sigma_1 (h)^{-1} \, u \, | \, dg \leq \bar{c}_3 \, | \, u \, | \tag{2.4}$$

for all $u \in E_1$, with a constant $\bar{c}_3$ independent of ϱ. In fact, by (i) and (ii) the left hand side is majorized by

$$\int\limits_{|g| > \varrho/2\varkappa} | \, [\sigma_2(h) k (h^{-1} \, g) \sigma_1(h)^{-1} - k(g)] u | dg + \int\limits_{\varrho/2\varkappa < |g| < 2\varkappa\varrho} | \, k (g) \, u \, | \, dg \leq c_3 \, | \, u \, | + c' \, | \, u |.$$

Next we note that, if $2\varkappa\,|\,h\,|\leq|\,g\,|$, then

$$\frac{|\,g\,|}{2\varkappa}\leq|\,h^{-1}\,g\,|\leq 2\varkappa\,|\,g\,|. \tag{2.5}$$

In fact, the second inequality is immediate from properties (ii), (iii) of the gauge; the first follows by $|\,g\,|\leq\varkappa\,(\,|\,h\,|+|\,h^{-1}\,g\,|)\leq|\,g\,|\,/2+\varkappa\,|\,h^{-1}\,g\,|$.

Now (2.5) shows that, for $2\varkappa\,|\,h\,|\leq|\,g\,|$,

$$\sigma_2(h)\,k^{\varepsilon,R}(h^{-1}\,g)\,\sigma_1(h)^{-1}-k^{\varepsilon,R}(g)=\begin{cases}\sigma_2(h)\,k(h^{-1}\,g)\,\sigma_1(h)^{-1}-k(g) & \text{if } 2\varkappa\varepsilon<|\,g\,|<R/2\varkappa\\ 0 \quad \text{if } |\,g\,|<\varepsilon/2\varkappa & \text{or } |\,g\,|>2\varkappa R\end{cases}$$

therefore

$$\int\limits_{|\,g\,|>2\varkappa\,|\,h\,|}|\,[\sigma_2(h)\,k^{\varepsilon,R}(h^{-1}\,g)\,\sigma_1(h)^{-1}-k^{\varepsilon,R}(g)]u\,|\,dg\leq\int\limits_{|\,g\,|>2\varkappa\,|\,h\,|}|[\sigma_2(h)k(h^{-1}\,g)\,\sigma_1(h)^{-1}-k(g)]u|dg+$$

$$\int\limits_{\substack{\varepsilon/2\varkappa<|\,g\,|<2\varkappa\varepsilon\\ |\,g\,|>2\varkappa\,|\,h\,|}}|\,\sigma_2(h)\,k\,(h^{-1}g)\,\sigma_1\,(h)^{-1}\,u\,|\,dg+\int\limits_{\substack{\varepsilon/2\varkappa<|\,g\,|<2\varkappa\varepsilon\\ |\,g\,|>2\varkappa\,|\,h\,|}}|\,k\,(g)\,u\,|\,dg+$$

$$\int\limits_{\substack{R/2\varkappa<|\,g\,|<2\varkappa R\\ |\,g\,|>2\varkappa\,|\,h\,|}}|\,\sigma_2\,(h)\,k\,(h^{-1}\,g)\,\sigma_1\,(h)^{-1}\,u\,|\,dg+\int\limits_{\substack{R/2\varkappa<|\,g\,|<2\varkappa R\\ |\,g\,|>2\varkappa\,|\,h\,|}}|\,k\,(g)\,u\,|\,dg$$

The first integral is $\leq c_3\,|\,u\,|$ by hypothesis. The third and fifth are $\leq c'\,|\,u\,|$ by our first remark (applied with $h=e$). To estimate the second we distinguish two cases: If $\varepsilon\geq 4\varkappa^2\,|\,h\,|$, then it is $\leq\bar{c}_3\,|\,u\,|$ by (2.4). If $\varepsilon<4\varkappa^2\,|\,h\,|$, then $2\varkappa\varepsilon<8\,\varkappa^3\,|\,h\,|$, so the integral is majorized by

$$\int\limits_{2\varkappa\,|\,h\,|<|\,g\,|<8\varkappa^3\,|\,h\,|}|\,\sigma_2\,(h)\,k\,(h^{-1}\,g)\,\sigma_1\,(h)^{-1}\,u\,|\,dg$$

and this is $\leq\bar{c}_3\,|\,u\,|$ by (2.4) applied with $\varrho=4\varkappa^2\,|\,h\,|$. The fourth integral is estimated in the same way as the second. It follows that (ii) holds for $k^{\varepsilon R}$ with $c_3'=c_3+4\bar{c}_3$.

The following theorem is our main result. As to its hypothesis concerning Λ_0^α, cf. Lemmas 1.7 and 1.8.

THEOREM 2.2. Let σ_1,σ_2 be uniformly bounded representations of G on the Banach spaces E_1,E_2. Assume that for some $\alpha>0$, $\Lambda_0^\alpha(G:K,E_1)$

is dense in every $L^p(G:K, E_1)$ $(1 < p < \infty)$. Let $k: G \to \mathcal{L}(E_1, E_2)$ satisfy (1.2), let k be integrable on all compact sets disjoint from K, and such that

(i) defining, for all $f \in L_0^\infty(G:K, E_1)$, $A^{\varepsilon,R}f(g) = \int \sigma_2(h)\, k^{\varepsilon,R}(h^{-1}g) \cdot \sigma_1(h)^{-1} f(h)\, dh$, for some $1 < r < \infty$ we have $\| A^{\varepsilon,R} f\|_r \leq c \|f\|_r$ with c independent of ε, R, f,

(ii) for all $h \in G$, $u \in E_1$, $v \in E_2'$,

$$\int_{|g| \geq 2\varkappa |h|} |[\sigma_2(h)\, k(h^{-1}g)\, \sigma_1(h)^{-1} - k(g)]\, u \,|\, dg \leq c\,|u|,$$

$$\int_{|g| \geq 2\varkappa |h|} |[\sigma_2(h)\, \check{k}(h^{-1}g)\, \sigma_1(h)^{-1} - \check{k}(g)]^*\, v\,|\, dg \leq c\,|v|,$$

(iii) for all $\varrho > 0$, $u \in E_1$, $v \in E_2'$, and for some (hence all) fixed $\beta > 0$,

$$\int_{|g| \leq \varrho} |g|^\beta\, |k(g)\, u|\, dg \leq c\varrho^\beta\, |u|,$$

$$\int_{|g| \leq \varrho} |g|^\beta\, |k(g)^*\, v|\, dg \leq c\varrho^\beta\, |v|,$$

(iv) $\lim\limits_{\varepsilon \to 0} \int_{\varepsilon < |h| < 1} k(h)\, u\, dh$ exists for all $u \in E_1$.

Define $A^\varepsilon f(g) = \int \sigma_2(h)\, k^\varepsilon(h^{-1}g)\, \sigma_1(h)^{-1} f(h)\, dh$ for all $f \in L_0^\infty(G:K, E_1)$. Then, for all $1 < p < \infty$, $\| A^\varepsilon f\|_p \leq c_p \|f\|_p$ with c_p depending only on p, and $Af = \lim\limits_{\varepsilon \to 0} A^\varepsilon f$ exists in $L^p(G:K, E_2)$.

PROOF. First we claim that $\| A^\varepsilon f\|_r \leq c \|f\|_r$ for all $\varepsilon > 0$ and $f \in L_0^\infty(G:K, E_1)$. In fact, let $\varrho > 0$ be such that $B(\varrho)$ contains the support of f. We have

$$A^\varepsilon f(g) - A^{\varepsilon,R} f(g) = \int_{|h^{-1}g| > R} \sigma_2(h)\, k(h^{-1}g)\, \sigma_1(h)^{-1} f(h)\, dh$$

If $|g| < M$, this expression is 0 for $R \geq \varkappa(\varrho + M)$ by property (iii) of the gauge. Therefore

$$\int_{B(M)} |A^\varepsilon f|^r\, d\mu = \int_{B(M)} |A^{\varepsilon,R} f|^r\, d\mu \leq c\, \|f\|_r^r .$$

Since this is true for every $M > 0$, the claim is proved.

For $1 < p < r$ the first assertion of the Theorem now follows from (ii) and (iii) by Theorem 2.1 and Lemmas 2.2 and 2.3. Since (i) is automatically satisfied for $(A^{\varepsilon,R})^*$ with the dual exponent r', it follows as above that $\| (A^\varepsilon)^* f \|_{p'} \leq c_{p'} \| f \|_{p'}$ for $1 < p' < r'$. This implies our assertion for $r < p < \infty$.

To prove the second assertion of the theorem it is now enough to prove that $\lim_{\varepsilon \to 0} A^\varepsilon f$ exists in the L^p-sense for all $f \in \Lambda_0^\alpha(G : K, E_1)$. For this let $0 < \varepsilon' < \varepsilon < \delta$ where δ belongs to f as in the definition of Λ^α. By a change of variable we have

$$| A^{\varepsilon'} f(g) - A^\varepsilon f(g) | = \left| \int_{\varepsilon' < |h^{-1} g| < \varepsilon} \sigma_2(h) k(h^{-1} g) \sigma_1(h)^{-1} f(h)\, dh \right| \leq | \varphi(g) | + | \psi(g) |$$

where

$$\varphi(g) = \int_{\varepsilon' < |h| < \varepsilon} \sigma_2(gh) k(h^{-1}) \sigma_1(gh)^{-1} [f(gh) - f(g)]\, dh$$

$$\psi(g) = \int_{\varepsilon' < |h| < \varepsilon} \sigma_2(gh) k(h^{-1}) \sigma_1(gh)^{-1} f(g)\, dh =$$

$$\sigma_2(g) \int_{\varepsilon' < |h| < \varepsilon} k(h) \sigma_1(g)^{-1} f(g)\, dh.$$

As $\varepsilon \to 0$, $\varphi(g) \to 0$ uniformly in g, since, using (iii),

$$| \varphi(g) | = \sup_{|v|=1} | \langle \varphi(g), v \rangle | \leq$$

$$\sup_{|v|=1} \int_{\varepsilon' < |h| < \varepsilon} | \langle f(gh) - f(g), [\sigma_2(gh) k(h^{-1}) \sigma_1(gh)^{-1}]^* v \rangle |\, dh \leq$$

$$\sup_{|v|=1} \int_{\varepsilon' < |h| < \varepsilon} M |h|^\alpha M_1 | \check{k}(h)^* \sigma_2(g)^* v |\, dh \leq$$

$$M M_1 M_2 c\, \varepsilon^\alpha .$$

Also $\varphi(g) = 0$ independently of ε for $|g|$ sufficiently large; this is clear, since $|h| < \varepsilon < \delta$ and $|g|$ large imply that $|gh|$ is large, and so g, gh are both outside of the support of f. It follows that $\varphi \to 0$ in $L^p(G : K, E_2)$ as $\varepsilon \to 0$.

To show that also $\psi \to 0$, note that by (iv) ψ tends to 0 pointwise. By (iv) and the Banach-Steinhaus theorem

$$|\psi(g)| \le M_1 M_2 \left| \int_{\varepsilon' < |h| < \varepsilon} \hat{k}(h^{-1})\, dh \right| \cdot |f(g)| \le M' |f(g)| .$$

By the Lebesgue dominated convergence theorem the assertion follows.

REMARKS. 1. If $k^\varepsilon \in L^p$ for all $1 < p < \infty$ with some (and hence all) $\varepsilon > 0$, then the formula

$$A^\varepsilon f(g) = \int \sigma_2(h)\, k^\varepsilon(h^{-1} g)\, \sigma_1(h)^{-1} f(h)\, dh$$

is valid for all $f \in L^p(G : K, E_1)$ $(1 < p < \infty)$. In this case we also have $\lim_{R \to \infty} A^{\varepsilon, R} f = A^\varepsilon f$ in the L^p-sense.

2. Assuming only the first inequality of (ii) the theorem still holds for $1 < p < r$, by the same proof as above.

3. Both inequalities in (iii) are implied by the stronger hypothesis

$$\int_{|g| < \varrho} |g|^\beta\, |k(g)|\, dg \le c\varrho^\beta .$$

Instead of (ii) one can also assume inequalities about operator norms, but there are still two independent inequalities to assume. An easy computation using the unimodularity of G shows that these operator norm inequalities are equivalent with

$$\int_{|g| > 2\varkappa |h|} |\check{k}(gh) - \check{k}(g)|\, dg, \qquad \int_{|g| > 2\varkappa |h|} |k(gh) - k(g)|\, dg \le c.$$

4. The conditions of the theorem are easy to translate to the case were our operators are given in terms of a kernel s on $X \times X$ as in the discussion after Definition 1.2. E.g. the inequalities in (ii) become

$$\int_{\gamma(x,p) > 2\varkappa \gamma(y,p)} |[s(x,y) - s(x,p)]\, u|\, dx \le c\, |u|$$

$$\int_{\gamma(x,p) > 2\varkappa \gamma(y,p)} |[s(y,x) - s(p,x)]^* v|\, dx \le c\, |v|$$

§ 3. An L^2-theorem.

The first lemma is a simple extension of a result of Cotlar, Knapp and Stein [5] [13].

LEMMA 3.1. Let $\mathcal{K}$, $\mathcal{H}$ be Hilbert spaces. Let $A_1, A_2, \ldots$ be uniformly bounded operators $\mathcal{K} \to \mathcal{H}$ such that, for all $i, j > 0$

$$\| A_{i+j}^* A_i \|, \qquad \| A_{i+j} A_i^* \| \leq c\varepsilon^j \tag{3.1}$$

with some $0 < \varepsilon < 1$. Then $\| A_1 + \ldots + A_N \| \leq c'$ for all N, with some $c' > 0$ independent of N.

PROOF. We may assume $\dim \mathcal{K} \leq \dim \mathcal{H}$; otherwise we consider A_1^*, $A_2^*, \ldots$ instead of $A_1, A_2, \ldots$. Let $\gamma : \mathcal{K} \to \mathcal{H}$ be an isometric injection, i. e. $\gamma^* \gamma = I_{\mathcal{K}}$. Let $T_i = A_i \gamma^*$ $(i = 1, 2, \ldots)$. Now the sequence $T_1, T_2, \ldots$ obviously satisfies (3.1), and hence, by [13, Lemma 1], $\| T_1 + \ldots + T_N \| \leq c'$ for all N. Since $A_1 + \ldots + A_N = (A_1 + \ldots + A_N)\gamma^* \gamma = (T_1 + \ldots + T_N)\gamma$, the assertion follows.

The next two lemmas generalize results of Cotlar [6, p. 38].

LEMMA 3.2. Let σ_1, σ_2 be unitary representations of G on the Hilbert spaces H_1, H_2. Assume that $q_1, q_2, \ldots$ are integrable functions $G \to \mathcal{L}(H_1, H_2)$ satisfying (1.2) and such that

(i) $\left| \int q_i \, d\mu \right|, \left| \int \check{q}_i \, d\mu \right| \leq M\delta^i$ with some $0 < \delta < 1$, for all i,

(ii) $\int | q_i | \, d\mu \leq M$ for all i,

(iii) $q_i(g) = 0$ for $| g | > c\eta^i$, with some $0 < \eta < 1$, $c > 0$,

(iv) $\int | q_i(gh) - q_i(g) | \, dg$, $\int | \check{q}_i(gh) - \check{q}_i(g) | \, dg \leq M \eta^{-i\alpha} | h |^\alpha$ whenever $| h | < a\eta^i$, with some $\alpha > 0$, $a > 0$.

Then the conclusion of Lemma 3.1 holds for the operators $A_i : L^2(G : K, H_1) \to L^2(G : K, H_2)$ defined by $A_i f(g) = \int \sigma_2(h) q_i(h^{-1} g) \sigma_1(h)^{-1} f(h) \, dh$.

Proof. By Lemma 1.3 and 1.4, $A_{i+j}^* A_i$ is given by the kernel $k(g) = \int \sigma_1(h) \check{q}_{i+j}(h^{-1} g)^* \sigma_2(h)^{-1} q_i(h) \, dh$. After a change of variable we can write $k = k' + k''$, with

$$k'(g) = \int \sigma_1(gh)\, \check{q}_{i+j}(h^{-1})^* \, \sigma_2(gh)^{-1} [q_i(gh) - q_i(g)]\, dh,$$

$$k''(g) = \int \sigma_1(gh)\, \check{q}_{i+j}(h^{-1})^* \, \sigma_2(gh)^{-1} q_i(g)\, dh.$$

By Lemma 1.2, it will suffice to estimate the L^1-norm of k' and k''. By Fubini's theorem,

$$\int |k'(g)|\, dg \leq \int (|\check{q}_{i+j}(h^{-1})^*| \int |q_i(gh) - q_i(g)|\, dg)\, dh.$$

By (iii) the h-integral is 0 for $|h| > c\eta^{i+j}$, therefore in the g-integral we have to consider only $|h| \leq c\eta^{i+j}$. This implies $|h| < a\eta^i$ if $j \geq j_0$ with j_0 such that $c\eta^{j_0} < a$. For such j (iv) can be applied to give, using also (iii) and (ii),

$$\int |k'(g)|\, dg \leq M\eta^{-i\alpha} \int |q_{i+j}(h^{-1})| \cdot |h|^\alpha\, dh \leq M\eta^{-i\alpha}(c\eta^{i+j})^\alpha M = M^2 c^\alpha \eta^{\alpha j}.$$

On the other hand, by (i) and (ii),

$$\int |k''(g)|\, dg = \int \left| \int \sigma_1(g)\, q_{i+j}(h)\, \sigma_2(g)^{-1} q_i(g)\, dh \right| dg \leq$$

$$\int \left| \int q_{i+j}\, d\mu \right| \cdot |q_i(g)|\, dg \leq M^2 \delta^{i+j} \leq M^2 \delta^j.$$

This shows that $\| A^*_{i+j} A_i \| < M' \varepsilon^j$ with some $0 < \varepsilon < 1$ whenever $j \geq j_0$. The same inequality for $A_{i+j} A^*_i$ follows by interchanging the roles of q_i with $\check{q}^*_i$, σ_1 with σ_2.

Lemma 3.1 can not be directly applied because of the condition $j \geq j_0$. However, we can apply Lemma 3.1 to each of the finitely many families $\{A_{1+kj_0}\}_{k=0}^\infty$, $\{A_{2+kj_0}\}_{k=0}^\infty$, ... and get the desired conclusion.

LEMMA 3.3. Let $H_1, H_2, \sigma_1, \sigma_2$ be as in Lemma 3.2. Assume that $k_1, k_2, \ldots$ are integrable functions $G \to \mathcal{L}(H_1, H_2)$ satisfying (1.2) and such that

(i) $\int k_i\, d\mu = \int \check{k}_i\, d\mu = 0$ for all i,

(ii) $\int |k_i| \, d\mu < M$ for all i,

(iii) $k_i(g) = 0$ for $|g| > c\eta^{-i}$, with some $0 < \eta < 1$, $c > 0$,

(iv) $\int |k_i(gh) - k_i(g)| \, dg, \quad \int |\check{k}_i(gh) - \check{k}_i(g)| \, dg < M \eta^{i\alpha} |h|^\alpha$

whenever $|h| < a\eta^{-i}$, with some $\alpha > 0$, $a > 0$. Then the conclusion of Lemma 3.1 holds for the operators A_i: $L^2(G:K, H_1) \to L^2(G:K, H_2)$ defined by $A_i f(g) = \int \sigma_2(h) \, k_i(h^{-1}g) \, \sigma_1(h)^{-1} f(h) \, dh$.

PROOF. By (i) the kernel of $A_{i+j}^* A_i$ can be written in the form

$$k(g) = \int [\sigma_1(h) \check{k}_{i+j}(h^{-1}g)^* \sigma_2(h)^{-1} - \check{k}_{i+j}(g)^*] \, k_i(h) \, dh =$$

$$\int \sigma_1(g) [k_{i+j}(g^{-1}h)^* - k_{i+j}(g^{-1})^*] \, \sigma_2(g)^{-1} k_i(h) \, dh.$$

As in Lemma 3.2, for $j \geq j_0$,

$$\int |k(g)| \, dg \leq \int \left(\int |k_{i+j}(gh) - k_{i+j}(g)| \, dg \, |k_i(h)| \right) dh \leq$$

$$M \eta^{(i+j)\alpha} \int |h|^\alpha \, |k_i(h)| \, dh \leq M \eta^{(i+j)\alpha} M (c\eta^{-i})^\alpha = M^2 c^\alpha \eta^{\alpha j}$$

which gives the necessary estimate of $\| A_{i+j}^* A_i \| \cdot A_{i+j} A_i^*$ can be dealt with similarly, and the assertion follows as in Lemma 3.2.

THEOREM 3.1. Assume that the gauge satisfies (L'_α) for some $\alpha > 0$. Let σ_1, σ_2 be unitary representations of G on the Hilbert spaces H_1, H_2, let $k: G \to (H_1, H_2)$ satisfy (1.2); let k be integrable on compact sets disjoint from K, and such that

(*a*) for some $0 < \lambda < 1$, for all $A > 0$ and $|h| < \lambda A$,

$$\left.\begin{aligned} &\int_{|g|>A} |\sigma_2(h) \, k(h^{-1}g) \, \sigma_1(h)^{-1} - k(g)| \, dg \\ &\int_{|g|>A} |\sigma_2(h) \, \check{k}(h^{-1}g) \, \sigma_1(h)^{-1} - \check{k}(g)| \, dg \end{aligned}\right\} \leq c \left(\frac{|h|}{A} \right)^\alpha$$

(*b*) $\int\limits_{|g|<\varrho} |g|^\beta \, |k(g)| \, dg \leq c\,\varrho^\beta$ for all $\varrho > 0$, and some (hence all) fixed $\beta > 0$,

(*c*) with some $\nu > 0$ we have for all small $s > 0$,

$$\left| \int\limits_{\varrho<|g|<\varrho(1+s)} k(g)\, dg \right|, \quad \left| \int\limits_{\varrho<|g|<\varrho(1+s)} \check{k}(g)\, dg \right| \begin{cases} \leq c\,\varrho^\nu s & \text{for small } \varrho > 0 \\ = 0 & \text{for large } \varrho > 0. \end{cases}$$

Then all conditions of Theorem 2.2 are satisfied.

PROOF. By Lemmas 1.6 and 1.7 the condition about the density of Λ^α is satisfied. Conditions (ii), (iii), (iv) follow in a trivial way from (*a*), (*b*), (*c*) (cf. Remark 3 after Theorem 2.2). We only have to prove that condition (i) is fulfilled.

Let φ be a non-negative C^∞-function with support in the interval $(1/2, 2)$ and such that, defining $\varphi_j(x) = \varphi(2^{-j}x)$, $\sum_{-\infty}^{\infty} \varphi_j = 1$ on $(0, \infty)$. The existence of such a φ is shown in [11, Lemma 2.3].

We define $q_i(g) = \varphi_{-i}(|g|)\, k(g)$, $k_i(g) = \varphi_i(|g|)\, k(g)$ $(i = 1, 2, \ldots)$ and show that, for sufficiently large i, the conditions of Lemmas 3.2 and 3.3 are satisfied. This will finish the proof since a finite number of q_i's and k_i's, being integrable, can be neglected, and since for any $0 < \varepsilon < R$ we have j', j'' such that $2^{j'-1} < \varepsilon \leq 2^{j'}$, $2^{j''} \leq R < 2^{j''+1}$; hence $k^{\varepsilon,R} = \sum_{j'}^{j''} k\varphi_i + k' + k''$ where k', k'' have their support in $2^{j'-2} \leq |g| \leq 2^{j'+1}$ and $2^{j''-1} \leq |g| \leq 2^{j''+2}$, respectively, and both are majorized by $|k|$. By condition (*b*) $\|k'\|_1$, $\|k''\|_1$ are bounded independently of j', j'', so the statement follows from Lemma 1.2.

We shall check only the conditions of Lemma 3.2 involving the q_i's; the case of the $\check{q}_i$'s and the case of Lemma 3.3 are entirely analogous. To check (i) note that, since the support of φ_{-i} is contained in $(2^{-i-1}, 2^{-i+1})$, $\int q_i \, d\mu$ can be approximated in norm arbitrarily closely by a sum

$$\sum_{l=0}^{n-1} \varphi_{-i}\left(2^{-i-1}\left(1 + l\frac{4}{n}\right)\right) \int\limits_{D_l} k \, d\mu$$

where D_l is the set $2^{-i-1}\left(1 + l\frac{4}{n}\right) < |g| < 2^{-i-1}\left(1 + (l+1)\frac{4}{n}\right)$. This sum can be majorized in norm (using that $|\varphi_{-i}| \leq 1$) by

$$\sum_{l=0}^{n-1} \left| \int_{D_l} k\, d\mu \right| \leq \sum_{l=0}^{n-1} c\, 2^{-\lambda i} \frac{4}{n} = c' \left(\frac{1}{2^\lambda}\right)^i$$

with some constants c, c', proving (i). Condition (ii) is immediate from (b) by the inequalities

$$\int |q_i|\, d\mu \leq \int_{2^{-i-1} < |g| < 2^{-i+1}} |k(g)|\, dg \leq (2^{-i-1})^\alpha \int_{2^{-i-1} < |g| < 2^{-i+1}} |g|^\alpha\, |k(g)|\, dg \leq 4^\alpha c.$$

Condition (iii) follows from the definitions, with $\eta = \frac{1}{2}$, $c = \eta$.

Now we have to check (iv) with $\eta = \frac{1}{2}$. Let

$$a = \min \left\{ \frac{1}{4\varkappa}, \frac{\lambda}{4\varkappa} \right\}$$

and let $|h| < a\eta^i = 2^{-i} a$. We have

$$\int |q_i(gh) - q_i(g)|\, dg \leq \int |k(gh) - k(g)|\, \varphi_{-i}(|gh|)\, dg +$$
$$\int |\varphi_{-i}(|gh|) - \varphi_{-i}(|g|)| \cdot |k(g)|\, dg = I_1 + I_2 .$$

The integrand in I_1 is 0 if $|gh| < 2^{-i-1}$. This is certainly the case if $|g| < 2^{-i-2}/\varkappa$, by the choice of a. Condition (a) with $A = 2^{-i-2}/\varkappa$ can be applied, since the choice of a guarantees $|h| < \lambda A$, and gives

$$I_1 \leq c\, (2^{i+2}\varkappa)^\alpha\, |h|^\alpha .$$

The integrand in I_2 is 0 if $|g| < 2^{-i-2}/\varkappa$, by the same remark as above. Similarly it is 0 if $|g|, |gh| > 2^{-i+1}$, which is certainly the case if $|g| > 2^{-i+2}\varkappa$. Now note that $\|\varphi'_{-i}\|_\infty = 2^i \|\varphi'\|_\infty$, and that $|g| > 2^{-i+1}$, $|h| < a$ imply $|h| < |g|$. So, by (L'_α),

$$|\varphi_{-i}(|gh|) - \varphi_{-i}(|g|)| \leq \|\varphi'_{-i}\|_\infty \, ||gh| - |g|| \leq 2^i M' |h|^\alpha |g|^{1-\alpha}$$

where we wrote M' for $\|\varphi'\|_\infty$. It follows that

$$I_2 \leq 2^i M' |h|^\alpha \int_{2^{-i-2}/\varkappa < |g| < 2^{-i+2\varkappa}} |g|^{1-\alpha} |k(g)|\, dg \leq 2^i M' |h|^\alpha \left(\frac{2^{-i-2}}{\varkappa}\right)^{1-\alpha} \int_{2^{-i-2}/\varkappa < |g| < 2^{-i+2\varkappa}} |k(g)|\, dg.$$

By condition (*b*) the last integral is bounded independently of i, so we have $I_2 \leq M'' 2^{i\alpha} \mid h \mid^{\alpha}$, which, together with the estimate on I_1, finishes the proof.

§ 4. Preservation of Lipschitz classes.

LEMMA 4.1. Assume all the hypotheses of Theorem 2.2. In addition assume that for some (hence all) fixed $\beta > 0$

$$\int_{|g|<\varrho} |g|^{\beta} \, |k(g)| \, dg = O(\varrho^{\beta}) \tag{4.1}$$

as $\varrho \to 0$. Let, for each small $\varrho > 0$, S_ϱ be a measurable subset of $B(\varrho)$. Then, for all $1 < p < \infty$ and all $f \in \Lambda^{\beta} \cap L^p(G : K, E_1)$ we have

$$\int_{h^{-1} g \in S_\varrho} \sigma_2(h)\, k(h^{-1}g)\, \sigma_1(h)^{-1} [f(h) - f(g)]\, dh = O(\varrho^{\beta}).$$

PROOF. Introducing the new variable $l = g^{-1}h$, the norm of the left hand side can be majorized as follows.

$$\left| \int_{l^{-1} \in S_\varrho} \sigma_2(gl)\, k(l^{-1})\, \sigma_1(gl)^{-1} [f(gl) - f(g)]\, dl \right| \leq$$

$$M_1 M_2 M \int_{l^{-1} \in S_\varrho} |k(l^{-1})| \cdot |l|^{\beta} dl \leq M_1 M_2 M \int_{B(\varrho)} |k(l)| \cdot |l|^{\beta} dl = O(\varrho^{\beta}),$$

In particular, it follows that the integral we have written down exists.

LEMMA 4.2. Assume all the hypotheses of Theorem 2.2, assume (4.1) and assume that $k^{\varepsilon} \in L^p$ for all $1 < p < \infty$ with some (hence all) $\varepsilon > 0$. Let A be defined as in Theorem 2.2. Then for all $f \in \Lambda^{\beta} \cap L^p(G : K, E_1)$ $(1 < p < \infty)$ and all $\varrho > 0$ we have

$$Af(g) = \int_{|h^{-1}g|<\varrho} \sigma_2(h)\, k(h^{-1}g)\, \sigma_1(h)^{-1} [f(h) - f(g)] +$$

$$\int_{|h^{-1}g|<\varrho} \sigma_2(h)\, k(h^{-1}g)\, \sigma_1(h)^{-1} f(g)\, dh + \int_{|h^{-1}g| \geq \varrho} \sigma_2(h)\, k(h^{-1}g)\, \sigma_1(h)^{-1} f(h)\, dh.$$

Proof. The first integral exists by Lemma 4.1, the second $\left(\text{to be interpreted as } \lim_{\delta \to 0} \int_{\delta < |h^{-1}g| < \varrho}\right)$ by condition (iv) of Theorem 2.2, the third by $k^{\varrho} \in L^{p'}$ (cf. Remark 1 after Theorem 2.2).

Let $0 < \varepsilon < \varrho$. Then, defining A^{ε} as in Theorem 2.2, the right hand side is easily seen to be equal to

$$\int_{|h^{-1}g| < \varepsilon} \sigma_2(h)\, k(h^{-1}g)\, \sigma_1(h)^{-1} [f(h) - f(g)]\, dh + \int_{|h^{-1}g| < \varepsilon} \sigma_2(h)\, k(h^{-1}g)\, \sigma_1(h)^{-1} f(g)\, dh + A^{\varepsilon} f(g).$$

Both of the integrals tend to 0 as $\varepsilon \to 0$; the first by Lemma 4.1, the second by a change of variable and condition (iv) of Theorem 2.2. Therefore $A^{\varepsilon} f(g)$ converges pointwise. Since $\lim_{\varepsilon \to 0} A^{\varepsilon} f = Af$ in L^p by Theorem 2.2, it follows that the same is true pointwise a. e. This proves the lemma.

Theorem 4.1. Assume that the gauge satisfies (L'_{α}) with some $\alpha > 0$. Let σ_1, σ_2 be uniformly bounded representations of G on the Banach spaces E_1, E_2, such that $|\sigma_i(g) - \sigma_i(gh)| = O(|h|^{\beta})$ for all $0 < \beta < \alpha$, $(i = 1, 2)$. Assume that k satisfies all conditions of Theorem 2.2, and in addition,

(i′) $k^{\varepsilon} \in L^p(G : K, \mathcal{L}(E_1, E_2))$ for all $1 < p < \infty$, $\varepsilon > 0$,

(ii′) for some $0 < \lambda < 1$, for all $A > 0$ and $|h| < \lambda A$,

$$\int_{|g| > A} |\sigma_2(h)\, \check{k}(h^{-1}g)\, \sigma_1(h)^{-1} - \check{k}(g)|\, dg \leq c \left(\frac{|h|}{A}\right)^{\alpha}$$

(iii′) $\int_{|g| < \varrho} |g|^{\beta} |k(g)|\, dg \leq c\, \varrho^{\beta}$ for all $\varrho > 0$, and for some (hence all) $\beta > 0$,

(iv′) $\left| \int_{1 < |h| < R} \check{k}(h)\, dh \right| \leq c$ for all $R > 1$.

Then, for all $0 < \beta < \alpha$ and $1 < p < \infty$, the operator A of Theorem 2.2 maps $\Lambda^{\beta} \cap L^p(G : K, E_1)$ into $\Lambda^{\beta} \cap L^p(G : K, E_2)$.

PROOF. Let $g, l \in G$, denote $\varrho = |l|/\lambda$. We have to show that $Af(gl) - Af(g) = O(\varrho^{\beta})$, uniformly in g.

We pick a large $R > 0$, and using Lemma 4.2 write

$$Af(g) = \int_{|h^{-1}g|<\varrho} \sigma_2(h)\, k(h^{-1}g)\, \sigma_1(h)^{-1} [f(h) - f(g)]\, dh + \int_{\varrho<|h^{-1}g|<R} \sigma_2(h)\, k(h^{-1}g)\, \sigma_1(h)^{-1} [f(h) - f(g)]\, dh +$$

$$\int_{|h^{-1}g|<R} \sigma_2(h)\, k(h^{-1}g)\, \sigma_1(h)^{-1} f(g)\, dh + \int_{|h^{-1}g|>R} \sigma_2(h)\, k(h^{-1}g)\, \sigma_1(h)^{-1} f(h)\, dh = I_1 + I_2 + I_3 + I_4,$$

$$Af(gl) = \int_{|h^{-1}gl|<2\varkappa\varrho} \sigma_2(h)\, k(h^{-1}gl)\, \sigma_1(h)^{-1} [f(h) - f(gl)]\, dh + \int_{2\varkappa\varrho<|h^{-1}gl|<R} \sigma_2(h)\, k(h^{-1}gl)\, \sigma_1(h)^{-1} [f(h) - f(g)]\, dh +$$

$$\int_{|h^{-1}gl|<2\varkappa\varrho} \sigma_2(h)\, k(h^{-1}gl)\, \sigma_1(h)^{-1} f(gl)\, dh + \int_{2\varkappa\varrho<|h^{-1}gl|<R} \sigma_2(h)\, k(h^{-1}gl)\, \sigma_1(h)^{-1} f(g)\, dh +$$

$$\int_{|h^{-1}gl|>R} \sigma_2(h)\, k(h^{-1}gl)\, \sigma_1(h^{-1})\, f(h)\, dh = J_1 + J_2 + J_3 + J_4 + J_5 .$$

I_1 and J_1 are $O(\varrho^{\beta})$, by Lemma 4.1. I_4, J_5 tend to 0 as $R \to \infty$, since $k \in L^{p'}$. Denoting

$$a = \int_{|h|<2\varkappa\varrho} \check{k}(h)\, dh, \qquad b = \int_{2\varkappa\varrho<|h|<R} k(h)\, dh$$

we have, by some variable changes,

$$J_3 + J_4 - I_3 = \sigma_2(gl)\, a\, \sigma_1(gl)^{-1} f(gl) + \sigma_2(gl)\, b\, \sigma_1(gl)^{-1} f(g) - \sigma_2(g)(a + b)\, \sigma_1(g)^{-1} f(g) =$$

$$[\sigma_2(gl) - \sigma_2(g)]\, a\, \sigma_1(gl)^{-1} f(gl) + \sigma_2(g)\, a\, [\sigma_1(gl)^{-1} - \sigma_1(g)^{-1}] f(gl) +$$

$$\sigma_2(g)\, a\, \sigma_1(g)^{-1} [f(gl) - f(g)] +$$

$$[\sigma_2(gl) - \sigma_2(g)]\, b\, \sigma_1(gl)^{-1} f(g) + \sigma_2(g)\, b\, [\sigma_1(gl)^{-1} - \sigma_1(g)^{-1}] f(g).$$

a, b are uniformly bounded by conditions (iv) and (iv′), σ_1, σ_2 by hypothesis, and $f(g)$, $f(gl)$ by Lemma 1.5.

Furthermore, each term on the right contains a factor which is $O(\varrho^\beta)$. Therefore the whole expression is $O(\varrho^\beta)$, uniformly in g and R.

It remains to show only that $J_2 - I_2 = O(\varrho^\beta)$. We have

$$(4.2) \qquad J_2 - I_2 = -\int\limits_{glB(2\varkappa\varrho) - gB(\varrho)} \sigma_2(h)\, k(h^{-1} g)\, \sigma_1(h)^{-1} [f(h) - f(g)]\, dh +$$

$$\int\limits_{gB(R) \cap glB(R) - glB(2\varkappa\varrho)} \sigma_2(h)\, [k(h^{-1} gl) - k(h^{-1} g)]\, \sigma_1(h)^{-1} [f(h) - f(g)]\, dh +$$

$$\int\limits_{glB(R) - gB(R)} \sigma_2(h)\, k(h^{-1} gl)\, \sigma_1(h)^{-1} [f(h) - f(g)]\, dh -$$

$$\int\limits_{gB(R) - glB(R)} \sigma_2(h)\, k(h^{-1} g)\, \sigma_1(h)^{-1} [f(h) - f(g)]\, dh.$$

Since $|l| = \lambda\varrho < \varrho$, property (iii) of the gauge gives $glB(2\varkappa\varrho) \subset gB(\varkappa(2\varkappa + 1)\varrho)$; therefore, by Lemma 4.1, the first integral is $O(\varrho^\beta)$. We majorize the second integral by taking norms under the integral sign and increasing the domain to $gB(R) - gB(\varrho)$. After a change of variable this gives

$$M_1 M_2 \int\limits_{\varrho < |h| < R} |k(h^{-1} l) - k(h^{-1})| \cdot |f(gh) - f(g)|\, dh.$$

By the Corollary of Lemma 1.5, $|f(gh) - f(g)| < M |h|^\beta$ for all g; this gives a further majorization which can be written in the form

$$-\int_\varrho^R s^\beta\, dm_l(s)$$

where

$$m_l(s) = M M_1 M_2 \int\limits_{|h| > s} |k(hl) - k(h)|\, dh.$$

Integration by parts gives

$$-\int_{\varrho}^{R} s^{\beta}\, dm_l(S) = -R^{\beta} m_l(R) + \varrho^{\beta} m_l(\varrho) + \beta \int_{\varrho}^{R} s^{\beta-1} m_l(s)\, ds.$$

Using (ii′) if follows that the first term tends to 0 as $R \to \infty$ and that the other two terms are $O(\varrho^{\beta})$.

To estimate the third integral in (4.2) we take norms inside, use that $|f(g)|$ is uniformly bounded, and make a change of variable to get

$$M' \int_{\substack{|h|<R \\ |lh|>R}} |\check{k}(h)|\, dh.$$

By (L'_α), if ϱ is small (exactly if $M\varrho^{\alpha} \leq 1$), the domain of integration is getting larger by taking

(4.3) $$M' \int_{R-R^{1-\alpha}<|h|<R} |\check{k}(h)|\, dh.$$

It is clear from (iii′) (cf. proof of Lemma 2.3) that $\int_{R/2<|h|<R} |\check{k}(h)|\, dh$ is bounded by a constant independent of R.

It follows that we can find a sequence $R_n \to \infty$ such that the corresponding sequence of integrals (4.3) tends to 0.

The fourth integral in (4.2) can be treated similarly to the third, finishing the proof of the theorem.

§ 5. Homogeneous gauges and kernels.

In this section we assume that G is a real Lie group, $K = \{e\}$, and that there is given a (multiplicatively written) one-parameter group $\{a(t)\}$ of automorphisms of G which is *contracting* in the sense that

(5.1) $$\lim_{t\to 0} a(t)\, g = e$$

for all $g \in G$. We also assume that G has a gauge which is *homogeneous* in the sense that

(5.2) $$|a(t)\, g| = t\, |g|$$

for all $t > 0$, $g \in G$. Uniqueness of the Haar measure implies the existence of a positive number, which we will denote by q throughout this section, such that

$$d(a(t)g) = t^q\, dg. \tag{5.3}$$

We assume furthermore that the induced group $\{a_*(t)\}$ of automorphism of $\mathfrak{g}$, the Lie algebra of G, is diagonalizable over the reals. This means that on an appropriate basis $\{X_i\}$ of $\mathfrak{g}$, $a_*(t)$ acts by multiplications $X_i \mapsto t^{\lambda_i} X_i$, $\lambda_i > 0$. Throughout this section we shall denote by α the smallest one of the λ_i. α is uniquely determined by the group $\{a(t)\}$. Reparametrizing $\{a(t)\}$ and taking an appropriate power of the gauge so that (5.2) should remain valid (cf. Remark 8 after Definition 1.1), we could always arrange $\alpha = 1$, but we prefer the present more flexible arrangement.

It is easy to see that the existence of a contracting group of automorphisms implies that G is a simply connected nilpotent group [17].

It is also easy to see that every symmetric relatively compact neighborhood U of e determines a homogeneous gauge on G by the formula $|g| = \sup\{r > 0 \mid g \notin a(t)\, U \text{ for all } t > r\}$ [17]. Every homogeneous gauge clearly satisfies the condition of Lemma 1.8, but, as one easily sees on examples in $G = \mathbf{R}^2$, not every homogeneous gauge has a property $(L_{\alpha\beta})$ or is continuous on G. On the other hand there always exist continuous or even smooth homogeneous gauges; one obtains these by starting with a sufficiently regular U in the construction above.

DEFINITION 5.1. Let s be a real number. We say that a function u on $G - \{e\}$ is homogeneous of degree s if

$$u(a(t)g) = t^s u(g) \tag{5.4}$$

for all $g \neq e$ and all $t > 0$.

LEMMA 5.1. *If u is real-valued, homogeneous of degree s and integrable on compact sets not containing e, then, for every $0 < a < b$,*

$$\int_{a<|g|<b} u(g)\, dg = \begin{cases} c\,(b^{q+s} - a^{q+s}) & \text{if } s \neq -q \\ c\,(\log b - \log a) & \text{if } s = -q \end{cases}$$

with some constant c depending on u.

PROOF. Let $r_0 > 0$, and for $r > r_0$ define

$$\omega(r) = \int_{r_0<|g|<r} u(g)\, dg.$$

By the integrability hypothesis on u, ω is absolutely continuous with respect to $\varphi(r) = \mu(B(r))$. By (5.2) and (5.3) we have $\varphi(r) = cr^q$, hence ω is absolutely continuous with respect to r, and

$$\int_{a<|g|<b} u(g)\, dg = \int_a^b \omega'(r)\, dr \tag{5.5}$$

(note that ω' is independent of the choice of r_0). From (5.3) and (5.4) it follows easily that ω' is a homogeneous function of degree $s+q-1$ on $(0, \infty)$. Hence $\omega'(r) = cr^{s+q-1}$, and the assertion follows from (5.5).

LEMMA 5.2. Suppose that the gauge is continuous on G. Let E be a Banach space, and let $u : G - \}e\{ \to E$ be a continuously differentiable homogeneous function of degree s. Then there exist numbers $M, N \geq 1$ such that

$$\left.\begin{array}{l} |u(gh) - u(g)| \\ |u(hg) - u(g)| \end{array}\right\} < M |h|^\alpha |g|^{s-\alpha}$$

whenever

$$N|h| \leq |g|.$$

PROOF. We use the basis of $\mathfrak{g}$ introduced at the beginning of this section; we identify G with $\mathfrak{g}$ via the exponential map, and denote the i'th coordinate of an element g by g_i.
$(\partial u/\partial g_i)$ is homogeneous of degree $s - \lambda_i$; this is clear for the difference quotient of u regarded as a function on $G \times G$, and hence true for its limit. Also $(gh)_i$ is homogeneous of degree λ_i on $G \times G$; it is also a polynomial in $g_1, g_2, \ldots, h_1, h_2, \ldots$ since G is nilpotent.

Using the vector space structure of $\mathfrak{g}$ we define the line segment

$$\sigma(t) = t\, gh + (1-t)\, g \qquad (0 \leq t \leq 1).$$

We have

$$|u(gh) - u(g)| = \left| \int_0^1 \sum_i \frac{\partial u}{\partial g_i} \frac{d\sigma_i}{dt}\, dt \right| \leq \sum_i \int_0^1 \left| \frac{\partial u}{\partial g_i}\bigg|_{\sigma(t)} \right| \cdot |(gh)_i - g_i|\, dt. \tag{5.6}$$

There exists $N \geq 1$ such that $N|h| < |g|$ implies $(1/2)|g| < |\sigma(t)| < 2|g|$ for all $0 \leq t \leq 1$. In fact, for given g such that $|g| = 1$ and given t there exists $\varepsilon_{g,t} > 0$ such that $|h| < \varepsilon_{g,t}$ implies $1/2 < |\sigma(t)| < 2$. By a compactness argument then there exists $\varepsilon > 0$ such that $|h| < \varepsilon$ implies $1/2 < |\sigma(t)| < 2$ for all g with $|g| = 1$, and all $0 \leq t \leq 1$. By homogeneity of the gauge our assertion follows with $N = 1/\varepsilon$.

Now, by the homogeneity of $\frac{\partial u}{\partial g_i}$ we have

$$\left| \frac{\partial u}{dg_i}\Big|_{\sigma(t)} \right| \leq \operatorname*{Max}_{\frac{1}{2}|g| \leq |l| \leq 2|g|} \left| \frac{\partial u}{\partial g_i}\Big|_l \right| \leq M_i |g|^{s-\lambda_i} \tag{5.7}$$

whenever $N|h| \leq |g|$.

Since $(gh)_i - g_i$ is a homogeneous polynomial of degree λ_i vanishing for $h = e$, we have

$$(gh)_i - g_i = \sum_j h_j P_{ij}(g, h)$$

where the P_{ij} are homogeneous polynomials of degree $\lambda_i - \lambda_j$.

We have $|h_j| \leq M_j' |h|^{\lambda_i}$, with the constant $M_j' = \operatorname*{Max}_{|h|=1} |h_j|$. We can majorize $|P_{ij}(g, h)|$ by the sum of the absolute values of its monomial terms. Each such term is of the form $c \prod_{k,l} g_k^{r_k} h_l^{s_l}$ where $\sum r_k \lambda_k + \sum s_l \lambda_l = \lambda_i - \lambda_j$, by the homogeneity of P_{ij}, and is therefore majorized by

$$|c| \prod_{k,l} (M_k' |g|^{\lambda_k})^{r_k} (M_l' |h|^{\lambda_l})^{s_l} \leq M_{ij} |g|^{\lambda_i - \lambda_j}.$$

It follows that

$$|(gh)_i - g_i| \leq \sum_j |h_j| \cdot |P_{ij}(g_i h)| \leq \tag{5.8}$$

$$\sum_j M_j' |h|^{\lambda_j} M_{ij} |g|^{\lambda_i - \lambda_j} \leq M_i'' |h|^{\alpha} |g|^{\lambda_i - \alpha}$$

The first inequality of the lemma now follows from (5.6), (5.7), (5.8). The proof of the second inequality is similar.

COROLLARY. (Knapp-Stein [13, Lemma 2].) If the gauge is continuously differentiable on $G - \{e\}$, then it satisfies (L_α').

In fact, the gauge being homogeneous of degree 1, the inequality of (L_α') holds for $N|h| \leq |g|$. By the Remark after Definition 1.5 this is enough.

THEOREM 5.1. Suppose that the gauge is continuously differentiable on $G - \{e\}$. Let σ_1, σ_2 be the trivial representations of G on the Banach spaces E_1, E_2. Let $k : G - \{e\} \rightarrow \mathcal{L}(E_1, E_2)$ be continuously differentiable, homogeneous of degree $-q$, and such that

$$\int_{a<|g|<b} k(g)\,dg = 0$$

for all $0 < a < b$. Then the conditions of Theorems 2.2 and 4.1 are satisfied. If E_1, E_2 are Hilbert spaces, then the conditions of Theorem 3.1 are also satisfied.

PROOF. We check the conditions of Theorem 3.1 first; they automatically imply also the conditions of Theorem. 2.2.

To show (*a*) we apply Lemma 5.2 to k, and then Lemma 5.1 to get, for every $A > 0$ and $N|h| \leq A$,

$$\int_{|g|>A} |k(h^{-1}g) - k(g)|\,dg \leq M|h|^{\alpha} \int_{|g|>A} |g|^{-q-\alpha}\,dg = Mc\frac{|h|^{\alpha}}{A^{\alpha}}\,.$$

The analogous inequality for $\check{k}$ follows similarly. (*b*) is immediate from Lemma 5.1. (*c*) is trivial.

In passing we have also shown conditions (ii′) and (iii′) of Theorem 4.1. (iv′) is trivial; to show (i′) we note that $|k|^p$ is homogeneous of degree $-qp$, and so by Lemma 5.1 $\int |k^{\varepsilon}|^p\,d\mu$ exists and is equal to $c\,\varepsilon^{q(1-p)}$ for every $p > 1$. This finishes the proof.

The following theorem is a generalization of a result in [1]. It will be used in § 8.

THEOREM 5.2. Let $\varphi : G \to \mathbf{C}$ be such that, with some $\gamma > 0$, for all $g, h \in G$,

(i) $$|\varphi(g)| \leq c(1 + |g|)^{-q-4\gamma}$$

(ii) $$\int \varphi\,d\mu = 0$$

(iii) $$\left.\begin{aligned}&\int |\varphi(h^{-1}g) - \varphi(g)|\,dg\\ &\int |\varphi(gh^{-1}) - \varphi(g)|\,dg\end{aligned}\right\} \leq c|h|^{4\gamma}\,.$$

For $e \neq g \in G$ let $k(g): \mathbf{C} \to L^2(0, \infty)$ be defined by $(k(g)z)(t) = t^{-q-\frac{1}{2}}\varphi(a(t^{-1})g)z$. Then conditions (ii), (iii), (iv) of Theorem 2.2 and the conditions of Theorem 4.1

are satisfied. If, in addition, $\gamma = \alpha$ and

$$\text{(ii}'\text{)} \qquad \int_{a<|g|<b} \varphi(g)\, dg = 0 \qquad \text{for all } 0 < a < b,$$

then the conditions of Theorem 3.1 are also satisfied.

PROOF. For $g \neq e$ and $z \in \mathbf{C}$ we have, using (i),

$$|k(g)z|^2 = \int_0^\infty t^{-2q-1} |\varphi(a(t^{-1})g)|^2\, dt\, |z|^2 \leq$$

$$c^2 \int_0^\infty \frac{t^{-2q-1}}{[1+(|g|/t)]^{2q+8\gamma}}\, dt\, |z|^2 = \frac{c^2}{|g|^{2q}} \int_0^\infty \frac{s^{8\gamma-1}}{(1+s)_{2q+8\gamma}}\, ds\, |z|^2$$

which shows that $|k(g)| \leq \bar{c}\, |g|^{-q}$ with a constant $\bar{c}$. This, by Lemma 5.1, implies $k^* \in L^p$ for all $1 < p < \infty$, and

$$\int_{|g|<\varrho} |g|^\beta\, |k(g)|\, dg \leq c\varrho^\beta$$

for all $\beta > 0$, $\varrho > 0$.

Next we note that $\varphi \in L^1(G)$ by (i) and by Lemma 5.1. It follows that the integrals in (iii) are bounded independently of h (by the number $2c\, \|\varphi\|_1$). Consequently the right hand side of (iii) can be replaced by

$$\bar{c}\, \frac{|h|^{4\gamma}}{(1+|h|)^{4\gamma}}\,.$$

Also, by (i) we have

$$|\varphi(h^{-1}g) - \varphi(g)| \leq \bar{\bar{c}}\, |g|^{-q-2\gamma}$$

whenever $2\varkappa\, |h| \leq |g|$ (since the latter inequality implies $|h^{-1}g| > |g|/2$).

Using first the Schwarz inequality and then the facts just mentioned we obtain, for all $A > 0$, $2\varkappa\, |h| < A$,

$$\int_{|g|>A} |k(h^{-1}g) - k(g)|\, dg = \int_{|g|>A} dg \left(\int_0^\infty t^{-2q-1} |\varphi(a(t^{-1})(h^{-1}g)) - \varphi(a(t^{-1})g)|^2 dt \right)^{\frac{1}{2}} \leq$$

$$\left(\int\limits_{|g|>A}\int\limits_0^\infty t^{-2q-1}\,|g|^{q+2\gamma}\,|\varphi(a(t^{-1})(h^{-1}g))-\varphi(a(t^{-1})g)|^2\,dt\,dg\right)^{\frac{1}{2}}\cdot\left(\int\limits_{|g|>A}|g|^{-q-2\gamma}\,dg\right)^{\frac{1}{2}}\leq$$

$$\frac{c}{A^\gamma}\left(\int\limits_0^\infty t^{-q+2\gamma-1}\,dt\int|\varphi(a(t^{-1})(h^{-1}g))-\varphi(a(t^{-1})g)|\,dg\right)^{\frac{1}{2}}\leq$$

$$\frac{c'}{A^\gamma}\left(\int\limits_0^\infty t^{2\gamma-1}\frac{(|h|/t)^{4\gamma}}{[1+(|h|/t)]^{4\gamma}}\,dt\right)^{\frac{1}{2}}=\frac{c'}{A^\alpha}\left(\int\limits_0^\infty|h|^{2\gamma}\frac{s^{2\gamma-1}}{(1+s)^{4\gamma}}\,ds\right)^{\frac{1}{2}}=c''\left(\frac{|h|}{A}\right)^\gamma.$$

A similar inequality follows for $\check{k}$ by the same proof.

Now let $0<a<b$. We have

$$I(a,b)=\left|\int\limits_{a<|g|<b}k(g)\,dg\right|^2=\int\limits_0^\infty\left|\int\limits_{a<|g|<b}t^{-q-\frac{1}{2}}\varphi(a(t^{-1})g)\,dg\right|^2dt=\int\limits_0^\infty\frac{dt}{t}\left|\int\limits_{a/t<|g|<b/t}\varphi(g)\,dg\right|^2.$$

It is now clear that (ii′) implies condition (c) of Theorem 3.1. If only (ii) is assumed, we observe that it implies

$$\int\limits_{|g|\leq r}\varphi(g)\,dg=-\int\limits_{|g|>r}\varphi(g)\,dg$$

whence, by (i) and by Lemma 5.1 it follows that

$$\left|\int\limits_{|g|\leq r}\varphi(g)\,dg\right|\leq c\,\frac{r^q}{(1+r)^{q+4\gamma}}\,.$$

This implies at once that $I(a,b)$ is bounded independently of a, b and converges as $a\to 0$. This finishes the proof.

Remarks 1. It is clear that $I(a,b)$ depends only on the ratio of a and b. Therefore condition (c) of Theorem 3.1 is satisfied if and only if (ii′) holds.

2. If we define a one-parameter group $\{U(s)\}$ of unitary transformations on $L^2(0,\infty)$ by $(U(s)f)(t)=s^{-1/2}f(t/s)$, we have

$$k(a(s)g)=s^{-q}\,U(s)\,k(g)$$

for all $s > 0, g \neq e$. This is a kind of generalized homogeneity for k similar to that which, in the case of $G = \mathbf{R}^n$, was studied in [21].

PART II. - APPLICATIONS.

§ 6. The Cauchy-Szegö integral for the generalized halfplane D.

Let $D \subset \mathbf{C}^n$, $n \geq 2$, be the generalized halfplane

$$D = \left\{ z = (z_1, \dots, z_n) \,\middle|\, \operatorname{Im} z_1 - \sum_2^n |z_k|^2 > 0 \right\}$$

and let B be its boundary in $\mathbf{C}^n$. D is the image of the complex unit ball under a generalized Cayley transformation T defined by

$$T : z_1 \longmapsto i\,\frac{1 - z_1}{1 + z_1}, \quad z_k \longmapsto i\,\frac{z_k}{1 + z_1} \quad (2 \leq k \leq n). \tag{6.1}$$

T is the same as the Cayley transformation of [15] preceded by the map $z_1 \longmapsto -z_1$. The transformation of [15] is, in turn, the same as the c of [14] followed by a map $z_k \longmapsto \sqrt{2}\, z_k \, (2 \leq k \leq n)$.

There is a group $\mathfrak{N}$ of holomorphic automorphisms of D, which as a set equals $\mathbf{R} \times \mathbf{C}^{n-1}$, an element $g = (\xi, \zeta)$ acting by

$$\begin{aligned} z_1 &\longmapsto z_1 + \xi + 2i \sum_2^n \bar{\zeta}_k z_k + i |\zeta|^2 \\ z_k &\longmapsto z_k + \zeta_k \qquad\qquad (2 \leq k \leq n) \end{aligned}$$

where $|\zeta|^2$ stands for $\sum_2^n |\zeta_k|^2$. $\mathfrak{N}$ is simply transitive on B, so $g \longmapsto g \cdot 0$ is a one-to-one map of $\mathfrak{N}$ onto B; the point $z = (z_1, \dots, z_n)$ corresponding to the element $g = (\xi, \zeta)$ is given by

$$z_1 = \xi + i |\zeta|^2, \quad z_k = \zeta_k \, (2 \leq k \leq n).$$

Multiplication in $\mathfrak{N}$ is given by

$$(\xi, \zeta)(\xi', \zeta') = \left(\xi + \xi' - 2 \operatorname{Im} \sum_2^n \zeta'_k \bar{\zeta}_k ,\ \zeta + \zeta' \right).$$

$\mathfrak{N}$ has a contracting one-parameter group $\{a(t)\}$ of automorphisms defined by

$$a(t)(\xi, \zeta) = (t\xi, t^{\frac{1}{2}}\zeta).$$

It is easy to see that ξ, $\operatorname{Re}\zeta_2$, $\operatorname{Im}\zeta_2, \ldots, \operatorname{Im}\zeta_n$ are canonical coordinates in $\mathfrak{N}$, therefore the minimal exponent α defined at the beginning of § 5 is now 1/2. For the Haar measure of $\mathfrak{N}$ we have

$$d(a(t)g) = t^n\, dg.$$

We define a homogeneous gauge in $\mathfrak{N}$ by

$$|g| = |(\xi, \zeta)| = (\xi^2 + |\zeta|^4)^{\frac{1}{2}}.$$

In [15] we used the gauge $\operatorname{Max}\{|\xi|, |\zeta|^2\}$ which is also homogeneous. The gauge used here has the advantage of being smooth on $\mathfrak{N} - \{e\}$, and that if $u \in B$ is the point corresponding to $g \in \mathfrak{N}$ (i. e. if $u = g \cdot 0$) then

$$|g| = |u_1| \tag{6.2}$$

with ordinary absolute value on the right hand side. It is also clear that the generalized distance function γ of Remark 1 after Definition 1.1 is now given by $\gamma(u, v) = |\varrho(u, v)|$, where $\varrho(u, v) = i(\bar{v}_1 - u_1) - 2\Sigma\, u_k \bar{v}_k$, as in [15].

The Szegö kernel of D is given by

$$S(z, w) = c_n\left(i(\bar{w}_1 - z_1) - 2\sum_2^n z_k \bar{w}_k\right)^{-n} \qquad c_n = \frac{2^{n-2}\,\Gamma(n)}{\pi^n}$$

(cf. 15]; the value of c_n given there is wrong, the fact that the Cayley transform used is not exactly the same as c in [14] having been overlooked). It is known [17] that for every $F \in L^2(B)$, $PF(z) = \int_B S(z, u)\, F(u)\, d\beta(u)$ defines an H^2 function on D and P is the orthogonal projection operator onto H^2 if we identify H^2 with a subspace of $L^2(B)$ by taking boundary values. The measure β on B is defined by $d\beta(u) = d(\operatorname{Re} u_1)\, d(\operatorname{Re} u_2)\, d(\operatorname{Im} u_2) \ldots$ $\ldots d(\operatorname{Im} u_n)$. It is also known that writing $(PF)|_{B+it} = (PF)_t$ and denoting by $\tilde{\varphi}$ the lift to $\mathfrak{N}$ of a function φ on B, i. e. $\tilde{\varphi}(g) = \varphi(g \cdot 0)$, we have, for all $t > 0$,

$$(PF)_t^{\sim} = \tilde{F} * k_t$$

where $k_t(g) = S(g \cdot 0, (it, 0))$; so, in the present case,

$$k_t(\xi, \zeta) = c_n (t + |\zeta|^2 - i\xi)^{-n}.$$

Since β lifts to a Haar measure on $\mathfrak{H}$, it follows that $\|f * k_t\|_2 \leq \|f\|_2$ for all $t > 0$, $f \in L^2(\mathfrak{H})$, and $\lim_{t \to 0} f * k_t$ is the orthogonal projection of f onto the subspace corresponding to boundary values of H^2-functions.

We wish to study the singular Cauchy-Szegö integral given by the kernel $k(g) = S(g \cdot 0, 0)$, i. e., explicitly, by

$$k(\xi, \zeta) = c_n (|\zeta|^2 - i\xi)^{-n}.$$

We will show that Theorem 5.1 can be applied to it, and we will also find the connections between the operators $f \mapsto f * k_\varepsilon$ and $f \mapsto f * k^\varepsilon$ (k^ε defined as in § 1).

Lemma 6.1.

$$\int_{a < |g| < b} k(g)\, dg = 0 \text{ for all } 0 < a < b.$$

Proof. Let dV_ζ denote the Euclidean volume element in $\mathbf{C}^{n-1}$. To compute the integral we introduce polar coordinates in $\mathbf{C}^{n-1}$

$$(6.3) \qquad \varrho = |\zeta|, \quad \zeta' = \frac{\zeta}{|\zeta|}$$
$$dV_\zeta = \varrho^{2n-3}\, d\varrho\, d\zeta'$$

where $d\zeta'$ is the surface element on S^{2n-3}, the unit sphere in $\mathbf{C}^{n-1}$. Next we make the variable change

$$(6.4) \qquad u = \varrho^2$$
$$du = 2\varrho\, d\varrho$$

and then introduce polar coordinates in the ξ, u plane in which we have to integrate on the upper halfplane,

$$(6.5) \qquad \xi = s\cos\theta, \quad u = s\sin\theta$$
$$du\, d\xi = s\, ds\, d\theta.$$

We find

$$\int_{a<|g|<b} k(g)\,dg = c_n \int_{a^2<|\zeta|^4+\xi^2<b^2} \frac{d\xi\, dV_\zeta}{(|\zeta|^2 - i\xi)^n} =$$

$$c_n \frac{|S^{2n-3}|}{2} \int_a^b \frac{ds}{s} \int_0^\pi (-\,ie^{i\theta})^{-n} \sin^{n-2}\theta\, d\theta$$

and this is equal to 0 since by an easy computation we have, for all $n \geq 2$,

$$\int_0^\pi e^{-in\theta} \sin^{n-2}\theta\, d\theta = 0. \tag{6.6}$$

LEMMA 6.2. Let, for $0 < a < b$, $n \geq 2$

$$J(a, b) = \int_0^\pi d\theta \sin^{n-2}\theta \int_a^b \frac{d\varrho}{\varrho(\varrho - ie^{i\theta})^n} .$$

Then

$$\lim_{\substack{a \to 0 \\ b \to \infty}} J(a, b) = \frac{\pi}{2^{n-1}(n-1)} .$$

PROOF. Differentiating $n-1$ times with respect to x the identity

$$\frac{1}{\varrho(\varrho - x)} = \left(-\frac{1}{\varrho} + \frac{1}{\varrho - x}\right)\frac{1}{x}$$

by Leibniz's rule we obtain

$$\frac{1}{\varrho(\varrho - x)^n} = \frac{(-1)^n}{\varrho x^n} + \sum_{k=0}^{n-1} \frac{(-1)^k}{x^{k+1}(\varrho - x)^{n-k}} .$$

Substituting this identity with $x = ie^{i\theta}$ into the definition of $J(a, b)$, we find

$$J(a, b) = \int_0^\pi d\theta \sin^{n-2}\theta \Bigg[(-\,ie^{i\theta})^{-n} \log\frac{b}{a} +$$

$$+ \sum_{k=0}^{n-2} \frac{(-1)^k (ie^{i\theta})^{-k-1}}{k+1-n} \left(\frac{1}{(b - ie^{i\theta})^{n-k-1}} - \frac{1}{(a - ie^{i\theta})^{n-k-1}}\right)\Bigg]$$

$$+ (-1)^{-1} \int_0^\pi d\theta \sin^{n-2}\theta\, (ie^{i\theta})^{-n} \int_a^b \frac{d\varrho}{\varrho - ie^{i\theta}} .$$

The integral of the term containing $\log \frac{b}{a}$ is 0 by (6.6). In the terms under the summation sign we let $b \to \infty$ and $a \to 0$; in the limit each term is a constant multiple of $e^{-in\theta}$, so the integral is again 0 by (6.6). We denote the last term by $J'(a, b)$ and rewrite it as

$$J'(a, b) = -i^n \int_a^b d\varrho \int_0^\pi e^{-in\theta} \sin^{n-2}\theta \frac{d\theta}{\varrho - ie^{i\theta}} = -i^n \int_a^b I(\varrho)\, d\varrho.$$

We rewrite $I(\varrho)$ as a complex line integral setting $z = e^{-i\theta}$, $dz = -ie^{i\theta}\, d\theta$ and denoting by Γ the lower half of the unit circle from 1 to -1:

$$I(\varrho) = \frac{1}{(2i)^{n-2}} \int_\Gamma \frac{(1-z^2)^{n-2} z^2}{\varrho z - i}\, dz.$$

The denominator has a zero in the upper halfplane only; therefore by Cauchy's theorem Γ can be changed to the straight line segment from 1 to -1. Now we have

$$I(\varrho) = \frac{-i}{(2i)^{n-2}} \left[i \int_{-1}^{1} \frac{(1-x^2)^{n-2} x^2}{(\varrho x)^2 + 1}\, dx + \varrho \int_{-1}^{1} \frac{(1-x^2)^{n-2} x^3}{(\varrho x)^2 + 1}\, dx \right].$$

The second integral is zero since the integrand is odd; the first integrand is even. So we have

$$J'(a, b) = \frac{1}{2^{n-3}} \int_0^1 dx\, (1-x^2)^{n-2} x \int_a^b \frac{x\, d\varrho}{(x\varrho)^2 + 1}.$$

The ϱ-integral equals $\tan^{-1} bx - \tan^{-1} ax$, and tends boundedly to $\pi/2$ as $a \to 0$, $b \to \infty$. Therefore

$$\lim_{\substack{a \to 0 \\ b \to \infty}} J'(a, b) = \frac{\pi}{2^{n-2}} \int_0^1 (1-x^2)^{n-2} x\, dx = \frac{\pi}{2^{n-1}(n-1)}$$

finishing the proof.

COROLLARY. For every fixed $R > 0$, $\lim_{\varepsilon \to 0} \int_{|g| < R} k_\varepsilon(g)\, dg = \frac{1}{2}$.

PROOF. Let $0 < \delta < \varepsilon < R$. By the variable changes (6.3), (6.4), (6.5) we have

$$\int\limits_{\delta < |g| < R} k_\varepsilon(g)\, dg = c_n \frac{|S^{2n-3}|}{2} \int\limits_0^\pi \sin^{n-2}\theta\, d\theta \int\limits_\delta^R \frac{s^{n-1}\, ds}{(\varepsilon - is e^{i\theta})^n}.$$

After the variable change $s = \varepsilon/\varrho$ this is equal to $\frac{2^{n-2}(n-1)}{\pi} J(\varepsilon/R,\ \varepsilon/\delta)$. Letting first $\delta \to 0$, then $\varepsilon \to 0$, the Corollary follows.

LEMMA 6.3. For every $f \in L^p(\mathfrak{H})$ $(1 \leq p < \infty)$ we have $\|f * k_\varepsilon - f * k^\varepsilon\|_p \leq \leq c \|f\|_p$ with some c independent of f and ε, furthermore

$$\lim_{\varepsilon \to 0} (f * k_\varepsilon - f * k^\varepsilon) = \frac{1}{2} f$$

in the L^p - sense and pointwise a. e.

PROOF. We adapt the argument of [22] to our case. We note first that, for $h = (\xi, \zeta)$ such that $|h| > \varepsilon$, we have

$$|k_\varepsilon(h) - k(h)| = \left| c_n \frac{(\varepsilon + |\zeta|^2 + i\xi)^n}{((\varepsilon + |\zeta|^2)^2 + \xi^2)^n} - c_n \frac{(|\zeta|^2 + i\xi)^n}{(|\zeta|^4 + \xi^2)^n} \right| \leq$$

$$\frac{c_n}{|h|^{4n}} \left| (\varepsilon + |\zeta|^2 + i\xi)^n (|\zeta|^4 + \xi^2)^n - (|\zeta|^2 + i\xi)^n ((\varepsilon + |\zeta|^2)^2 + \xi^2)^n \right| =$$

$$\frac{\varepsilon\, |P(\varepsilon, |\zeta|^2, \xi)|}{|h|^{4n}}$$

where P is a homogeneous polynomial of degree $3n - 1$. Majorizing P by the sum of the moduli of its terms, then majorizing ε, $|\zeta|^2$, $|\xi|$ by $|h|$, we obtain

$$(6.7) \qquad |k_\varepsilon(h) - k(h)| \leq c \frac{\varepsilon}{|h|^{n+1}} \quad (|h| > \varepsilon).$$

It is immediate that the function $k'_\varepsilon = k_\varepsilon - k^\varepsilon$ has the homogeneity property

$$(6.8) \qquad k'_\varepsilon(g) = \varepsilon^{-n} k'_1(a(1/\varepsilon)\, g).$$

From this and (6.7) it follows that $\|k'_\varepsilon\|_1$ is finite and independent of ε. This implies the first assertion of the Lemma.

To prove the L^p part of the second statement, let $0 < \varepsilon < R$. By Lemma 6.1 we can write, for any $g \in G$,

$$(6.9)\qquad f * k_\varepsilon(g) - f * k^\varepsilon(g) = \int_{|h|<\varepsilon} [f(gh^{-1}) - f(g)]\, k_\varepsilon(h)\, dh + \int_{\varepsilon<|h|<R} [f(gh^{-1}) - f(g)]\,[k_\varepsilon(h) - k^\varepsilon(h)]\, dh + f(g)\int_{|h|<R} k_\varepsilon(h)\, dh + \int_{|h|>R} f(gh^{-1})\,[k_\varepsilon(h) - k^\varepsilon(h)]\, dh.$$

The last integral tends to 0 in L^p as $R \to \infty$ by (6.8). The third integral tends to $1/2\, f(g)$ by the Corollary of Lemma 1.2. We will show that the first two integrals tend to 0 in L^p, uniformly in R.

By Minkowski's inequality we have

$$\left(\int \left| \int_{|h|<\varepsilon} [f(gh^{-1}) - f(g)]\, k_\varepsilon(h)\, dh \right|^p dg\right)^{1/p} \leq \int_{|h|<\varepsilon} |k_\varepsilon(h)| \cdot \| R_{h^{-1}} f - f \|_p\, dh$$

where $R_{h^{-1}} f$ denotes the right translate of f by h^{-1}. By the trivial inequality $|k_\varepsilon(h)| < 3^n c_n \varepsilon^{-n}$ for $|h| < \varepsilon$, this is majorized by

$$\frac{3^n c_n}{\varepsilon^n} \int_{|h|<\varepsilon} \| R_{h^{-1}} f - f \|_p\, dh \leq c \sup_{|h|<\varepsilon} \| R_{h^{-1}} f - f \|_p$$

which tends to 0 as $\varepsilon \to 0$.

To estimate the second integral we choose $\varepsilon < \eta < R$ and use (6.7) to get the majorization

$$c\,\varepsilon \left\{ \int_{\varepsilon<|h|<\eta} + \int_{\eta<|h|<R} \right\} \frac{|f(gh^{-1}) - f(g)|}{|h|^{n+1}}\, dh.$$

The L^p-norm of the first term here is majorized by Minkowski's inequality and by Lemma 5.1 by

$$c\,\varepsilon \int_{\varepsilon<|h|<\eta} \| R_{h^{-1}} f - f \|_p \frac{dh}{|h|^{n+1}} \leq c\,\varepsilon \sup_{|h|<\eta} \| R_{h^{-1}} f - f \|_p \left(\frac{c'}{\varepsilon} - \frac{c'}{\delta}\right) \leq c\,c' \sup_{|h|<\eta} \| R_{h^{-1}} f - f \|_p .$$

The L^p-norm of the second term is similarly majorized by

$$2 \|f\|_p \, c \, \varepsilon \int_{|h|>\eta} \frac{dh}{|h|^{n+1}} = c'' \frac{\varepsilon}{\eta} .$$

By choosing first η then ε conveniently, both terms can be made arbitrarily small.

To prove that $f * k_\varepsilon - f * k^\varepsilon$ tends to $\frac{1}{2} f$ a. e. we let R tend to ∞ in (6.9) to obtain

$$f * k_\varepsilon (g) - f * k^\varepsilon (g) = \int_{|h|<\varepsilon} [f(gh^{-1}) - f(g)] \, k_\varepsilon (h) \, dh +$$

$$\int_{|h|>\varepsilon} [f(gh^{-1}) - f(g)] \, [k_\varepsilon (h) - k^\varepsilon (h)] \, dh + \frac{1}{2} f(g).$$

Using again the estimate $|k_\varepsilon (h)| \leq 3^n c_n \varepsilon^{-n}$ for $|h| < \varepsilon$ it follows by Lebesgue's theorem (cf. [7] and the remarks following our Lemma 1.1) that the first term tends to 0 a. e. as $\varepsilon \to 0$.

Let now $\eta > \varepsilon$. Split the second term as follows:

$$\int_{|h| \geq \eta} f(gh^{-1}) [k_\varepsilon (h) - k^\varepsilon (h)] \, dh + \int_{\varepsilon \leq |h| < \eta} [f(gh^{-1}) - f(g)] \, [k_\varepsilon (h) - k^\varepsilon (h)] \, dh -$$

$$\int_{|h| \geq \eta} f(g) \, [k_\varepsilon (h) - k^\varepsilon (h)] \, dh = J_1 + J_2 + J_3 .$$

By (6.7), $J_1 \to 0$ as $\varepsilon \to 0$. Let, for $r > 0$,

$$M_g (r) = \int_{|h|<r} |f(gh^{-1}) - f(g)| \, dh$$

which is an increasing function of r. Using (6.7) we have

$$|J_2| \leq \varepsilon c \int_{\varepsilon \leq |h| \leq \eta} |f(gh^{-1}) - f(g)| \frac{dh}{|h|^{n+1}} = \varepsilon c \int_\varepsilon^\eta \frac{dM_g (r)}{r^{n+1}} =$$

$$\varepsilon \frac{c}{n} \left(\frac{M_g (\varepsilon)}{\varepsilon^{n+1}} - \frac{M_g (\eta)}{\eta^{n+1}} \right) - \varepsilon c \frac{n+1}{n} \int_\varepsilon^\eta \frac{M_g (r)}{r^n} \frac{dr}{r^2} ,$$

where in the last step we integrated by parts. The last expression in absolute value does not exceed a constant multiple of

$$\sup_{r \leq \eta} \frac{M_g(r)}{r^n}.$$

For g in the Lebesgue set of f this can be made arbitrarily small by choosing η small enough. Finally, by (6.8) and a change of variable we have

$$J_3 = \varepsilon^{-n} \int_{|g| \geq \eta} k_1'(a(1/\varepsilon)\, g)\, dg = \int_{|h| \geq \eta/\varepsilon} k_1'(h)\, dh.$$

Since k_1' is integrable, this tends to 0 as $\varepsilon \to 0$.

THEOREM 6.1. Let $1 < p < \infty$. For all $F \in L^p(B)$ the limit

$$(PF)(v) = \frac{1}{2} F(v) + \lim_{\varepsilon \to 0} \int_{|u_1| > \varepsilon} S(v, u)\, F(u)\, d\beta(u) \tag{6.10}$$

exists in $L^p(B)$ and a. e. P is a bounded projection in $L^p(B)$ whose range is $H^p(D)$ regarded as a subspace of $L^p(B)$, and PF is the boundary function of the Cauchy Szegö integral $\int_B S(z, u)\, F(u)\, d\beta(u)$. P maps the classes $\Lambda^\beta \cap L^p(B)$ into themselves for all $0 < \beta < 1/2$, $1 < p < \infty$.

PROOF. $(PF)^\sim = \frac{1}{2} \widetilde{F} + \lim_{\varepsilon \to 0} \widetilde{F} * k^\varepsilon$. Therefore we have to show that k satisfies the conditions of Theorems 2.2 and 4.1. There are two ways of doing this: From Lemma 6.1 it is clear that k satisfies the conditions of Theorem 5.1, which implies the other two theorems. The other way avoids the general L^2-theory of § 3; Lemma 6.3 together with the results of [17] mentioned earlier implies condition (i) of Theorem 2.2; the other conditions and those of Theorem 4.1 follow by the same simple arguments which we used in § 5.

To prove the remaining statements we note that $\int_B S(z, u)\, F(u)\, d\beta(u)$ is a function in H^p, since its restriction to $it + B\, (t > 0)$ lifted to $\mathfrak{N}$ equals $\widetilde{F} * k_t$ and $\| \widetilde{F} * k_t \|_p \leq c\, \| \widetilde{F} \|_p = c\, \| F \|_p$ by the first statement of Lemma 6.3. Still by Lemma 6.3, PF is the boundary function of this H^p-

function. Since it is known [15] that every H^p-function converges to its boundary function a. e., Lemma 6.3 also implies that (6.10) holds a. e.

To see that P is a projection, it suffices to see that $P^2F = PF$ for all F in a dense subset of $L^p(B)$; this is clearly true for $F \in L^p \cap L^2$ by the results of [17]. To see that the range of P is exactly H^p, it suffices to see that $PF = F$ for all F in a dense subset of H^p. Now $PF = F$ holds for $F \in H^p \cap H^2$ again by the results of [17], and $H^p \cap H^2$ is dense in H^p by the following argument implicitly contained in [23]: Given $F \in H^p$, it can be approximated in H^p by F_t $(t > 0)$. $F_t \in H^\infty \cap H^p$, hence, and since $S(z, ip)$ is in $H^2 \cap H^\infty$, the function $S(0, ip)^{-1} S(\alpha(\varepsilon) z, ip) F_t(z)$, where $\alpha(\varepsilon) z = (\varepsilon z_1, \varepsilon^{1/2} z_2, \dots, \varepsilon^{1/2} z^n)$, is in $H^2 \cap H^p$. For small $\varepsilon > 0$ this function approximates F_t in H^p by the Lebesgue dominated convergence theorem.

§ 7. The Cauchy-Szegö integral for the complex unit ball.

Let $\mathcal{D}$ be the open unit ball in $\mathbf{C}^n$, let $\mathcal{B}$ be its boundary. The unitary group $G = U(n)$ acts on $\mathcal{B}$ transitively; the isotropy group K at $p = (1, 0, \dots, 0)$ is isomorphic with $U(n-1)$. We identify $\mathcal{B}$ with G/K whenever convenient.

There are several ways of defining a gauge for (G, K), or, what by Remark 1 after Definition 1.1 amounts to the same, a generalized distance γ on $\mathcal{B}$. Gong and Sun [10] use the definition $\gamma_G(u, v) = |1 - u \cdot \bar{v}|$; in [15] we used $\gamma_H(u, v) = \operatorname{Max}\{\pi^{-1} |\arg u \cdot \bar{v}|, 1 - |u \cdot \bar{v}|^2\}$. Here we will use γ defined by

$$\gamma(u, v) = \begin{cases} \dfrac{|1 - u \cdot \bar{v}|}{|1 + u \cdot \bar{v}|} & \text{if} \quad \operatorname{Re} u \cdot \bar{v} \geq 0 \\ 1 & \text{if} \quad \operatorname{Re} u \cdot \bar{v} < 0. \end{cases} \tag{7.1}$$

This has the advantage that, when $\gamma(u, p) \leq 1$, we have

$$\gamma(u, p) = |(Tu)_1| \tag{7.2}$$

by the formulas (6.1). So a « ball » of radius $\varrho < 1$ in $\mathcal{B}$ is transformed by T onto a ball of the same radius in B.

In checking that γ really determines a gauge only properties (iii) and (iv) of Definition 1.1 are not entirely trivial. (iii) can be checked by a very easy direct computation, or in the following way: Denoting, for $\varrho > 0$,

$$\mathcal{B}_\varrho = \{u \in \mathcal{B} \mid \gamma_H(u, p) < \varrho\} \text{ and } \mathcal{A}_\varrho = \{u \in \mathcal{B} \mid \gamma(u, p) < \varrho\}$$

there exist constants $c_1, c_2 > 0$ such that

$$\mathcal{B}_{c_1 \varrho} \subset \mathcal{A}_\varrho \subset \mathcal{B}_{c_2 \varrho} \tag{7.3}$$

for all ϱ. This is easy to see, since $u \in \mathcal{A}_\varrho$ is equivalent to $\left|\frac{1-u_1}{1+u_1}\right| < \varrho$ (when $\varrho \leq 1$), i. e. to u_1 being in the intersection of the one-dimensional complex unit disc with the disc of center $\frac{1+\varrho^2}{1-\varrho^2}$ and radius $\frac{2\varrho}{1-\varrho^2}$, while $u \in \mathcal{B}_\varrho$ is equivalent to $|\arg u_1| < \pi\varrho$, $|u_1| > (1-\varrho)^{1/2}$. It was shown in [15] that γ_H has property (iii); from (7.3) it follows that γ has it too. As for property (iv), in [15] it was shown that $\mu(\mathcal{B}_\varrho) = \varrho^n$ for $\varrho \leq 1$. This implies (iv) for γ_H, and then, by (7.3), it also holds for γ. Another way of checking (iv) consists in applying the Cayley transform T and computing on B.

It is also easy to see that a relation of the type (7.3) holds between γ_H and γ_G, so γ_G also defines a gauge.

LEMMA 7.1. The gauge determined by γ has the property $(L'_{\frac{1}{2}})$.

PROOF. The inequality (1.5) to be proved can be reformulated in terms of γ as follows:

$$|\gamma(u,v) - \gamma(u,p)| \leq M\gamma(u,p)^{\frac{1}{2}}\gamma(v,p)^{\frac{1}{2}} \tag{7.4}$$

whenever $\gamma(v,p) \leq \gamma(u,p)$. It is clearly enough to prove this for the case $\gamma(u,p) \leq \delta$ with some $\delta > 0$; since γ is bounded, (7.4) will then automatically be true without restriction. We choose $\delta > 0$ such that $\gamma(u,p) \leq \delta$ implies $\operatorname{Re} u_1$, $\operatorname{Re} v_1$, $\operatorname{Re} u \cdot \bar{v} \geq 0$.

Denoting $u' = p - u$, $v' = p - v$, we have

$$|u'_1| = |1 - u_1| \leq 2\gamma(u,p), \tag{7.5}$$

$$|u'|^2 = 2 \operatorname{Re} u'_1 \leq 2|u'_1|. \tag{7.6}$$

Now

$$\gamma(u,v) - \gamma(u,p) = \frac{1}{|1+u\cdot\bar{v}|\,|1+u_1|}(|1-u\cdot\bar{v}|\,|1+u_1| - |1-u_1|\,|1+u\cdot\bar{v}|).$$

By the choice of δ the denominator is ≥ 1 and hence can be omitted. It follows that

$$|\gamma(u,v) - \gamma(u,p)| \leq \frac{\left|\,|1-u\cdot\bar{v}|^2|1+u_1|^2 - |1-u_1|^2|1+u\cdot\bar{v}|^2\right|}{|1-u\cdot\bar{v}|\,|1+u_1| + |1-u_1|\,|1+u\cdot\bar{v}|} = \frac{|a|}{b}.$$

It is clear that $b \geq |1 - u_1| \geq \gamma(u, p)$. Writing $s = \overline{v_1'} - u' \cdot \overline{v'}$ we have by (7.5), (7.6) and $\gamma(v, p) \leq \gamma(u, p)$,

$$|s| \leq |v_1'| + |u'||v'| \leq 6\gamma(u, p)^{\frac{1}{2}} \gamma(v, p)^{\frac{1}{2}} . \tag{7.7}$$

A direct computation gives

$$a = 4[2 \operatorname{Re}(u_1' \overline{s}) + |s|^2 - 2 \operatorname{Re}(u_1' \overline{s}) \operatorname{Re} u_1' - |s|^2 \operatorname{Re} u_1 + |u'|^2 \operatorname{Re} s] \leq$$

$$4[2|u_1'||s| + |s|^2 + 2|u_1'|^2|s| + |s|^2|u_1| + |u_1'|^2|s|].$$

By (7.5), (7.7) and $\gamma(v, p) \leq \gamma(u, p)$ this implies $|a| \leq c\gamma(u, p)^{3/2} \gamma(v, p)^{1/2}$. Together with our estimate on b this implies the Lemma.

REMARK. A very similar computation proves the property $(L'_{\frac{1}{2}})$ for γ_G.

The Szegö kernel of $\mathcal{D}$ is given [15], [14] for $z, w \in \mathcal{D}$ by

$$\mathcal{S}(z, w) = \frac{1}{(1 - z \cdot \overline{w})^n} .$$

For every $F \in L^2(\mathcal{B})$, $PF(z) = \int_{\mathcal{B}} \mathcal{S}(z, u)\, d\mu(u)$ defines an H^2 function on $\mathcal{D}$, and P is the orthogonal projection onto the subspace of boundary functions of H^2-functions. μ is the normalized G-invariant measure on $\mathcal{B}$; the Haar measure of G can also be denoted by μ without leading to any confusion. Writing, for $0 < r < 1$, $(PF)_r$ for $PF|_{r\mathcal{B}}$ and denoting by $\widetilde{\varphi}$ the lift to G of any function φ on $\mathcal{B}$ (i. e $\widetilde{\varphi}(g) = \varphi(gp)$) we have

$$(PF)_r^{\sim} = \widetilde{F} * \mathcal{K}_r$$

where $\mathcal{K}_r(g) = \mathcal{S}(rgp, p)$. $\widetilde{F} \to \lim_{r \to 1} \widetilde{F} * \mathcal{K}_r$ is then the orthogonal projection operator on H^2; we shall investigate the connection of this with the singular integral operator given by the kernel

$$\mathcal{K}(g) = \mathcal{S}(gp, p).$$

LEMMA 7.2. $\int \mathcal{K}_r(g)\, dg = 1$ for all $0 \leq r < 1$, and $\lim_{\varepsilon \to 0} \int \mathcal{K}^{\varepsilon}(g)\, dg = \frac{1}{2}$.

PROOF. The first statement is immediate by applying the Cauchy-Szegö integral to the function identically 1, which is in $H^2(\mathcal{D})$.

To prove the second statement we write the integral as an integral on $\mathcal{B}$, then use the Cayley transform (6.1) to transform it into an integral on B where it is easier to compute. We have by [14] formula (4.1) and a formula on p. 342,

$$d\mu(u) = \frac{|S(Tu, ip)|^2}{S(ip, ip)} d\beta(Tu), \tag{7.8}$$

$$\mathcal{S}(z, w) = \frac{S(ip, ip)\ S(Tz, Tw)}{S(Tz, ip)\ S(ip, Tw)}. \tag{7.9}$$

These formulas are also easy to check directly. Using these, noting that $Tp = 0$, writing $Tu = v$ and using (7.2), (6.2) we have

$$\int \mathcal{K}^\varepsilon(g)\, dg = \int_{\gamma(u,p) > \varepsilon} \mathcal{S}(u, p)\, d\mu(u) = \int_{|v_1| > \varepsilon} \frac{S(v, 0)\, S(ip, v)}{S(ip, 0)} d\beta(v) =$$

$$c_n \int_{(\xi^2 + |\zeta|^4)^{1/2} > \varepsilon} (|\zeta|^2 - i\xi)^{-n} (1 + |\zeta|^2 + i\xi)^{-n}\, d\xi\, dV_\zeta.$$

Performing the variable changes (6.3), (6.4), (6.5) in the last integral we find that it equals (in the notation of Lemma 6.2)

$$c_n \frac{|S^{2n-3}|}{2} J(\varepsilon, \infty).$$

By Lemma 6.2 this tends to 1/2 as $\varepsilon \to 0$.

LEMMA 7.3. For every $f \in L^p(G:K)$ $(1 \leq p < \infty)$,

$$\lim_{r \to 1} (f * \mathcal{K}_r - f * \mathcal{K}^{1-r}) = \frac{1}{2} f$$

in L^p and a. e.

PROOF. For all $g \in G$,

$$f * \mathcal{K}_r(g) - f * \mathcal{K}^{1-r}(g) = \int_{|h| < 1-r} [f(gh^{-1}) - f(g)]\, \mathcal{K}_r(h)\, dh +$$

$$\int_{|h| \geq 1-r} [f(gh^{-1}) - f(g)] [\mathcal{K}_r(h) - \mathcal{K}^{1-r}(h)]\, dh +$$

$$f(g) \int \mathcal{K}_r(h)\, dh - f(g) \int_{|h| > 1-r} \mathcal{K}^{1-r}(h)\, dh$$

By Lemma 7.2 the third term equals $f(g)$ and the last term tends to $f(g)/2$ as $r \to 1$. We have to show only that the first two terms tend to 0 in L^p. We majorize the L^p-norm of the first term using Minkowski's inequality:

$$\left(\int \left| \int_{|h| < 1-r} [f(gh^{-1}) - f(g)] \mathcal{K}_r(h)\, dh \right|^p dg \right)^{1/p} \leq \int_{|h| < 1-r} | \mathcal{K}_r(h) | \cdot \| R_{h^{-1}} f - f \|_p\, dh.$$

By the explicit formula for $\mathcal{S}$ we have $\| \mathcal{K}_r \|_\infty = (1 - r)^{-n}$. Also, we find $\int_{|h| < 1-r} dh \leq c (1 - r)^n$ by transforming the integral to B by T as in Lemma 7.2. Hence we have the further majorization by

$$c \sup_{|h| < 1-r} \| R_{h^{-1}} f - f \|_p .$$

This number tends to 0 as $r \to 1$.

To majorize the second term we apply Minkowki's inequality, then, taking $1 - r < \eta < 1$, we split the resulting integral into a sum $I_1 + I_2 + I_3$ where

$$I_j = \int_{D_j} | \mathcal{K}_r(h) - \mathcal{K}^{1-r}(h) | \cdot \| R_{h^{-1}} f - f \|_p\, dh$$

and

$$D_1 = \{ h \mid 1 - r < | h | \leq \eta \}$$

$$D_2 = \{ h \mid \eta < | h | < 1 \}$$

$$D_3 = \{ h \mid |h| = 1 \}.$$

Writing $hp = u$ we have

$$\text{(7.10)} \qquad \mathcal{K}_r(h) - \mathcal{K}^{1-r}(h) = \frac{1}{(1 - r u_1)^n} - \frac{1}{(1 - u_1)^n} =$$

$$- (1 - r)\, u_1 \sum_{k=0}^{n-2} \frac{1}{(1 - r u_1)^{k+1} (1 - u_1)^{n-k}} .$$

If $h \in D_3$, then $\operatorname{Re} u_1 \leq 0$, hence $|1 - ru_1| \geq 1$. Therefore the sum in (7.10) is majorized on D_3 by an integrable function φ independent of r. Hence

$$I_3 \leq 2\,\|f\|_p\,(1-r) \int\limits_{D_3} \varphi\, d\mu$$

which tends to 0 as $r \to 1$.

On D_1 and D_2, $\operatorname{Re} u_1 > 0$. Therefore $|1-ru_1| > r|1-u_1|$ and $|1+u_1| > 1$. Hence, by (7.10),

$$|\mathcal{K}_r(h) - \mathcal{K}^{1-r}(h)| > \frac{(1-r)(n-1)}{r^{n-1}\,|1-u_1|^{n+1}} < \frac{(1-r)(n-1)}{r^{n-1}}\,\gamma(u,p)^{-n-1}.$$

For I_1 this gives the estimate

$$I_1 \leq \frac{(1-r)(n-1)}{r^{n-1}} \sup_{|h|<\eta} \|R_{h^{-1}}f - f\|_p \int\limits_{1-r<|h|<\eta} \gamma(hp,p)^{-n-1}\,dh.$$

The integral here can be computed by applying T as in Lemma 7.2 and then lifting the integral from B to $\mathfrak{N}$; it is majorized by

$$c \int\limits_{|g|>1-r} |g|^{-n-1}\,dg = \frac{c'}{1-r},$$

the last equality by Lemma 5.1. This shows that I_1 can be made arbitrarily small, uniformly for $1-r<\eta$, by choosing η small enough.

For I_2 we have, by the same transformations and by Lemma 5.1,

$$I_2 \leq 2\,\|f\|_p\,(n-1)\,\frac{1-r}{r^{n-1}} \int\limits_{\eta<|h|<1} \gamma(hp,p)^{-n-1}\,dh = c\,\|f\|_p\,\frac{1-r}{r^{n-1}}\left(\frac{1}{\eta}-1\right).$$

This also tends to 0 as $r \to 1$, finishing the proof of the statement on L^p-convergence.

Convergence a.e. is proved similarly as in Lemma 6.3.

Lemma 7.4. The gauge induced by γ and the kernel $\mathcal{K}$ satisfy all conditions of Theorems 2.2 and 4.1.

Proof. We have shown in Lemma 7.1 that the gauge satisfies $(L'^{1/2})$. Condition (i) of Theorem 2.2 follows from Lemma 7.3. Instead of checking conditions (ii), (iii), (iv) we will, with almost no extra effort, check the more stringent conditions (*a*), (*b*), (*c*) of Theorem 3.1. In this way we also have an other proof of condition (i), avoiding Lemma 7.3 but involving the L^2 theory of § 3.

It is enough to check (*a*) for small $A > 0$ since G is compact, and it is enough to check the first inequality in (*a*) since $\check{\mathcal{K}} = \bar{\mathcal{K}}$. We write $u = gp$, $v = hp$ and use (7.8) and (7.9) writing again u, v instead of Tu, Tv,

$$\int_{|g|>A} |\mathcal{K}(h^{-1}g) - \mathcal{K}(g)|\, dg = \int_{\gamma(u,p)>A} |\mathcal{S}(u, v) - \mathcal{S}(u, p)|\, d\mu(u) =$$

$$\int_{|u_1|>A} |S(ip, u)| \left| \frac{S(u, v)}{S(ip, v)} - \frac{S(u, 0)}{S(ip, 0)} \right| d\beta(u) \le$$

$$\int_{|u_1|>A} \left| \frac{S(ip, u)}{S(ip, v)} \right| |S(u, v) - S(u, 0)|\, d\beta(u) +$$

$$\left| \frac{1}{S(ip, v)} - \frac{1}{S(ip, 0)} \right| \int_{|u_1|>A} |S(u, 0)| \cdot |S(ip, u)|\, d\beta(u) = I_1 + I_2.$$

These integrals can be rewritten as integrals on $\mathfrak{H}$ using the kernel k of § 6. Since we consider small A only and $|h| \le A$, v is close to 0. Therefore

$$I_1 \le 2 \int_{|u_1|>A} |S(u, v) - S(u, 0)|\, d\beta(u) = 2 \int_{|g|>A} |k(l^{-1} g) - k(g)|\, dg$$

where $l \in \mathfrak{H}$ is such that $l \cdot 0 = v$, therefore $|l| = |h|$. Lemma 5.2 and Lemma 5.1 now give, for $N|h| \le A$

$$I_1 \le 2M |h|^{\frac{1}{2}} \int_{|g|>A} |g|^{-n-\frac{1}{2}}\, dg \le M' \left(\frac{|h|}{A}\right)^{\frac{1}{2}}.$$

To estimate I_2 we again write $v = l \cdot 0$; if $l = (\xi', \zeta')$, the expression in front of the integral is $c_n^{-1} |(1 + |\zeta'|^2 + i\xi')^n - 1|$ which is less than $M(\xi'^2 + |\xi'|^4)^{1/2} = M|l| = M|h| < M|h|^{1/2}$ (since $|h|$ is smal). Therefore we have

$$I_2 \leq M \, |h|^{\frac{1}{2}} \int\limits_{(\xi^2+|\zeta|^4)^{1/2} > A} (|\zeta|^4 + \xi^2)^{n/2} ((1+|\zeta|^2)^2 + \xi^2)^{n/2} \, d\xi \, dV_\zeta \leq$$

$$M \, |h|^{\frac{1}{2}} \int\limits_{|g|>A} |g|^{-n-\frac{1}{2}} \, dg = M' \left(\frac{|h|}{A}\right)^{\frac{1}{2}}$$

by Lemma 5.1.

To check (b) and at the same time condition (iii′) of Theorem 4.1, let $\beta > 0$, $0 < \varrho < 1$. Applying T as usual, we find

$$\int\limits_{|g|<\varrho} |g|^\beta \, |\mathcal{K}(g)| \, dg = \int\limits_{\gamma(u,p)<\varrho} \gamma(u,p)^\beta \, |\mathcal{S}(u,p)| \, d\mu(u) =$$

$$\int\limits_{|u_1|<\varrho} |u_1|^\beta \, |S(u,0)| \left| \frac{S(ip,u)}{S(ip,0)} \right| d\beta(u) \leq$$

$$\int\limits_{|u_1|<\varrho} |u_1|^\beta \, |S(u,0)| \, d\beta(u) = \int\limits_{|g|<\varrho} |g|^\beta \, |\mathcal{K}(g)| \, dg = c' \varrho^\beta,$$

the last equality by Lemma 5.1.

To check (c), by $\check{\mathcal{K}} = \overline{\mathcal{K}}$ it is again sufficient to consider $\mathcal{K}$. The computation of Lemma 7.2 gives, for small $0 < a < b$,

$$\int\limits_{a<|g|<b} \mathcal{K}(g) \, dg = c' J(a,b) = c' \int_a^b R(\varrho) \frac{d\varrho}{\varrho},$$

where

$$R(\varrho) = \int_0^\pi \frac{\sin^{n-2}\theta}{(\varrho - ie^{i\theta})^n} \, d\theta.$$

$R(\varrho)$ is a continuously differentiable function of ϱ on the interval $[0, 1/2]$, so $R'(\varrho)$ is bounded on this interval. $R(0) = 0$ by (6.6). It follows that $|R(\varrho)| \leq M\varrho$, and hence $J(a,b) \leq M(b-a)$. This implies (c) with $\nu = 1$ for small ϱ; by compactness of G this is all that is needed.

By compactness the conditions (i′) and (iv′) of Theorem 4.1 are trivial, so the proof of the Lemma is finished.

THEOREM 7.1. Let $1 < p < \infty$. For all $F \in L^p(\mathcal{B})$ the limit

$$(PF)(v) = \frac{1}{2} F(v) + \lim_{\varepsilon \to 0} \int_{\gamma(u,p) > \varepsilon} \mathcal{S}(v,u) F(u) \, d\mu(u)$$

exists in $L^p(\mathcal{B})$ and a. e. P is a bounded projection in $L^p(\mathcal{B})$ whose range is $H^p(\mathcal{D})$ regarded as a subspace of $L^p(\mathcal{B})$ and PF is the boundary function of the Cauchy-Szegö integral $\int_{\mathcal{B}} S(z,u) F(u) \, d\mu(u)$. P maps the classes $\Lambda^\beta(\mathcal{B})$ into themselves for all $0 < \beta < 1/2$.

PROOF. The proof follows from Lemmas 7.3 and 7.4 in analogy to the proof of Theorem 6.1. One detail requires further mention: in the case $p = 2$, P is the orthogonal projection onto $H^2(\mathcal{D})$. In fact, we have $P^2 = P$ by the reproducing property of the Szegö kernel, and P is selfadjoint by $\mathcal{S}(w,z) = \overline{\mathcal{S}(z,w)}$.

REMARK. The analogue of Theorem 7.1 holds also if we use γ_G instead of γ. To see this, denote by $|g|_G$ the gauge induced by γ_G. An elementary computation shows that if $|g|_G = \varepsilon < 1$ then $\varphi(\varepsilon) = \varepsilon(4 - \varepsilon^2)^{-\frac{1}{2}} \leq |g| \leq \varepsilon(2 - \varepsilon)^{-1} = \psi(\varepsilon)$. Hence, setting

$$\mathcal{K}_G^\varepsilon(g) = \begin{cases} \mathcal{K}(g) & \text{if } |g|_G < \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

we have

$$\|\mathcal{K}_G^\varepsilon - \mathcal{K}^{\varphi(\varepsilon)}\|_1 \leq \|\mathcal{K}^{\varphi(\varepsilon)} - \mathcal{K}^{\psi(\varepsilon)}\|_1$$

Transforming the integral by the Cayley transform and using Lemma 5.1 we get a majorization by $c[\log \psi(\varepsilon) - \log \varphi(\varepsilon)] = \frac{c}{2} \log \frac{2+\varepsilon}{2-\varepsilon}$. This tends to 0 as $\varepsilon \to 0$, and it follows that $\lim_{\varepsilon \to 0} f * \mathcal{K}_G^\varepsilon = \lim_{\varepsilon \to 0} f * \mathcal{K}^{\varphi(\varepsilon)} = \lim_{\varepsilon \to 0} f \; \mathcal{K}^\varepsilon$ for all $f \in L^p (1 < p < \infty)$.

§ 8. The functions of Littlewood-Paley and Lusin on D.

We consider the generalized halfplane D of § 6. As we have seen, the group $\mathfrak{N}$ acts on D by holomorphic automorphisms. There is also a one-parameter group $\{\alpha(t)\}$ $(t > 0)$ of holomorphic automorphisms acting by

$$z_1 \longmapsto tz_1$$

$$z_k \longmapsto t^{\frac{1}{2}} z_k \quad (2 \leq k < n)$$

(It may be noted that the automorphism group $\{a(t)\}$ of § 6 acts on $\mathfrak{N}$ by $a(t)\, g = \alpha(t)\, g\, \alpha(t^{-1})$).

Every point $z \in D$ can be uniquely written as $g\, \alpha(t) \cdot ip$ $(g \in \mathfrak{N}\, t > 0)$, where $ip = (i, 0, \dots, 0)$. The pair (t, g) or (t, ξ, ζ) $(t > 0, \xi \in \mathbf{R}, \zeta \in \mathbf{C}^{n-1})$ can be used as coordinates on D. The formulas for the coordinate change are

$$\text{(8.1)} \qquad \begin{aligned} t &= \operatorname{Im} z_1 - \sum_2^n |z_k|^2 \\ \xi &= \operatorname{Re} z_1 \\ \zeta &= (z_2, \dots, z_n). \end{aligned}$$

In this section we will use the new coordinates throughout. Also we will identify the boundary B of D with $\mathfrak{N}$ under the map $g \longmapsto g \cdot 0$.

We write $\zeta_k = \xi_k + i\eta_k$ $(2 \leq k \leq n)$ and use the usual notation $\frac{\partial}{\partial \zeta_k} = \frac{1}{2}\left(\frac{\partial}{\partial \xi_k} - i\frac{\partial}{\partial \eta_k}\right)$.

LEMMA 8.1. $\Xi = \frac{\partial}{\partial \xi}$, $Z_k = \frac{\partial}{\partial \zeta_k} + i\bar{\zeta}_k \frac{\partial}{\partial \xi}$ $(2 \leq k \leq n)$ are left-invariant vector fields on $\mathfrak{N}$.

PROOF. Trivial computation based on the composition rule of $\mathfrak{N}$.

LEMMA 8.2. The vector fields $t\frac{\partial}{\partial t}$, $t\,\Xi$, $t^{\frac{1}{2}} \operatorname{Re} Z_k$, $t^{\frac{1}{2}} \operatorname{Im} Z_k$ $(2 \leq k \leq n)$ on D are invariant under $\mathfrak{N}$ and $\{\alpha(t)\}$. At every point of D they form an orthonormal basis with respect to the Bergman metric of D.

PROOF. The statement about group-invariance follows by a trivial verification. Since every holomorphic automorphism is an isometry in the Bergman metric, and since $\mathfrak{N}\cdot\{\alpha(t)\}$ is transitive on D, it suffices to check orthonormality at one point, e. g. at ip.

The values of our vector fields at ip are $\frac{\partial}{\partial t}, \frac{\partial}{\partial \xi}, \frac{1}{2}\frac{\partial}{\partial \xi_k}, \frac{1}{2}\frac{\partial}{\partial \eta_k}$ $(2\leq k\leq n)$. Expressed in terms of the coordinates $z_j = x_j + iy_j$ $(1 \leq j \leq n)$ of $\mathbf{C}^n$ they kave the form $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{1}{2}\frac{\partial}{\partial x_k}, \frac{1}{2}\frac{\partial}{\partial y_k}$ $(2 \leq k \leq n)$. There are several ways to see that these are orthonormal. The most direct way is to compute explicitly the coefficients $g_{j\bar{k}}(z) = \frac{\partial}{\partial z_j \partial \bar{z}_k} K(z, z)$ of the Bergman metric at $z = ip$. It is known [19] that $K(z, z) = c\,(\mathrm{Im} z_1 - \sum_2^n |z_k|^2)^{-n-1}$ in our case, so we find $(g_{j\bar{k}}(ip)) = c'$ diag $\left\{\frac{1}{4}, 1, \ldots, 1\right\}$ proving the assertion. Another way is to use that $(g_{j\bar{k}})$ is the inverse matrix of the coefficient matrix $(g^{j\bar{k}})$ of the invariant Laplacian; for the latter there is an explicit formula in [15, p. 511]. A third way consists in computing the differential of the Cayley transform T at 0; the metric matrix of the unit ball at 0 is cI by rotation invariance, and T carries it to the metric matrix of D at ip.

Let us recall from [15] that every real-valued L^p-function f on B (or, what is the same, on $\mathfrak{N}$) has a «Poisson integral» F defined on D by $F(t, g) = (f * P_t)(g)$, where

$$P_t(\xi, \zeta) = c_n \frac{t^n}{[\xi^2 + (t + |\zeta|^2)^2]^n}.$$

F is harmonic with respect to the Begman metric.

DEFINITION 8.1. Let $f \in L^p(\mathfrak{N}, \mathbf{R})$. The Littlewood-Paley function $G[f]$: $\mathfrak{N} \to \mathbf{R}$ is defined by

$$G[f](\xi, \zeta) = \left(\int_0^\infty |\nabla F(t, \xi, \zeta)|^2 \frac{dt}{t}\right)^{\frac{1}{2}}$$

where F is the Poisson integral of f, ∇F is the gradient of F with respect to the Bergman metric, and $|\ |$ denotes lengh with respect to the Bergman metric. We also define the following auxiliary functions:

$$G_0[f](\xi,\zeta) = \left(\int_0^\infty \left(\frac{\partial F}{\partial t}\right)^2 t\,dt\right)^{\frac{1}{2}}$$

$$G_1[f](\xi,\zeta) = \left(\int_0^\infty (\Xi F)^2 t\,dt\right)^{\frac{1}{2}}$$

$$G_k[f](\xi,\zeta) = \left(\int_0^\infty |Z_k F|^2\,dt\right)^{\frac{1}{2}} \qquad (2 \leq k \leq n)$$

REMARKS 1. By Lemmma 8.2 we have

$$G[f]^2 = G_0[f]^2 + \dots + G_n[f]^2 .$$

2. Since $F = f * P_t$, by Lemma 8.1 we have $\frac{\partial F}{\partial t} = f * \frac{\partial}{\partial t} P_t$, $\Xi F = f * \Xi P_t$, $Z_k F = f * Z_k P_t \quad (2 \leq k \leq n)$.

DEFINITION 8.2. On $\mathfrak{H}$ we define the functions $\varphi_0 = \frac{\partial}{\partial t} P_t|_{t=1}$, $\varphi_1 = \Xi P_1$, $\varphi_k = Z_k P_1 \, (2 \leq k \leq n)$. Explicitly, we have

$$\varphi_0(\xi,\zeta) = nc_n \frac{\xi^2 + |\zeta|^4 - 1}{[\xi^2 + (1 + |\zeta|^2)^2]^{n+1}}$$

$$\varphi_1(\xi,\zeta) = -2nc_n \frac{\xi}{[\xi^2 + (1 + |\zeta|^2)^2]^{n+1}}$$

$$\varphi_k(\xi,\zeta) = -2nc\bar{\zeta}_k \frac{1 + |\zeta|^2 + i\xi}{[\xi^2 + (1 + |\zeta|^2)^2]^{n+1}} \qquad (2 \leq k \leq n)$$

LEMMA 8.3. For all $t > 0$, $g \in \mathfrak{H}$,

$$t \frac{\partial}{\partial t} P_t(g) = t^{-n} \varphi_0(a(1/t)\,g)$$

$$t\, \Xi P_t(g) = t^{-n} \varphi_1(a(1/t)\,g)$$

$$t^{\frac{1}{2}} Z_k P_t(g) = t^{-n} \varphi_k(a(1/t)\,g) \qquad (2 \leq k \leq n).$$

PROOF. Immediate by direct calculation. The calculation can be somewhat shortened by using $P_t(g) = t^{-n} P(a(1/t)g)$, which is a known general identity [14] and also easy to check directly.

LEMMA 8.4. For all $f \in L^2(\mathfrak{N}, \mathbf{R})$, $\| G[f] \|_2 = \|f\|_2$ with a constant c independent of f.

PROOF. By continuity it suffices to prove the Lemma for all smooth f with compact support. Let $0 < \varepsilon < R$, and let $D_{\varepsilon R}$ be the subset of D determined by the inequalities

$$\varepsilon \leq t \leq R$$

$$|\xi| \leq R$$

$$|\xi_k|, |\eta_k| \leq R^{\frac{1}{2}} \qquad (2 \leq k \leq n).$$

We have Green's formula

$$(8.2) \qquad \int_{D_{\varepsilon R}} (u\Delta v - v\Delta u) = \int_{\partial D_{\varepsilon R}} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)$$

where the Laplacian, the unit normal and the volume elements are taken with respect to the Bergman metric. We set $u = t^n$ and $v = F^2$ where F is the Poisson integral of f. It is easy to see that $\Delta t^n = 0$ (in [15, p. 515] it is erroneously stated that $\Delta t = 0$). Also, by the well known formula $\operatorname{div}(\alpha X) = \alpha \operatorname{div} X + \nabla\alpha \cdot X$ and by $\operatorname{div} \nabla F = \Delta F = 0$ it follows that $\Delta F^2 = 2|\nabla F|^2$. Taking into account that the invariant volume on D is proportional to $t^{-n-1} dt\, d\xi\, dV_\zeta$ we find that the left hand side of (8.2) equals

$$c \int_{D_{\varepsilon R}} |\nabla F|^2 \frac{dt}{t}\, d\xi\, dV_\zeta$$

with some constant c. As $\varepsilon \to 0$ and $R \to \infty$ this tends to $c\,\| G[f]\|_2^2$.

The right hand side of (8.2) has to be considered separately on the different faces of $\partial D_{\varepsilon R}$. On the piece where $t = R$ the unit normal pointing outward is $R \dfrac{\partial}{\partial t}$ by Lemma 8.2. The surface element on this set is $cR^{-n}\, d\xi\, dV_\zeta$ as it follows from the formula for the volume element on D

and again from Lemma 8.2. This gives us

$$(8.3) \qquad c \int\limits_{\substack{|\xi| \le R \\ |\xi_k|, |\eta_k| \le R^{\frac{1}{2}}}} \left(R \frac{\partial F^2}{\partial t}(R, \xi, \zeta) - nF^2(R, \xi, \zeta) \right) d\xi \, dV_\zeta .$$

We have $\frac{\partial F^2}{\partial t} = 2F\frac{\partial F}{\partial t}$. By the obvious estimates $P_t(g) \le \frac{c}{t^n}$, $\left|\frac{\partial}{\partial t} P_t(g)\right| \le \frac{c}{t^{n+1}}$ it follows that $|F(R, \xi, \zeta)| \le \frac{c'}{R^n}$, $\left|\frac{\partial}{\partial t} F(R, \xi, \zeta)\right| \le \frac{c'}{R^{n+1}}$. Hence (8.3) tends to 0 as $R \to \infty$.

On the piece of the boundary where $t = \varepsilon$ we get (8.3) with R in the integrand replaced by ε and with signs changed due to the fact that the normal pointing outwards is now $-\varepsilon\frac{\partial}{\partial t}$. As $\varepsilon \to 0$ (and then $R \to \infty$) the second term tends to $cn\|f\|_2^2$ by elementary properties of the Poisson integral (cf. [14]). We claim that $\lim_{\varepsilon \to 0} \varepsilon \frac{\partial}{\partial t} F(\varepsilon, \xi, \zeta) = 0$ uniformly, whence the first term tends to 0.

To prove the claim we note that $C^\infty(\mathfrak{N}) \subset \Lambda^{\frac{1}{2}}(\mathfrak{N}, \mathbf{R})$ by Taylor's formula (cf. Lemma 1.8), and hence, e.g. by Lemma 1.5 we have $|f(gh) - f(g)| \le M|h|^{\frac{1}{2}}$ for all $g, h \in \mathfrak{N}$, with some $M > 0$. Clearly $\int \frac{\partial}{\partial t} P_t(g)\, dg = 0$, and thus we have

$$\left| \varepsilon \frac{\partial}{\partial t} F(\varepsilon, g) \right| = \varepsilon \left| \int\limits_{\mathfrak{N}} \frac{\partial}{\partial t} P_t(h^{-1})\big|_{t=\varepsilon} \cdot f(gh)\, dh \right| =$$

$$\varepsilon \left| \int\limits_{\mathfrak{N}} \frac{\partial}{\partial t} P_t(h^{-1})\big|_{t=\varepsilon} (f(gh) - f(g))\, dh \right| \le \varepsilon M \int\limits_{\mathfrak{N}} \left| \frac{\partial}{\partial t} P_t(h^{-1})\big|_{t=\varepsilon} \right| \cdot |h|^{\frac{1}{2}}\, dh.$$

Making the change of variable $h = a(1/\varepsilon)\, g$ and using Lemma 8.3 the last expression is seen to be equal to

$$\varepsilon^{\frac{1}{2}} M \int\limits_{\mathfrak{N}} |\varphi_0(g)| \cdot |g|^{\frac{1}{2}}\, dg$$

and the claim follows.

We still have to consider the lateral surfaces of $D_{\varepsilon R}$, i. e. those where one of the variables ξ, $\xi_k\, \eta_k$ is constant. $t\frac{\partial}{\partial t}$ is tangent to these surfaces, hence by Lemma 8.2 the unit normal is a linear combination of $t\,\Xi$, $t^{\frac{1}{2}}\,\mathrm{Re}\,Z_k$, $t^{\frac{1}{2}}\,\mathrm{Im}\,Z_k$. Hence the second term on the right hand side of (8.2) is 0. The integrand in the first term is $t^n F \frac{\partial F}{\partial n}$; the arguments (t, g) of F that occur satisfy the inequalities $0 < t \leq R \leq |g|$. It is easy to see from our explicit formulas that $P_t(g) \leq c \frac{t^n}{|g|^{2n}}$ and that $|t\Xi P_t(g)|$, $|t^{\frac{1}{2}} Z_k P_t(g)| \leq$ $\leq c \frac{t^n}{|g|^{2n}}$ if $t \leq |g|$. Since F is the Poisson integral of a function with compact support, F and $\frac{\partial F}{\partial n}$ satisfy the same inequalities for $|g|$ large enough. Thus, for large R, our integrand is majorized by $c\,\frac{t^{3n}}{R^{4n}}$.

The integrand has to be multiplied by the surface element expressed in the coordinates t, ξ, ξ_k, η_k, which we now proceed to estimate. Let A denote the matrix transforming the orthonormal basis $t\frac{\partial}{\partial t}$, $t\,\Xi, \ldots$ to the basis $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial \xi}$, $\frac{\partial}{\partial \xi_k}$, $\frac{\partial}{\partial \eta_k}$. A^{-1} is known from the definition of Ξ, Z_k, and an easy computation gives

$$A = \begin{bmatrix} t^{-1} & & & & & & & \\ & t^{-1}, & -2\eta_2 t^{-1}, & \ldots, & -2\eta_n t^{-1}, & 2\xi_2 t^{-1}, & \ldots, & 2\xi_n t^{-1} \\ & & 2t^{-\frac{1}{2}} & & & & & \\ & & & \ddots & & & & \\ & & & & 2t^{-\frac{1}{2}} & & & \\ & & & & & -2t^{-\frac{1}{2}} & & \\ & & & & & & \ddots & \\ & & & & & & & -2t^{-\frac{1}{2}} \end{bmatrix}$$

with zero everywhere else. The metric matrix is clearly $A\,{}^tA$. The principal minors of order $2n - 1$ of this are majorized by $|g|\,t^{-2n}$ or $|g|\,t^{-2n+1}$, where $g = (\xi, \zeta)$. The surface element is the square root of such a minor, and therefore majorized on the lateral surfaces of $D_{\epsilon R}$ by $c\,R^{\frac{1}{2}}\,t^{-n}$ or $cR^{\frac{1}{2}}\,t^{-n+\frac{1}{2}}$ where c is some constant. Carrying out the integration with respect to t, all ξ_k, η_k, and ξ except one, we find that our integral is majorized by $\frac{c}{R^{n-1}}$ and tends to 0 as $R \to \infty$. This finishes the proof of the Lemma.

THEOREM 8.1. For every $1 < p < \infty$ there exists a constant c_p such that $\| G[f] \|_p \leq c_p \|f\|_p$ for all $f \in L^p(\mathfrak{N})$.

PROOF. Each $\varphi_i (0 \leq i \leq n)$ satisfies conditions (i)-(iii) of Theorem 5.2 with $\gamma = 1/8$. For (i) and (ii) this is trivial. For (iii) it is enough to consider the case of small $|h|$; putting $\varphi_i(h^{-1}g) - \varphi_i(g)$ over a common denominator the terms of the numerator not involving the coordinates of h cancel out and one gets a majorization by $c\,|h|^{1/2}(1 + |g|^2)^{-n}$. This is integrable with respect to g by Lemma 5.1 and so we get the first inequality of (iii). The proof of the second inequality is the same.

Let k_i be the kernel defined with the aid of φ_i $(0 \leq i \leq n)$, as in Theorem 5.2. Then for all smooth f with compact support we have $\| G_i[f] \|_2 = \lim_{\varepsilon \to 0} \|f * k_i^\varepsilon \|_2$. In fact, noting that $k_i^\varepsilon \in L^a$ for all $1 < a < \infty$, we have

$$\|f * k_i^\varepsilon\|_2^2 = \int_{\mathfrak{N}} dg \int_0^\infty dt \left| \int_{|h^{-1}g| > \varepsilon} t^{-n-\frac{1}{2}} \varphi_i(a(1/t)(h^{-1}g)) f(h)\, dh \right|^2 .$$

The innermost integral has the value $(f * k_i^\varepsilon)(g)(t)$ and, since $\varphi_i \in L^1$, converges for every fixed g and t to the value of the innermost integral in

$$\| G_i[f] \|_2^2 = \int_{\mathfrak{N}} dg \int_0^\infty dt \left| \int_{\mathfrak{N}} t^{-n-\frac{1}{2}} \varphi_i(a(1/t)(h^{-1}g)) f(h)\, dh \right|^2 .$$

On the other hand, $f * k^\varepsilon$ converges as an element of $L^2(\mathfrak{N}, L^2(0, \infty))$ by the argument used at the end of the proof of Theorem 2.2. The limit must coincide with the pointwise limit a. e. and it follows that $\lim_{\varepsilon \to 0} \|f * k_i^\varepsilon\|_2^2 = \| G_i[f] \|_2^2$.

This shows that k_i satisfies all conditions of Theorem 2.2. It follows, therefore, that $\lim_{\varepsilon \to 0} \|f * k_i^\varepsilon \|_p \leq c_p \| f \|_p$ for all $1 < p < \infty$. It is also clear that $\| G_i[f] \|_p \leq \lim_{\varepsilon \to 0} \|f * k_i^\varepsilon\|_p$, and so the proof is finished.

We recall from [15] the definition of an admissible domain in D at $g \cdot 0 \in B$,

$$\Gamma_\lambda(g) = \{(t, g') \in D \mid |g^{-1}g'| < \lambda t\}$$

where $\lambda > 0$. (The fact that we use here a slightly different gauge from [15] is irrelevant).

DEFINITION 8.3. Let $f \in L^p(\mathfrak{N}, \mathbf{R})$ $(1 \leq p < \infty)$. The Lusin function $S[f]$ of f is defined for all $g \in N$ by

$$S[f](g) = \left(\int_{\Gamma_\lambda(g)} |\nabla F|^2 \right)^{\frac{1}{2}}$$

where λ is some fixed positive number, F is the Poisson integral of f and the integral is taken with respect to the invariant volume of D.

THEOREM 8.2. For every $1 < p < \infty$ there exists a constant c_p such that $\| S[f] \|_p \leq c_p \|f\|_p$ for all $f \in L^p(\mathfrak{N}, \mathbf{R})$.

PROOF. Since the invariant volume on D is $\dfrac{dt\, dg'}{t^{n+1}}$, we have

$$S[f](g)^2 = \int_{\substack{0<t<\infty \\ |g^{-1}g'|<\lambda t}} \frac{dt\, dg'}{t^{n+1}} \sum_{i=0}^{n} \left| \int_{\mathfrak{N}} t^{-n} \varphi_i (a(1/t)(h^{-1}g')) f(h)\, dh \right|^2 .$$

After the change of variable $l = a(1/t)(g^{-1}g')$ this is equal to

$$\int_0^\infty dt \int_{|l| \leq \lambda} dl \sum_{i=0}^{n} \left| \int_{\mathfrak{N}} t^{-n-\frac{1}{2}} (R_l \varphi_i)(a(1/t)(h^{-1}g)) f(h)\, dh \right|^2$$

where R_l denotes right translation by l. By invariance of the Haar measure it follows that $\| S[f] \|_2^2 = c \| G[f] \|_2^2$, where c is the measure of the set $B(\lambda) = \{ l \in \mathfrak{N} \mid |l| \leq \lambda \}$. Hence the analogue of Lemma 8.4 holds for $S[f]$.

Now let $\varphi_i (0 \leq i \leq n)$ be as in Definition 8.2. For each $g \in \mathfrak{N}$ define $k_i(g) : \mathbf{C} \to L^2((0, \infty) \times B(\lambda))$ by $k_i(g) z = t^{-n-\frac{1}{2}} \varphi_i(a(1/t)\, gl^{-1}) z$. As in the proof of Theorem 8.1 one sees that

$$\| S[f] \|_2^2 = \sum_{i=0}^{n} \lim_{\varepsilon \to 0} \| f * k_i^\varepsilon \|_2^2$$

which shows that condition (i) of Theorem 2.2 is satisfied for each of the kernels k_i. An obvious modification of the proof of Theorem 5.2 (cf. [1, p. 365]) shows that its conclusions remain true for the kernels k_i, and hence our Theorem follows.

§ 9. The Riesz transform on S^{n-1}.

Let S^{n-1} be the unit sphere in $\mathbf{R}^n$ $(n \geq 3)$ and let U^n be its interior. We want to consider the following problem: Suppose that $f \in L^p(S^{n-1}, \mathbf{R})$ $(1 < p < \infty)$ is (the limit on S^{n-1} of) the normal component of the gradient ∇u of a harmonic function u in U^n. Is it true that the tangential component of ∇u also has boundary values on S^{n-1} and the mapping R, of f to these boundary values is a bounded linear transformation from $L^p(S^{n-1}, \mathbf{R})$ to $L^p(S^{n-1}, \mathbf{R}^n)$?

If the same question is formulated for a hyperplane and a half-space instead of S^{n-1} and U^n, the analogue of R is the Riesz transform in the sense of M. Riesz and J. Horváth [12], and the affirmative answer follows from classical results on singular integral operators [3].

Here we will show that R is a singular integral operator on S^{n-1} regarded as a homogeneous space of $SO(n)$, and the answer which is affirmative in this case, too, (5) will follow from Theorem 3.1.

It is known [8, pp. 261-2] that the solution of the Neumann problem for S^{n-1} with continuous boundary datum f $\left(\text{subject to the condition } \int_{S^{n-1}} f = 0\right)$ is given by (6)

$$u(\xi) = \int_{S^{n-1}} K(\xi, y) f(y)\, dy. \tag{9.1}$$

Here K is the kernel function of U^n. i.e. the reproducing kernel of the Hilbert space $\mathfrak{D}$ which consists of the functions u harmonic in U^n, vanishing at the origin, and having finite Dirichlet norm $\left(\int_{U^n} |\nabla u|^2\right)^{\frac{1}{2}}$. dy denotes the measure induced on S^{n-1} by the natural Euclidean structure of $\mathbf{R}^n$.

---

(5) E. M. Stein informs us that this result is also obtainable by other methods, cf. D. A. LEVINE, *Systems of singular integral operators on spheres,* Trans. Amer. Math. Soc., **144**, (1969), 493-522, and E. M. STEIN, *Topics in harmonic analysis related to the L^p-theory,* Ch. 3, Ann. of Math. Studies, Princeton, 1970. For the case $n = 3$ see also V. Morley, *Singular integrals,* Thesis, University of Chicago, 1956. Cf. also [24].

(6) The sign is different from the sign in [8] because we look at f as the derivative in the direction of the normal pointing *outward.* Also note that $K(y, \xi) = K(\xi, y)$ by a well-known property of K.

LEMMA 9.1. For $\xi \in U^n$, $y \in S^{n-1}$, we have

$$K(\xi, y) = \frac{2}{(n-2)\,|S^{n-1}|}\left(\frac{1}{|\xi - y|^{n-2}} - 1\right) + \frac{1}{|S^{n-1}|}\int_0^{|\xi|}\left(\frac{1}{|\varrho\xi' - y|^{n-2}} - 1\right)\frac{d\varrho}{\varrho} \tag{9.2}$$

where ξ' denotes $|\xi|^{-1}\xi$.

PROOF. For $\xi, \eta \in U^n$, $K(\xi, \eta) = \Sigma\, u_\alpha(\xi)\, u_\alpha(\eta)$ where $\{u_\alpha\}$ is a complete orthonormal system in $\mathfrak{D}$ (cf. [8, p. 262]).

Let u be a homogeneous harmonic polynomial of degree k; then $u(\xi) = |\xi|^k S_k(\xi')$ where S_k is a spherical function of degree k on S^{n-1}. By the divergence theorem and by the relation $\operatorname{div}(u \nabla u) = \nabla u + |\nabla u|^2$ we have

$$\int_{U^n} |\nabla u|^2 = \int_{S^{n-1}} u\frac{\partial u}{\partial n} = k\int_{S^{n-1}} |S_k|^2$$

where $\partial/\partial n$ is differentiation with respect to the normal pointing outward.

It follows that if $\{S_{k,l}\}$ is a complete orthonormal system of spherical functions with respect to the usual norm, with k denoting the degree, then

$$K(\xi, \eta) = \sum_{k=1}^{\infty} \frac{1}{k}\,|\xi|^k\,|\eta|^k \sum_{l=1}^{N(n,k)} S_{k,l}(\xi')\, S_{k,l}(\eta').$$

By the addition theorem for spherical functions [19, p. 10] this becomes

$$K(\xi, \eta) = \frac{1}{|S^{n-1}|}\sum_{k=1}^{\infty}\frac{N(n,k)}{k}\,|\xi|^k\,|\eta|^k\, P_k(\langle \xi', \eta'\rangle) \tag{9.3}$$

Here P_k is defined by

$$\sum_{k=0}^{\infty} c_k(n)\,\varrho^k P_k(t) = (1 - 2\varrho t + \varrho^2)^{1-\frac{n}{2}}, \quad c_k(n) = \frac{\Gamma(k+n-2)}{\Gamma(n-2)\,\Gamma(k+1)}$$

[19, p. 33]. Since $n \geq 3$ is fixed once and for all we do not display the dependence (7) of P_k on n.

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(7) We have $P_k(t) = c_k(n)\, C^{\frac{n-2}{2}}(t)$, i.e. P_k is a Gegenbauer polynomial renormalized so that $P_k(1) = 1$.

By an explicit formula for $N(n,k)$ [19, p. 4] it is clear that

$$\frac{N(n,k)}{k}=\left(\frac{2}{n-2}+\frac{1}{k}\right)c_k(n).$$

It follows that

$$K(\xi,\eta)=\frac{2}{(n-2)\,|S^{n-1}|}\left(\frac{1}{(1-2\langle\xi,\eta\rangle+|\xi|^2|\eta|^2)^{\frac{n-2}{2}}}-1\right)+ \tag{9.4}$$

$$\frac{1}{|S^{n-1}|}\int_0^{|\xi||\eta|}\left(\frac{1}{(1-2\varrho\langle\xi',\eta'\rangle+\varrho^2)^{\frac{n-2}{2}}}-1\right)\frac{d\varrho}{\varrho}.$$

Setting $\eta=y$, $|y|=1$, this formula reduces to (9.2).

Remarks. 1. Using a different standard formula [19, p. 30] to sum (9.3) we find

$$K(\xi,\eta)=\frac{1}{|S^{n-1}|}\int_0^{|\xi||\eta|}\left(\frac{1-\varrho^2}{(1-2\varrho\langle\xi',\eta'\rangle+\varrho^2)^{n/2}}-1\right)\frac{d\varrho}{\varrho} \tag{9.5}$$

whose equivalence with (9.4) can also be checked directly.

2. The radial component of the gradient of $K(\xi,y)$ with respect to ξ and for fixed $y\in S^{n-1}$ is

$$\frac{\partial}{\partial|\xi|}K(\xi,y)=\frac{1}{|\xi|}\left(\frac{1-|\xi|^2}{|y-\xi|^n}-1\right)$$

as it is immediate from (9.5). The first term inside the parenthesis is just the Poisson kernel of S^{n-1}. By well known results it follows that if $f\in L^p(S^{n-1},\mathbf{R})$ such that $\int_{S^{n-1}} f=0$ and u is given by (9.1), then the normal derivative of u on rS^{n-1} converges to f in L^p and a. e. as r tends to 1.

Lemma 9.2. For all $f\in L^p(S^{n-1},\mathbf{R})$ and $0<r<1$, define $R_rf\colon S^{n-1}\to\mathbf{R}^n$ by $(R_rf)(x)=(\nabla u)_{\tan}(rx)$ where u is as in (9.1) and $(\nabla u)_{\tan}$ denotes the tangential component (i. e. the component perpendicular to rx) of the gradient of u. Then

$$(R_rf)(x)=\int_{S^{n-1}} q_r(x,y)f(y)\,dy$$

where $q_r = s_r + t_r$ and

$$s_r(x, y) = \frac{2}{|S^{n-1}|} \frac{y - \langle y, x \rangle x}{|y - rx|^n}$$

$$t_r(x, y) = \frac{n-2}{2r} \int_0^r s_\varrho(x, y)\, d\varrho.$$

PROOF. If a function Φ on $\mathbf{R}^n$ is of the form $\Phi(\xi) = \varphi(|\xi|, \langle \xi', a \rangle)$ with some fixed vector a, then the « tangential » and radial components of the gradient, at $\xi \neq 0$, are

$$(\nabla \Phi)_{\tan} = (D_2 \varphi) \frac{a - \langle \xi', a \rangle \xi'}{|\xi|},$$

$$(\nabla \Phi)_{\mathrm{rad}} = (D_1 \varphi)\, \xi'.$$

Using this, the assertions follow from (9.2) by an easy computation.

LEMMA 9.3. Let for $x, y \in S^{n-1}$, $x \neq y$,

$$s(x, y) = \frac{2}{|S^{n-1}|} \frac{y - \langle x, y \rangle x}{|x - y|^n}$$

and let $1 < p < \infty$. Then, for all $f \in L^p(S^{n-1}, \mathbf{R})$

$$(Sf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} s(x, y) f(y)\, dy$$

exists in $L^p(S^{n-1}, \mathbf{R}^n)$, and S is a bounded linear transformation.

PROOF. We regard S^{n-1} as a homogeneous space of $G = SO(n)$. $\gamma(x, y) = |x - y|$ is a G-invariant distance on S^{n-1}, which gives us a gauge with $\varkappa = 1$. We denote by p the point $(0, \ldots, 0, 1)$. For any $x \in S^{n-1}$ we define.

$$x^{(v)} = \langle x, p \rangle p$$

$$x^{(h)} = x - \langle x, p \rangle p,$$

so x is an orthogonal sum, $x = x^{(v)} + x^{(h)}$. The measure dx induced on S^{n-1} by the Euclidean measure of $\mathbf{R}^n$ is G-invariant and by an easy computation is seen to be equal to

$$dx = \frac{dx_1 \dots dx_{n-1}}{(1-|x^{(h)}|^2)^{\frac{1}{2}}} = \frac{|x^{(h)}|^{n-2}\, d\,|x^{(h)}|\, d\Sigma}{(1-|x^{(h)}|^2)^{\frac{1}{2}}} \tag{9.6}$$

where $d\Sigma$ is the volume element on S^{n-2}.

Clearly we have $s(gx, gy) = g\,s(x, y)$ for all $g \in G$. Therefore by the discussion after Definition 1.2 the operator S is of the type considered in Theorems 2.2 and 3.1, with $k(g) = s(gp, p)$, σ_1 the trivial representation and σ_2 the identity representation of $SO(n)$ on $\mathbf{R}^n$. We are going to check that the conditions of Theorem 3.1 are satisfied; this will finish the proof.

To check (*a*), let $\lambda = 1/2$. The first integral in (*a*) is now (cf. Remark 4 after Theorem 2.2).

$$\int_{|x-p|>A} |s(x, y) - s(x, p)|\, dx$$

and the integrand is clearly majorized by

$$\frac{2}{|S^{n-1}|}\,|y - \langle x, y\rangle x|\,\left|\frac{1}{|x-y|^n} - \frac{1}{|x-p|^n}\right| + \frac{2}{|S^{n-1}|}\,\frac{1}{|x-p|^n}\,|y - p - \langle x, y-p\rangle x|.$$

Now, for every $x, y \in S^{n-1}$ we have

$$|y - \langle x, y\rangle x| = |y - x + \langle y, y-x\rangle x| \leq 2\,|y - x| \tag{9.7}$$

furthermore, for $|x - p| > A$ and $|y - p| < \lambda A = A/2$,

$$|x - y| \leq |x - p| + |y - p| < 2\,|x - p|, \tag{9.8}$$

$$|x - y| \geq |x - p| - |y - p| > \frac{1}{2}\,|x - p|. \tag{9.9}$$

By these inequalities our integral is majorized by

$$c\,|y - p| \int_{|x-p|>A} \frac{dx}{|x-p|^n} \leq c\,\frac{|y-p|}{A}$$

where the last inequality follows from (9.6) and from $|x - p| \geq |x^{(h)}|$.

We have shown that the first inequality in Theorem 3.1 (*a*) holds. For the second one we have to estimate

$$\int_{|x-p|>A} |s(y,x) - s(p,x)|\, dx.$$

Here the integrand is majorized by

$$\frac{2}{|S^{n-1}|}|x - \langle x,y\rangle y|\left|\frac{1}{|x-y|^n} - \frac{1}{|x-p|^n}\right| + \\ + \frac{2}{|S^{n-1}|}\,\frac{1}{|x-p|^n}|\langle x,p\rangle p - \langle x,y\rangle y|.$$

The first term is estimated as before. The last factor of the second term can be rewritten as $|\langle x,p-y\rangle p + \langle x,y\rangle (p-y)|$ which is clearly $\leq 2|y-p|$. So we get the same estimate as for the first integral, this finishes the proof for (*a*).

To check condition (*b*), and at the same time (iii′) of Theorem 4.1, let $\beta > 0$. Using (9.7) we have

$$\int_{|x-p|<\varrho} |x-p|^{\beta}|s(x,p)|\, dx \leq \frac{4}{|S^{n-1}|}\int_{|x-p|<\varrho} |x-p|^{\beta+1-n}\, dx$$

and this is $\leq c\varrho^{\beta}$ by (9.6).

To check (*c*), by compactness of G it suffices to consider small $0 < \sigma < \varrho$. In order to estimate

$$\int_{\varrho<|x-p|<\varrho(1+\sigma)} s(x,p)\, dx \tag{9.10}$$

we note that

$$\frac{|S^{n-1}|}{2}s(x,p) = \frac{p - \langle x,p\rangle (x^{(h)} + x^{(v)})}{|x-p|^n} = -\frac{\langle x,p\rangle x^{(h)}}{|x-p|^n} + \frac{(1-\langle x,p\rangle^2)p}{|x-p|^n}.$$

The first term on the right is an odd function of $x^{(h)}$, therefore its integral is 0. Since $|x-p|^2 = 2(1-\langle x,p\rangle)$, the second term is majorized in norm by $|x-p|^{2-n} \leq |x^{(h)}|^{2-n}$. So (9.10) is majorized in norm by

$$\frac{2}{|S^{n-1}|}\int_{\varrho<|x-p|<\varrho(1+\sigma)} \frac{dx}{|x^{(h)}|^{n-2}}.$$

Since $4\,|\,x^{(h)}\,|^2 = 4\,|\,x - p\,|^2 - |\,x - p\,|^4$, the domain of integration is equivalently described by the inequalities

$$\varrho\left(1 - \frac{\varrho^2}{4}\right)^{\frac{1}{2}} < |\,x^{(h)}\,| < \varrho\,(1+\sigma)\left(1 - \frac{(\varrho^2\,(1+\sigma)^2)}{4}\right)^{\frac{1}{2}}.$$

By (9.6) this shows that our integral is majorized by a constant multiple of $\varrho\,\sigma$, which finishes the proof of the first inequality in (*c*). As for the second, we have, for any $0 < a < b$,

$$\int\limits_{a<|x-p|<b} s\,(p,x)\,dx = \frac{2}{|\,S^{n-1}|}\int\limits_{a<|x-p|<b} \frac{x^{(h)}}{|\,x-p\,|^n}\,dx = 0 \tag{9.11}$$

since the integrand is an odd function of $x^{(h)}$.

Remarks 1. If f is thought of as a mass distribution on S^{n-1}, Sf is just the tangential derivative of the potential of f, i. e. the tangential component of the force on S^{n-1}.

2. In the proof we have also checked condition (iii′) of Theorem 4.1; the other conditions are trivial. This gives a new proof of the known fact that S preserves Lipschitz classes.

Lemma 9.4. For $x, y \in S^{n-1}$, $\varepsilon > 0$, let $s^\varepsilon(x,y) = s\,(x,y)$ if $|\,x - y\,| > \varepsilon$ and $s^\varepsilon(x,y) = 0$ otherwise. For all $f \in L^p\,(S^{n-1}, \mathbf{R})$ $(1 < p < \infty)$ we have

$$\lim_{r\to 1}\int (s_r\,(x,y) - s^{1-r}\,(x,y))\,f\,(y)\,dy = 0$$

in L^p and a. e.

Proof. The proof is analogous to that of Lemmas 6.3 and 7.3; we only sketch the proof of the L^p-convergence. Let $\tilde f$ denote the lift of f to G, and let $k_r\,(g) = s_r\,(gp, p)$, $k^\varepsilon(g) = s^\varepsilon(gp, p)$. By a change of variable we have

$$\int (s_r\,(x,y) - s^{1-r}\,(x,y))\,f\,(y)\,dy = \int (h\,k_r\,(h^{-1}\,g) - h\,k^{1-r}\,(h^{-1}\,g))\,\tilde f\,(h)\,dh =$$

$$\int g l^{-1}\,(k_r\,(l) - k^{1-r}\,(l))\,\tilde f\,(g l^{-1})\,dl.$$

Taking into account (9.11) and the analogous equality for s_r, this is further equal to

$$(9.12)\qquad \int_{|l|<1-r} gl^{-1}\, k_r(l)\, (\tilde{f}(gl^{-1}) - \tilde{f}(g))\, dl + \int_{|l|\geq 1-r} gl^{-1}\, (k_r(l) - k(l))\, (\tilde{f}(gl^{-1}) - \tilde{f}(g))\, dl\,.$$

The L^p-norm of the first integral is majorized, by Minkowski's integral inequality, by

$$\int_{|l|<1-r} |\, k_r(l)\,|\ \|\, R_{l^{-1}} \tilde{f} - \tilde{f}\,\|_p\, dl$$

where $R_{l^{-1}}$ is right translation by l^{-1}. It is easy to see that for all x, $y \in S^{n-1}$, $0 \leq r \leq 1$ we have the inequalities

$$(9.13)\qquad |\, x - ry\,| \geq 1 - r$$

$$(9.14)\qquad |\, x - ry\,| \geq r\, |\, x - y\,|.$$

By (9.7) and (9.13) we have, writing x for lp,

$$|\, k_r(l)\,| = \frac{2\,|\, p - \langle x, p\rangle x\,|}{|\, S^{n-1}|\, |\, p - rx|^n} \leq \frac{4}{|S^{n-1}|\,(1-r)^{n-1}}\,.$$

Now (9.6) gives a further majorization by the quantity

$$c \sup_{|l|<1-r} \|\, R_{l^{-1}} \tilde{f} - \tilde{f}\,\|_p$$

which tends to 0 as $r \to 1$.

The second integral in (9.12) is majorized by

$$\Bigg(\int_{1-r<|l|\leq\eta} + \int_{\eta\leq|l|} \Bigg) |\, k_r(l) - k^{1-r}(l)\,|\ \|\, R_{l^{-1}} \tilde{f} - \tilde{f}\,\|_p\, dl = I_1 + I_2$$

where η is to be determined later. Writing $lp = x$, using (9.7), and in the last step (9.14), we have

$$|\, k_r(l) - k^{1-r}(l)\,| = \frac{2}{|S^{n-1}|}\, |\, p - \langle p, x\rangle x\,| \left| \frac{1}{|p - rx\,|^n} - \frac{1}{|\, p - x\,|^n} \right| \leq$$

$$\frac{4\,|p-x|}{|S^{n-1}|}\;\frac{|p-x|^n-|p-rx|^n}{|p-x|^n\,|p-rx|^n}=$$

$$\frac{4}{|S^{n-1}|}\,\big|\,|p-x|-|p-rx|\,\big|\,\Big|\sum_{k=0}^{n=1}|p-x|^{-k}\,|p-rx|^{k-n}\leq$$

$$\frac{4}{|S^{n-1}|}(1-r)\sum_{k=0}^{n-1}|p-x|^{-k}\,|p-rx|^{k-n}\leq\frac{4n}{|S^{n-1}|}\,\frac{1-r}{r^n\,|p-x|^n}\,.$$

It follows using (9.6) that

$$I_1\leq\frac{4n}{|S^{n-1}|}\,\frac{1-r}{r}\sup_{|l|<\eta}\|R_{l-1}\tilde f-\tilde f\|_p\int\limits_{1-r<|x-p|<\eta}\frac{1}{|x-p|^n}\,dx\leq$$

$$c\sup_{|l|<\eta}\|R_{l-1}\tilde f-\tilde f\|_p$$

which can be made arbitrarily small by choosing η appropriately. On the set $|l|>\eta$, i. e. $|x-p|>\eta$, we have by the estimate above that $|k_r(l)-k^{1-r}(l)|$ tends to 0 uniformly as $r\to 1$. Hence I_2 tends to 0, finishing the proof.

Lemma 9.5. Define for $x, y\in S^{n-1}$, $x\neq y$,

$$t(x,y)=\frac{n-2}{2}\int_0^1 s_\varrho(x,y)\,d\varrho.$$

Then $t(\cdot,p)\in L^1(S^{n-1})$ and $\lim_{r\to 1}t_r(\cdot,p)=t(\cdot,p)$ in $L^1(S^{n-1})$.

Proof. Using (9.7), (9.13) and (9.14) we obtain, for small $|x-p|$,

$$\frac{n-2}{2}\int_0^1|s_\varrho(x,p)|\,d\varrho=\frac{2}{|S^{n-1}|}\Bigg(\int_0^{1-|x-p|}+\int_{1-|x-p|}^1\Bigg)\frac{|p-\langle x,p\rangle x|}{|p-\varrho x|^n}\,d\varrho\leq$$

$$\frac{4\,|x-p|}{|S^{n-1}|}\Bigg(\int_0^{1-|x-p|}\frac{d\varrho}{(1-\varrho)^n}+\frac{1}{|x-p|^n}\int_{1-|x-p|}^1\frac{d\varrho}{\varrho^n}\Bigg)\,.$$

It follows that $t(x,p)=O(|x-p|^{2-n})$, and thus, by (9.6), $t(\cdot,p)\in L^1(S^{n-1})$.

As for the second statement, we have

$$\| t(\cdot, p) - t_r(\cdot, p) \|_1 = \frac{n-2}{2} \int_{S^{n-1}} \left| \left(\int_0^1 - \frac{1}{r} \int_0^r \right) s_\varrho(x, p)\, d\varrho \right| dx \leq$$

$$\frac{n-2}{2} \left(\int_{S^{n-1}} \int_0^r | s_\varrho(x, p) |\, d\varrho\, dx \left| 1 - \frac{1}{r} \right| + \frac{1}{r} \int_{S^{n-1}} \int_r^1 | s_\varrho(x, p) |\, d\varrho\, dx \right).$$

Both terms on the right tend to 0 as $r \to 1$ since

$$\int_{S^{n-1}} \int_0^1 | s_\varrho(x, p) |\, d\varrho\, dx < \infty.$$

by the first part of the proof.

THEOREM 9.1. Let $q(x, y) = s(x, y) + t(x, y)$. For all $f \in L^p(S^{n-1}, \mathbf{R})$ $(1 < p < \infty)$ the limit

$$(Rf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} q(x, y) f(y)\, dy$$

exists in L^p and a. e. R is a bounded linear map from $L^p(S^{n-1}, \mathbf{R})$ to $L^p(S^{n-1}, \mathbf{R}^n)$ and maps every class Λ^β $(0 < \beta < 1)$ into itself. Furthermore, $\lim_{r \to 1} R_r f = Rf$ in L^p and a. e.

PROOF. The limit defining Rf exists in L^p and R is bounded by Lemmas 9.3, 9.5, and 1.2. $R_r f$ tends to Rf in L^p by Lemmas 9.4 and 9.5. To prove the assertions about convergence a. e. we note that if u is defined by (9.1) then the radial component of ∇u is given by the Poisson integral of f as in Remark 2 after Lemma 9.1, and its tangential component on rS^{n-1} is $R_r f$. It follows that ∇u is in the space H^p studied in [9], and hence has boundary values a. e. on S^{n-1}. This shows that $R_r f$ tends to Rf a. e. and the convergence a. e. of the limit defining Rf follows by Lemmas 9.4 and 9.5.

REMARKS 1. The subspace of $L^p(S^{n-1}, \mathbf{R})$ formed by functions f having cylindrical symmetry with respect to p is isometrically isomorphic with the space $L^p([0, \pi], \sin^{n-2}\theta\, d\theta)$ under the correspondence $f(x) = F(\theta)$ where $\langle x, p \rangle = \cos\theta$. For such a function, (9.1), (9.3), and the Funk-Hecke formula [19, p. 20] give

$$u(x) = \frac{1}{|S^{n-1}|} \sum_{k=1}^{\infty} \lambda_k \frac{N(n,k)}{k} r^k P_k(\langle x, p \rangle)$$

where

$$\lambda_k = |S^{n-2}| \int_0^{\pi} F(\theta)\, P_k(\cos\theta) \sin^{n-2}\theta \, d\theta.$$

Hence, by definition of R_r, we have

$$(R_r f)(x) = \left(\frac{1}{|S^{n-1}|} \sum_{k=1}^{\infty} \lambda_k \frac{N(n,k)}{k} r^{k-1} P_k'(\cos\theta) \sin\theta \right) \nu_x = (\widetilde{R}_r F)(\theta)\, \nu_x$$

where P_k' denotes the derivative of P_k and ν_x is the unit vector $(p - \langle x, p \rangle x)'$ in the plane of x and p. It follows now from Theorem 9.1 that the operators $\widetilde{R}_r$ $(0 < r < 1)$ are uniformly bounded on $L^p([0,\pi], \sin^{n-2}\theta\, d\theta)$ for every fixed $1 < p < \infty$, and that $\lim_{r \to 1} \widetilde{R}_r$ exists in the strong operator topology.

2. For $F \in L^p([0,\pi], \sin^{n-2}\theta\, d\theta)$ Muckenhoupt and Stein [18, p. 34] define the conjugate Poisson integral by

$$\widetilde{F}(r,\theta) = \frac{1}{|S^{n-1}|} \sum_{k=0}^{\infty} \lambda_k \frac{N(n,k)}{k+n-2} r^k P_k'(\cos\theta) \sin\theta$$

(here we have rewritten everything in our notation). A simple computation now gives

$$\frac{1}{r} \widetilde{F}(r,\theta) = (\widetilde{R}_r F)(\theta) - \frac{n-2}{r^{n-1}} \int_0^r \varrho^{n-2} (\widetilde{R}_\varrho F)(\theta)\, d\varrho.$$

Since $\| \widetilde{R}_r \|_p$ is bounded uniformly in r (Remark 1), it follows by Minkowski's inequality that the maps $F \longmapsto \widetilde{F}(r, \cdot)$ are also uniformly bounded linear transformations on $L^p([0,\pi], \sin^{n-2}\theta\, d\theta)$. This gives a new proof of [18, Theorem 4, part *a*] for the special values of the index λ of that paper for which $\lambda = \frac{n-2}{2}$; $n = 3, 4, 5, \ldots$

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