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CESARI-WEIERSTRASS SURFACE INTEGRALS AND LOWER k-AREA

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This paper contains a discussion on existence, representation, and convergence theorems for Cesari-Weierstrass integrals over continuous k-dimensional parametric surfaces in n-space. Theorems of this type were first proved by Cesari [3] for 2-dimensional surfaces in 3-space. The present theorems extend those of Cesari and are, for the most part, consequences of the general theory of quasi additive set functions. This theory was formulated by Cesari [6], [7] and has been further developed by Nishiura [14], Warner [18], [19], and the author [1].

We consider surfaces having finite lower k area in the sense of Radó [16] or, equivalently, finite Geöcze k-area in the sense of Cesari [5]. After summarizing a few definitions and well known results in Section 1, we turn, in Section 2, to various hypotheses under which C-W integrals and, more generally, quasi additive set functions are currently being introduced in surface area theory. It is shown that these hypotheses all lead to the lower k-area functional and, more generally, to equivalent C-W integrals. The size of the class of surfaces to which each hypothesis applies varies, however, and it is shown that the class $R^*(k, n)$ recently introduced by Nishiura [15] is the largest. In particular, $R^*(k, n)$ contains the class $T^*(k, n)$ which leads to the Geöcze k-area. The Geöcze and lower k-areas coincide, of course, for surfaces in $T^*(k, n)$, and when $k \leq 2$ or $k = n$ every surface of finite k-area belongs to $T^*(k, n)$.

In Section 3 we show that C-W integrals over a surface in $R^*(k, n)$ can be represented as Lebesgue integrals with respect to an area measure and Radon-Nikodym derivatives induced by the surface. The measure, R N derivatives, and representation coincide with those studied in [4], [8], [14],

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[10], and [2] in case the surface belongs to $T^*(k, n)$. The area measure has been further studied by Nishiura [15] for surfaces in $R^*(k, n)$.

$R^*(k, n)$ has been characterized by Nishiura [15] as the class of surfaces of finite k-area which satisfy the author's significant cylindrical condition ([2]). This characterization is independent of any quasi additivity hypothesis. Further information including an additional characterization of $R^*(k, k+1)$ is given in Section 4.

Representations of C-W integrals as Lebesgue integrals with respect to Lebesgue measure and generalized Jacobians are proved in Section 5 for AC surfaces in $R^*(k, n)$. These extend a representation proved by Cesari [3] for 2-dimensional AC surfaces in 3-space. The present proof uses the representation of Section 3 together with a new result showing an equivalence between the R-N derivatives and the generalized Jacobians for AC surfaces.

In Section 6 we prove a number of convergence theorems for C-W integrals corresponding to a sequence of surfaces in $R^*(k, n)$ converging to a given surface in $R^*(k, n)$. These are all special cases of a general convergence theorem proved by Warner for C-W integrals in an abstract setting. Included are extensions of a well known convergence theorem of Cesari [3] and a theorem concerning the weak convergence of current-valued measures. An alternate proof of the latter theorem has also been given by Gariepy [10] for surfaces in $T^*(k, n)$.

1. Preliminaries.

We denote by E^0 and E^* the interior and frontier, respectively, of a set E in the Euclidean k space E_k, and by L_k the k-dimensional Lebesgue measure. The Euclidean norm is denoted by $|\cdot|$ and, unless indicated otherwise, we set $a^{\pm} = (|a| \pm a)/2$ for any real number a.

By a *polyhedral region* we mean the compact point set R covered by a strongly connected k-complex situated in E_k; R is said to be *simple* if $k = 1$ or if $k \geq 2$ and $E_k - R$ is connected. A finite union $F = \cup R$ of nonoverlapping polyhedral regions R is called a *figure* if $F^0 = \cup R^0$, and a nonempty set A in E_k is said to be *admissible* if A is either open or open in a homeomorph of a figure. By a *domain* we mean a nonempty connected open set in E_k.

Let T be a continuous mapping from an admissible set A in E_k into E_n, $1 \leq k \leq n$; such a mapping is said to belong to the class $T(k, n)$. If $k = n$, then T is said to be *flat* and to belong to the class $T(k, k)$. The restriction of T to an arbitrary subset M of A will be denoted by (T, M).

We shall denote by I any polyhedral region contained in A or any bounded domain whose closure is contained in A, and by $D=[I]$ any finite system of nonoverlapping sets of this type. If $M \subset A$, we set $s(I, M)=1$ or 0 according as I is or is not contained in M.

We assume first that T is flat. $0(x, T, I)$ will denote the topological index of a point x in the range space E_k' of T relative to (T, I). As a function of x, this index is integer valued, constant on each component of $E_k' - T(I^*)$, and vanishes on $E_k' - T(I)$ and on $T(I^*)$. Further properties are given in [5] and [17].

For each $M \subset A$, the multiplicity function

$$N(x, T, M)=\sup_{D} \sum_{I \in D} s(I, M)\,|\,0(x, T, I)\,|, \qquad x \in E_k', \tag{1}$$

where the supremum is taken over all systems D, is lower semicontinuous in x, and the integral

$$V(T, M)=\int_{E_k'} N(x, T, M)\, dx \tag{2}$$

is called the (essential) *total variation* of (T, M). Multiplicity functions $N^{\pm}(x, T, M)$ are defined by replacing 0 by $0^{\pm}$ in (1), and *positive* and *negative variations* $V^{\pm}(T, M)$ are defined by replacing N by $N^{\pm}$ in (2). L_k-equivalent multiplicity functions, and hence the same variations, arise if the supremum in (1) is restricted to systems of polyhedral or simple polyhedral regions ([13]).

The equality $N=N^{+}+N^{-}$ holds for almost all x in E_k'; thus $V(T, M)=V^{+}(T, M)+V^{-}(T, M)$. If M is admissible, then $V(T, M)$ coincides with the Lebesgue and Geöcze k-areas of (T, M) ([5], [13], [14]).

1.1. Proposition ([5], [13]). Let T be flat and let $M \subset A$.

(*a*) $V(T, M^0)=V(T, M)$.

(*b*) $V(T, M)=\lim_{i \to \infty} V(T, A_i)$ for any sequence of admissible sets A_i *invading* M $\left(\text{i. e., } A_i \subset A_{i+1} \subset M \text{ and } M^0=\bigcup_{i=1}^{\infty} A_i^0\right)$.

(*c*) $V(T, M) \geq \sum_{i=1}^{\infty} V(T, M_i)$ for any sequence of nonoverlapping sets M_i whose union is contined in M; equality holds if the M_i are open and their union is M.

(*d*) $V(T, M)=\sup_{D} \sum_{I \in D} s(I, M)\; V(T, I)$.

These properties hold also for $V^{\pm}$.

Property (d) holds also if the supremum is restricted to systems D of simple polyhedral regions.

Suppose now that $T \in T(k, n)$, $k \leq n$. It is convenient to denote by T_r, $r = 1, \dots, m = \binom{n}{k}$, the flat mappings $T_r = P_r\, T : A \to E_{kr}$ obtained by composing T with the usual orthogonal projections P_r of E_n onto its k-dimensional coordinate planes E_{kr}. We shall write $V_r(T, M)$ and $V_r^{\pm}(T, M)$ in place of $V(T_r, M)$ and $V^{\pm}(T_r, M)$.

The *lower k-area* of (T, M), $M \subset A$, is defined as

$$R(T, M) = \sup_{D} \sum_{I \in D} s(I, M) \left[\sum_{r=1}^{m} V_r^2(T, I) \right]^{1/2}.$$

In view of (1.1) and the fact that domains may be invaded by polyhedral regions, we may restrict the supremum to range over all systems D of polyhedral regions. If T is flat, then $R(T, M) = V(T, M)$.

The functional R also satisfies the properties in (1.1). Also,

$$V_r(T, M) \leq \left[\sum_{r=1}^{m} V_r^2(T, M) \right]^{1/2} \leq R(T, M) \leq \sum_{r=1}^{m} V_r(T, M) \tag{3}$$

for each $r = 1, \dots, m$ and $M \subset A$. Note that by (1.1) it suffices to prove the second inequality in (3) for any set I, and in this case the inequality is a simple consequence of the definition.

T is said to be $\underline{BV}$ if $R(T, A) < \infty$. By (3), T is BV iff the flat mapping T_r, $r = 1, \dots, m$, are all BV. If T is BV, then the *relative variations*

$$\mathcal{V}_r(T, M) = V_r^+(T, M) - V_r^-(T, M), \qquad \mathcal{V} = (\mathcal{V}_1, \dots, \mathcal{V}_m),$$

and the integrals

$$u_r(T, I) = \int_{E_{kr}} 0(x, T_r, I)\, dx, \qquad u = (u_1, \dots, u_m),$$

are defined and finite for each set M and I.

2. Quasi additivity classes.

Let $T \in T(k, n)$ be BV, and let $z(T, \cdot)\, z = (z_1, \dots, z_m)$, $m = \binom{n}{k}$, be any vector function satisfying $z_r^{\pm}(T, I) \leq V_r^{\pm}(T, I)$ for each set I and each

$r = 1, \dots, m$. For each system D we define a mesh

$$\delta(z, T, D) = \max\{\text{diameter } T(I) : I \in D\}$$

$$+ \max\{V_r(T, A) - \sum_{I \in D} |z_r(T, I)| : r = 1, \dots, m\}.$$

We have $0 \leq \delta(z, T, D) < \infty$, and if $\delta(z, T, D) = 0$ for some system D, then $z(T, I)$ is the zero vector for every set of the type I.

We shall say that T belongs to the *quasi additivity class* $T(z, k, n)$ iff T is BV and $\inf_D \delta(z, T, D) = 0$. Applying this definition to the flat mappings T_r, we see that T_r belongs to the class $T(z_r, k, k)$ iff T_r is BV and the mesh

$$\delta(z_r, T_r, D) = \max\{\text{diameter } T_r(I) : I \in D\} + V_r(T, A) - \sum_{I \in D} |z_r(T, I)|$$

can be made arbitrarily small. Note that if $T \in T(z, k, n)$, then $T_r \in T(z_r, k, k)$ for each r.

Special notations are reserved for classes corresponding to the functions $\mathcal{V}(T, \cdot)$ and $u(T, \cdot)$. Following [14] and [15], we say that $T \in R^*(k, n)$ iff $T \in T(\mathcal{V}, k, n)$ and that $T \in T^*(k, n)$ iff $T \in T(u, k, n)$ and $\delta(u, T, D)$ can be made arbitrarily small for systems D of simple polyhedral regions. Classes $R^*(k, k)$ and $T^*(k, k)$ are defined analogously for BV flat mappings. Note that if $T \in R^*(k, n)$, then $\delta(\mathcal{V}, T, D)$ can be made arbitrarily small for systems D of polyhedral regions.

2.1. THEOREM. Let $T \in T(k, n)$ be BV, and let H_n^k be the k-dimensional Hausdorff measure on E_n. If $k \leq 2$, or $k = n$, or $H_n^{k+1}[T(A^0)] = 0$, then $T \in T^*(k, n)$. Moreover, if $T \in T(z, k, n)$ for any function z as above, then $T \in R^*(k, n)$.

The first statement is proved in [5], [12], [13]. To prove the second, we observe that

$$V_r(T, A) - \Sigma\, |\mathcal{V}_r(T, I)| \leq V_r(T, A) - \Sigma\, |z_r(T, I)|$$

$$+ \Sigma\, |z_r(T, I) - \mathcal{V}_r(T, I)|$$

and

$$\Sigma\, |z_r(T, I) - \mathcal{V}_r(T, I)|$$

$$= \Sigma\, |z_r^+(T, I) - z_r^-(T, I) - V_r^+(T, I) + V_r^-(T, I)|$$

$$\leq \Sigma\, [V_r^+(T, I) - z_r^+(T, I) + V_r^-(T, I) - z_r^-(T, I)]$$

$$= \Sigma\, [V_r(T, I) - |z_r(T, I)|] \leq V_r(T, A) - \Sigma\, |z_r(T, I)|,$$

where we have written Σ in place of $\sum_{I \in D}$. These inequalities imply that $\delta(\mathcal{V}, T, D) \leq 2\,\delta(z, T, D)$ for each system D.

The following theorem shows that if $T \in T(z, k, n)$, then the function $z(T, \cdot)$ is quasi additive in the sense of Cesari, and that the functionals V_r, $V_r^{\pm}$, $\mathcal{V}_r$, $\mathcal{V}$, and R are Burkill-Cesari integrals. (See [6], [1]).

2.2. THEOREM. Let $T \in T(z, k, n)$ and let $M \subset A$. Then

(a) $$V_r(T, M) = \lim \sum_{I \in D} s(I, M)\,|\,z_r(T, I)\,|,$$

(b) $$V_r^{\pm}(T, M) = \lim \sum_{I \in D} s(I, M)\, z_r^{\pm}(T, I),$$

(c) $$\mathcal{V}_r(T, M) = \lim \sum_{I \in D} s(I, M)\, z_r(T, I),$$

(d) $$\mathcal{V}(T, M) = \lim \sum_{I \in D} s(I, M)\, z(T, I),$$

(e) $$R(T, M) = \lim \sum_{I \in D} s(I, M)\,|\,z(T, I)\,|,$$

all limits being taken as $\delta(z, T, D) \to 0$. Moreover, if $\varepsilon > 0$ and $D_0 = [I]$ is any system with $\delta(z, T, D_0) < \varepsilon/2m$, then there exists $\lambda = \lambda(\varepsilon, D_0) > 0$ such that the quasi additivity relations

$$(qa):\quad \begin{cases} \displaystyle\sum_{I \in D_0} \Big|\, z(T, I) - \sum_{J \in D} s(J, I)\, z(T, J)\,\Big| < \varepsilon, \\ \displaystyle\sum_{J \in D} \Big[1 - \sum_{I \in D_0} s(J, I)\Big]\,|\,z(T, J)\,| < \varepsilon, \end{cases}$$

hold for every system $D = [J]$ with $\delta(z, T, D) < \lambda$.

The theorem may be proved by using simple modifications in Nishiura's proof ([14]) for mappings in $T^*(k, n)$. Observe first that $(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d)$. The implication $(a) \Longrightarrow (e)$ and the final statement of the theorem are obtained by replacing u by z in the proofs of [14, 5.7, 5.9]. To prove (a), we observe that $\sum_{I \in D} |z_r(T, I)| \leq \sum_{I \in D} V_r(T, I) \leq \int_E N(x, T_r, A)\, dx$ whenever $T_r(I) \subset E$ for all $I \in D$, and then proceed as in the proof of [14, 5.4].

Applying (2.2) to the mappings T_r, we see that the limits in (a)-(c) may be taken as the smaller mesh $\delta(z_r, T_r, D)$ tends to zero; relations (qa) also hold for $z_r(T, \cdot)$ and this mesh. We also remark that it follows from (e) that $R(T, M)$ coincides with the Geöcze k-area (see [5], [13], [14]) $V(T, M)$ whenever $T \in T^*(k, n)$.

Assume now that $T \in T(z, k, n)$, that ϱ is a non-negative function satisfying $\inf_D [\delta(z, T, D) + \varrho(D)] = 0$, and that $S: A \to K$, $p = S(w)$, is a transformation from A into a metric space K such that

$$\max \{\text{diameter } S(I): I \in D\} \leq \delta(z, T, D) + \varrho(D)$$

for each system D. Let $S^{m-1} = \{q \in E_m: |q| = 1\}$ denote the unit sphere in E_m, and let $f: K \times E_m \to E_1$, $a = f(p, q)$, be a real valued function satisfying the conditions

(f_1): f is bounded and uniformly continuous on $K \times S^{m-1}$,

(f_2): $f(p, tq) = tf(p, q)$ for all $t \geq 0$, $p \in K$, $q \in E_m$,

of a parametric integrand. In view of (2.2), the following theorem of Cesari [6] holds.

2.3. THEOREM. Under the above conditions, the *Cesari-Weierstrass integral*

$$\int [f(S, z), A] = \lim \sum_{I \in D} f[S(w_I), z(T, I)], \ w_I \in I,$$

exists and is finite as $\delta(z, T, D) + \varrho(D) \to 0$, and its value is independent of the choice of points $w_I \in I$.

We may, in particular, take $S = T$, $K = T(A)$, $\varrho = 0$, and consider the integral $\int [f(T, z), A]$ whenever $T \in T(z, k, n)$. The integral $\int [f(T, u), A]$ has been studied in [3], [5], [8], [14] for $T \in T^*(k, n)$.

2.4. REMARK. If ϱ is of the form

$$\varrho(D) = \varrho_1(D) + R(T, A) - \sum_{I \in D} |z(T, I)|, \ \varrho_1 \geq 0,$$

then for each $\varepsilon > 0$ there exists $\eta > 0$ depending only on ε, on an upper bound for $R(T, A)$, and on f, such that if $\delta(z, T, D) + \varrho(D) < \eta$, then $\left| \int [f(S, z), A] - \sum_{I \in D} f[S(w_I), z(T, I)] \right| < \varepsilon$ for all choices of $w_I \in I$. If T is flat, then the term $R(T, A) - \sum_{I \in D} |z(T, I)|$ is already incorporated in $\delta(z, T, D)$ and may be omitted from ϱ. See [6].

3. Induced measures.

Let $T \in T(k, n)$. In addition to the meshes of the preceding section, it is convenient to define

$$\delta(z_r, T, D) = \max\{\text{diameter } T(I) : I \in D\} + V_r(T, A) - \sum_{I \in D} |z_r(T, I)|$$

whenever T_r is BV and z_r satisfies $z_r^{\pm}(T, I) \leq V_r^{\pm}(T, I)$ for each set of the type I.

By a *component of constancy* for T we mean a component of $T^{-1}(x)$ for some $x \in T(A)$; a compact component of constancy will be termed an m. m. c. (maximal model continuum). Let $\mathcal{J} = \mathcal{J}(T, A)$ be the topology of all sets open in A which are unions of components of constancy for T, and let t denote the interior operator for this topology.

3.1. PROPOSITION. Let $T \in T(k, n)$.

(*a*) $R(T, M^t) = R(T, M)$ for every $M \subset A$; analogous statements hold for V_r and V_r^{+}.

(*b*) If $G \in \mathcal{J}$, then $R(T, G) = \sup R(T, G')$, where the supremum is taken over all G' in $\mathcal{J}$ whose $\mathcal{J}$-closure is contained in G; analogous statements hold for V_r and $V_r^{\pm}$.

(*c*) The set functions $V_r(T, \cdot)$ and $V_r^{\pm}(T, \cdot)$ are countably subadditive on the topology $\mathcal{J}$ provided T_r is BV and the mesh $\delta(\mathcal{V}_r, T, D)$ can be made arbitrarily small; if $T \in R^*(k, n)$, then $R(T, \cdot)$ is also countably subadditive on $\mathcal{J}$.

The proofs are analogous to those of [14, 6.4 - 6.9].

Let $\mathcal{B} = \mathcal{B}(T, A)$ be the σ-algebra generated by the topology $\mathcal{J}$. For each set B in $\mathcal{B}$, define

$$\mu(T, B) = \inf R(T, G), \quad \mu_r(T, B) = \inf V_r(T, G), \quad \mu_r^{\pm}(T, B) = \inf V_r^{\pm}(T, G),$$

where the infima are taken over all sets G in $\mathcal{J}(T, A)$ containing B. If T is BV, we may also define

$$\nu_r(T, B) = \mu_r^{+}(T, B) - \mu_r^{-}(T, B), \quad \nu = (\nu_1, \dots, \nu_m).$$

These set functions agree with the corresponding k areas on the sets of $\mathcal{J}(T, A)$.

3.2. THEOREM. If T_r is BV and $\delta(\mathcal{V}_r, T, D)$ can be made arbitrarily small. then μ_r and $\mu_r^{\pm}$ are finite regular measures on $\mathcal{B}(T, A)$, and $\nu_r = \mu_r^{+} - \mu_r^{-}$ is a Jordan decomposition. If $T \in R^*(k, n)$, then μ is also a finite regular measure on $\mathcal{B}(T, A)$, μ is the total variation of ν, and each ν_r is absolutely continuous with respect to μ; moreover, if $\theta_r = d\nu_r/d\mu$ and $\theta = (\theta_1, \dots, \theta_m)$, the $|\theta| = 1$ μ-a.e. in A. Finally, if $T \in T(z, k, n)$, then under the hypotheses of (2.3) the function $f[S(w), \theta(w)]$, $w \in A$, is μ-integrable on A, and

$$\int [f(S, z), A] = \int_A f[S(w), \theta(w)]\, d\mu.$$

The theorem is an immediate consequence of (2.2), (3.1), and the theory of quasi additive set functions (see [7], [1]). Various forms of the theorem have been studied in [4], [5], [8], [14], [2], and [10].

In Section 6 we shall use the following variant of Theorem 3.2. Let T_r be BV and suppose that the mesh $\delta(z_r, T, D) + \varrho(D)$ can be made arbitrarily small. Let $S: A \longrightarrow K$ satisfy

$$\max\{\text{diameter } S(I) : I \in D\} \leq \delta(z_r, T, D) + \varrho(D)$$

for each system D, and let $f: K \times E_1 \rightarrow E_1$ satisfy the conditions (f_1) and (f_2) with $m = 1$. Since $\delta(z_r, T_r, D) \leq \delta(z_r, T, D)$, an application of Theorem 2.3 to the mapping T_r shows that the integral

$$\int [f(S, z_r), A] = \lim \sum_{I \in D} f[S(w_I), z_r(T, I)]$$

exists as $\delta(z_r, T, D) + \varrho(D) \longrightarrow 0$. The theorems of [7] and [1] guarantee, further, that the function $f[S(w), d\nu_r/d\mu_r(w)]$ is μ_r-integrable on A, and that

$$\int [f(S, z_r), A] = \int_A f[S(w), d\nu_r/d\mu_r(w)]\, d\mu_r,$$

where ν_r and μ_r are the signed measure and measure of (3.2).

3.3. Theorem. If $T \in R^*(k, n)$ and the above conditions hold, then

$$\int [f(S, z_r), A] = \int_A f[S(w), \theta_r(w)]\, d\mu.$$

Proof. In view of (3.2), the obvious absolute continuity relations, (f_2), and the chain rule for Radon-Nikodym derivatives, we have, upon setting $\sigma_r = d\nu_r/d\mu_r$ and $\eta_r = d\mu_r/d\mu$,

$$\int [f(S, z_r), A] = \int_A f[S(w), \sigma_r(w)]\, d\mu_r$$

$$= \int_A f[S(w), \sigma_r(w)] \eta_r(w) \, d\mu$$

$$= \int_A f[S(w), \sigma_r(w) \eta_r(w)] \, d\mu$$

$$= \int_A f[S(w), \theta_r(w)] \, d\mu.$$

4. The significant cylindrical condition.

Let $T \in T(k, n)$ be BV. Then, by (2.1), each flat mapping T_r belongs to the class $R^*(k, k)$ and, in view of (3.2), induces a measure

$$\mu(T_r, B) = \inf\{V_r(T, G) : B \subset G \in \mathcal{T}(T_r, A)\}$$

on the σ algebra $\mathcal{B}(T_r, A)$ generated by the topology $\mathcal{T}(T_r, A)$ of all sets open in A which are unions of components of constancy for T_r. Note that $\mathcal{B}(T_r, A)$ and $\mathcal{T}(T_r, A)$ are in general coarser than the corresponding families $\mathcal{B}(T, A)$ and $\mathcal{T}(T, A)$; thus $\mu(T_r, \cdot)$ need not be the same at the measure $\mu_r(T, \cdot)$ considered in the preceding section.

A set X in $\mathcal{B}(T_r, A)$ is said to be *significant* (or essential) for T_r if (a) $\mu(T_r, X) = \mu(T_r, A)$, and ($b$) $L_k[T_r(B \cap X)] = 0$ for every set B in $\mathcal{B}(T_r, A)$ satisfying $\mu(T_r, B) = 0$.

Suppose X_r is significant for T_r, and let W_r denote the union of all m.m.c.s g for T_r such that $g \subset X_r$ and g is not an m.m.c. for T. T is said to satisfy the *significant cylindrical condition* if $\mu(T_r, W_r) = 0$ for each $r = 1, \dots, m$. This definition clearly does not depend on the choice of significant sets X_r.

We may rephrase the preceding condition in terms of L_k and the topological index as follows. Let A_{2r} denote the union of all m.m.c.s g for T_r such that $g \subset A^0$ and such that for each open set U with $g \subset U \subset A$ there exists a simple polyhedral region I with $g \subset I^0 \subset I \subset U$ and $0(T_r(g), T_r, I) \neq 0$. Then A_{2r} is significant for T_r (see [2]). Since $L_k(E) = 0$ implies $\mu(T_r, T_r^{-1}(E)) = 0$ for any Borel set $E \subset E_{kr}$ (see [2]), we conclude that, with W_r defined relative to A_{2r} as above, T satisfies the significant cylindrical condition iff $L_k[T_r(W_r)] = 0$ for each $r = 1, \dots, m$.

We remark that the significant cylindrical condition is implied by the global cylindrical condition of Cesari [5, 16.10]; this latter condition is satisfied by all BV mappings in $T(2, n)$, but not by all mappings in $T^*(k, n)$ with $2 < k < n$ (see [12]).

4.1. **Theorem.** Let $T \in T(k, n)$ be BV. Then $T \in R^*(k, n)$ iff T satisfies the significant cylindrical condition.

The necessity of the condition is proved in [2] for mappings in $T^*(k, n)$; a modification the proof applies to the class $R^*(k, n)$. The sufficiency of the condition, as well as an alternate proof of its necessity, is due to Nishiura [15].

A combination of (4.1) and [2, 6.iv] yields the following theorem.

4.2. Theorem. Let $T \in T(k,n)$ be BV, and let $M = \bigcup_{r=1}^{m} A_{2r}$. If $H_n^{k+1}[T(M)] = 0$, then $T \in R^*(k, n)$. Further, if $n = k + 1$, then $T \in R^*(k, k+1)$ iff $L_{k+1}[T(M)] = 0$.

5. AC mappings.

We shall assume throughout this section that the mapping $T \in T(k, n)$ is BV.

We say that T is $\underline{AC}$ provided (*a*) $R(T, I) = \sum_j R(T, I_j)$ for every polyhedral region $I \subset A$ and every finite subdivision of I into nonoverlapping polyhedral regions I_j, and (*b*) for each $\varepsilon > 0$ there exists $\eta > 0$ such that $\sum_{I \in D} R(T, I) < \varepsilon$ for every system D of polyhedral regions satisfying $\sum_{I \in D} L_k(I) < \eta$. These two conditions are independent [5, p. 216].

For each $r = 1, \dots, m$, let A_{3r} denote the union of all m.m.c.s g for T_r such that $g \subset A_{2r}$ and g is a single point. As in the 2-dimensional case ([4, p. 229]), A_{3r} is a Borel set. Let $A_3 = \bigcup_{r=1}^{m} A_{3r}$. Each point of A_3 is an m. m. c. for at least one of the mappings T_r, and hence also an m. m. c. for T. Thus each Borel subset of A_3 belongs to the σ algebra $\mathcal{B}(T, A)$.

For $w \in A$, let Q denote a k-cube with faces parallel to the coordinate hyperplanes of E_k and with $w \in Q^0$. Define

$$D_r^{\pm}(w) = \lim_{L_k(Q) \to 0} V_r^{\pm}(T, Q) / L_k(Q) \tag{4}$$

provided $w \in A_{3r}$ and these limits exist and are finite; set $D_r^{\pm}(w) = 0$ otherwise, and set $J_r = D_r^+ - D_r^-$, $J = (J_1, \dots, J_m)$. J_r is called the *generalized Jacobian* of T_r.

5.1. REMARK. As in the 2-dimensional case [5, p. 429], L_k-equivalent derivatives result from using (4) at all points $w \in A^0$ for which the limits exist and are finite. Under the present definition, the derivatives $D_r^{\pm}$ and J are not only Borel measurable on A, but also measurable relative to the σ-algebra $\mathcal{B}(T, A)$.

5.2. THEOREM. Let $T \in T(k, n)$ be BV.

(*a*) The inequalities

$$V_r(T, G) \geq \int_G |J_r(w)|\, dw, \tag{5}$$

$$V_r^{\pm}(T, G) \geq \int_G D_r^{\pm}(w)\, dw, \tag{6}$$

hold for every set G open in A. Equality holds for $G = A$ in (5) iff T_r is AC. If T_r is AC, then equality holds in (5), (6), and

$$\mathcal{V}_r(T, G) = \int_G J_r(w)\, dw \tag{7}$$

for every G open in A.

(*b*) The mappings T_r, $r = 1, \ldots, m$, are all AC iff T is AC and belongs to the class $R^*(k, n)$.

(*c*) The inequality

$$R(T, G) \geq \int_G |J(w)|\, dw \tag{8}$$

holds for every set G open in A. If $T \in R^*(k, n)$, then equality holds for $G = A$ in (8) iff T is AC and, in this case, equality holds in (8) for every G open in A.

PROOF. In view of (1.1) and the fact that the derivatives vanish on $A - A^0$, it suffices to prove (*a*) for the case in which A is an open set. Since $T_r \in R^*(k, k)$, (*a*) in this case is an immediate consequence of (2.2), (5,1), and [1, 3.3].

If the mappings T_r are all AC, then T satisfies the significant cylindrical condition by [2, 6.iv]; hence $T \in R^*(k, n)$ by (4.1). Statement (*b*) is now a consequence of (2.2) and [1, 3.4].

In view of (1.1), (2.2), and the fact that J vanishes on $A - A^0$, it suffices to prove (*c*) for the case in which A is an open set. Inequality (8) follows from (3) and standard differentiation theorems (see [5, 27.5]). The remaining parts of (*c*) follow from (2.2), (5.1), and [1, 3.6]. This completes the proof.

Alternate proofs of (*a*) are given in [5] and [17]. Parts (*b*) and (*c*) extend to higher dimensions results of [5, p. 445] for BV mappings in $T(2,3)$.

5.3. REMARK. If T is BV and the mappings T_r all possess weak total differentials (in the sense of [17]) L_k-a.e. in A^0, then J coincides L_k-a.e. in A^0 with the vector $j = (j_1, \dots, j_m)$ of ordinary Jacobians of the mappings T_r ([17, p. 351]). (We assume here that the spaces E_k and E_{kr} are oriented by the ordering of their cartesian coordinates). Thus, under additional differentiability hypotheses, (5.2) yields conditions sufficient for $R(T, A)$ to be the value of the classical k-area integral.

5.4 THEOREM. If T is AC and belongs to the class $R^*(k, n)$, then $J = \theta \, | J | \; L_k$-a.e. in A, where θ is the vector of Radon-Nikodym derivatives defined in (3.2).

PROOF. Let λ_k denote the restriction of L_k to the σ algebra $\mathcal{B}(T, A)$. Then $\lambda_k = L_k$ on the Borel subsets of A_3, J vanishes on $A - A_3$, and by (3.2) and (5.2) we have

$$\mu(T, G) = R(T, G) = \int_G |J(w)| \, dw = \int_G |J(w)| \, d\lambda_k$$

for every set G in $\mathcal{G}(T, A)$. It follows that

$$\mu(T, B) = \int_B |J(w)| \, d\lambda_k \tag{9}$$

for every set B in $\mathcal{B}(T, A)$. Similarly, $\mu_r^{\pm}(T, B) = \int_B D_r^{\pm}(w) \, d\lambda_k$ and

$$\nu_r(T, B) = \int_B J_r(w) \, d\lambda_k \tag{10}$$

for every B in $\mathcal{B}(T, A)$. Thus

$$J_r = d\nu_r / d\lambda_k = [d\nu_r / d\mu] \, [d\mu / d\lambda_k] = \theta_r \, | J |$$

λ_k - a.e. in A by (9), (10), (3.2), and the chain rule for Radon-Nikodym derivatives. Since $\lambda_k = L_k$ on the Borel subsets of A_3 and $| J | = 0$ on $A - A_3$, we conclude that $J = \theta \, | J | \, L_k$ - a.e. in A.

5.5 THEOREM. In addition to the hypotheses of (2.3), assume that T is AC. Then

$$\int [f(S, z), A] = \int_A f[S(w), J(w)]\, dw.$$

If, in addition, the mappings T_r all possess weak total differentials L^k - a.e. in A^0, then

$$\int [f(S, z), A] = \int_{A^0} f[S(w), j(w)]\, dw.$$

Analogous statements hold under the conditions of (3.3).

PROOF. The second statement follows from the first by (5.3) To prove the first, we use (3.2), condition (f_2), and (5.4) to obtain

$$\int [f(S, z), A] = \int_A f[S(w), \theta(w)]\, d\mu$$

$$= \int_A f[S(w), \theta(w)][d\mu/d\lambda_k]\, d\lambda_k$$

$$= \int_A f[S(w), \theta(w)] \mid J(w) \mid d\lambda_k$$

$$= \int_A f[S(w), \theta(w) \mid J(w) \mid]\, d\lambda_k$$

$$= \int_A f[S(w), J(w)]\, d\lambda_k$$

$$= \int_A f[S(w), J(w)]\, dw.$$

This theorem extends a representation theorem of Cesari [3] for the integral $\int [f(T, u), A]$ relative to BV and AC mappings in $T(2, 3)$.

6. Convergence theorems.

We shall assume throughout this section that $T_i: A_i \to E_n$, $i = 1, 2, \dots$, is a sequence of mappings in $T(k, n)$ converging to the mapping T. This means that the admissible sets A_i invade A and that

$$\lim_{i \to \infty} \sup \{ | T_i(w) - T(w) | : w \in A_i \} = 0.$$

The functionals R, V_r, and $V_r^{\pm}$ are known to be lower semicontinuous with respect to this convergence: $R(T, A) \leq \underline{\lim}_{i \to \infty} R(T_i, A_i)$, and similarly for V_r and $V_r^{\pm}$. We shall further assume that the mappings T and T_i all belong to the class $R^*(k, n)$. To shorten notations, we shall write $\mathcal{V}(I)$, $\mathcal{V} = (\mathcal{V}_1, \dots, \mathcal{V}_m)$, and $\mathcal{V}_i(I)$, $\mathcal{V}_i = (\mathcal{V}_{i1}, \dots, \mathcal{V}_{im})$, in place of $\mathcal{V}(T, I)$ and $\mathcal{V}(T_i, I)$.

We shall first state a special case of Warner's convergence theorem [19, 3.ii] for abstract Cesari-Weierstrass integrals Let S and S_i be transformations from A and A_i, respectively, into a metric space K such that

$$\lim_{i \to \infty} \sup \{ d[S_i(w), S(w)] : w \in A_i \} = 0, \tag{11}$$

where d is the metric of K. Given a system $D = [I]$, let

$$\begin{cases} \Delta(D) = \delta(\mathcal{V}, T, D) + \varrho(D) + R(T, A) - \sum_{I \in D} | \mathcal{V}(I) |, \\ \Delta_i(D) = \delta(\mathcal{V}_i, T_i, D) + \varrho_i(D) + R(T_i, A_i) - \sum_{I \in D} | \mathcal{V}_i(I) |, \end{cases} \tag{12}$$

where in the second line we consider only systems of sets whose closures are contained in A_i. We assume that the non-negative functions ϱ, ϱ_i and the transformations S, S_i are chosen so that

$$\inf_D \Delta(D) = 0, \qquad \inf_D \Delta_i(D) = 0, \tag{13}$$

$$\begin{cases} \max \{\text{diameter } S(I) : I \in D\} \leq \Delta(D), \\ \max \{\text{diameter } S_i(I) : I \in D\} \leq \Delta_i(D), \end{cases} \tag{14}$$

where in the second lines of (13) and (14) we again consider only systems of sets whose closures are contained in A_i. Finally, let $f: K \times E_m \to E_1$

satisfy conditions (f_1) and (f_2). Then the integrals $\int [f(S, \mathcal{V}), A]$ and $\int [f(S_i, \mathcal{V}_i), A_i]$, which are defined relative to Δ and Δ_i, respectively, exist by Theorem 2.3.

6.1. THEOREM. In addition to the above assumptions, suppose that for each $\varepsilon > 0$ there exists a system D and an integer N such that

(*a*) the closure of each $I \in D$ is contained in A_i for all $i \geq N$,

(*b*) $\Delta(D) < \Delta_i(D) < \varepsilon$ for all $i \geq N$,

(c) $\sum_{I \in D} |\mathcal{V}(I) - \mathcal{V}_i(I)| < \varepsilon$ for all $i \geq N$.

Then

$$\int [f(S, \mathcal{V}), A] = \lim_{i \to \infty} \int [f(S_i, \mathcal{V}_i), A_i]$$

provided $R(T, A) = \lim_{i \to \infty} R(T_i, A_i)$.

In view of hypothesis (*b*), Remark 2.4, and the fact that the *k* areas $R(T, A)$ and $R(T_i, A_i)$ are uniformly bounded, D and N may be chosen such that

$$(15)\qquad \begin{aligned} &\left| \int [f(S, \mathcal{V}), A] - \sum_{I \in D} f[S(w_I), \mathcal{V}(I)] \right| < \varepsilon, \\ &\left| \int [f(S_i, \mathcal{V}_i), A_i] - \sum_{I \in D} f[S_i(w_I), \mathcal{V}_i(I)] \right| < \varepsilon, \end{aligned}$$

for all $i \geq N$ and all choices of $w_I \in I$.

It will be convenient to state explicitly the following variant of (6.1). Let S and S_i satisfy (11) and let $f: K \times E_1 \to E_1$ satisfy (f_1) and (f_2) with $m = 1$. Replace (12) by

$$(12)'\qquad \begin{cases} \Delta(D) = \delta(\mathcal{V}_r, T, D) + \varrho(D), \\ \Delta_i(D) = \delta(\mathcal{V}_{ir}, T, D) + \varrho_i(D), \end{cases}$$

and assume that ϱ, ϱ_i and S, S_i are chosen so that (13) and (14) hold relative to the meshes (12)′. Then, as noted in (3.3), the integrals $\int [f(S, \mathcal{V}_r), A]$ and $\int [f(S_i, \mathcal{V}_{ir}), A_i]$ exist relative to these meshes.

6.2. THEOREM. In addition to these assumptions, suppose that for each $\varepsilon > 0$ there exists a system D and an integer N such that

(*a*) the closure of each $I \in D$ is contained in A_i for all $i \geq N$, (*b*) $\Delta(D) < \varepsilon$ and $\Delta_i(D) < \varepsilon$ for all $i \geq N$,

(*c*) $\sum_{I \in D} | \mathcal{V}_r(I) - \mathcal{V}_{ir}(I) | < \varepsilon$ for all $i \geq N$.

Then

$$\int [f(S, \mathcal{V}_r), A] = \lim_{i \to \infty} \int [f(S_i, \mathcal{V}_{ir}), A_i]$$

provided $V_r(T, A) = \lim_{i \to \infty} V_r(T_i, A_i)$.

Theorem 6.2 is also a special case of Warner's convergence theorem. Since the k areas $V_r(T, A)$ and $V_r(T_i, A_i)$ are uniformly bounded, D and N can be chosen so that inequalities analogous to those in (15) are satisfied

6.3. REMARK. In the mappings T_i are all quasi linear, then the hypothesis $R(T_i, A_i) \to R(T, A)$ in(6.1) implies that $R(T, A)$ equals the Lebesgue k-area $L(T, A)$ of T. While only the inequality $R \leq L$ is known for arbitrary mappings in $T(k, n)$, equality has been established in the following important cases: $k = 1$ as is well known; $(k, n) = (2, 3)$ [5]; $k = n$ [14]; $k = 2$ or $H_n^{k+1}[T(A)] = 0$ if A is a polyhedral region [10].

In the remainder of this section we shall discuss some important special cases of Theorems 6.1 and 6.2. The following two propositions will be needed.

6.4. PROPOSITION. If the mappings T_i are all AC, and if $R(T, A) = \lim_{i \to \infty} R(T_i, A_i)$, then $V_r(T, A) = \lim_{i \to \infty} V_r(T_i, A_i)$ for each $r = 1, \dots, m$.

The proof is similar to that of [5, 9.9] and is valid if the mapping T is simply assumed to be BV and the mappings T_i are BV and AC. Moreover, the AC hypothesis on the mappings T_i may be dropped if R coincides with L for all mappings concerned (cf. [5, 24.3]).

6.5. PROPOSITION. If $V_r(T, A) = \lim_{i \to \infty} V_r(T_i, A_i)$ and D is any system such that the closure of each $I \in D$ is contained in A^0, then

$$\overline{\lim_{i \to \infty}} \sum_{I \in D} | \mathcal{V}_r(I) - \mathcal{V}_{ir}(I) | \leq V_r(T, A) - \sum_{I \in D} V_r(T, I).$$

PROOF. Since the sets A_i invade A and the sets $I \in D$ have compact closures, the closure of each $I \in D$ is contained in A_i for all sufficiently large i. Let $\varepsilon > 0$ be given, and let N be the number of sets I in D. By the lower semicontinuity of $V_r^{\pm}$ we have

$$V_r^{\pm}(T, I) - \varepsilon/3N < V_r^{\pm}(T_i, I)$$

for all $I \in D$ and all sufficiently large i. Since $\mathcal{V}_r = V_r^+ - V_r^-$ and $V_r = V_r^+ + V_r^-$, we have

$$\sum_{I \in D} | \mathcal{V}_r(I) - \mathcal{V}_{ir}(I) |$$

$$\leq \sum_{I \in D} | V_r^+(T, I) - V_r^+(T_i, I) | + \sum_{I \in D} | V_r^-(T, I) - V_r^-(T_i, I) |$$

$$\leq \sum_{I \in D} [V_r^+(T_i, I) - V_r^+(T, I) + \varepsilon/N]$$

$$+ \sum_{I \in D} [V_r^-(T_i, I) - V_r^-(T, I) + \varepsilon/N]$$

$$= 2\varepsilon + \sum_{I \in D} [V_r(T_i, I) - V_r(T, I)]$$

$$\leq 2\varepsilon + V_r(T_i, A_i) - \sum_{I \in D} V_r(T, I)$$

for all sufficiently large i. The proof is completed by letting $i \to \infty$.

In the following theorem we take K to be a subset of E_n containing the sets $T(A)$ and $T_i(A_i)$, and we assume that $f: K \times E_m \to E_1$ satisfies (f_1) and (f_2). It is convenient to denote

$$I(f, T, A) = \int_A f[T(w), \theta(w, T)] \, d\mu,$$

$$I(f, T_i, A_i) = \int_{A_i} f[T_i(w), \theta(w, T_i)] \, d\mu_i,$$

where $\mu = \mu(T, \cdot)$ and $\mu_i = \mu(T_i, \cdot)$ are the measures induced by T and T_i in (3.2).

6.6. Theorem. If $R(T, A) = \lim_{i \to \infty} R(T_i, A_i)$ and the mappings T_i are all AC, then $I(f, T, A) = \lim_{i \to \infty} I(f, T_i, A_i)$.

Proof. We take $S = T$, $S_i = T_i$, and use the meshes (12) with $\varrho = \varrho_i = 0$. Then (11) and (14) clearly hold, and (13) holds by (2.2). Let $\varepsilon > 0$ be given, and let η satisfy $0 < \eta < \varepsilon$. With the help of (1.1) and (2.2) we choose a system $D = [I]$ such that $\Delta(D) < \eta$ and the closure of each $I \in D$ is contained in A^0. Since $R(T_i, A_i) \to R(T, A)$, we may use (6.4) and (6.5) to find an N such that

(i) $| R(T, A) - R(T_i, A_i) | < \eta,$

(ii) $| V_r(T, A) - V_r(T_i, A_i) | < \eta,$

(iii) the closure of each $I \in D$ is contained in A_i,

(iv) $$\sum_{I \in D} | \mathcal{V}_r(I) - \mathcal{V}_{ir}(I) | \leq \eta + V_r(T, A) - \sum_{I \in D} V_r(T, I)$$

$$\leq \eta + \Delta(D) < 2\eta,$$

for all $i \geq N$ and each $r = 1, \ldots, m$. By (11) we may also assume that

(v) $\max \{\text{diameter } T_i(I) : I \in D\} < 2\eta$

for all $i \geq N$. Thus

(vi) $$0 \leq V_r(T_i, A_i) - \sum_{I \in D} | \mathcal{V}_{ir}(I) |$$

$$\leq | V_r(T_i, A_i) - V_r(T, A) | + V_r(T, A) - \sum_{I \in D} | \mathcal{V}_r(I) |$$

$$+ \sum_{I \in D} | \mathcal{V}_r(I) - \mathcal{V}_{ir}(I) |$$

$$< 4\eta,$$

and

(vii) $$0 \leq R(T_i, A_i) - \sum_{I \in D} | \mathcal{V}_i(I) |$$

$$\leq | R(T_i, A_i) - R(T, A) | + R(T, A) - \sum_{I \in D} | \mathcal{V}(I) |$$

$$+ \sum_{r=1}^{m} \sum_{I \in D} | \mathcal{V}_r(I) - \mathcal{V}_{ir}(I) |$$

$$< (2 + 2m)\eta,$$

for all $i \geq N$.

To complete the proof we choose η such that $(6 + 2m)\eta < \varepsilon$, and refer to Theorems 6.1 and 3.2.

If $R = L$ for all mappings concerned, then the T_i need not be assumed AC in the preceding theorem.

In the following variant of (6.6) we again take K to be a subset of E_n containing the sets $T(A)$ and $T_i(A_i)$, but now assume that $f : K \times E_1 \rightarrow E_1$ satisfies (f_1) and (f_2) with $m = 1$. We fix r and consider the integrals

$$I_r(f, T, A) = \int_A f[T(w), \theta_r(w, T)]\, d\mu,$$

$$I_r(f, T_i, A_i) = \int_{A_i} f[T_i(w), \theta_r(w, T_i)]\, d\mu_i .$$

6.7. THEOREM. If $V_r(T, A) = \lim_{i \to \infty} V_r(T_i, A_i)$, then $I_r(f, T, A) = \lim_{i \to \infty} I_r(f, T_i, A_i)$.

PROOF. We again take $S = T$, $S_i = T_i$, and $\varrho = \varrho_i = 0$, but now use the meshes Δ and Δ_i of (12)′; conditions (13) and (14) hold relative to these meshes, and we proceed as in the proof of (6.6) to produce a system D and an integer N such that (ii)-(vi) hold. The proof is completed by references to Theorems 6.2 and 3.3.

For the remainder of this section we shall assume that the set A is compact and that

$$T = l_T m_T, \quad m_T \colon A \longrightarrow \Gamma, \quad l_T \colon \Gamma \longrightarrow E_n,$$

is a monotone-light factorization of T with middle space Γ (see [5]). Γ is compact and may be metrized so that

$$\text{diameter } m_T(I) \leq \text{diameter } T(I)$$

for each set I. We shall take K' to be a subset of E_n containing the sets $T(A)$ and $T_i(A_i)$, and shall take $K = \Gamma \times K'$.

For the next theorem, let $f \colon K \times E_m \longrightarrow E_1$ satisfy (f_1) and (f_2), and denote

$$I(f, m_T, T, A) = \int_A f[m_T(w), T(w), \theta(w, T)]\, d\mu,$$

$$I(f, m_T, T_i, A_i) = \int_{A_i} f[m_T(w), T_i(w), \theta(w, T_i)]\, d\mu_i .$$

6.8. THEOREM. If the mappings T_i are all AC, and if $R(T_i, A_i) \longrightarrow R(T, A)$ as $i \longrightarrow \infty$, then $I(f, m_T, T_i, A_i) \longrightarrow I(f, m_T, T, A)$ as $i \longrightarrow \infty$.

PROOF. We take $S = (m_T, T)$, $S_i = (m_T, T_i)$, and use the meshes (12) with $\varrho(D) = \varrho_i(D) = \max\{\text{diameter } T(I) \colon I \in D\}$. Then (11) and (14) hold, and (13) holds for the mesh Δ.

To show that (13) holds for Δ_i, let i be fixed and let $\varepsilon > 0$ be given. By (2.2) there is a system $D = [J]$ of nonoverlapping polyhedral regions $J \subset A_i$ such that

$$\delta(\mathcal{V}_i, T_i, D) + R(T_i, A_i) - \sum_{J \in D} |\mathcal{V}_i(J)| < \varepsilon/8.$$

Thus

$$0 \leq R(T_i, A_i) - \sum_{J \in D} R(T_i, J)$$

$$\leq R(T_i, A_i) - \sum_{J \in D} |\mathcal{V}_i(J)| < \varepsilon/8,$$

and analogous relations hold for V_r. Since T_i is AC, we may subdivide each region $J \in D$ into a finite number of nonoverlapping polyhedral regions J' such that $R(T_i, J) = \sum_{J' \subset J} R(T_i, J')$; an analogous relation holds for each V_r by (5.2). Since the mapping T is uniformly continuous on the union of the sets $J \in D$, we may require that diameter $T(J') < \varepsilon/2$ for each of the sets J'. With the help of (2.2), it is easily seen that each mapping (T_i, J') belongs to the class $R^*(k, n)$. Thus in each J' there exists a system $D_{J'} = [I]$ of nonoverlapping polyhedral regions I such that

$$\delta[\mathcal{V}_i, (T_i, J'), D_{J'}] + R(T_i, J') - \sum_{I \in D_{J'}} |\mathcal{V}_i(I)| < \varepsilon/8N,$$

where N is the total number of sets J'. Let $D_0 = [I]$ be the system of all sets $I \in D_{J'}$ corresponding to the various sets J'. Then

$$\max\{\text{diameter } T_i(I) : I \in D_0\} < \varepsilon/8N < \varepsilon/2,$$

$$\max\{\text{diameter } T(I) : I \in D_0\} < \varepsilon/2.$$

Also,

$$0 \leq R(T_i, A_i) - \sum_{I \in D_0} |\mathcal{V}_i(I)|$$

$$\leq R(T_i, A_i) - \sum_{J \in D} \sum_{J' \subset J} R(T_i, J')$$

$$+ \sum_{J \in D} \sum_{J' \subset J} [R(T_i, J') - \sum_{I \in D_0} s(I, J') |\mathcal{V}_i(I)|]$$

$$< \varepsilon/8 + N(\varepsilon/8) = \varepsilon/4,$$

and an analogous relation holds for each V_r. Thus $\Delta_i(D_0) < \varepsilon$, and (13) holds for Δ_i.

The hypotheses of Theorem 6.1 may now be verified as in the proof of (6.7), and the proof is completed by a reference to Theorem 3.2.

In the following variant of (6.8) we assume that $f: K \times E_1 \to E_1$ satisfies (f_1) and (f_2) with $m = 1$. For fixed r we denote

$$I_r(f, m_T, T, A) = \int_A f[m_T(w), T(w), \theta_r(w, T)]\, d\mu,$$

$$I_r(f, m_T, T_i, A_i) = \int_{A_i} f[m_T(w), T_i(w), \theta_r(w, T_i)]\, d\mu_i .$$

6.9. **Theorem.** If the mappings T_{ir} are all AC, and if $V_r(T_i, A_i) \to V_r(T, A)$ as $i \to \infty$, then $I_r(f, m_T, T_i, A_i) \to I_r(f, m_T, T, A)$ as $i \to \infty$.

Proof. Take S, S_i, ϱ, and ϱ_i as in the proof of (6.8). Conditions (11), (13), and (14) are verified with respect to the meshes (12)′ as in (6.8), the hypotheses of (6.2) are verified as in (6.7), and the proof is completed by a reference to (3.3).

As an application of (6.9), suppose that the mappings T_i are all AC and that $V_r(T_i, A_i) \to V_r(T, A)$ as $i \to \infty$ for each $r = 1, \dots, m$. For each continuous real valued function ψ on Γ, and each C^∞ vector valued function $\varphi = (\varphi_1, \dots, \varphi_m)$ on E_n, let

$$T_{\#}(\psi, \varphi) = \int_A \psi[m_T(w)]\, \{\varphi[T(w)] \cdot \theta(w, T)\}\, d\mu,$$

$$T_{i\#}(\psi, \varphi) = \int_{A_i} \psi[m_T(w)]\, \{\varphi[T_i(w)] \cdot \theta(w, T_i)\}\, d\mu_i ,$$

where the dot denotes the Euclidean inner product. Setting

$$f(g, x, q) = \psi(g) \sum_{r=1}^{m} \varphi_r(x)\, q_r , \quad g \in \Gamma, \quad x \in E_n , \quad q = (q_1, \dots, q_m) \in E_m ,$$

we see that these integrals are sums of integrals of the type discussed in (6.9). Moreover, if we identify φ with the differential k-form $\varphi = \sum_{r=1}^{m} \varphi_r e^r$ and θ with the k-vector $\theta = \sum_{r=1}^{m} \theta_r e_r$, where $\{e^1, \dots, e^m\}$ and $\{e_1, \dots, e_m\}$ are the standard bases for the spaces of k-covectors and k-vectors, respectively, on E_n, then $T_\#$ and $T_{i\#}$ are current-valued measures (see [9], [10], [11]) on the middle space of T. Note that if T_i is smooth or quasi linear, then in view of (5.5),

$$T_{i\#}(\psi, \varphi) = \int_{A_i^0} \psi\,[m_T(w)]\,\{\varphi\,[T_i(w)]\cdot j\,(w, T_i)\}\,dw,$$

and so $T_{i\#}$ is the usual current-valued measure induced by T_i on the middle space of T. As a consequence of (6.9) we have the following theorem.

6.10. THEOREM. If the mappings T_i are all AC, and if $V_r(T_i, A_i) \longrightarrow V_r(T, A)$ as $i \longrightarrow \infty$ for each $r = 1, \dots, m$, then $T_{i\#}$ converges weakly to $T_\#$, i. e., $T_{i\#}(\psi, \varphi) \longrightarrow T_\#(\psi, \varphi)$ for each ψ and φ as above.

REMARKS. Theorems 6.1 and 6.2 can be rephrased in terms of any vector function z of the type discussed in Section 2 provided the mappings T and T_i belong to the quasi additivity class $T(z, k, n)$.

Proofs of Theorems 6.6-6.10 can also be based on the vector function u provided the mappings T and T_i belong to the class $T(u, k, n)$; the only major change required is the replacement of (6.5) by an approximation theorem of Gariepy [10].

Theorem 6.6 was proved by Cesari [3] in the form $\int [f(T_i, u_i), A_i] \longrightarrow \int [f(T, u), A]$ for BV mappings in $T(2, 3)$. Another proof of (6.10) has been given by Gariepy [10] for the case in which the T_i are quasi linear or smooth and $T \in T^*(k, n)$. Other forms of (6.10) as well as its implications are discussed in [9], [10], and [11].

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