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ON LOCAL IMAGES OF HOLOMORPHIC MAPPINGS

A. T. HUCKLEBERRY

1. Preliminary Remarks and Examples.

There are many known results concerning images of holomorphic mappings. The following is a brief summary of some of these. Let X and Y be analytic spaces and

$$f : X \rightarrow Y$$

be a holomorphic mapping. $f[X]$ will be an analytic set in Y if f has one of the following properties:

(α) $f^{-1}[K]$ is compact for every compact $K \subset Y$ [14,8].

(β) For every $y \in f[X]$, there is a relatively compact neighborhood of y, U, and a compact set $K \subset X$ so that $f[K] = f[X] \cap U$ [11].

(γ) X is pure dimensional, and for every $x \in X$, $f^{-1}[f(x)]$ has constant dimension [13].

The theorem associated with condition (α) will be called the proper mapping theorem, (β) the semi-proper mapping theorem, and (γ) the constant fiber dimension mapping theorem. We remark that (β) $\Longleftarrow$ (α). (γ) is only a local theorem in the sense that for every $x \in X$, there is a fundamental system of neighborhoods of x whose images are analytic sets.

Unfortunately the above theorems are not applicable in some important cases. This is particularly true in the case of algebras of holomorphic functions. For example, one would like to give necessary and sufficient conditions for a point in a Grauert scheme to have a neighborhood which is an analytic variety [7, 12]. This is equivalent to characterizing local images of holomorphic mappings. Furthermore, it would be interesting to know exactly when local holomorphic images of analytic varieties are *not* analytic varieties. Knowing this, one might be able to define the smallest category of sets

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with analytic structure which is preserved under holomorphic mappings. We prove here a theorem giving necessary and sufficient algebraic conditions for a map from one 2-dimensional complex manifold to another to be locally open at a point. A conjecture for the n-dimensional version of this theorem is outlined in the last section.

Let U be a neighborhood of the origin in $\mathbb{C}^2$. Suppose

$$F : U \longrightarrow \mathbb{C}^2$$

is a holomorphic mapping with

$$F(z, w) = (f(z, w), \ g(z, w)),$$

where f and g are holomorphic functions which vanish at the origin. Let $\mathcal{O}$ be the germs of holomorphic functions at the origin in $\mathbb{C}^2$. Define $F^*[\mathcal{O}]$ to be the pull-back of $\mathcal{O}$ by F and $Q[F^*[\mathcal{O}]]$ its quotient field.

DEFINITION. Let F be as above. F is said to be subflat if and only if

$$Q(F^*[\mathcal{O}]) \cap \mathcal{O} = F^*[\mathcal{O}].$$

If $F = (f, g)$, where f and g are polynomials, we replace $\mathcal{O}$ with the sheaf of germs of holomorphic rational functions at Θ, the origin.

Note that subflatness is equivalent to the property that every holomorphic solution of the equation $H_1 \cdot x = H_2$, for arbitrary non-zero H_1 and H_2 in $F^*[\mathcal{O}]$, must again be in $F^*[\mathcal{O}]$. This condition is implied by, but is weaker than, the condition that $\mathcal{O}$ is flat over $F^*[\mathcal{O}]$. The comparison with flatness is carried out in detail in the last section.

DEFINITION. Let F be as above. F is said to be open at the origin if and only if there is a fundamental system of neighborhoods $\{U_\alpha\}_{\alpha \in A}$ at the origin with $F[U_\alpha]$ open for every $\alpha \in A$.

Open mappings on analytic spaces are well understood. In fact, with X pure dimensional and Y locally irreducible, $f : X \longrightarrow Y$ is open if and only if f has constant fiber dimension $m - p$ where dimension $X = m$ and dimension $Y = p$ [13]. We will refer to this as the open mapping theorem.

At this point it is relevant to look at several examples.

EXAMPLE. Let

$$F : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

be defined by

$$F(z, w) = (z, zw).$$

F abviously does not satisfy any of the conditions mentioned above. This is rightly so as $F[\mathbb{C}^2]$ is not an analytic set at the origin. This follows from noting that $F[\mathbb{C}^2] = \mathbb{C}^2 - S$, where $S = \{(z, w) : z = 0, w \neq 0\}$.

We remark that F is not subflat as $g/f \in Q(F^*[\mathcal{O}]) \cap \mathcal{O} - F^*[\mathcal{O}]$.

EXAMPLE. Let

$$F: \mathbb{C}^2 \to \mathbb{C}^2$$

be defined by

$$F(z, w) = (zw, z(w^2 + z)).$$

Again, F does not satisfy any of the conditions for the image to be an analytic space or open. We remark that F is subflat. It can be shown that F is open at the origin. We do this explicitly here to indicate the difficulty of such a calculation in this particularly simple algebraic case.

We wish to show that $F(z, w) = (zw, z(w^2 + z))$ is open at the origin. This is equivalent to showing that the system

$$zw = x, \tag{1.1}$$

$$z(w^2 + z) = y \tag{1.2}$$

has small solutions for values of x and y prescribed small enough. If $x = 0$, then we take $w = 0$ and $z = y^{\frac{1}{2}}$. For $x \neq 0$, we substitute $w = x/z$ in (1.2) and get

$$z^3 - yz + x^2 = 0. \tag{1.3}$$

If z_0 is any solution of (1.3), then

$$|z_0|(|z_0|^2 - |y|) \leq |x|^2.$$

Thus

$$|z_0| \leq |x|$$

or

$$|z_0|^2 \leq |y| + |x|.$$

Thus

$$|z_0| \leq |x| \vee (|y| + |x|)^{\frac{1}{2}}.$$

Take z_0 to be a solution of (1.3) with maximal absolute value. Since the product of the absolute values of the roots of (1.3) is equal to x, we have

$$|z_0| \geq |x|^{2/3}.$$

Taking $w_0 = x/z_0$, we have

$$|w_0| \leq |x|^{1/3}$$

and

$$|z_0| \leq |x| \vee (|y| + |x|)^{\frac{1}{2}}.$$

Since equations (1.1) and (1.2) are satisfied by (z_0, w_0), F is open at the origin. We remark that the calculations for many other examples are more difficult than the above. This is due to the fact that the equation which is obtained by eliminationg one variable is not so simple as (1.3). It is not necessarily true that every solution of that equation tends to zero as x and y tend to zero. For this reason, one must be cautious when choosing the appropriate root.

Such examples as these lead us to conjecture that F is open at the origin if and only if F is subflat. This, in fact, is our main result.

Parts of this paper were contained in my Stanford University dissertation, which was directed by Professor H. Royden. I wish to hank him for many helpful conversations. Also, I thank Professor A. Andreotti for suggesting that I make a comparison between flatness and subflatness.

2. A Necessary and Sufficient Algebraic Condition for Local Openness at the Origin.

For $F = (f, g)$, a holomorphic map of the type considered above, denote by $J_x(f, g)$ the jacobian of (f, g) at the point x. We say that F is of generic rank 2 if $J_x(f, g) \not\equiv 0$.

THEOREM. Let U be a neighborhood of the origin, Θ, in $\mathbb{C}^2$ and

$$F : U \longrightarrow \mathbb{C}^2$$

a holomorphic mapping of generic rank 2. Suppose $F(\Theta) = \Theta$. F is locally open at Θ if and only if F is subflat.

We note that if F does not have generic rank 2 then the theorem is false. However, it is only relevant to discuss case of generic rank 2, as $J(f, g) \equiv 0$ trivially implies $F = (f, g)$ is not locally open.

PROOF OF THE THEOREM. If F is not subflat then there are non-unit germs of holomorphic functions at the origin, H_1 and H_2, so that

$$\frac{H_1(f, g)}{H_2(f, g)} \in \mathcal{O} - F^*[\mathcal{O}] \tag{2.1}$$

where f and g are germs at the origin of f and g respectively. Without loss of generality we may assume that H_1 and H_2 are relatively prime. Taking representatives of the germs in (2.1), we have

$$\frac{H_1(F(z, w))}{H_2(F(z, w))} = h(z, w),$$

where h is holomorphic in some neighborhood of Θ. If F were locally open at Θ then we could choose W, an open neighborhood of Θ, such that

1) $F[W]$ is open
2) h is bounded on W
3) $\{H_1 = 0\} \cap \{H_2 = 0\} \cap F[W] = \{\Theta\}$.

By 1) there would be a $p \in W$ such that $F(p) \in \{H_2 = 0\} - \{H_1 = 0\}$. We would then have

$$h(p) = \frac{H_1(F(p))}{H_2(F(p))} = \infty.$$

But this is contrary to 2). Therefore F is not locally open. Hence, F locally open at Θ implies F is subflat. The proof of the converse is given by the following remarks and lemmas.

Since the theorem is local in nature, we may restrict our attention to as small a neighborhood of the origin as we like. From here on, U will denote such a neighborhood, no matter how small we have taken it. We remark that since $\mathcal{O}$ is a unique factorization domain, we may write

$$\mathbf{f} = \mathbf{PR}$$

and

$$\mathbf{g} = \mathbf{QR}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are relatively prime. Thus there exists a neighborhood U and representatives of the above germs so that

$$f = PR$$

$$g = QR$$

where P and Q are relatively prime on U [8].

Let

$$L_x(F) = \{y \in U \mid F(y) = F(x)\}.$$

We remark that $L_x(F)$ is a subvariety of U of dimension 0 or 1. Since P and Q are relatively prime, they can only vanish simultaneously on a discrete set. Thus if R is a unit at the origin, then

$$\dim_\Theta L_\Theta(F) = 0. \tag{2.2}$$

If (2.2) is the case, then the semi-continuity of $\dim L_x(F)$ gives $\dim L_x(F) = 0$ for $x \in U$ where U is taken sufficiently small [8]. In this case, F is in fact an open map on U [13]. We thus assume that R is a non-unit at the origin.

Let $\mathfrak{g} = \{x \mid J_x(f, g) = 0\}$. Since $\mathfrak{g}$ is 1-dimensional, its singular points are 0-dimensional and thus discrete. Hence, we may take U small enough so that $\mathfrak{g}$ has at most the origin as a singular point and is connected. Let $\mathcal{R} = \{x \in U \mid R(x) = 0\}$. $\mathcal{R}$ is a 1-dimensional sub-variety of $\mathfrak{g}$. If $x \in U$ is such that $\dim_x L_x(F) = 1$ then $x \in \mathfrak{g}$. Thus $L_x(F) \subset \mathfrak{g}$ as a subvariety. Hence $L_x(F)$ must contain one of the irreducible components of $\mathfrak{g}$ and as a result contains the origin. Therefore, $\dim_x L_x(F) = 1$ implies $F(x) = 0$.

Since $\dim_x L_x(F) = 1$ for every $x \in \mathcal{R}$, we have

$$\mathcal{R} = \{x \in U \mid \dim_x L_x(F) = 1\}. \tag{2.3}$$

By the semi-continuity of $\dim_x L_x(F)$ and the constant fiber-dimension mapping theorem, we conclude that if $\dim_x L_x(F) = 0$, then F is open in a neighborhood of x. From this fact and (2.3), we see that

$$F[U] = V \cup \{\Theta\},$$

where $V = F[U - \mathcal{R}]$ is an open set in $\mathbb{C}^2$.

We say that Θ is an isolated boundary point of V if and only if it is not an accumulation point of the boundary points of V. If Θ is an isolated boundary point of V, then we can take B to be a small ball with center at the origin so that

$$B \cup \partial V = \{\Theta\}.$$

Thus $B' = B - \{\Theta\}$ satisfies

$$B' \cap \partial V = \emptyset.$$

B' is connected. $V \cap B'$ and $\complement(\overline{V}) \cap B'$ are disjoint open sets covering B'. $B' \cap V \neq \emptyset$. Thus $B' \subset V$. As a result,

$$B \subset F[U],$$

and F is open at the origin.

As a consequence of the above remarks we will have proved the theorem if we show that if F is subflat then the origin is an isolated boundary point of $F[U - \mathcal{R}]$.

DEFINITION. Let $p, q \in U$ where U is prepared as above. We say that p and q are associates (denoted by $p \backsim q$) if

(i) p or $q \in U - \mathcal{R}$ and $F(p) = F(q)$;

or

(ii) p and $q \in \mathcal{R}$, and there exist sequences $\{p_n\}$ and $\{q_n\}$ contained in $U - \mathcal{R}$ with

$$p_n \longrightarrow p,\ q_n \longrightarrow q \text{ and } F(p_n) = F(q_n).$$

LEMMA 2.1. If F is subflat then every $p \in \mathcal{R}$ is an associate of Θ.

PROOF. Suppose p and Θ are not associates. Then for S and T sufficiently small disjoint neighborhoods of p and Θ respectively, we have

$$F(s) \neq F(t)$$

for every $s \in S - \mathcal{R}$ and $t \in T - \mathcal{R}$. There exists a 1-dimensional complex line $\mathcal{L}$ through p such that $\mathcal{L} \cap \mathcal{R} = \{p\}$ [13]. F has fiber dimension equal to zero on $\mathcal{L}$. By the constant fiber dimension mapping theorem, there exists a neighborhood of p, S' such that $S' \subset S$, and $F[\mathcal{L} \cap S']$ is a 1 dimensional subvariety containing Θ in an open set Ω in $\mathbb{C}^2$. For notational purposes, take $S' = S$. Let $\mathcal{C}$ be an irreducible component at the origin of $F(\mathcal{L} \cap S)$ so that $F^{-1}[\mathcal{C}] \supset \mathcal{L} \cap S$. Then

$$\mathcal{C} = \{x \in \Omega \mid H(x) = 0\}$$

where H is a prime holomorphic function which vanishes at the origin. By taking S and T small enough, we may assume that $F[S], F[T] \subset \Omega$.

We now restrict our attention to T. Let

$$\mathcal{V} = F^{-1}[\mathcal{C}] \cap T.$$

Then $\mathcal{V}$ is a subvariety of T containing the origin. We have $F(s) \neq F(t)$ for every $s \in S - \mathcal{R}$ and $t \in T - \mathcal{R}$. We claim that for every $t \in \mathcal{V}$, $F(t) = 0$. This follows by noting that if there exists $t \in \mathcal{V}$ such that $F(t) \neq 0$, then $F(t) \in F[(\mathcal{L} \cap S) - \mathcal{R}]$. Hence, there exists $s \in \mathcal{L} \cap S - \mathcal{R}$ such that $F(s) = F(t)$. But $F(s) \neq F(t)$ for every $s \in S - \mathcal{R}$ and $t \in T - \mathcal{R}$. Thus on

T, $H(f, g) = 0$ if and only if $R = 0$. Hence for N large enough,

$$\frac{R^N}{H(f, g)}$$

is holomorphic in a neighborhood of the origin. Thus

$$\mathfrak{q}_1 = \frac{\mathfrak{f}^N}{\mathbf{H}(\mathfrak{f}, \mathfrak{g})} \text{ and } \mathfrak{q}_2 = \frac{\mathfrak{g}^N}{\mathbf{H}(\mathfrak{f}, \mathfrak{g})}$$

are elements of $Q[F^*[\mathcal{O}]] \cap \mathcal{O}$.

Suppose

$$\mathfrak{q}_1 \in F^*[\mathcal{O}].$$

Then

$$q_1 = \frac{f^N}{H(f, g)} = H_1(f, g)$$

where $H_1 = H_1(s, t)$ is holomorphic in some neighborhood of the origin. Thus

$$f^N - A_1(f, g) = 0$$

where

$$A_1(s, t) = H(s, t)\, H_1(s, t)$$

is holomorphic in some neighborhood of the origin. If A_1 is not a function of s alone, then we have a non-trivial analytic relation

$$f^N - A_1(f, g) = 0$$

which involves both f and g. This can only happen if $J(f, g) \equiv 0$, which is not the case. Thus A_1 is a function of s alone. We apply the same argument to q_2 with the result that there are holomorphic functions A_2 and H_2 so that

$$A_2(s, t) = H(s, t)\, H_2(s, t)$$

and $g^N - A_2(f, g) = 0$. Hence A_2 must be a function of t alone. But

$$\{H = 0\} \subseteq \{A_1 = 0\} \cap \{A_2 = 0\} \subseteq \{\Theta\}.$$

Thus H is a unit. This is absurd. Thus, $Q[F^*[\mathcal{O}]] \cap \mathcal{O} - F^*[\mathcal{O}] \neq \emptyset$ and F is not subflat.

In the case that f and g are polynomials it is enough to show that H can be taken to be a polynomial. This will follow if $\mathcal{C}$ is an algebraic variety. But this is indeed the case as it is the image by a polynomial map of a complex line.

LEMMA 2.2. Let A be the algebra of polynomials in f and g where f and g are holomorphic on $U \subset \mathbf{C}^2$ (U, f and g taken as above). Let $\mathcal{R}$ be as above, and $\{\mathcal{R}_\alpha\}_{\alpha=1}^n$ be the irreducible components of $\mathcal{R}$. If A^* is the quotient field of A, then there exists an open neighborhood of the origin, U', contained in U and $q_\alpha \in A^*$, $\alpha = 1, \dots, n$, with q_α having at most the origin as a point of indeterminacy and having discrete level sets on $\mathcal{R}_\alpha - \{(0, 0)\}$.

PROOF. Recall that $\mathcal{R}$ has at most the origin as a singular point. Each $\mathcal{R}_\alpha$, being irreducible, has the property that $\mathcal{R}_\alpha - \{(0, 0)\}$ is a connected 1-dimensional complex manifold. Thus, if we can construct q_α which is holomorphic on $\mathcal{R}_\alpha - \{(0, 0)\}$ and takes more than one value on $\mathcal{R}_\alpha - \{(0, 0)\}$, then its level sets must be discrete.

As before,

$$f = PR \text{ and } g = QR$$

where P and Q are relatively prime.

$$R = \prod_{\alpha=1}^{n} R_\alpha^{m_\alpha}$$

where the R_α are prime and

$$\mathcal{R}_\alpha = \{x \in U \mid R_\alpha(x) = 0\}.$$

It suffices to construct q_1, as the construction of the other $q's$ is the same except for taking U' smaller.

Since P and Q are relatively prime, R_1 does not divide both P and Q. Thus either $\{P = 0\} \cap \mathcal{R}_1$ or $\{Q = 0\} \cap \mathcal{R}_1$ is discrete. Hence, without loss of generality, we assume that U' has been chosen small enough so that $\{P = 0\} \cap \mathcal{R}_1 \subseteq \{(0, 0)\}$.

Now, let $p \in \mathcal{R}_1 - \{(0, 0)\}$. Since $\mathcal{R}_1$ is regular away from the origin, we may take R_1 as the first of the coordinates at p. By a holomorphic change of variables in a neighborhood of p and translation to Θ, we write

$$f = z^n f_1(z, w)$$

and

$$g = z^m g_1(z, w)$$

where f_1 and g_1 are holomorphic near Θ. f_1 and g_1 are taken to have no factors of z. Since f vanishes only on $\mathcal{R}_1$ near p, f_1 is a unit at Θ. We desire to construct a quotient of polynomials in f and g which is holomorphic in a neighborhood of the origin, and is not constant on $\{z = 0\}$.

If g_1 is a non-unit at Θ, the proof is complete as $g_1(z, w) = g_1'(w) + + zg_1''(z, w)$ where $g_1'(0) = 0$. In this case,

$$q = \frac{g^n}{f^m} = \frac{(g_1'(w) + zg_1''(z, w))^n}{f_1^m}.$$

Since f_1 is a unit, q is holomorphic, and

$$q(0, w) = \frac{g_1'(w)^n}{f_1^m(0, w)}$$

is non constant.

Now suppose g is a unit. Let Q be the subset of A^* the elements of which can be written as $z^j u$ in the above coordinate system with u being a unit. Let $q_0 = z^d u$ be an element of Q with minimum positive exponent of z. There are integers s and t such that

$$\frac{f^s}{q_0^t} = z^k u$$

where u is a unit and k is the greatest common divisor of d and n. $k < d$ and

$$\frac{f^s}{q_0^t} \in Q.$$

Thus $k = 0$. Hence $n = k_1 d$ and

$$\frac{f}{q_0^{k_1}} = u_1$$

where u_1 is a unit. If $u_1(0, w) = a_1$ is constant, then

$$\frac{f}{q_0^{k_1}} = a_1 + z^{n_1} h$$

where h is holomorphic in some fixed polycylindrical neighborhood of the origin, Δ, and is not divisible by z. Thus

$$f - a_0 q_0^{k_1} = z^{k_1 d + n_1} \cdot q_1$$

where all functions mentioned are holomorphic in Δ and z does not divide q_1. We now repeat the above procedure for $f - a_1 q_0^{k_1}$ instead of f. Successively applying this procedure, we either get a quotient non-constant on $\{z = 0\}$ at some stage or

$$f = \sum_{j=1}^{\infty} a_j q_0^{k_j} \tag{2.4}$$

formally.

Since we have assumed that g_1 is a unit, we can apply the same procedure as the above to g with the result that we either complete the proof by finding a quotient which is non-constant on $\{z = 0\}$ or

$$g = \sum_{j=1}^{\infty} b_j q_0^{l_j} \tag{2.5}$$

formally. However, (2.4) and (2.5) cannot simultaneously hold. This follows by noting that in this case, $J(f, g)$ is formally identically zero, and thus the Taylor expansion at the origin of $J(f, g)$ has all coefficients zero. As a result, $J(f, g) \equiv 0$, which is contrary to our assumptions. Hence, there is a quotient q_1 which is holomorphic at the origin and non-constant on $\{z = 0\}$.

In our original coordinates, we have constructed an element of A^*, q_1, which is holomorphic in a neighborhood of p and is non-constant on $\mathcal{R}_1$. Let $\mathcal{P}$ be the polar set of q_1. Since q_1 is holomorphic near p, $\mathcal{P} \cap \mathcal{R}_1$ is discrete. Now

$$M = \mathcal{R}_1 - (\mathcal{P} \cup \{(0, 0)\})$$

is a connected 1-dimensional complex manifold. q_1 restricted to M is holomorphic on M and is non-constant on M in a neighborhood of p. Thus q_1 has discrete level sets on M. We take U' small enough so that $\mathcal{R}_1 \cap \cap (\mathcal{P} \cup \{(0, 0)\}) = \{(0, 0)\}$, and q_1 has at most $(0, 0)$ as a point of indeterminacy in U'.

For notational purposes, denote U' in the above lemma by U.

COROLLARY 2.3. For $p \neq \Theta$, let $S_p = \{w \in U \mid p \backsim w,\ w \neq \Theta\}$. S_p is discrete.

PROOF. For $p \in U - \mathcal{R}$, this is clear as F is discrete fibered at such points. If $p \in \mathcal{R}$, then $S_p \subset \mathcal{R}$. In this case, let $w \in S_p$. Then $w \in \mathcal{R}_i$ for some i. Thus, by Lemma 2.2., there exists q_i which is holomorphic in a neighborhood of w, W, and q_i has discrete level sets on $\mathcal{R}_i$. W can be taken small enough so that q_i takes the value $q_i(w)$ on $\mathcal{R}_i \cap W$ only at w.

$p \backsim w$ implies that there exist sequences $\{w_n\}$ and $\{p_n\}$ contained in $U - \mathcal{R}$ converging to p and w, respectively, with $F(p_n) = F(w_n)$. Since q_i is a quotient of relatively prime polynomials in f and g and is not indeterminate on $U - \mathcal{R}$, $q_i(w_n) = q_i(p_n)$. Due to the fact that q_i is not indeterminant at p and w, it is continuous at these points, and thus

$$q_i(p) = q_i(w).$$

Further, if $w' \in W \cap \mathcal{R}_i$ is an associate of p, then by the same argument as above,

$$q_i(p) = q_i(w').$$

Thus

$$q_i(w) = q_i(w')$$

and $w = w'$.

Lemma 2.4. Suppose $p, w \in U$ and $p \backsim w$. Assume that there are relatively compact disjoint neighborhoods N_p and N_w of p and w respectively such that there are no associates of p in $\bar{N}_w$ other than w and no associates of p in $\bar{N}_p$ other than p. Let $\{p_n\} \subset U - \mathcal{R}$ be a sequence which converges to p. Then there is a sequence $\{w_{n_k}\} \subset U - \mathcal{R}$ which converges to w with

$$F(w_{n_k}) = F(p_{n_k}).$$

Proof. If $p \in U - \mathcal{R}$ then $w \in U - \mathcal{R}$ and F is an open mapping near p and w. $F(p) = F(w) = a$. We may take N_p and N_w as above and such that $F[N_p] = F[N_w] = \Omega$ is an open neighborhood of a. We may assume that $F(p_n) \in \Omega$ for every n. Let $w_n \in N_w$ such that $F(w_n) = F(p_n)$. Let $\{w_{n_k}\}$ be a subsequence of $\{w_n\}$ which converges to $w' \in \bar{N}_w$. Thus $F(p) = F(w')$ and $p \backsim w'$. Hence, by assumption, $w' = w$. $\{w_{n_k}\}$ is therefore the desired sequence.

Suppose $p, w \in \mathcal{R}$. Since $p \backsim w$ there are sequences $\{s_n\} \subset N_p - \mathcal{R}$ and $\{t_n\} \subset N_w - \mathcal{R}$ with $F(s_n) = F(t_n)$. Now $F(N_p - \mathcal{R}) = \Omega_p$ and $F(N_w - \mathcal{R}) = \Omega_w$ are open sets, each having Θ as a boundary point. Let $\Omega = \Omega_p \cap \Omega_w$. Since $F(s_n) = F(t_n)$, $\Omega \neq \emptyset$. Let $N_p' = F^{-1}[\Omega] \cap N_p$ and $N_w' = F^{-1}[\Omega] \cap N_w$. N_p' and N_w' are open sets having p and w, respectively, as boundary points.

Suppose $\{x_n\} \subset \partial N_p' \cap (N_p - \mathcal{R})$ and $x_n \to p$. Then $F(x_n) \in \partial \Omega \subseteq \partial \Omega_p \cup \partial \Omega_w$. Since $F(x_n) \in \Omega_p$, $F(x_n) \notin \Omega_w$. Now, $\partial \Omega_w = \partial F[N_w - \mathcal{R}] \subseteq F[\partial N_w]$. Thus $F(x_n) \in F[\partial N_w]$ for every n. Since ∂N_w is compact, there is a sequence $\{y_{n_k}\} \subset \partial N_w$ such that $y_{n_k} \to w' \in \partial N_w$ and $F(y_{n_k}) = F(x_{n_k})$. Thus $w' \backsim p$. Therefore, by assumption, $w' = w$. But $w' \in \partial N_w$. This is contrary to N_w being a neighborhood of w. Thus no such sequence $\{x_n\}$ can exist. We conclude that the boundary points of N_p' which are in $N_p - \mathcal{R}$ are bounded away from p. Let T be the interior of $N_p' \cup \mathcal{R}$. By the above remarks, T is an open neighborhood of p.

Let $\{p_n\} \subset U - \mathcal{R}$ be any sequence converging to p. We may assume that $\{p_n\} \subseteq T$. Since $\{p_n\} \cap \mathcal{R} = \emptyset$, $\{p_n\} \subset N_p'$. Thus $F[\{p_n\}] \subset \Omega \subset F[N_w]$. Hence there exists $\{w_n\} \subset N_w$ such that $F(w_n) = F(p_n)$. Let $\{w_{n_k}\}$ be a subsequence of $\{w_n\}$ which converges to $w' \in \overline{N}_w$. Thus $w' \backsim p$. By assumption, $w' = w$.

COROLLARY 2.5. Let W be a neighborhood of Θ such that $W \subset\subset U$. Suppose $p \in \partial W$ and $p \backsim \Theta$. Let $\{p_n\} \subset \partial W - \mathcal{R}$ be a sequence converging to p. Then there exists $w \in W \cap \mathcal{R}$ and a sequence $\{w_{n_k}\} \subset W - \mathcal{R}$ converging to w with

$$F(w_{n_k}) = F(p_{n_k}).$$

PROOF. If $S_p \cap W \neq \emptyset$ then we are finished as Corollary 2.3 gives the neighborhoods required in Lemma 2.4 for the construction of such a sequence converging to any given $w \in S_p \cap W$. If $S_p \cap W = \emptyset$ then p has no associates in W other than Θ. N_p is again given by Corollary 2.3. We may take N_Θ to be any open neighborhood of Θ such that $\overline{N}_\Theta \cap \overline{N}_p = \emptyset$. We again apply Lemma 2.4.

The proof of the Theorem is now immediate. Assume that F is subflat. Thus, by Lemma 2.1, every $p \in \mathcal{R}$ is an associate of Θ. Let U be prepared as above and W be any neighborhood of Θ such that $W \subset\subset U$. By a previous remark, it is enough to show that Θ is an isolated boundary point of $F[W - \mathcal{R}]$. Suppose this is not the case. Then there exists a sequence $\{a_n\} \subset \partial F[W - \mathcal{R}]$ such that $a_n \neq \Theta$ for every n and $a_n \to \Theta$. Since $F[\partial [W - \mathcal{R}]] \supseteq \partial F[W - \mathcal{R}]$, there is a sequence $\{p_n\} \subset \partial [W - \mathcal{R}]$ with

$$F(p_n) = a_n .$$

Since $a_n \neq \Theta$, $p_n \in \partial W - \mathcal{R} \subset U - \mathcal{R}$. By taking a subsequence we may assume that $\{p_n\}$ converges to $p \in \partial W$. Since $a_n \to \Theta$, $F(p) = \Theta$. Thus $p \in \mathcal{R}$ and therefore is an associate of Θ.

By Corollary 2.5 there is a sequence $\{w_{n_k}\} \subset W - \mathcal{R}$ with $w_{n_k} \to w \in W$ and

$$F(w_{n_k}) = F(p_{n_k}) = a_{n_k} .$$

But F is an open mapping near each w_{n_k}. This is contrary to $a_{n_k} \in \partial F[W - \mathcal{R}]$. Thus Θ is an isolated boundary point of $F[W - \mathcal{R}]$. This completes the proof of the theorem.

3. Flatness and Subflatness.

In this section we wish to compare flatness properties of a mapping to subflatness. As we will show, flatness alone can be of little hel pin proving local openness theorems. However the similarity of the two properties gives us hope that the algebraic and functorial tools of type used by Douady [3,4] in the flat case can also be utilized in the subflat case.

We begin with a review of terms. All rings will be assumed to be commutative with 1.

DEFINITION. An A-module E is said to be A-flat if for any other A-modules, F' and F, such that

$$0 \longrightarrow F' \xrightarrow{\ \alpha\ } F'$$

is exact, it follows that

$$0 \longrightarrow E \otimes E' \xrightarrow{\ 1 \otimes \alpha\ } E \otimes F$$

is exact.

Note that, since $\otimes$ is a right exact functor, E is A-flat if and only if $E \otimes$ sends short exact sequences of A-modules into short exact sequences of A-modules.

EXAMPLE. Let $A = \mathbb{Z}$, $E = \mathbb{Z}_2$, and $F = F' = \mathbb{Z}$. Then

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ \alpha\ } \mathbb{Z}$$

is exact, where $\alpha(x) = 2x$. However, $1 \otimes \alpha : \mathbb{Z}_2 \otimes \mathbb{Z} \to \mathbb{Z}_2 \otimes \mathbb{Z}$ is the zero mapping. This suggests that flatness and torsion might be related. Indeed, there is a strong tie between flatness and the « Tor » functor [1].

The following proposition [1] clarifies the meaning of flatness.

PROPOSITION 3.1. A necessary and sufficient condition for E to be a flat A-module is the following: Every solution $(y_k)_{1 \leq k \leq n}$ formed from elements of E, of a homogeneous linear system of equations

$$\sum_{k=1}^{n} c_{ki}\, y_k = 0 \qquad (1 \leq i \leq m) \tag{3.1}$$

with coefficients $c_{ki}\, A$, is a linear combination

$$y_k = \sum_{j=1}^{q} b_j\, z_{jk} \qquad (1 \leq k \leq n)$$

with coefficients $b_j \in E$ of solutions $(z_{jk})_{1\leq k\leq n}$ of (3.1) formed from elements of A.

From now on $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ will denote reduced, locally irreducible complex spaces.

DEFINITION. Let $F: X \longrightarrow Y$ be a holomorphic mapping and $\mathcal{S}$ an analytic sheaf on X. $\mathcal{S}$ is said to be F-flat at $p \in X$ if $\mathcal{S}_p$ is a flat $F^*[\mathcal{O}_{Y, F(p)}]$-module. F is called a flat morphism if $\mathcal{O}_X$ is F-flat on X.

This ends our review.

Recall that the main observation which is needed to prove the 2-dimensional local openness theorem is that a mapping is not locally open only if there is a non-discrete analytic variety in the complement of the image set.

DEFINITION. Let $F: X \longrightarrow Y$ be a holomorphic mapping. F is said to omit a variety at $p \in X$ if, for a sufficiently small open neighborhood, U, of p, there is a non-discrete analytic subvariety V in a neighborhood Ω of $F(p)$ such that $V \cap F[U] = \{F(p)\}$.

Our general philosophy is that F is not locally open at p only if F omits a variety at p. Of course the situation cannot be as simple as the 2-dimensional case in which there is only one possible interesting type of set to omit, a 1-dimensional variety.

DEFINITION. Suppose $F: X \longrightarrow Y$ is a holomorphic mapping. F is said to be *subflat* at $p \in X$ if for every prime ideal $I \subset \mathcal{O}_{Y, F(p)}$ such that $\dim V(I) > 0$,

$$(F^*[I]) \cap F^*[\mathcal{O}_{Y, F(p)}] = F^*[I]$$

where $(F^*[I])$ denotes the ideal generated by $F^*[I]$ in $\mathcal{O}_{X,p}$.

We remark that in the 2-dimensional case this definition of subflatness is the same as that in section 1.

CONJECTURE. Assume $F: X \longrightarrow Y$ is a holomorphic mapping. Suppose $\dim X \geq \dim Y$ and F has *generic* maximal rank. Then, F is subflat at p if and only if F is locally open at p.

Later on in this paper we will prove the «easy» direction of this conjecture. In a forthcaming paper we will prove the conjecture under a certain uniform semi-continuity assumption on the fiber dimension of F in a neighborhood of p. We now prove that the conjecture makes sense on the basis of previous heuristic remarks.

PROPOSITION 3.2. Let $F: X \longrightarrow Y$ be a holomorphic mapping of generic maximal rank. Then, F is subflat at p if and only if it does not omit a variety at p.

PROOF. We may assume that X and Y are subvarieties in open sets of $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. Assume that $F = (f_1, \dots, f_m)$, where each f_i is a holomorphic function on X. Furthermore we may suppose that p is the origin, Θ_n, and $f_i(p) = 0$ for $i = 1, \dots, m$.

Suppose F omits a variety at Θ_n. Let V be an irreducible component of that variety at Θ_m and $I = (H_1, \dots, H_k)$ be the prime ideal of germs of holomorphic functions on Y vanishing on V. By assumption $\dim_{\Theta_m} V \geqq 1$. It is clear that

$$V(H_1 \circ F, \dots, H_k \circ F) = V(f_1, \dots, f_m).$$

Thus by the Nullstellensatz,

$$\sqrt{(H_1 \circ F, \dots, H_k \circ F)} = \sqrt{(f_1, \dots, f_m)}\,.$$

If F is subslat at Θ_n there exist $h_{ij} \in \mathcal{O}_{Y, \Theta_n}$ and $N_i \in \mathbb{Z}$ such that at every point $(y_1, \dots, y_m) \in F[X]$

$$\sum_{j=1}^{k} (H_j \cdot h_{ij})(y_1, \dots, y_m) = y_i^{N_i}. \tag{3.2}$$

Since $\dim_{\Theta_n} X \geqq \dim_{\Theta_m} Y$ and F is of generic maximal rank, $F[X]$ contains an open subset of Y. Thus, (3.2) holds on all of Y. The right hand sides of the m equations in (3.2) vanish simultaneously only at Θ_m. But the left and sides vanish simultaneously on V. Since $\dim_{\Theta_m} V \geqq 1$, this is absurd. Thus F is not subflat at Θ_n.

Suppose F is not subflat at Θ_n. Let I be a prime ideal which exhibits this fact. Suppose $I = (H_1, \dots, H_k)$. Thus there are germs $x_j \in \mathcal{O}_{X, \Theta_n}$ and $H \in \mathcal{O}_{Y, \Theta_m}$ such that

$$(H_1 \circ F)\, x_1 + \dots + (H_k \circ F)\, x_k = H \circ F, \tag{3.3}$$

but the x_j'*s cannot* be taken to be in $F^*[\mathcal{O}_{Y, \Theta_m}]$. If

$$\{H = 0\} \supset \{H_1 = \dots = H_k = 0\}$$

at Θ_m then $H \in \sqrt{(H_1, \dots, H_k)} = \sqrt{I} = I$. This is impossible, because (3.3) has no solution in $F^*[\mathcal{O}_{Y, \Theta_m}]$. Thus

$$\{H = 0\} \not\supset \{H_1 = \dots = H_k = 0\}$$

at Θ_m. Hence there is an irreducible component, V_1, of $V(I)$ such that $\dim_{\Theta_m} V_1 = N$ and $\dim_{\Theta_m}(V_1 \cap V(H)) \leq N - 1$. By the usual dimension arguments [13] there is a complex subspace P containing Θ_m such that

$\dim_{\Theta_m} P \geq m-(N-1)$ and $P \cap (V_1 \cap V(H)) = \{\Theta_m\}$. Since $\dim_{\Theta_m} V_1 + \dim_{\Theta_m} P \geq m+1$, $\dim_{\Theta_m}(V_1 \cap P) \geq 1$. Let $\widetilde{V} = V_1 \cap P$. Then $\widetilde{V} \cap V(H) = P \cap (V_1 \cap V(H)) = \{\Theta_m\}$.

Let U be a neighborhood of Θ_n such that (3.3) holds on U. For $u \in U$, $F(u) \in V(I)$ implies $F(u) \in V(H)$. Thus $F[U] \cap \widetilde{V} = \{\Theta_m\}$. Therefore F omits $\widetilde{V}$.

The following is the «easy» direction of the conjecture.

PROPOSITION 3.4. Let $F: X \to Y$ be a holomorphic mapping which is generically of maximal rank. If F is locally open at p then F is subflat at p.

PROOF. Suppose F is not subflat at p. Assume $I = (H_1, \dots, H_k)$ is a prime ideal which exhibits this. Then, for some $H \in \mathcal{O}_{Y, \Theta_m}$, we have

$$(H_1 \circ F)\, x_1 + \dots + (H_k \circ F)\, x_k = H \circ F,$$

where $x_j \in \mathcal{O}_{X, \Theta_n}$ and $H \circ F \notin F^*[I]$. If F is locally open at then $H \in \operatorname{id}\operatorname{loc}(I) = \sqrt{I} = I$, since I is prime. This is contrary to $H \circ F \notin F^*[I]$. Thus F is not locally open at Θ_n.

We now procede with our comparison to flatness.

PROPOSITION 3.4. Let $F: X \to Y$ be a holomorphic mapping. Suppose there is a coherent sheaf $\mathcal{S}$ on a neighborhood U of $p \in X$ such that $\mathcal{S}$ is F-flat at p. Suppose that the support of $\mathcal{S}$ contains an open neighborhood of p. Then F is subflat at p.

PROOF. Frisch [5,6] proves that the set of points in U at which $\mathcal{S}$ is F-flat is the complement of an analytic subvariety in U. Thus, since $\mathcal{S}$ is F flat at p, $\mathcal{S}$ is F-flat on a neighborhood of p, U'. We might as well assume that $\operatorname{supp} \mathcal{S} \supset U'$.

Douady [3] proves, under the above assumptions, that F is an open mapping on $\operatorname{supp} \mathcal{S} \cap \{x \in X : \mathcal{S} \text{ is } F\text{-flat at } x\}$. Thus F is an open mapping on U'. In particular, F is locally open at p and Proposition 3.3 implies F is subflat at p.

Note that F a flat morphism implies F is subflat follows easily from Proposition 3.1. To see this let $\{H_1, \dots, H_k, H\} \subset \mathcal{O}_{Y, \Theta_m}$ and

$$(H_1 \circ F)\, x_1 + \dots + (H_k \circ F)\, x_k = H \circ F, \tag{3.4}$$

where $x_j \in \mathcal{O}_{X,\Theta_n}$. By the flatness assumption

$$x_i = \sum_{l=1}^{m} a_{li} z_{li} \qquad (1 \leq i \leq k)$$

and

$$-1 = \sum_{l=1}^{m} a_{l(k+1)} z_{l(k+1)},$$

where $z_{li} \in F^* [\mathcal{O}_{Y,\Theta_m}]$, $a_{li} \in \mathcal{O}_{Y,\Theta_n}$ and

$$(H_1 \circ F) z_{l1} + (H_2 \circ F) z_{12} + \dots + (H_k \circ F) z_{lk} = (H \circ F)(-z_{l(k+1)}).$$

For some l (say $l = 1$), $z_{1(k+1)}$ must be a unit. Thus $y_i = \dfrac{z_{1i}}{-z_{1(k+1)}}$, for $i = 1, \dots, k$, satisfies

$$(H_1 \circ F) y_1 + \dots + (H_k \circ F) y_k = H \circ F.$$

Since $y_i \in F^* [\mathcal{O}]$, F is subflat.

Next we point out that flatness alone can be of little help in the cases of real interest. For simplicity we restrict our remarks to mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$.

DEFINITION. Let $F : U \rightarrow \mathbb{C}^n$ be a holomorphic mapping, where U is an open set in $\mathbb{C}^n$. A point $a \in U$ is said to be exceptional if $\dim_a L_a(F) \geq 1$.

It easily follows, as in the arguments of della Riccia [2], that if a is exceptional then there is a neighborhood W of a such that $\{a\} \supset \{u \in U : F$ is locally open at $u\} \cap (W \cap L_a(F))$. To prove any kind of openness theorem for F at $a \in U$ using the F-flatness of a coherent sheaf $\mathcal{S}$, supp $\mathcal{S}$ must have interior. Since the support of a coherent sheaf is an analytic set, supp $\mathcal{S}$ must contain an open neighborhood of a. If $\mathcal{S}$ is F-flat at a then supp $\mathcal{S} \cap \{u \in U : \mathcal{S}$ is F-flat at $u\}$ contains a neighborhood of a. Thus F is an open mapping at a. Hence a is not an exceptional point. Thus flatness alone is of little use at exceptional points.

In section 1 we gave an example of a map from $\mathbb{C}^2$ to $\mathbb{C}^2$ which is locally open at an exceptional point. Since no coherent sheaf with support on an open set can be F-flat at an exceptional point, this is a 2-dimensional example of the converse of Proposition 3.4. being false. We conclude this paper with an example of this phenomenon in $\mathbb{C}^n$. This example is a generalization of an example of della Riccia [2].

EXAMPLE. Let F: $\mathbb{C}^n \to \mathbb{C}^n$ be defined by $F = (f_1, \dots, f_n)$, where

$$\begin{aligned}
f_1(x_1, \dots, x_n) &= x_1(x_n + x_1)\\
f_2(x_1, \dots, x_n) &= x_1(x_n + x_2)\\
\vdots\quad & \qquad \vdots\\
f_{n-1}(x_1, \dots, x_n) &= x_1(x_n + x_{n-1})\\
f_n(x_1, \dots, x_n) &= x_1 x_2 \dots x_n .
\end{aligned}$$

For n odd, F is locally open at Θ_n. For n even, F is *not* locally open at Θ_n. Define F': $\mathbb{C}^n \to \mathbb{C}^n$ by $F' = (f'_1, \dots, f'_n)$, where $f'_i = f_i$ for $i = 1, \dots, n-1$ and $f'_n(x_1, \dots, x_n) = x_1^2 x_2 \dots x_n$. For n even, F' is locally open at Θ_n.

PROOF. We first consider F and n odd. Let S_r be the sphere with radius r and center Θ_n in $\mathbb{C}^n$. Let $0 < \varepsilon < 1$ and $(y_1, \dots, y_n) \in S_{\varepsilon^n}$. Consider the equations

$$\begin{aligned}
y_1 &= x_1(x_n + x_1)\\
y_2 &= x_1(x_n + x_2)\\
\vdots\ & \qquad \vdots\\
y_{n-1} &= x_1(x_n + x_{n-1})\\
y_n &= x_1 x_2 \dots x_n .
\end{aligned}$$

These imply

$$\begin{aligned}
x_n &= \frac{y_1}{x_1} - x_1\\
x_2 &= \frac{y_2 - y_1}{x_1} + x_1\\
\vdots\ & \qquad \vdots\\
x_{n-1} &= \frac{y_{n-1} - y_1}{x_1} + x_1\\
-y_n x_1^{n-2} &= \left[\prod_2^{n-1}(y_k - y_1 + x_1^2)\right](-y_1 + x_1^2).
\end{aligned} \tag{3.5}$$

Let s_j^k be the symmetric polynomial in k variables taken j at a time, $s_j^k(t_1, \dots \dots, t_k) = \Sigma t_{i_1} t_{i_2} \dots t_{i_j}$. The last equation of (3.5) can be written

$$x_1^{2n-2} + s_1^{n-1}(y_2 - y_1, \dots, y_{n-1} - y_1, -y_1)\, x_1^{2n-4} + \dots + \\ + s_{n-1}^{n-1}(y_2 - y_1, \dots, y_{n-1} - y_1, -y_1) + y_n x_1^{n-2} = 0. \tag{3.6}$$

Let x_1^m be a solution of (3.6) with maximal absolute value and C a universal positive constant independent of ε. Then $|x_1^m| \geq C|y_n|^{1/n}$ and

$$|x_1^m| \geq C(|s_{n-k}^{n-1}(y_2 - y_1, \ldots, y_{n-1} - y_1, -y_1)|^{1/n-k})^{\frac{1}{2}}.$$

Let

$$f(y_1, \ldots, y_{n-1}) = \sum_1^{n-1} |s_{n-k}^{n-1}(y_2 - y_1, \ldots, y_{n-1} - y_1, -y_1)|^{1/n-k}.$$

Averaging the above estimates and using the convexity of the square root, we have

$$|x_1^m| \geq C(f^{\frac{1}{2}} + |y_n|^{1/n}).$$

On $S_{\varepsilon^n} \cap \{|y_n| \geq \varepsilon^n/2\}$ we have $|x_1^m| \geq C\varepsilon$. On $S_{\varepsilon^n} \cap \{|y_n| \leq \varepsilon^n/2\}$

$$|y_1|^2 + \cdots + |y_{n-1}|^2 \geq \frac{3\varepsilon^{2n}}{4}$$

Note that $f(\lambda y_1, \ldots, \lambda y_{n-1}) = |\lambda| f(y_1, \ldots, y_{n-1})$ for all complex numbers λ. Thus, for $|y_1|^2 + \cdots + |y_{n-1}|^2 = r^2$, $f(y_1, \ldots, y_{n-1}) \geq r \cdot M$, where M is the minimum of f on the unit sphere in $\mathbb{C}^{n-1}$. $f = 0$ if and only if $y_1 = \ldots = = y_{n-1} = 0$. Thus $M > 0$. Hence, on $S_{\varepsilon^n} \cap \{|y_n| \leq \varepsilon^n/2\}$, $|x_1^m| \geq C\varepsilon^{n/2}$. It is easy to show that $|x_1^m| \leq C\varepsilon^{\frac{1}{2}}$. Thus (3.5) and these estimates prove that $(x_1^m, \ldots, x_n^m) \in B_{C\varepsilon^{1/2}}$, the ball with center Θ_n and radius $C\varepsilon^{\frac{1}{2}}$. Hence F is locally open at Θ_n.

If n is even then the term in (3.6) involving y_n combines with one of the symmetric polynomials in the other variables. In this case F is not locally open. For example, when $n = 4$, the variety defined by $(y_1 - y_2, y_4 - - (y_3 - y_1) y_1, 2y_1 - y_3)$ is omitted. If we modify F by changing f_n to f_n' where $f_n'(x_1, \ldots, x_n) = x_1^2 x_2 \ldots x_n$, then everything remains the same except the last term of (3.6) is changed to $y_n x_1^{n-3}$. Since n is even, there is no combination with the terms involving the symmetric polynomials. Thus the same arguments as the above go through and F' is locally open at Θ_n.

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