ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

C. BÀNICÀ
O. STÀNÀSILÀ

Some results on the extension of analytic entities defined out of a compact

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 25, nº 2 (1971), p. 347-376
<http://www.numdam.org/item?id=ASNSP_1971_3_25_2_347_0>

SOME RESULTS ON THE EXTENSION OF ANALYTIC ENTITIES DEFINED OUT OF A COMPACT (*)

by C. BÀNICÀ and O. STÀNÀSILÀ

Introduction.

In this paper, some results on the cohomology with compact supports of analytic coherent sheaves are given. The main facts were obtained in the period april-june 1969 and were presented at the Seminar on analytic spaces — Bucharest, september 1969.

In § 1 we give two duality theorems on Stein manifolds, which are essentially used in the proof we give for the topological characterisation of the depth of an analytic coherent sheaf on a Stein space (§ 2).

In § 3 some applications are given: relative to Hartogs and Cousin problems, relative to the boundary of a Stein space and the category of analytic coherent sheaves defined around the boundary of a Stein space. The theorems (1.3) and (2.1, i) are proved (in a slightly different form) in [15], th. (2.8), th. (2.13). We have obtained them independently and in another way, first passing by the theorem (1.1). We want to mention that R. Harvey's results proceed ours.

The statement of the theorems (2.4) and (c. 3.2), which have been at the origin of this work, were suggested to us by A. Grothendieck during his visit in 1969 to Bucharest. We are very grateful to him, and to A. Andreotti for their help and encouragement.

1. Two results on the duality for coherent sheaves on Stein manifolds.

We first recall some known facts [9], which will be used in this paper. Let X be a topological space. A family Φ of closed subset of X is called

---

Pervenuto alla Redazione il 27 Giugno 1970.

(*) A part of this work was done during the author's visit at the Institut of Mathematics Pisa (as invited by C. N. R.).

a family of supports if any subset of an element of Φ is also in Φ and the union of two elements of Φ is in Φ. In this paper, only paracompact spaces will be considered and Φ is taken as one of the following families of the space: the family of all closed sets, of all compact sets or that of all closed sets contained in a given compact.

Let $\mathcal{F}$ be a sheaf of abelian groups on X and $\Gamma_\Phi(X, \mathcal{F})$ be the subgroup of $\Gamma(X, \mathcal{F})$ consisting of all sections whose support is in Φ; thus, a functor $\mathcal{F} \to \Gamma_\Phi(X, \mathcal{F})$ is obtained. Its derived functors are denothed by $H_\Phi^\cdot(X, \cdot)$ and they could be calculated by means of flabby resolntions. If X is paracompact and Φ is the family of all compacts of X, then one obtains $H_c^\cdot(X, \cdot)$ (the cohomology groups with compact supports). Clearly, there are isomorphisms $H_c^\cdot(X, \mathcal{F}) \simeq \varinjlim_U H_c^\cdot(U, \mathcal{F})$, where the direct limit is over the open sets of any exhaustion of X. If Φ is the family of the closed subset of a compact $K \subset X$, X paracompact, then the corresponding derived functors are called cohomology groups with supports in K and are denoted by $H_K^\cdot(X, \cdot)$. It is easy to verify the isomorphisms $H_c^\cdot(X, \mathcal{F}) \simeq \varinjlim_K H_K^\cdot(X, \mathcal{F})$, where the direct limit is taken over any exhaustion with compacts of X.

Now, let $(X, \mathcal{O})$ be a ringed space and Φ a family of supports. For any two $\mathcal{O}_X$-Modules $\mathcal{F}$, $\mathcal{G}$, we put $\mathrm{Hom}_{\Phi, \mathcal{O}}(\mathcal{F}, \mathcal{G}) = \Gamma_\Phi(X, \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}))$ and thus, we obtain a functor whose derived functors are called the Ext'*s* with supports in Φ, $\mathrm{Ext}_{\Phi, \mathcal{O}}^\cdot(\cdot, \cdot)$. If $\mathcal{F}$ is a locally free of finite rank $\mathcal{O}$-Module, then $\Gamma_\Phi(X, \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})) \simeq \Gamma_\Phi(X, \check{\mathcal{F}} \otimes_{\mathcal{O}} \mathcal{G})$, and an isomorphism $\mathrm{Ext}_{\Phi, \mathcal{O}}(\mathcal{F}, \mathcal{G}) \simeq H_\Phi(X, \check{\mathcal{F}} \otimes_{\mathcal{O}} \mathcal{G})$ is so induced. As usual $\check{\mathcal{F}}$ stands for the dual $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{F}, \mathcal{O})$ of $\mathcal{F}$.

We now recall the definition of functors $\underline{\mathrm{Ext}}$; let $\underline{\mathrm{Ext}}^\cdot(\mathcal{F}, \mathcal{G})$ be the sheaf associated to the presheaf $U \to \mathrm{Ext}_{\mathcal{O}/U}(\mathcal{F}/U, \mathcal{G}/U)$, where U is an arbitrary open set of X. If $(X, \mathcal{O})$ is a complex space and $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}\, X$ (i. e. two analytic coherent sheaves on X), then it is easily seen that for any $q \geq 0$, $\underline{\mathrm{Ext}}_{\mathcal{O}}^q(\mathcal{F}, \mathcal{G})$ is a coherent $\mathcal{O}$-Module and $\underline{\mathrm{Ext}}_{\mathcal{O}}^q(\mathcal{F}, \mathcal{G})_x \simeq \mathrm{Ext}_{\mathcal{O}_x}^q(\mathcal{F}_x, \mathcal{G}_x)$, for any point $x \in X$.

Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be three $\mathcal{O}$-Modules over an arbitrary ringed space $(X, \mathcal{O})$. For any two families of supports Φ, Ψ, the composition of homomorhisms defines a linear map $\mathrm{Hom}_{\Phi, \mathcal{O}}(\mathcal{F}, \mathcal{G}) \otimes \mathrm{Hom}_{\Psi, \mathcal{O}}(\mathcal{G}. \mathcal{H}) \to \mathrm{Hom}_{\Phi \cap \Psi, \mathcal{O}}(\mathcal{F}, \mathcal{H})$, which induces a canonical map between Ext'*s* with supports, by Yoneda'*s* method of calculus of Ext'*s*. Let $I^\cdot$ and $J^\cdot$ be injective resolutions of $\mathcal{F}$ and $\mathcal{G}$ and $\mathrm{Hom}_{\Phi, \mathcal{O}}^\cdot(I^\cdot, J^\cdot)$ be the complex given by $\mathrm{Hom}_{\Phi, \mathcal{O}}^q(I^\cdot, J^\cdot) = \prod_{p=0}^{\infty} \mathrm{Hom}_{\Phi, \mathcal{O}}(I^p, J^{p+q})$, and whose differential is given by the formula

$df = (d^{p+q} \cdot f^p + (-1)^{q+1} f^{p+1} \cdot d^p)_{p \geq 0}$, for any $f = (f^p)_{p \geq 0} \in \mathrm{Hom}^q_{\Phi, \mathcal{O}}(I^{\cdot}, J^{\cdot})$. The canonical map $\mathrm{Hom}^{\cdot}_{\Phi, \mathcal{O}}(I^{\cdot}, J^{\cdot}) \to \mathrm{Hom}_{\Phi, \mathcal{O}}(\mathcal{F}, J^{\cdot})$ induces a isomorphism to cohomology $H^q(\mathrm{Hom}^{\cdot}_{\Phi, \mathcal{O}}(I^{\cdot}, J^{\cdot})) \simeq \mathrm{Ext}^q_{\Phi, \mathcal{O}}(\mathcal{F}, \mathcal{G})$. Let $\mathcal{K}$ be an injective resolution of the third $\mathcal{O}$-Module $\mathcal{H}$. The composition of homomorphisms gives rise to a morphism

$$\mathrm{Hom}^{\cdot}_{\Phi, \mathcal{O}}(I^{\cdot}, J^{\cdot}) \otimes \mathrm{Hom}^{\cdot}_{\Psi, \mathcal{O}}(J^{\cdot}, \mathcal{K}^{\cdot}) \to \mathrm{Hom}^{\cdot}_{\Phi \cap \Psi, \mathcal{O}}(I^{\cdot}, \mathcal{K}^{\cdot}),$$

which is compatible with the differentials. Thus, for any $p, q \geq 0$, the following bilinear maps are obtained: $\mathrm{Ext}^p_{\Phi, \mathcal{O}}(\mathcal{F}, \mathcal{G}) \times \mathrm{Ext}^q_{\Psi, \mathcal{O}}(\mathcal{G}, \mathcal{H}) \to \mathrm{Ext}^{p+q}_{\Phi \cap \Psi, \mathcal{O}}(\mathcal{F}, \mathcal{H})$. For the duality results of this paragraph, we shall consider $\Phi =$ the family of all compact sets of a locally compact space and $\Psi =$ the familly of all closed sets of the space. In this case, we have the following bilinear maps, which are called Yoneda's maps:

$$\mathrm{Ext}^p_{c, \mathcal{O}}(\mathcal{F}, \mathcal{G}) \times \mathrm{Ext}^q_{\mathcal{O}}(\mathcal{F}, \mathcal{H}) \to \mathrm{Ext}^{p+q}_{c, \mathcal{O}}(\mathcal{F}, \mathcal{H}).$$

The first result of this section is the following:

THEOREM (1.1): *Let $(X, \mathcal{O})$ be a Stein manifold of dimension n and ω the sheaf of germs of differential forms af type $(n, 0)$ on X with holomorphic coefficients. Then for any $\mathcal{F} \in \mathrm{Coh}\, X$ and for any integer $q \geq 0$, $\mathrm{Ext}^{n-q}_{\mathcal{O}}(X; \mathcal{F}, \omega)$ has a structure of Fréchet space, whose topological dual is algebrically isomorphic to $H^q_c(X, \mathcal{F})$.*

We first prove the following:

LEMMA (1.2): *Let $(X, \mathcal{O})$ be a Stein space and $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}\, X$. Then the canonical map $\mathrm{Ext}^q_{\mathcal{O}}(X; \mathcal{F}, \mathcal{G}) \to \Gamma(X, \underline{\mathrm{Ext}}^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G}))$ is an isomorphism, for any $q \geq 0$.*

PROOF. According to [9], [11], there is a spectral sequence which converges to $\mathrm{Ext}^{\cdot}_{\mathcal{O}}(X; \mathcal{F}, \mathcal{G})$, with $E_2^{p,q} = H^p(X, \underline{\mathrm{Ext}}^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G}))$. But every $\mathcal{O}$-Module $\underline{\mathrm{Ext}}^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ is coherent, hence by theorem B, $E_2^{p,q} = 0$ for any $p \geq 1$. Since the spectral sequence is degenerate the lemma follows.

This lemma allows us to consider a topology of Fréchet space on every $\mathrm{Ext}^q_{\mathcal{O}}(X; \mathcal{F}, \mathcal{G})$. Particulary, the topological structure asserted in theorem (1.1) is obtained in this way.

In what follows, all considered complex spaces are not necessarily reduced.

THE PROOF OF THE THEOREM (1.1). We have $\mathbb{C}$-bilinear Yoneda's maps $\mathrm{Ext}^q_c(X; \mathcal{O}, \mathcal{F}) \times \mathrm{Ext}^{n-q}_{\mathcal{O}}(X; \mathcal{F}, \omega) \to \mathrm{Ext}^n_c(X; \mathcal{O}, \omega)$. Since $\mathrm{Ext}^q_c(X; \mathcal{O}, \mathcal{F}) \simeq \simeq H^q_c(X, \mathcal{F})$ and $\mathrm{Ext}^n_c(X; \mathcal{O}, \omega) \simeq H^n_c(X, \omega)$, we then obtain a $\mathbb{C}$-bilinear map $H^q_c(X, \mathcal{F}) \times \mathrm{Ext}^{n-q}_{\mathcal{O}}(X; \mathcal{F}, \omega) \to H^n_c(X, \omega)$ for any $q \geq 0$, and by composition with the « trace » $H^n_c(X, \omega) \to \mathbb{C}$ (defined by an integral, by using a resolution of ω with sheaves of germs of differential forms with coefficients distributions), the following $\mathbb{C}$-linear maps are obtained:

$$(1) \qquad H^q_c(X, \mathcal{F}) \to \mathrm{Hom}_{\mathbb{C}}(\mathrm{Ext}^{n-q}_{\mathcal{O}}(X; \mathcal{F}, \omega), \mathbb{C}).$$

For a cohomology-class $\xi \in H^q_c(X, \mathcal{F})$, we denote by L_ξ its image by the map (1). Let $(U_r)_{r \geq 1}$ be an exhaustion with relatively compact Stein open sets of X, such that the restrictions $\Gamma(X, \mathcal{O}) \to \Gamma(U_r, \mathcal{O})$ are dense. Then the restrictions $\Gamma(X, \mathcal{F}) \to \mathcal{F}(U_r, \mathcal{F})$ are also dense, for any $\mathcal{F} \in \mathrm{Coh}\, X$. (This fact will be applied for the sheaves $\underline{\mathrm{Ext}}^q \ldots$). For any integer q, there is a canonical isomorphism $\varinjlim_r H^q_c(U_r, \mathcal{F}) \xrightarrow{\sim} H^q_c(X, \mathcal{F})$, hence for any $\xi \in H^q_c(X, \mathcal{F})$ there exist an integer r and a cohomology class $\eta \in H^q_c(U_r, \mathcal{F})$ so that $\mathrm{Im}(\eta) = \xi$ and the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Ext}^{n-q}_{\mathcal{O}}(X; \mathcal{F}, \omega) & \longrightarrow & \mathrm{Ext}^{n-q}_{\mathcal{O}}(U_r; \mathcal{F}, \omega) \\ & {\scriptstyle L_\xi} \searrow \quad \swarrow {\scriptstyle L_\eta} & \\ & \mathbb{C} & \end{array}$$

By lemma (1.1) and by the considered topologies, the horizontal arrow is continuous. We shall now prove that the maps L_ξ are continuous for any ξ. By the above remark we can suppose the existence of an exact sequence of the form:

$$(2) \qquad 0 \to \mathcal{G} \to \mathcal{O}^r \to \mathcal{F} \to 0.$$

If $q = n$, then we have the exact sequence $H^n_c(X, \mathcal{O}^r) \to H^n_c(X, \mathcal{F}) \to \to H^{n+1}_c(X, \mathcal{G}) = 0$ (the last cohomology group is null, because it can be calculated by means of Dolbeault resolution of $\mathcal{O}$, tensored by $\otimes_{\mathcal{O}} \mathcal{G}$. Here we apply a result on flatness due to Malgrange, [17]). Let $\eta \in H^q_c(X, \mathcal{O}^r)$ be an element such that $\mathrm{Im}(\eta) = \xi$. In the commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}^0_{\mathcal{O}}(X; \mathcal{F}, \omega) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \omega) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^r, \omega) = \mathrm{Ext}^0_{\mathcal{O}}(X; \mathcal{O}, \omega) \\ & {\scriptstyle L_\xi} \searrow \quad \swarrow {\scriptstyle L_\eta} & \\ & \mathbb{C} & \end{array}$$

the horizontal arrow is continuous and the map L_η is continuous, since the integral is so [21]. Therefore, L_ξ is continuous for $q = n$. In the general case, we proced by induction on $q\ (q < n)$. Let $\eta \in H_c^{q+1}(X, \mathcal{G})$ be the image of ξ by the boundary morphism. We then have the following commutative diagram:

$$(3)\qquad \begin{array}{ccc} \operatorname{Ext}_{\mathcal{O}}^{n-q-1}(X; \mathcal{G}, \omega) & \longrightarrow & \operatorname{Ext}_{\mathcal{O}}^{n-q}(X; \mathcal{F}, \omega) \\ & (-1)^{q+1} L_\eta \searrow \quad \swarrow L_\xi & \\ & \mathbb{C} & \end{array}$$

From (2) one obtains the exact sequence $\operatorname{Ext}_{\mathcal{O}}^{n-q-1}(X; \mathcal{G}, w) \longrightarrow \operatorname{Ext}_{\mathcal{O}}^{n-q}(X; \mathcal{F}, \omega) \longrightarrow$ $\longrightarrow \operatorname{Ext}_{\mathcal{O}}^{n-q}(X; \mathcal{O}^r, \omega)$. But $\underline{\operatorname{Ext}}_{\mathcal{O}}^{n-q}(\mathcal{O}^r, \omega) = 0$ (all its stalks vanish), hence by lemma (1.2) the horizontal arrow from the diagram (3) is a strict surjective map. The induction assumption then implies the continuouity of L_ξ. Therefore, the map (1) gives rise to a $\mathbb{C}$-linear map:

$$(4)\qquad H_c^q(X, \mathcal{F}) \longrightarrow (\operatorname{Ext}_{\mathcal{O}}^{n-q}(X; \mathcal{F}, \omega))',$$

where the right hand means topological dual. We shall prove that this map is an isomorphism. If $\mathcal{F}$ is locally free of finite rank, this fact is well-known, [21].

According to lemma (1.2), the maps $\operatorname{Ext}_{\mathcal{O}}^{n-q}(X; \mathcal{F}, \omega) \longrightarrow \operatorname{Ext}_{\mathcal{O}}^{n-q}(U_r; \mathcal{F}, \omega)$ are all dense, hence the canonical maps

$$(5)\qquad \varinjlim_r (\operatorname{Ext}_{\mathcal{O}}^{n-q}(U_r; \mathcal{F}, \omega))' \longrightarrow (\operatorname{Ext}_{\mathcal{O}}^{n-q}(X; \mathcal{F}, \omega))'$$

are injective. If $L : \operatorname{Ext}_{\mathcal{O}}^{n-q}(X; \mathcal{F}, \omega) \longrightarrow \mathbb{C}$ is a continuous linear form, then there are a constant $\alpha > 0$ and a compact $K \subset X$ such that $|L(s)| \leq \alpha p_K(s)$, for any $s \in \operatorname{Ext}_{\mathcal{O}}^{n-q}(X; \mathcal{F}, \omega) = \Gamma(X, \underline{\operatorname{Ext}}_{\mathcal{O}}^{n-q}(\mathcal{F}, \omega))$. In this inequality, p_K is the seminorm « $\sup_K$ » given by K and by a suitable surjective map over a Stein open set containing K, $\mathcal{O}^r \longrightarrow \underline{\operatorname{Ext}}^{n-q}(\mathcal{F}, \omega)$. (For convenience, we have omitted to design the consideration of such a surjection by an index). It is easily to be seen that L is factorized by any $\operatorname{Ext}_{\mathcal{O}}^{n-q}(U; \mathcal{F}, \omega)$, where $U \supset K$ is a Stein open set of X. Thus, the map (5) is also surjective, that is an isomorphism. By virtue of the isomorphisms $\varinjlim_r H_c^q(U_r, \mathcal{F}) \xrightarrow{\sim} H_c^q(X, \mathcal{F})$, it is clear that we have only to prove the isomorphism (4) for U_r, therefore we can suppose $\mathcal{F}$ admits a finite resolution with locally free of finite rank sheaves. By induction on the length of the resolution, we are reduced to prove that, whenever $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of analytic coherent sheaves on X and the theorem holds for $\mathcal{F}$ and $\mathcal{G}$, then same is true for $\mathcal{H}$. Or, we have the following exact cohomology sequences:

$$\ldots \to H_c^q(X,\mathcal{F}) \to H_c^q(X,\mathcal{G}) \to H_c^q(X,\mathcal{H}) \to H_c^{q+1}(X,\mathcal{F}) \to H_c^{q+1}(X,\mathcal{G}) \to \ldots$$

$$\ldots \leftarrow \mathrm{Ext}_{\mathcal{O}}^{n-q}(X;\mathcal{F},\omega) \leftarrow \mathrm{Ext}_{\mathcal{O}}^{n-q}(X;\mathcal{G},\omega) \leftarrow \mathrm{Ext}_{\mathcal{O}}^{n-q}(X;\mathcal{H},\omega) \leftarrow$$

$$\leftarrow \mathrm{Ext}_{\mathcal{O}}^{n-q-1}(X;\mathcal{F},\omega) \leftarrow \mathrm{Ext}_{\mathcal{O}}^{n-q-1}(X;\mathcal{G},\omega) \leftarrow \ldots$$

The maps of the second sequence are topological homomorphisms (for they are obtained by (1.2) from the sequence associated to the functor $\underline{\mathrm{Ext}}$). Hence, a new exact sequence of $\mathbb{C}$-vectorial spaces is obtained for topological duals and our assertion is a consequence of the lemma of the five homomorphisms. This completes the proof of theorem (1.1).

We now give a more precise results than (1.1).

THEOREM (1.3): *Let* $(X,\mathcal{O})$ *be a Stein manifold of dimension* n, $K \subset X$ *a holomorphically-convex compact and* ω *the sheaf of germs of differential forms of type* $(n, 0)$ *with holomorphic coefficients. Then, for any* $\mathcal{F} \in \mathrm{Coh}\, X$ *and for any* $q \geq 0$, $\mathrm{Ext}_{\mathcal{O}}^{n-q}(K;\mathcal{F},\omega)$ *has a structure of LF-space, whose topological dual is algebrically isomorphic to* $H_K^q(X,\mathcal{F})$.

For proving this, we need two preliminary lemmas:

LEMMA (1.4): *Let* $(X,\mathcal{O})$ *be a Stein space,* $K \subset X$ *a holomorphically-convex compact and* $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}\, X$. *Then, for any integer* $q \geq 0$, *there is a canonical isomorphism* $\mathrm{Ext}_{\mathcal{O}}^q(K;\mathcal{F},\mathcal{G}) \to \Gamma(K, \underline{\mathrm{Ext}}_{\mathcal{O}}^q(\mathcal{F},\mathcal{G}))$.

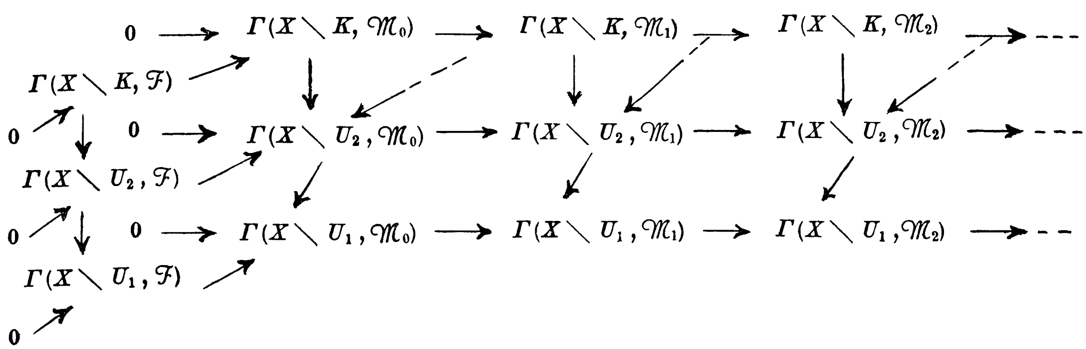
PROOF. The canonical morphism $\underline{\mathrm{Ext}}_{\mathcal{O}}^q(\mathcal{F},\mathcal{G})|_K \to \underline{\mathrm{Ext}}_{\mathcal{O}}^q|_K(\mathcal{F}|K, \mathcal{G}|K)$ is an isomorphism: we could prove this either passing to stalks or calculating functor $\underline{\mathrm{Ext}}^{\cdot}$ by means of a resolution — on a neighbourhood of K — for $\mathcal{F}$ with locally free sheaves, the existence of such a resolution being assured by theorems A and B. Then, the morphism asserted in the statement is given by the morphism $\mathrm{Ext}_{\mathcal{O}|K}^q(K;\mathcal{F}|K, \mathcal{G}|K) \to \Gamma(K, \underline{\mathrm{Ext}}_{\mathcal{O}|K}^q(\mathcal{F}|K, \mathcal{G}|K))$, induced by passing from a presheaf to its associated sheaf. By [9], [11] there is a spectral sequence with $E_2^{pq} = H^p(K, \underline{\mathrm{Ext}}_{\mathcal{O}|K}^q(\mathcal{F}|K, \mathcal{G}|K)) = H^p(K, \underline{\mathrm{Ext}}_{\mathcal{O}}^q(\mathcal{F},\mathcal{G}))$ which converges to $\mathrm{Ext}_{\mathcal{O}}^{\cdot}(K;\mathcal{F},\mathcal{G})$.

Since $\underline{\mathrm{Ext}}^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ is a coherent $\mathcal{O}$-Module, then theorem B shows that the spectral sequence degenerates and the lemma is immediate.

Lemma (1.4) gives the possibility to define a topology of LF space on every $\mathrm{Ext}^q_{\mathcal{O}}(K; \mathcal{F}, \mathcal{G})$, ([16]), and particularly to deduce the topology asserted in the statement of the theorem (1.3).

LEMMA (1.5): *Let X be a paracompact topological space with a countable base, $K \subset X$ a compact, $\mathcal{F}$ a sheaf of abelian groups on X and $q \geq 0$ an integer. Then the canonical morphism $H^q(X \setminus K, \mathcal{F}) \to \varprojlim_{V \supset K,\ V \text{ open}} H^q(X \setminus V, \mathcal{F})$ is an epimorphism for any q and an isomorphism for any $q \neq 1$; moreover, if the restrictions $\Gamma(X, \mathcal{F}) \to \Gamma(X \setminus U, \mathcal{F})$ are surjective for any open U belonging to a foundamental system of neighbourhoods of K, then that morphism is an isomorphism for $q = 1$ too.*

PROOF. We consider a soft resolution on $X \setminus K$ of the sheaf $\mathcal{F} | X \setminus K$: $0 \to \mathcal{F} | X \setminus K \to \mathcal{M}_0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \ldots$; this resolution permits us to calculate the groups $H^{\cdot}(X \setminus K, \mathcal{F})$ and $H^{\cdot}(X \setminus U, \mathcal{F})$, where U is a neighbouhood of K. One can choose a countable fundamental system of neighbourhoods $(U_r)_{r \geq 1}$ of K and the lemma will result by an elementary reasoning on projective systems, from the following commutative diagram in which all maps $\Gamma(X \setminus U_{r+1}, \mathcal{M}_i) \to \Gamma(X \setminus U_r, \mathcal{M}_i)$ are surjective $(i \geq 0, r \geq 1)$:



For case $q = 1$ it suffices to use the surjectivity of the maps $\Gamma(X \setminus U_{r+1}, \mathcal{F}) \to \Gamma(X \setminus U_r, \mathcal{F})$ which fact results by assumption.

THE PROOF OF THE THEOREM (1.3). There is a canonical morphism:

$$H^q_K(X, \mathcal{F}) \to \varprojlim_{U \supset K} H^q_c(U, \mathcal{F}). \tag{1}$$

Since K is holomorphically-convex, the projective limit can be considered over a countable fundamental system of Stein neihbourhoods of K. By (1.1), every $\mathbb{C}$-linear space $\mathrm{Ext}_{\mathcal{O}}^{n-q}(U; \mathcal{F}, \omega)$ has a structure of Fréchet space, for which the topological dual is algebraically isomorphic to $H_c^q(U, \mathcal{F})$ by an isomorphism compatible with the maps induced by inclusions $V \subset U$. We then obtain a natural isomorphism

$$\lim_{\substack{\longleftarrow \\ U \supset K}} H_c^q(U, \mathcal{F}) \simeq \lim_{\substack{\longleftarrow \\ U \supset K}} (\mathrm{Ext}_{\mathcal{O}}^{n-q}(U; \mathcal{F}, \omega))' \simeq (\mathrm{Ext}_{\mathcal{O}}^{n-q}(K; \mathcal{F}, \omega))'.$$

The last isomorphism follows by lemmas (1.4), (1.2) and by definition of considered topologies. By composition of this isomorphism with the map (1) we obtain the following functorial in $\mathcal{F}$ morphism

$$H_K^q(X, \mathcal{F}) \to (\mathrm{Ext}_{\mathcal{O}}^{n-q}(K; \mathcal{F}, \omega))', \tag{2}$$

which is compatible with the coboundaries, for any exact short sequence. We first prove that (2) is an isomorphism for any $\mathcal{F}$ locally free and for this, it suffices to show that (1) is an isomorphism; or, the case $i = 0$ is trivial and for $i = 1$ we can write the exact sequences:

$$0 \to \Gamma_K(X, \mathcal{F}) = 0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X \setminus K, \mathcal{F}) \to H_K^1(X, \mathcal{F}) \to 0$$

$$0 \to \Gamma_c(U, \mathcal{F}) = 0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X \setminus U, \mathcal{F}) \to H_c^1(U, \mathcal{F}) \to 0,$$

noting that the map $\lim\limits_{\substack{\longleftarrow \\ U}} \Gamma(X \setminus U, \mathcal{F}) \to \lim\limits_{\substack{\longleftarrow \\ U}} H_c^1(U, \mathcal{F})$ is surjective ($\lim\limits_{\longleftarrow}$ is to be considered countably indexed). For $i \geq 2$ we consider the following exact sequences:

$$\ldots \to H^{q-1}(X, \mathcal{F}) \to H^{q-1}(X \setminus K, \mathcal{F}) \to H_K^q(X, \mathcal{F}) \to H^q(X, \mathcal{F}) \to \ldots$$

$$\ldots \to H^{q-1}(X, \mathcal{F}) \to H^{q-1}(X \setminus U, \mathcal{F}) \to H_c^q(U, \mathcal{F}) \to H^q(X, \mathcal{F}) \to \ldots$$

If $\dim X = 1$, then these sequences show that $H_K^q(X, \mathcal{F}) = H_U^q(U, \mathcal{F}) = 0$ ($q \geq 2$). If $\dim X \geq 2$, then the maps $\Gamma(X, \mathcal{F}) \to \Gamma(X \setminus U, \mathcal{F})$ are surjective (their cokernels are $H_c^1(U, \mathcal{F})$, which are null ([21]) and the conclusion follows by (1.5) and theorem B.

Let us prove now the isomorphism (2) for any $\mathcal{F} \in \mathrm{Coh}\, X$; we consider a finite resolution of $\mathcal{F}$ on a neighbourhood of K with locally free of finite rank sheaves. This resolution could be considered on K itself, replacing X

by a neighbourhood of K and noting that the invariants of the statement are not changed. The proof of this theorem is to be completed as in the end of the proof of (1.1).

REMARKS. *a*) The duality theorem for analytic coherent sheaves on complex manifolds ([21], [17], [23]) gives, in some conditions, a duality between $H^q(X, \mathcal{F})$ and $\mathrm{Ext}^{n-q}_{\mathcal{O}, c}(X; \mathcal{F}, \omega)$. If X is a Stein manifold, then $\mathrm{Ext}^{q}_{\mathcal{O}, c}(X; \mathcal{F}, \omega) = 0$ for any $q < \dim X$. We have established the theorems (1.1), (1.3) having in mind the results from the following section.

b) From the proof of (1.3), it is clear that this theorem also holds for any complex manifold and for any compact $K \subset X$, which admits a foundamental system of Stein neighbourhoods. Such a compact we shall call a Stein compact.

To finish with this section we give some consequences of the above theorems.

COROLLARY (1.6): *Let* $(X, \mathcal{O})$ *be a complex manifold of dimension* n, $\mathcal{F} \in \mathrm{Coh}\, X$ *and* x *a point of* X. *For any integer* $q \geq 0$, *the topological dual of the analytic module* $\mathrm{Ext}^{n-q}_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{O}_x)$, *considered with the canonical topology deduced from the uniform convergence of germs* [16], *is algebrically isomorphic to* $H^q_x(X, \mathcal{F}) \cdot (H^{\cdot}_x(X, \mathcal{F}) = H^{\cdot}_{\{x\}}(X, \mathcal{F})$ *are cohomology groups of* $\mathcal{F}$ *with supports in* $\{x\}$).

The proof is immediate from theorem (1.3) applied to a Stein neighbourhood of x, taking $K = \{x\}$.

COROLLARY (1.7): *Let* X *be a Stein space,* $K \subset X$ *a holomorphically convex compact and* $\mathcal{F} \in \mathrm{Coh}\, X$. *The the canonical morphism* $H^q_K(X, \mathcal{F}) \longrightarrow$

$$\longrightarrow \varprojlim_{U \supset K,\ U \text{ open}} H^q_c(U, \mathcal{F})$$

is an isomorphism for any integer $q \geq 0$.

PROOF. If X is a Stein manifold, then this morphism is exactly the isomorphism (1) from the proof of (1.3). In the general case, the problem concerns only the neighbourhood of K; thus we can suppose $\sup_{x \in X} \dim m_x / m_x^2 < \infty$ and we are immediately reduced to the case of a numerical space, (by a suitable closed immersion, i. e. embedding, [26]), that is to the case of manifolds.

COROLLARY (1.8): *In the same hypothesis as in* (1.7), *the canonical morphism* $H^q_K(X, \mathcal{F}) \longrightarrow H^q_c(X, \mathcal{F})$ *is injective, for any* $q \geq 0$.

PROOF. In the case X is a Stein manifold, the theorem (1.1), (1.3) give us the following canonical isomorphism : $H_c^q(X, \mathcal{F}) \simeq (\mathrm{Ext}_{\mathcal{O}}^{n-q}(X; \mathcal{F}, \omega))'$, $H_K^q(X, \mathcal{F}) \simeq (\mathrm{Ext}_{\mathcal{O}}^{n-q}(K; \mathcal{F}, \omega))'$. Since $\mathrm{Ext}_{\mathcal{O}}^{n-q}(X; \mathcal{F}, \omega) = \Gamma(X, \underline{\mathrm{Ext}}_{\mathcal{O}}^{n-q}(\mathcal{F}, \omega))$ and $\mathrm{Ext}_{\mathcal{O}}^{n-q}(K; \mathcal{F}, \omega) = \Gamma(K, \underline{\mathrm{Ext}}_{\mathcal{O}}^{n-q}(\mathcal{F}, \omega))$, the corollary is immediate for the restrictions $\Gamma(X, \mathcal{H}) \longrightarrow \Gamma(K, \mathcal{H})$ are dense for any $\mathcal{H} \in \mathrm{Coh}\, X$. In the case of a general Stein space, we choose an exhaustion $(U_r)_{r \geq 1}$ with relatively compact Stein open sets of X, containing K. We have $H_c^q(X, \mathcal{F}) = \varinjlim_r H_c^q(U, \mathcal{F})$ and we are first reduced to the case $\sup_{x \in X} \dim_{\mathbb{C}} m_x/m_x^2 < \infty$ and then by a closed immersion $X \longrightarrow \mathbb{C}^N$, to the case of manifolds.

COROLLARY (1.9): *Suppose $(X, \mathcal{O})$ a Stein space, $U \subset X$ a Stein open set such that the restriction map $\Gamma(X, \mathcal{O}) \longrightarrow \Gamma(U, \mathcal{O})$ is dense. Then for any $\mathcal{F} \in \mathrm{Coh}\, X$ and for any integer $q \geq 0$, the natural maps $H_c^q(U, \mathcal{F}) \longrightarrow H_c^q(X, \mathcal{F})$ are injective.*

PROOF. If $K \subset U$ is a holomorphically-convex compact in U, then the hypothesis (U, X) to be a Runge pair implied that K is holomorphically convex in X. We have $H_c^q(U, \mathcal{F}) = \varinjlim_K H_K^q(U, \mathcal{F})$, where $\varinjlim$ is taken over all holomorphically-convex compacts of U. Since $H_K^q(U, \mathcal{F}) \simeq H_K^q(X, \mathcal{F})$, we can apply the preceeding corollary.

2. A topological characterisation of the depth of an analytic coherent sheaf on a Stein space.

In this section, some theorems on the depth of an analytic coherent sheaf on a complex space are given. We first recall some results of commutative Algebra which will be necessary in the following.

Let A be a noetherian local ring, $\underline{m}$ its maximal ideal, $k = A/\underline{m}$ the residual field and M a finitely generated A-module. A finite sequence of elements $x_1, \dots, x_n \in \underline{m}$ such that every x_i is a nonzerodivisor in $M/\sum_{j=1}^{i-1} x_j M$ is called M-regular. The *depth of M* is by definition the integer $\mathrm{prof}_A(M) = \sup\{r \mid$ there is an M-regular sequence having r elements$\}$. (If $M = 0$, one takes $\mathrm{prof}_A(M) = \infty$). In [1], [12], [22] the following facts are proved :

— $\mathrm{prof}_A(M) \leq \dim(M)$

— all maximal M-regular sequences have the same length, equal to $\mathrm{prof}_A(M)$

— $\operatorname{prof}_A(M) = \inf\{i \mid \operatorname{Ext}^i_A(k, M) \neq 0\}$

— if A is a regular ring of dimension n and q a natural number, then we have $q < \operatorname{prof}_A(M)$ if and only if $\operatorname{Ext}^r_A(M, A) = 0$ for any $r \geq n - q$

— if $A \to B$ is a finite local morphism of noetherian local ring, then for any finitely generated B-module N, $\operatorname{prof}_B(N) = \operatorname{prof}_A(N)$

— if A is a normal noetherian local ring and $\dim(A) \geq 2$, then $\operatorname{prof}_A(A) \geq 2$.

A noetherian local ring A for which $\operatorname{prof}_A(A) = \dim(A)$ is called a Cohen-Macauley ring.

For the algebraic affine case is well-known ([12]) the following characterisation of the depth in terms of some relative cohomological invariants: « Let A be a noetherian local ring, $\underline{m}$ its maximal ideal, M a finitely-generated A-module and $N \geq 0$ an integer. Let $X = \operatorname{Spec} A$, $Y = \{\underline{m}\}$ the unique closed point of X, $U = X \setminus Y$ an $\mathcal{F}$ the sheaf on X associated to M. Then the following assertions are equivalent:

1) $\operatorname{prof}_A(M) \geq N + 1$

2) the natural morphism $H^q(X, \mathcal{F}) \to H^q(U, \mathcal{F})$ is injective for $q = N$ and bijective for any $q < N$

3) $\operatorname{Ext}^q_A(k, M) = 0$ for any $q \leq N$ $(k = A/\underline{m})$

4) $H^q_Y(X, \mathcal{F}) = 0$ for any $q \leq N$ ».

We shall prove an analytic analogous result, in which the unique closed point of Spec A is replaced, for a Stein space, by the boundary on the space and the invariants $H^\cdot_Y$ are replaced by $H^\cdot_c$.

In the algebraic case, the following result also holds: « Keeping the above notations, if A is a factor-ring of a regular local ring, then the following assertion are equivalent:

1) $\operatorname{prof}(\mathcal{F}_x) \geq N + 1 - \dim \overline{\{x\}}$, for any $x \in X \setminus \{\underline{m}\}$

2) $H^q_Y(X, \mathcal{F})$ are A-modules of finite length, for any $q \leq N$ ».

In the complex case, we shall give a similar result, in which the first assertion is replaced by the following: $\operatorname{prof}(\mathcal{F}_x) \geq N + 1$ for all points of the complementary of a compact and in the assertion 2, we put $H^\cdot_c$ instead of $H^\cdot_Y$.

Let now $(X, \mathcal{O}_X)$ be a complex space and $\mathcal{F} \in \operatorname{Coh} X$. We shall use the notation $\operatorname{prof}(\mathcal{F}) = \inf_{x \in X} \operatorname{prof}_{\mathcal{O}_x}(\mathcal{F}_x)$, introduced in [1]. We shall often write $\operatorname{prof} \mathcal{F}_x$ instead of $\operatorname{prof}_{\mathcal{O}_x}(\mathcal{F}_x)$. The first result we prove is the following:

THEOREM (2.1): *Let $(X, \mathcal{O})$ be a Stein space, $K \subset X$ a holomorphically-convex compact, $\mathcal{F} \in \operatorname{Coh} X$ and $N \geq 0$ an integer. Then:*

(i) *$\operatorname{prof}(\mathcal{F}_x) \geq N + 1$ for any $x \in K$ if and only if $H^q_K(X, \mathcal{F}) = 0$ for any $q \leq N$.*

(ii) *there is a neighbourhood U of K such that prof$(\mathcal{F}_x) \geq N+1$ for any $x \in U \setminus K$ if and only if $\mathbb{C}$-linear spaces $H^q_K(X, \mathcal{F})$ are finite dimensional, for any $q \leq N$.*

PROOF. (i) We first suppose X a Stein manifold of dimension n. By (1.3), (1.4) and theorem A we have the following sequence of equivalent assertion: $H^q_K(X, \mathcal{F}) = 0 \iff \text{Ext}^{n-q}_{\mathcal{O}}(K; \mathcal{F}, \omega) = 0 \iff \underline{\text{Ext}}^{n-q}_{\mathcal{O}}(\mathcal{F}, \omega)_x = 0$ for any $x \in K \iff \text{Ext}^{n-q}_{\mathcal{O}_x}(\mathcal{F}_x, \omega_x) = 0$ for any $x \in K$. Since $\mathcal{O}_x$ is a regular local ring of dimension n and $\omega_x \simeq \mathcal{O}_x$, the theorem is an immediate consequence of the characterisation of the depth of a finitely generated module over the regular local ring $\mathcal{O}_x$ in terms of $\text{Ext}_{\mathcal{O}_x}(\cdot, \mathcal{O}_x)$, that we already recalled.

For the general case, we may replace X by a relatively compact Stein neighbourhood of K and suppose that a closed immersion $X \xrightarrow{i} \mathbb{C}^p$ does exist, with p suitable. The image K' of K by i is a holomorphically-convex compact in $\mathbb{C}^p$. If we denote $\mathcal{F}^* = i_*(\mathcal{F})$, then obviously $H^q_K(X, \mathcal{F}) \simeq \simeq H^q_{K'}(\mathbb{C}^p, \mathcal{F}^*)$ and prof$(\mathcal{F}_x) = $ prof$(\mathcal{F}^*_{i(x)})$ for any $x \in X$. Therefore, we are reduced to the nonsingular case.

(ii) The proof is similar, noting that for the sheaves $\underline{\text{Ext}}$ we have to apply the following simple lemma:

LEMMA (2.2): *Let X be a Stein space, $K \subset X$ a compact and $\mathcal{H} \in$ Coh X. Then $\dim_{\mathbb{C}} \Gamma(K, \mathcal{H}) < \infty$ if and only if there is a neighbourhood U of K such that $\mathcal{H}_x = 0$ for any $x \in U \setminus K$.*

PROOF. Let U be a neighbourhood of K and suppose $\mathcal{H}_x = 0$ for any $x \in U \setminus K$; then Supp $\mathcal{H}|_U \subset K$ is a finite set, Supp $\mathcal{H}|_U = \{x_1, \dots, x_p\}$. By the « Nullstellensatz », every stalk $\mathcal{H}_{x_i}$ is annihilated by a power of the maximal ideal $\underline{m}_{x_i}$, hence $\dim_{\mathbb{C}} \mathcal{H}_{x_i} < \infty$. But $\Gamma(K, \mathcal{H}) = \prod_{i=1}^{p} \mathcal{H}_{x_i}$, therefore $\dim_{\mathbb{C}} \Gamma(X, \mathcal{H}) < \infty$.

Conversely, for any point $x \in K$ we consider the ideal $\underline{m}(x)$ of the ring $\Gamma(X, \mathcal{O})$, which corresponds to x. The hypothesis $\dim_{\mathbb{C}} \Gamma(X, \mathcal{H}) < \infty$ implies the existence of a number r such that $\underline{m}(x)^r \Gamma(X, \mathcal{H}) = \underline{m}(x)^{r+1} \cdot \Gamma(X, \mathcal{H}) = \dots$, hence $\underline{m}_x^r \mathcal{H}_x = \underline{m}_x^{r+1} \mathcal{H}_x = \dots$ (by theorem A) and Krull's theorem gives $\underline{m}_x^r \mathcal{H}_x = 0$. Therefore, the stalks of $\mathcal{H}$ in the points of K are annihilated by suitable powers of the corresponding maximal ideal and this property also holds in the points of a neighbourhood of K, where Supp $\mathcal{H}$ is discrete. We can find an open U with the required property, by a suitable restriction of that neighbourhood.

COROLLARY (2.3): *Let X be a complex space, $x \in X$ and $\mathcal{F} \in \operatorname{Coh} X$. Then the following assertions hold:*

(i) *$Prof(\mathcal{F}_x) \geq N+1$ if and only if $H^q_x(X, \mathcal{F}) = 0$ for any $q \leq N$.*

(ii) *There is a neighbourhood U of x such that $prof(\mathcal{F}_y) \geq N+1$ for any $y \in U \setminus \{x\}$ if and only if $H^q_x(X, \mathcal{F})$ are finite-dimensional for any $q \leq N$.*

For proving this, it suffices to apply the theorem for a Stein neighbourhood of x, taking $K = \{x\}$.

REMARKS. *a*) From the proof of the theorem (2·1), it is clear that this theorem also holds for an arbitrary complex space and K a Stein compact.

b) By use of the long cohomology sequence, it is easy to show that the condition (i) of the theorem (2.1) is equivalent to the following: for any open set U containing K, the maps $H^q(U, \mathcal{F}) \longrightarrow H^q(U \setminus K, \mathcal{F})$ are bijective, for any $q \leq N-1$. In this way, the corollary (2.3) (i) is exactly the theorem 1.1 from [24], which thus receives another proof. The corollary (2.3) (ii) is a particular case of a deep result of Trautmann [25], in which the set $\{x\}$ is replaced by an analytic subset and the depth, by the depth relative to that analytic set.

c) Without any difficulties, one shows that the condition (ii) of the theorem (2.1) is still equivalent to the following: $\dim H^q(X \setminus K, \mathcal{F}) < \infty$ for any $1 \leq q \leq N+1$ and the kernel and the cokernel of the restriction map $\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X \setminus K, \mathcal{F})$ are finite-dimensional.

We now give some global results relative to the depth of a coherent Module.

THEOREM (2.4): *Let $(X, \mathcal{O})$ be a Stein space, $\mathcal{F} \in \operatorname{Coh} X$ and $N \geq 0$ an integer. Then the following assertion are equivalent:*

a) *$prof(\mathcal{F}) \geq N+1$*

b) *$H^q_K(X, \mathcal{F}) = 0$ for any holomorphically-convex compact $K \subset X$ and for any $q \leq N$ (one can replace such compacts by the points of X).*

c) *$H^q_c(X, \mathcal{F}) = 0$ for any $q \leq N$.*

PROOF. The equivalence $(a) \Longleftrightarrow (b)$ is immediate from theorem (2.1) (i). Then, we have the isomorphisms $H^q_c(X, \mathcal{F}) \simeq \varinjlim_K H^q_K(X, \mathcal{F})$, where the direct limit is taken over all holomorphically-convex compacts (which are sufficiently large). By (1.8) the equivalence of the conditions (*b*), (*c*) will result immediately from these isomorphisms.

REMARK. We can deduce the following fact for an arbitrary complex space X and $\mathcal{F} \in \operatorname{Coh} X$: $\operatorname{prof}(\mathcal{F}) \geq N+1$ if and only if $H^q_c(U, \mathcal{F}) = 0$ for any $q \leq N$ and for any Stein open set U of X (it suffices to consider Stein open sets sufficiently small). These two conditions are still equivalent to each of the following ones:

— $\mathrm{Ext}^q_{\mathcal{O}}(k(x), \mathcal{F}) = 0$ for any $q \leq N$ and for any $x \in X$ ($k(x)$ stands for the coherent sheaf concentrated on x, whose stalk is the field $k(x) = \mathcal{O}_x/m_x \simeq \mathbb{C}$)

— for any relatively compact Stein open set $U \subset X$, the maps $H^q(X, \mathcal{F}) \to H^q(X \setminus U, \mathcal{F})$ are injective for $q = N$ and bijective for $q \leq N - 1$.

(Only the definitions, the theorem (2.4) for U and the cohomology sequence relative to the pair (U, X) are to be used for proving all these equivalences).

We now give an extension of the implication $(a) \Longrightarrow (c)$:

PROPOSITION (2.5): *Let X be a complex space having a covering with $c + 1$ Stein open sets. For any $\mathcal{F} \in \mathrm{Coh}\, X$ such that $prof(\mathcal{F}) \geq N + 1$ and for any $q \leq N - c$, $H^q_c(X, \mathcal{F}) = 0$.*

For the proof of this proposition, we preceed by induction on c, by means of the following lemma, which provides a result of Mayer-Victoris type ([1], pg. 251, where is considered the general case of the families of arbitrary supports; here we give — in the particular case of compact supports — another proof):

LEMMA (2.6): *Let X be a paracompact space and $\mathcal{F}$ a sheaf of abelian groups on X. If X is an union of two open subset U, V, then we have an exact cohomology sequence:*

$$\ldots \to H^q_c(U \cap V, \mathcal{F}) \to H^q_c(U, \mathcal{F}) \oplus H^q_c(V, \mathcal{F}) \to$$
$$\to H^q_c(X, \mathcal{F}) \to H^{q+1}_c(U \cap V, \mathcal{F}) \to \ldots$$

PROOF. For any flabby sheaf $\mathcal{G}$ on X, we clearly have the exact sequence: $0 \to \Gamma_c(U \cap V, \mathcal{G}) \to \Gamma_c(U, \mathcal{G}) \oplus \Gamma_c(V, \mathcal{G}) \xrightarrow{\theta} \Gamma_c(X, \mathcal{G})$, where the first morphism is deduced from the diagonal map and the second is the map $\theta: (s, t) \to s - t$ (in both of them, one utilies the trivial extension of sections). We now prove the surjectivity of θ. Let $v \in \Gamma_c(X, \mathcal{G})$ and $K = \mathrm{Supp}\, v$, a compact subset of X. We consider the compact sets $K_1 = K \cap (X \setminus U)$, $K_2 = K \cap (X \setminus V)$, $K_1 \cap K_2 = \varnothing$ and we can suppose them nonempty (otherwise, one immediately finds a preimage of v). Let U_1, U_2 be neighbourhoods of K_1, K_2 such that $U_1 \cap U_2 = \varnothing$ and define

$$s = \begin{cases} 0 & \text{on } (X \setminus K) \cup U_1 \\ v & \text{on } U_2 . \end{cases}$$

This is a section of $\mathcal{G}$ on the open set $(X \setminus K) \cup U_1 \cup U_2$. hence there is a section $s' \in \Gamma(X, \mathcal{G})$ which extends it ($\mathcal{G}$ is supposed flabby). Obviously, $s'|_U \in \Gamma_c(U, \mathcal{G})$ and we put $s_1 = s'|_U$, $s_2 = s' - v|_V$. It is easily to be seen that $s_2 \in \Gamma_c(V, \mathcal{G})$ and $s_1 - s_2 = v$, therefore $\theta(s_1, s_2) = v$.

If $\mathcal{G}^{\cdot}$ is a flabby resolution of $\mathcal{F}$, then we have an exact sequence of complexes $0 \to \Gamma_c(U \cap V, \mathcal{G}^{\cdot}) \to \Gamma_c(U, \mathcal{G}^{\cdot}) \oplus \Gamma_c(V, \mathcal{G}^{\cdot}) \to \Gamma_c(X, \mathcal{G}^{\cdot}) \to 0$, whence the locked for exact sequence.

As a consequence of (2.5) we have:

COROLLARY (2.7): *Let $(X, \mathcal{O})$ be a compact complex space, $n = Prof(\mathcal{O})$ (i. e., dih X in notation from* [1]). *Then X cannot be covered by less than $n+1$ Stein open sets.*

If we suppose the contrary, then there results $\Gamma_c(X, \mathcal{O}) = \Gamma(X, \mathcal{O}) = 0$, a contradiction.

REMARKS. *a*) For a compact complex manifold $(X, \mathcal{O})$ we have $\dim(X) = \text{Prof}(\mathcal{O})$, hence such a manifold cannot be covered by less than $\dim(X)+1$ Stein open sets. This conclusion could be also obtained directly from the usual relations Mayer-Victotis [1] and the duality between $\Gamma(X, \mathcal{O})$ and $H^n(X, \omega)$.

The corollary (2.7) is also applyable to normal compact complex spaces of dimension ≥ 2: it is impossible to cover such a space by two Stein open sets.

b) Given a compact complex manifold V, in [2] is denoted by $d(V)$ the minimal number of Stein open sets by which V can be covered. In the same paper, the following is proved: « If $(\mathcal{V}, w, M)$ is a family of deformations of compact complex manifold, $\mathcal{V} = (V_t)_{t \in M}$, then $d(V_t)$ is an upper semicontinuous function of t for $t \in M$ ».

If we similary denote by $d(X)$ the minimal number of Stein open sets by which X can be covered, then the corollary (2.7) shows that $d(X) \geq \text{dih}\, X + 1$.

THEOREM (2.8): *Let X be a Stein space, $\mathcal{F} \in \text{Coh}\, X$, $N \geq 0$ an integer. Then the following statements take place:*

(i) *there is a finite set $A \subset X$ such that $Prof(\mathcal{F}|_{X \setminus A}) \geq N+1$ if and only if $\mathbb{C}$-linear spaces $H_c^q(X, \mathcal{F})$ are finite dimensional for any $q \leq N$.*

(ii) *the following assertion are equivalent:*

a) *there is a discrete set $A \subset X$ such that $Prof(\mathcal{F}|_{X \setminus A}) \geq N+1$.*

b) *the linear spaces $H_K^q(X, \mathcal{F})$ are finite dimensional for any $q \leq N$ and for any holomorphically convex compact $K \subset X$.*

c) *the linear spaces $H_c^q(X, \mathcal{F})$ are of at most countable dimension for any $q \leq N$.*

Before giving the proof of this theorem, we need two simple lemmas:

LEMMA (2.9): *If X is a Stein space and $\mathcal{H} \in \mathrm{Coh}\, X$, then $\dim_{\mathbb{C}} \Gamma(X, \mathcal{H}) < \infty$ if and only if Supp $\mathcal{H}$ is a finite set.*

This lemma can proved similary to (2.2) or by use of a result from [7], as follows: if $\dim_{\mathbb{C}} \Gamma(X, \mathcal{F}) < \infty$, then the $\Gamma(X, \mathcal{O})$-module

$$\mathrm{Hom}_{\Gamma(X, \mathcal{O})} (\Gamma(X, \mathcal{H}), \Gamma(X, \mathcal{H}))$$

and its submodule $\Gamma(X, \mathcal{O})/\mathrm{Ann}\, \Gamma(X, \mathcal{H})$ are noetherian, hence Stein algebra $\Gamma(X, \mathcal{O})/\mathrm{Ann}\, \Gamma(X, \mathcal{H})$ is noetherian, that is $\mathrm{Supp}\, \mathcal{H} = \mathrm{Supp}\, \mathcal{O}/\mathrm{Ann}\, \mathcal{H}$ is a finite set.

LEMMA (2.10): *Let $(X, \mathcal{O})$ be a Stein space and $\mathcal{H} \in \mathrm{Coh}\, X$. The topological dual of Fréchet space $\Gamma(X, \mathcal{H})$ has at most countable dimension over $\mathbb{C}$ if and only if Supp $\mathcal{H}$ is a discrete set.*

PROOF. Let $(U_r)_{r \geq 1}$ be an exhaustion of X by relatively compact Stein open sets, such that the restrictions $\Gamma(X, \mathcal{O}) \to \Gamma(U_r, \mathcal{O})$ are all dense. If Supp $\mathcal{H}$ is discrete, then the preceeding lemma shows that every $\Gamma(U_r, \mathcal{H})$ is finite-dimensional. But $(\Gamma(X, \mathcal{H}))' \simeq \varinjlim_r (\Gamma(U_r, \mathcal{H}))'$ and an implication is proved.

Conversely, by (2.9) it suffices to prove that $\dim_{\mathbb{C}} \Gamma(U_r, \mathcal{H}) < \infty$ for any r. Or, let $U \Subset X$ be a Stein open set so that the restriction $\Gamma(U, \mathcal{H}) \to$ $\to \Gamma(U, \mathcal{H})$ is dense and K be a holomorphically-convex compact containing U. For a Stein neighbourhood of K and for a surjection $\mathcal{O}^r \to \mathcal{H}$ on it, one can define a seminorm $p = \text{«}\sup_K\text{»}$ and we denote by $\Gamma(\widehat{X, \mathcal{H}})$ the completion of $\Gamma(X, \mathcal{H})/p^{-1}(0)$ in the topology given by p. $\Gamma(\widehat{K, \mathcal{H}})$ is a Banach space and, since the map $\Gamma(X, \mathcal{H}) \to \Gamma(K, \mathcal{H})$ is dense, the map $(\Gamma(\widehat{K, \mathcal{H}}))' \to (\Gamma(X, \mathcal{H}))'$ is injective. $(\Gamma(\widehat{K, \mathcal{H}}))'$ is a Banach space and by hypothesis it has a finite dimension, therefore $\dim_{\mathbb{C}} \Gamma(\widehat{K, \mathcal{H}}) < \infty$. Since the natural map $\Gamma(\widehat{X, \mathcal{H}}) \to \Gamma(U, \mathcal{H})$ is dense, there result $\dim_{\mathbb{C}} \Gamma(U, \mathcal{H}) < \infty$ and this completes the proof.

THE PROOF OF THE THEOREME (2.8). (i) Let us first consider the special case X manifold. Let $A \subset X$ be a finite set such that $\mathrm{Prof}(\mathcal{F}|_{X \setminus A}) \geq$ $\geq N + 1$. By the characterisation of the depth of finitely generated modules over regular local rings we have $\mathrm{Ext}_{\mathcal{O}_x}^{n-q} (\mathcal{F}_x, \omega_x) = 0$ for any $x \notin A$ and

$q \leq N$, hence $\underline{\mathrm{Ext}}_{\mathcal{O}}^{n-q}(\mathcal{F}, \omega)|_{X \setminus A} = 0$ $(q \leq N)$, hence $\mathrm{Supp}\,(\underline{\mathrm{Ext}}_{\mathcal{O}}^{n-q}(\mathcal{F}, \omega))$ is a finite set. By (1.2) and (2.9), $\mathrm{Ext}_{\mathcal{O}}^{n-q}(X; \mathcal{F}, \omega) = \Gamma(X, \underline{\mathrm{Ext}}_{\mathcal{O}}^{n-q}(\mathcal{F}, \omega))$ is a $\mathbb{C}$-linear space of finite dimension and the conclusion results from (1.1).

For the other implication we make an inverse reasoning, again using (2.9).

Let now X be a Stein space of finite embedding dimension, i. e. $\sup_{x \in X} \dim_{\mathbb{C}} \underline{m}_x/\underline{m}_x^2 < \infty$, and $X \xrightarrow{i} \mathbb{C}^p$ a suitable closed imersion. If we put $\mathcal{F}^* = i_*(\mathcal{F})$, then $H_c^{\cdot}(X, \mathcal{F}) \simeq H_c^{\cdot}(\mathbb{C}^p, \mathcal{F}^*)$ and for any $x \in \mathbb{C}^p \setminus i(X)$, $\mathcal{F}_x^* = 0$, that is $\mathrm{Prof}(\mathcal{F}_x^*) = \infty$ and we are immediately reduced to the first considered case.

Finally, let X be an arbitrary space. If $A \subset X$ is a finite set such that $\mathrm{Prof}(\mathcal{F}|_{X \setminus A}) \geq N + 1$, then for any relatively compact Stein open set U including A, we have the isomorphisms

$$H_c^q(U, \mathcal{F}) \simeq \prod_{x \in A} (\mathrm{Ext}^{n_U - q}_{\mathcal{O}_{\mathbb{C}^{n_U}, i_U(x)}}(\mathcal{F}_x, \mathcal{O}_{\mathbb{C}^{n_U}, i_U(x)}))' \quad (q \leq N),$$

where $i_U \colon U \to \mathbb{C}^{n_U}$ is a suitable imersion. (We have used here (1.1) for $\mathbb{C}^{n_U}$ and $i_U(\mathcal{F})$). It is not difficult to note that the invariants which appear in that product are independent on the imersion i_U. (If $x \in U$, then

$$(\mathrm{Ext}^{n_U - q}_{\mathcal{O}_{\mathbb{C}^{n_U}, i_U(x)}}(\mathcal{F}_x, \mathcal{O}_{\mathbb{C}^{n_U}, i_U(x)}))' \simeq H_{i(x)}^q(\mathbb{C}^{n_U}, i_U(\mathcal{F})) \simeq H_x^q(X, \mathcal{F}),$$

by the corollary (1.6)). For any two relatively compact Stein open sets including A, the canonical maps $H_c^q(U, \mathcal{F}) \to H_c^q(V, \mathcal{F})$ are isomorphisms for any $q \leq N$ and the conclusion is immediate from the above considered case of embeddable Stein spaces.

We are now proving the converse assertion of (i). Let $A = \{x \in X \mid \mathrm{Prof}(\mathcal{F}_x) \leq N\}$. If U is a relatively compact Stein open set of X, then $A \cap U$ is a finite set. Let (U_r) be an exhaustion of X by relatively compact Stein open sets.

Obviously, $H_c^q(X, \mathcal{F}) \simeq \varinjlim_r H_c^q(U_r, \mathcal{F})$ and as above, $A \cap U_r = A \cap U_{r+1} = \dots$, whenewer the maps $H_c^q(U_r, \mathcal{F}) \to H_c^q(U_{r+1}, \mathcal{F})$ are isomorphisms for $q \leq N$. Thus, A is a finite set and the assertion (i) of (2.8) is completely proved.

In what concerns the assertion (ii), it could be proved by a similar reasoning, by means of (2.10).

REMARKS. *a*) In the poiuts $x \in A$ (where A is the set from (2.8) (ii)), we assert the following supplementary property:

(*) For any prime ideal $\underline{p} \in \operatorname{Spec} \mathcal{O}_x$, $\underline{p} \neq \underline{m}_x$, $\operatorname{Prof}_{(\mathcal{O}_x)_{\underline{p}}}((\mathcal{F}_x)_{\underline{p}}) \geq N+1 - \dim \mathcal{O}_x/\underline{p}$. Indeed, the problem is local, so we can first suppose X a closed subspace of a Stein manifold and immediately, we are reduced to the case X a Stein manifold. Since $\operatorname{Ext}^{n-q}_{\mathcal{O}_y}(\mathcal{F}_y, \omega_y) = 0$ for $q \leq N$ and for any y sufficiently near x, then by « Nullstellensatz » a power of $\underline{m}_x$ annihilates $\operatorname{Ext}^{n-q}_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{O}_x)$, for any $q \leq N$. Then the support (in Spec $\mathcal{O}_x$) of the module $\operatorname{Ext}^{n-q}_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{O}_x)$ contains at most the point $\underline{m}_x$, hence for any $\underline{p} \in \operatorname{Spec} \mathcal{O}_x$, $\underline{p} \neq \underline{m}_x$, $\operatorname{Ext}^{n-q}_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{O}_x)_{\underline{p}} \simeq \operatorname{Ext}^{n-q}_{(\mathcal{O}_x)_{\underline{p}}}((\mathcal{F}_x)_{\underline{p}}, (\mathcal{O}_x)_{\underline{p}}) = 0$ for any $q \leq N$.

The proporty (*) then results from the caracterisation of the depth of a regular local ring $(\mathcal{O}_x)_{\underline{p}}$ in terms of Ext.

Conversely, if the property (*) holds for any $x \in X$, then a discrete set $A \subset X$ does exist, such that $\operatorname{Prof}(\mathcal{F}|_{X \setminus A}) \geq N+1$. For, it suffices to consider X a n-dimensional Stein manifold, in which case the condition (*) exhibits that for any $x \in X$, $\operatorname{Ext}^{n-q}_{(\mathcal{O}_x)_{\underline{p}}}((\mathcal{F}_x)_{\underline{p}}, (\mathcal{O}_x)_{\underline{p}}) \simeq (\operatorname{Ext}^{n-q}_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{O}_x))_{\underline{p}} = 0$, with $\underline{p} \in \operatorname{Spec} \mathcal{O}_x$ arbitrary by $\underline{p} \neq \underline{m}_x$.

Since a power of $\underline{m}_x$ annihilates $\operatorname{Ext}^{n-q}_{\mathcal{O}_x}(\mathcal{F}_x, \omega_x)$, the sheaves $\operatorname{Ext}^{n-q}_{\mathcal{O}}(\mathcal{F}, \omega)$ $(q \leq N)$ have their supports discrete, etc.

According to [12], the property (*) is also equivalent to the fact that for any $q \leq N$, all $\mathcal{O}_x$-modules $H^q_x(\mathcal{F}_x)$ have a finite length (where $H^q_x(\cdot)$ stands for the cohomology goup of Spec $\mathcal{O}_x$ relative to the closed set $\{\underline{m}_x\}$).

b) If $\mathcal{F}$ is an analytic coherent sheaf on X, $\mathcal{I}$ is a coherent Ideal such that $\operatorname{Prof}(\mathcal{F}) \geq N+1$ and $\operatorname{Supp} \mathcal{O}_x/\mathcal{I}$ is finite (resp: discrete), then the sheaf $\mathcal{I}\mathcal{F}$ verifies the condition of the theorem (2.8). The same theorem could be applied for those $\mathcal{F} \in \operatorname{Coh} X$ for which $\operatorname{Prof}(\mathcal{F}) \geq N+1$, out of a compact set.

3. Applications.

The application given in (a), (b), (c) do justify the following principle: on a Stein space « can be not taken into account » a relatively compact Stein open set (resp: a Stein compact), whewever the depth is sufficiently large. On the other hand, from (d) one can deduce the following: on a compact complex space and in suitable conditions on the depth, a Stein open set (resp: a Stein compact) » can be not taken into account ».

(a) *Interpretation of the depth in small dimensions: Hartogs and Cousin type results.*

The first two corollaries we give, characterise the analytic coherent sheaves (on Stein space) having their depth greater than one or two.

COROLLARY (a.3.1): *Let X be a Stein space and $\mathcal{F} \in \mathrm{Coh}\, X$. We have $Prof(\mathcal{F}) \geq 1$ if and only if any section of $\Gamma(X, \mathcal{F})$ with a compact support is null.*

PROOF. By theorem (2.4) $(a) \Longleftrightarrow (c)$, for $N = 0$.

COROLLARY (a.3.2): *Let X be a Stein space and $\mathcal{F} \in \mathrm{Coh}\, X$. Then the following assertion are equivalent:*

a) $Prof(\mathcal{F}) \geq 2$

b) for any relatively compact Stein open set $U \subset X$, the restriction map $\Gamma(X, \mathcal{F}) \to \Gamma(X \setminus U, \mathcal{F})$ is an isomorphism.

c) for any holomorphically-convex compact $K \subset X$, the restriction $\Gamma(X, \mathcal{F}) \to \Gamma(X \setminus K, \mathcal{F})$ is an isomorphism.

PROOF. It suffices to apply the theorem (2.4) by using the following exact sequences:

$$0 \to \Gamma_c(U, \mathcal{F}) \to \Gamma(X, \mathcal{F}) \to \Gamma(X \setminus U, \mathcal{F}) \to H^1_c(U, \mathcal{F}) \to H^1(X, \mathcal{F}),$$

$$0 \to \Gamma_K(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}) \to \Gamma(X \setminus K, \mathcal{F}) \to H^1_K(X, \mathcal{F}) \to H^1(X, \mathcal{F}).$$

The implications $(a) \Longrightarrow (b)$, $(a) \Longrightarrow (c)$ are more general: if X is an arbitrary compelx space, $U (\Subset X$ (resp: $K \subset X$) a Stein open set (resp: a Stein compact) and $\mathrm{Prof}(\mathcal{F}|_U) \geq 2$ (resp: $\mathrm{Prof}(\mathcal{F}|_K) \geq 2$), then the map $\Gamma(X, \mathcal{F}) \to \Gamma(X \setminus U, \mathcal{F})$ (resp: $\Gamma(X, \mathcal{F}) \to (X \setminus K, \mathcal{F})$) is surjective.

The corollary which follows is a Hartogs type result.

COROLLARY (a.3.3): *Suppose X a Stein space, $K \subset X$ a holomorphically-convex compact and $\mathcal{F} \in \mathrm{Coh}\, X$. Then the canonical map $\Gamma(X, \mathcal{F}) \to \Gamma(X \setminus K, \mathcal{F})$ is bijective if and only if $Prof(\mathcal{F}_x) \geq 2$ for any $x \in K$.*

The proof is immediate by theorem (2.1) (i).

REMARKS. *a)* The corollaries (a.3.2), (a.3.3) can be applied for the structural sheaf of a normal space of dimension ≥ 2 (resp: normal space of dimension ≥ 2, in the points of K). One can also consider the case of the complex spaces having their stalks Cohen-Macauley rings of dimension ≥ 2.

b) If K is an arbitrary compact of a Stein space and $\operatorname{Prof}(\mathcal{F}) \geq 2$, then by (a.3.3) one can deduce that there is a compact K' such that for any section $s \in \Gamma(X \setminus K, \mathcal{F})$ a unique section $s' \in \Gamma(X, \mathcal{F})$ does exist so that $s' = s$ on $X \setminus K'$ (one takes $K' = \widehat{K}$). In this way, the corollary (a.3.3) can be interpreted in terms of the boundary ∂X of X. We denote by $\Gamma(\partial X, \mathcal{F}) = \varinjlim_{U} \Gamma(U, \mathcal{F})$, where $\varinjlim$ is taken over all neighbourhoods U of the boundary (open sets with a compact complementary). Thus, we can obtaine the following result: if X is a Stein space and $\mathcal{F} \in \operatorname{Coh} X$ such that $\operatorname{Prof}(\mathcal{F}) \geq 2$, then the canonical map $\Gamma(X, \mathcal{F}) \to \Gamma(\partial X, \mathcal{F})$ is an isomorphism. We thus obtain a generalisation of [13], Ch. VII, D. 5.

The following applications deal with Cousin problems and they are consequences of the results from § 2, for $N \geq 3$. For the case of manifolds they are proved in [20].

First, we formulate Cousin problems for spaces (non necessarily reduced). Let $(X, \mathcal{O})$ be a complex space. For any open set $U \subset X$, we denote $S_U = \{f \in \Gamma(U, \mathcal{O}_x) \mid f_x$ is a nonzerodivisor in $\mathcal{O}_x$ for any $x \in U\}$. Clearly, S_U is a multiplicative system and a presheaf $U \to \Gamma(U, \mathcal{O}_U)_{S_U}$ (quotient-ring with denominators in S_U) is so obtained. The associated sheaf $\mathcal{M}$ is called the sheaf of germs of meromorphic section on X. It is easily to be seen that we have a canonical inclusion of sheaves $\mathcal{O} \subset \mathcal{M}$. If $s_x \in \mathcal{O}_x$ is a nonzerodivisor, then by multiplication with it, an injective morphism $\mathcal{O}_x \to \mathcal{O}_x$ is so defined. If $s \in \Gamma(U, \mathcal{O})$ is a section whose germ in x is s_x, then the map $\mathcal{O}|_U \xrightarrow{s} \mathcal{O}|_U$ is injective in x and since $\mathcal{O}$ is coherent, there is a neighbourhood $V \subset U$ of x, such that the restriction $\mathcal{O}|_V \to \mathcal{O}|_V$ is also injective. Therefore, $s|_V \in S_V$. By this remark, it is immediate that the stalk $\mathcal{M}_x$ is canonically identified with the total ring of quotient of $\mathcal{O}_x$. In the particular case X reduced, we have $S_U = \{f \in \Gamma(U, \mathcal{O}) \mid f$ does not vanish on any nonempty open subset of $U\}$ and we just obtain the definition of $\mathcal{M}$ given in the book « An Introduction to the theory of analytic spaces » by R. Narasimhan, Springer, 1966.

Cousin problems are to be formulated exactly as in the case of manifolds. The additive (resp: multiplicative) problem consists in giving conditions of surjectivity for the canonical map $\Gamma(X, \mathcal{M}) \to \Gamma(X, \mathcal{M}/\mathcal{O})$ (resp: $\Gamma(X, \mathcal{M}^*) \to \Gamma(X, \mathcal{M}^*/\mathcal{O}^*)$). (As usual $\mathcal{O}^*$ (resp: $\mathcal{M}^*$) stands for the subsheaf of $\mathcal{O}$ (resp: $\mathcal{M}$) of all invertible sections).

COROLLARY (a.3.4): *Let X be a Stein space, $K \subset X$ a holomorphically-convex compact. If $\operatorname{Prof}(\mathcal{O}_x) \geq 3$ for any $x \in K$, then Cousin additive problem has a solution on $X \setminus K$. If moreover, $H^2(X \setminus K, Z) = 0$, then Cousin multiplicative problem is soluble on $X \setminus K$.*

Proof. We have an exact sequence $H^1(X, \mathcal{O}) \to H^1(X \setminus K, \mathcal{O}) \to \to H_K^2(X, \mathcal{O})$. By (2.1) (i), $H^1(X \setminus K, \mathcal{O}) = 0$ and the first stetement of the corollary is immediate from the exact sequence

$$\Gamma(X \setminus K, \mathcal{M}) \to \Gamma(X \setminus K, \mathcal{M}/\mathcal{O}) \to H^1(X \setminus K, \mathcal{O}).$$

For the second part, we have the exact sequence $H^1(X \setminus K, \mathcal{O}) \to \to H^1(X \setminus K, \mathcal{O}^*) \to H^2(X \setminus K, \mathbb{Z})$ deduced from the exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$ (where $\mathbb{Z}$ is the simple sheaf of stalk the ring $\mathbb{Z}$ of integers and the second arrow is $\varphi \mapsto e^{2\pi i \varphi}$). Therefore, $H^1(X \setminus K, \mathcal{O}^*) = 0$ and the conclusion is immediate from another exact sequence $\Gamma(X \setminus K, \mathcal{M}^*) \to \to \Gamma(X \setminus K, \mathcal{M}^*/\mathcal{O}^*) \to H^1(X \setminus K, \mathcal{O}^*)$.

Corollary (a.3.5): *Let X be a Stein space and $U \Subset X$ a Stein open set such that $Prof(\mathcal{O}|_U) \geq 3$. Then the Cousin additive problem has a solution on $X \setminus U$. Moreover, if $H^2(X \setminus U, \mathbb{Z}) = 0$, then the same is true for the multiplicative problem.*

The proof is similar to the preceeding.

Remark. It is clear that (a.3.4), (a.3.5) also hold for X arbitrary, but $H^1(X, \mathcal{O}) = 0$. In (a.3.4) we must only suppose K a Stein compact.

Elements of $\Gamma(X, \mathcal{M}^*/\mathcal{O}^*)$ are called Cartier divisors on X. For a section $m \in \Gamma(X, \mathcal{M}^*)$, we denote by (m) the image of m by the canonical map $\Gamma(X, \mathcal{M}^*) \to \Gamma(X, \mathcal{M}^*/\mathcal{O}^*)$.

Corollary (a.3.6): *In the same hypothesis as in* (a.3.4) (*resp:* (a.3.5)), *we suppose all stalks of X factorial rings. Then any meromorphic section m on $X \setminus K$ (resp: $X \setminus U$) can be extended to a meromorphic section on X.*

Proof. We can suppose $m \in \Gamma(X \setminus K, \mathcal{M}^*)$ and let $D = (m)$ be the associated divisor on $X \setminus K$. Since X is a factorial space, on the multiplicative group $\Gamma(X, \mathcal{M}^*/\mathcal{O}^*)$ (whose composition law is additively denoted), an order relation can be introduced, such that every divisor D is decomposable in two positive divisors, $D = D_+ - D_-$. By (a.3.4), let $f, g \in \Gamma(X \setminus K, \mathcal{M}^*)$ be such that $(f) = D_+$, $(g) = D_-$; since D_+, D_- are positive divisors, then f, g are holomorphic sections. By (a.3.3), there are $F, G \in \Gamma(X, \mathcal{O})$ such that $F|_{X \setminus K} = f$, $G|_{X \setminus K} = g$ and it is clear that F, G do not vanish on any nonempty open set, that is $F, G \in S_X$ and moreover, F/G extends m. A similar proof for the other case.

(b) *Some topological consequences for the boundary of a Stein space.*

A complex space $(X, \mathcal{O})$ is called connected to the boundary if for any compact $K \subset X$, the complement in $X \setminus K$ of the union of all relatively compact components of $X \setminus K$ is connected ([13], Ch. VII, D). We also recall the notion of a germ of an analytic set to the boundary: it is a pair (U, S), where U is a neighbourhood of the boundary (i. e. $X \setminus U$ is compact) and S a closed analytic subset of U. We shall identify two such pairs (U_1, S_1), (U_2, S_2) whenever a neighbourhood of the boundary $U \subset U_1 \cap U_2$ does exist such that $S_1 \cap U = S_2 \cap U$. Let ∂X be the germ defined by any neighbourhood of the boundary.

We call a complex space $(X, \mathcal{O})$ *irreducible to the boundary* if ∂X cannot be written as an union of two proper germs. (Unions, intersections, etc. of germs of analytic sets to the boundary are naturally defined). Any such a space is connected to the boundary. Indeed, for any compact K, we shall have $X = K \cup (\bigcup_i V_i) \cup (\bigcup_\alpha U_\alpha)$, where V_i, U_α are the connected components of $X \setminus K$, V_i those relatively compact and U_α those relatively non compact. For any compact $K' \supset K$ and for any α, $U_\alpha \cap (X \setminus K') \neq \emptyset$; hence, from the definition of the irredudibility, the set of α's is reduced to a single element.

COROLLARY (b.3.1): *Let $(X, \mathcal{O})$ be a connected Stein space such that $Prof(\mathcal{O}) \geq 2$. Then X is connected to the boundary.*

PROOF. Let K be an arbitrary compact and $X = K \cup (\bigcup_i V_i) \cup (\bigcup_\alpha U_\alpha)$ as above. Since for any α, $\widehat{K} \cap U_\alpha \neq \emptyset$, it suffices to show that $X \setminus \widehat{K}$ is connected and then will result that the set of $\alpha' s$ is reduced to a single element. Indeed, by corollary (a.3.3), the natural map $\Gamma(X, \mathcal{O}) \to \Gamma(X \setminus \widehat{K}, \mathcal{O})$ is an isomorphism and our assertion is then a consequence of the following general fact:

LEMMA (b.3.2): *A ringed space in local rings $(X, \mathcal{O})$ is connected if and only if $\Gamma(X, \mathcal{O})$ cannot be represented as a product of unitary commutative rings.*

PROOF. If we write $X = U \cup V$, where U, V are disjoint nonempty open sets, then $\Gamma(X, \mathcal{O}) = \Gamma(U, \mathcal{O}) \times \Gamma(V, \mathcal{O})$. For the other implication, suppose $\Gamma(X, \mathcal{O}) = A \times B$ (A, B-rings). Let $U = \{x \in X \mid f(x) \neq 0$ for at least an element f of $A\}$, $V = \{x \in X \mid g(x) \neq 0$ for at least an element g of $B\}$. As usual, for a section $f \in \Gamma(X, \mathcal{O})$ we have denoted by $f(x)$ its image by the composition map $\Gamma(X, \mathcal{O}) \to \mathcal{O}_x \to \mathcal{O}_x / \underline{m}_x \simeq \mathbb{C}$ · ($\underline{m}_x$ the maximal ideal of $\mathcal{O}_x$). The condition $f(x) \neq 0$ means $f_x \notin \underline{m}_x$, that is the germ of f in x is invertible; therefore U, V are open sets. Since the unit section can be written $1 = e_1 + e_2$, $e_1 \in A$, $e_2 \in B$, then for any $x \in X$ we cannot have simultaneously $e_1(x) = e_2(x) = 0$, hence $X = U \cup V$. On the other hand, $U \cap V = \emptyset$ and lemma follows.

The following corollary extends [13], ch VII, D. 2,

COROLLARY (b.3.3): *Let X be an irreducible Stein space of dimension ≥ 2. Then X is irreducible to the boundary and particulary, connected to the boundary.*

PROOF. It suffices to probe that for any holomorphically-convex compact $K \subset X$, $X \setminus K$ is an irreducible space. Let X' be the normalisation of X and K' be the inverse image of K by the projection morphism. The space X' verifies the conditions of the preceeding corollary and K' is holomorphically-convex, therefore $X' \setminus K'$ is connected. The conclusion is immediate if we note that $X' \setminus K'$ is the normalisation of $X \setminus K$.

The following statement give some other connections between Stein spaces and their boundaries.

COROLLARY (b.3.4): *Let X, Y be two complex spaces.*

(i) *If Y is a Stein space and Prof $(\mathcal{O}_Y) \geq 1$, then the natural map Hom $(X, Y) \to$ Hom $(\partial X, \partial Y)$ is injective.* (*Hom $(\partial X, \partial Y)$ means naturally $\varinjlim_U$ Hom (U, Y), U an arbitrary neighbourhood of the boundary of X*).

(ii) *If Y is a Stein space and Prof $(\mathcal{O}_Y) \geq 2$, then the natural map Hom $(X, Y) \to$ Hom $(\partial X, \partial Y)$ is bijective.*

PROOF. (i) Let $\varphi, \psi \in \mathrm{Hom}(X, Y)$ be such that $\varphi|_U = \psi|_U$, where U is a neighbourhood of ∂X. We could suppose $X \setminus U = K$ be holomorphically-convex. We then have a commutative canonical diagram:

$$
\begin{array}{ccc}
 & \mathrm{Hom}\,(X, Y) \longrightarrow & \mathrm{Hom}\,(U, Y) \\
 & \swarrow & \Big\downarrow \\
\mathrm{Hom}_{\mathbb{C}\text{-alg. top.}}\,(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X)) \to & & \\
 & & \longrightarrow \mathrm{Hom}_{\mathbb{C}\text{-alg. top.}}\,(\Gamma(Y, \mathcal{O}_Y), \Gamma(U = X \setminus K, \mathcal{O}_X)).
\end{array}
$$

The first vertical arrow is an isomorphism ([7]). The second horizontal arrow is injective, as the map $\Gamma(X, \mathcal{O}_X) \to \Gamma(X \setminus K, \mathcal{O}_X)$ has this property (by (a.3.1)). Therefore, $\varphi = \psi$.

For proving (ii), the above diagram and corollary (a.3.4) for $\mathcal{F} = \mathcal{O}$ are to be used.

COROLLARY (b.3.5): *Suppose X, Y two irreducible Stein spaees, each of them of depth ≥ 2. If their boundaries are isomorphic, then the spaces themselves are isomorphic.*

PROOF. Let $U \subset X$, $V \subset Y$ be two neighbourhoods of the boundaries and $\varphi : U \to V$ an isomorphism between them. By the preceeding corollary, there are maps $f : X \to Y$, $G : Y \to X$ and $U' \subset X$, $V' \subset Y$ neighbourhoods of $\partial X, \partial Y$ respectively, such that $F = \varphi\,|_{U'}$, $G = \varphi^{-1}\,|_{V'}$ (one can suppose $U' \subset U$, $V' \subset V$). Then $G \circ F = \mathrm{id}$ on the nonempty open set $U' \cap \varphi^{-1}(V')$ and since X is irreducible, $G \circ F = \mathrm{id}$ on X (X can be embedded in a numerical space and $G \circ F$ is then identified with a system of holomorphic functions, etc.). Similary, $F \circ G = \mathrm{id}$ on Y and this completes the proof.

REMARK. For Stein manifolds of dimension ≥ 2, this corollary was proved in [8] and [18].

(c) *The category of analytic coherent sheaves defined around the boundary of a Stein space.*

Let $(X, \mathcal{O})$ be a complex space. We shall consider pairs $(U, \mathcal{F})$, where U is a neighbourhood of the boundary of X and $\mathcal{F} \in \mathrm{Coh}\, U$. Two such pairs $(U_1, \mathcal{F}_1), (U_2, \mathcal{F}_2)$, for which there is a neighbourhood of the boundary $U \subset U_1 \cap U_2$ such that $\mathcal{F}_1\,|_U = \mathcal{F}_2\,|_U$ are to be identified. The equivalence classes define a cathegory — the morphisms are naturally defined — called the category of analytic coherent sheaves around the boundary of X (or category of germs of analytic coherent sheaves defined to infinity) and we shall denote it by $\mathrm{Coh}\, \partial X$. The restriction of sheaves provides a functor $|\, \partial X : \mathrm{Coh}\, X \to \mathrm{Coh}\, \partial X$.

We shortly recall some definitions. A functor $F : C \to D$ between to arbitrary categories is called *faithful* (resp : *faithfully full*) if for any two objects M, N of C, the map $\mathrm{Hom}_C(M, N) \to \mathrm{Hom}_D(F(M), F(N))$, $\varphi \to F(\varphi)$ is injective (resp : bijective). The functor F is called *essentially surjective* if any object of D is isomorphic to the image by F of an object of C.

Before giving the main result, we prove an useful lemma :

LEMMA (c.3.1): *Let A be a noetherian local ring and M, N two finitely generated A-modules.*

(i) *If $Prof_A(N) \geq 1$, then $Prof_A(Hom_A(M, N)) \geq 1$.*

(ii) *If $Prof_A(N) \geq 2$, then $Prof_A(Hom_A(M, N)) \geq 2$.*

PROOF. (i) Let $a \in \underline{m}$ (the maximal ideal of A) be a nonzerodivisor in N. The multiplication with a in N, $N \xrightarrow{a} N$ is then injective, hence the map $\mathrm{Hom}(M, a) : \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N)$ is also injective. But it coincides with the multiplication with a in the A-module $\mathrm{Hom}_A(M, N)$.

(ii) Let (a_1, a_2) be a N-regular sequence of two elements of $\underline{m}$. The exact sequence $0 \to N \xrightarrow{a_1} N \to N/a_1 N \to 0$ gives the exact sequence $0 \to \mathrm{Hom}_A(M, N) \xrightarrow{a_1} \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N/a_1 N)$, whence a natural injective map $\mathrm{Hom}_A(M, N)/a_1 \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N/a_1 N)$ is deduced. The conclusion is then immediate from the following commutative diagram, in which the two horizontal and the second vertical arrows are injective:

$$\begin{array}{ccc} \mathrm{Hom}_A(M, N)/a_1 \mathrm{Hom}_A(M, N) & \longrightarrow & \mathrm{Hom}_A(M, N/a_1 N) \\ a_2 \downarrow & & a_2 \downarrow \\ \mathrm{Hom}_A(M, N)/a_1 \mathrm{Hom}_A(M, N) & \longrightarrow & \mathrm{Hom}_A(M, N/a_1 N). \end{array}$$

THEOREM (c.3.2): *Let X be a Stein space. Then the restriction functor $|\partial X : Coh\, X \to Coh\, \partial X$ has the following properties:*

(i) *is faithul on the sheaves of depth ≥ 1*

(ii) *is faithfully-full on the sheaves of depth ≥ 2*

(iii) *is essentially surjective on the sheaves of depth ≥ 3.*

The properties (i), (ii) are direct consequences of the following more precise result:

THEOREM (c.3.2) (i), (ii): *Let $(X, \mathcal{O})$ be a Stein space, $K \subset X$ a holomorphically-convex compact and $\mathcal{F}, \mathcal{G} \in Coh\, X$.*

(i) *if $Prof_{\mathcal{O}_x}(\mathcal{G}_x) \geq 1$ for any $x \in K$, then the canonical map $Hom_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \to Hom_{\mathcal{O}}(\mathcal{F}|_{X \setminus K}, \mathcal{G}|_{X \setminus K})$ is injective.*

(ii) *if $Prof_{\mathcal{O}_x}(\mathcal{G}_x) \geq 2$ for any $x \in K$, then the canonical map $Hom_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \to Hom_{\mathcal{O}}(\mathcal{F}|_{X \setminus K}, \mathcal{G}|_{X \setminus K})$ is bijective.*

PROOF. We consider the coherent $\mathcal{O}$-Module $\mathcal{H} = \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$. By the preceeding lemma and the isomorphism $\mathcal{H}_x \simeq \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{G}_x)$, if $\mathrm{Prof}(\mathcal{G}) \geq 1$ (resp: $\mathrm{Prof}(\mathcal{G}) \geq 2$) in the points of K, then $\mathrm{Prof}(\mathcal{H}_x) \geq 1$ (resp: $\mathrm{Prof}(\mathcal{H}_x) \geq 2$) for any $x \in K$. We have the exact sequence $0 \to \Gamma_K(X, \mathcal{H}) \to \Gamma(X, \mathcal{H}) \to \to \Gamma(X \setminus K, \mathcal{H}) \to H^1_K(X, \mathcal{H}) \to \dots$ and since $\Gamma\, \underline{\mathrm{Hom}} = \mathrm{Hom}$, the theorem follows by virtue of (2.1)

Property (iii) of (c.3.2) result from a more precise fact, which is essentially a consequence of [24], th. 3.1.

THEOREM (c.3.2) (iii): *Let X be a Stein space, $K \subset X$ a holomorphically-convex compact and $\mathcal{F} \in Coh\ (X \setminus K)$. If $Prof_{\mathcal{O}_x}(\mathcal{F}_x) \geq 3$ for any $x \in X$ sufficiently near to K, then there is an unique sheaf $\widehat{\mathcal{F}} \in Coh\ X$ such that $\widehat{\mathcal{F}}|_{X \setminus K} \simeq \mathcal{F}$ and $Prof_{\mathcal{O}_x}(\widehat{\mathcal{F}}_x) \geq 2$ for any $x \in K$.*

PROOF. The uniqueness of the sheaves of depth ≥ 2 (in the points of K) which extend $\mathcal{F}$, follows immediately by the preceeding theorem. For the proof of the existence, since K has a fundamental system of Stein neighbourhoods, we can suppose Prof$(\mathcal{F}) \geq 3$. For a moment, let us suppose that there is a sequence of compacts $T_i \supset K$, which converges to K, a sequence $\widehat{\mathcal{F}}_i$ of coherent $\mathcal{O}_X$-Modules of depth ≥ 2 and a sequence of isomorphisms $\xi_i : \widehat{\mathcal{F}}_i|_{X \setminus T_i} \simeq \mathcal{F}|_{X \setminus T_i}$. For each pair (i, j) and for each holomorphically convex compact L which contains $\widehat{T}_i \cup \widehat{T}_j$ ($\widehat{T}_i$, $\widehat{T}_j$ are holomorphic-convex hulls of T_i, T_j), we have an isomorphism $\xi_i^{-1} \circ \xi_j : \widehat{\mathcal{F}}_j|_{X \setminus L} \simeq \widehat{\mathcal{F}}_i|_{X \setminus L}$; by virtue of the preceeding theorem, we obtain an isomorphism $\xi_{ij} : \widehat{\mathcal{F}}_j \xrightarrow{\sim} \widehat{\mathcal{F}}_i$. Since all bijective maps from (c.3.2.ii) are functorial, one easily sees that ξ_{ij} does not depend on L and for any triplet (i, j, k) the compatibility relations $\xi_{ii} = 1$, $\xi_{ij}\, \xi_{jk}\, \xi_{ki} = 1$ take place. In this way, one obtains the looked for sheaf $\widehat{\mathcal{F}}$.

It remains only to prove the existence of such sequences $\{T_i\}$, $\{\widehat{\mathcal{F}}_i\}$, $\{\xi_i\}$. Let V be a neighbourhood of K, $U \subset V$ a relatively compact Stein neighbohrhood of K, $\Phi : U \longrightarrow \mathbb{C}^n$ a suitable closed imersion, [26]. $\Phi(K) = L$ is a compact in $\mathbb{C}^n$ and we can find a real number $\varrho > 0$ such that $L \subset B_\varrho$ (euclidian ball centered in origin of radius ϱ). We denote $T = \Phi^{-1}(B_{\varrho'})$ with $\varrho' > \varrho$. T is a compact set and obviously, $K \subset T \subset U$.

Let now $\mathcal{G}$ be the direct image of $\mathcal{F}$ by closed imersion $\Phi_{U \setminus K} : U \setminus K \longrightarrow \mathbb{C}^n \setminus L$; this is an analytic coherent sheaf on $\mathbb{C}^n \setminus L$ having a depth greater than 3. According to [24], th. 3.1, 4.1, there is an analytic coherent sheaf $\widehat{\mathcal{G}}$ on $\mathbb{C}^n$ such that Prof$(\widehat{\mathcal{G}}) \geq 2$ and $\widehat{\mathcal{G}} \simeq \mathcal{G}|_{\mathbb{C}^n \setminus B_{\varrho'}}$. The sheaf $\Phi^*(\widehat{\mathcal{G}}) \in$ $\in$ Coh U and it is isomorphic to $\mathcal{F}$ on $U \setminus T$. We can glue $\Phi^*(\widehat{\mathcal{G}})$ and $\mathcal{F}|_{X \setminus T}$ to each other and thus, one obtains a coherent $\mathcal{O}_X$ - Module, isomorphic (on $X \setminus T$) with $\mathcal{F}$ and having its depth ≥ 2 on $X \setminus T$. According to [24], th. 4.1 we can modify this sheaf in the points of depth ≤ 1 (the set of these points is a closed analytic subset contained in T, hence finite). Finally, an analytic coherent sheaf $\widehat{\mathcal{F}}_T$ is obtained and we have $\widehat{\mathcal{F}}_T|_{X \setminus T} \simeq$ $\simeq \mathcal{F}|_{X \setminus T}$, Prof$(\widehat{\mathcal{F}}_T) \geq 2$. This completes the proof.

(d) *Some applications to compact complex spaces.*

THEOREM (d.3.1): *Let X be a compact analytic space, $Y \subset X$ a closed set such that $X \setminus Y$ is a Stein open set and $\mathcal{F} \in Coh\, X$. Then in order that Prof $(\mathcal{F}|_{X\setminus Y}) \geq N+1$, it is necessary and sufficient that the canonical map $H^q(X, \mathcal{F}) \to H^q(Y, \mathcal{F}|_Y)$ to be bijective for $q \leq N-1$ and injective for $q = N$.*

PROOF. Let $0 \to \mathcal{F} \to \mathcal{M}^\cdot$ be a soft resolution of $\mathcal{F}$; then we have an exact sequence of complexes $0 \to \Gamma_c(X\setminus Y, \mathcal{M}^\cdot) \to \Gamma(X, \mathcal{M}^\cdot) \to \Gamma(Y, \mathcal{M}^\cdot|_Y) \to 0$, where the first arrow is given by the trivial extension of sections. Thus, one obtains the exact cofomology sequence:

$$\dots \to H_c^q(X \setminus Y, \mathcal{F}) \to H^q(X\ \mathcal{F}) \to H^q(Y, \mathcal{F}|_Y) \to H_c^{q+1}(X \setminus Y, \mathcal{F}) \to \dots$$

and the conclusion is immediate by (2.4).

We denote by Coh Y the category of germs of analytic coherent sheaves defined on Y: the objects of this category are classes (under an obvious equivalence relation) of analytic coherent sheaves defines on neighbourhood of Y. The restriction of sheaves defines a functor $\cdot|: \mathrm{Coh}\, X \to \mathrm{Coh}\, Y$.

THEOREM (d.3.2): *Let X be a compact space, $Y \subset X$ a closed subset such that $X \setminus Y$ is a Stein open set. The functor $\cdot|: Coh\, X \to Coh\, Y$ has the following properties:*

(i) *is faithful on the sheaves of depth ≥ 1 on $X \setminus Y$;*

(ii) *is faithfully-full on the sheaves of depth ≥ 2 on $X \setminus Y$;*

(iii) *is essentially surjective on the sheaves of Coh Y of depth ≥ 3 in the points of $X \setminus Y$.*

PROOF. (i) Let $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}\, X$ be two sheaves of depth ≥ 1 on $X \setminus Y$ and $\varphi, \psi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ be such that $\varphi = \psi|_Y$. Then there is a neighbourhood U of Y so that $\varphi = \psi|_U$. By (c.3.2) (i) applied to Stein space $X \setminus Y$, to the sheaves $\mathcal{F}|_{X\setminus Y}$, $\mathcal{G}|_{X\setminus Y}$ and to the holomorphic-convex hull (in $X \setminus Y$) of the compact $X \setminus Y$, we obtain $\varrho = \psi$. The assertion (ii), (iii) could be similary proved by utilisation of (c.3.2) (ii), (iii).

COROLLARY (d.3.3): *Let X be a compact complex space and Y a closed subset of X such that $X \setminus Y$ is a Stein open set. If Prof $(\mathcal{O}_x) \geq 2$ for any $x \in X \setminus Y$, then X is a connected if and only if Y is connected.*

PROOF. By (d.3.1), the canonical map $\Gamma(X, \mathcal{O}) \to \Gamma(Y, \mathcal{O})$ is bijective and the assertion is immediate from (b.3.2).

REMARK. Similar statements can be given by replacing Y by the complement of a Stein compact: one can consider, for istance, the case X is a projective space and Y the complement of a holomorphically-convex compact in an affine cart.

Institut of Mathematics
Bucharest

Added in proof:

1. The topogical characterisation of depth was firstly given by Y. T. Siu (using gap-sheaves), Com. Math., 21 (1969), 52-58.

2. We may add at the References, now, the following: J. P. Ramis et G. Ruget, Publ. Math. I. H. E. S., 38 (1970), 77-91 (where, beside the duality theorem of [19], one proves too a duality concerning the invariants $H_c(X, \cdot)$, and of which theorem (1.1) is a particular case).

REFERENCES

[1] ANDREOTTI, A. et GRAUERT, H. - *Théorèmes de finitude pour la cohomologie des espaces complexes* - Bul. Soc. Math. France 90, 193-259 (1962)

[2] ANDREOTTI, A. and VESENTINI, E. - *On the pseudo-rigidity of Stein manifolds* - Annali della Sc. Norm. Sup. Pisa 16, 213-223 (1962).

[3] BÀNICÀ, C. et STÀNÀSILÀ, O. - *Sur la profondeur d'un faisceauz analitique coherent sur un espace de Stein* - C. R. Acad. Sc. Paris 269, 636-639 (1969).

[4] BÀNICÀ, C. et STÀNÀSILÀ, O. - *Sur les germes de faisceaux analytiques cohérents definis autor de la frontière d'un espace de Stein* - C. R. Acad. Sc. Paris 270, 239-241 (1970).

[5] BÀNICÀ, C. et STÀNÀSILÀ, O. - *Sur la cohomologie des faisceaux analytiques cohérents à support dans un compact holomorph-convexe* - C. R. Acad. Sc. Paris 270, 1174-1177 (1970).

[6] CARTAN, H. - *Familles d'espaces complexes et fondaments de la géométrie analytique* - Séminaire E. N. S. Paris (1960-61).

[7] FORSTER, O. - *Zur Theorie der Steinschen Algebren und Moduln* - Math. Z. 97, 376-405 (1967).

[8] GHERARDELLI, F. - *Deformazioni rigide all'infinito di varietà di Stein* - Annali della Sc. Norm. Sup. Pisa - 20, 583-588 (1966).

[9] GODEMENT, R. - *Topologie algébrique et théorie des faisceaux* - Hermann, Paris (1958).

[10] GRAUERT, H. - *Ein Theorem der Analytischen Garbentheorie und die Modulräume Komplexer Strukturen* - Publ. Math. I. H. E. S. 5, 1-64 (1960).

[11] GROTHENDIECK, A. - *Sur quelques points d'algèbre homologique* - Tohoku Math. J. 9, 119-221 (1957).

[12] GROTHENDIECK, A. - *Cohomologie locale des faisceaux cohérents* - Séminaire de géométrie algébrique - Fasc. 2, Paris (1962).

[13] GUNNING, R. and ROSSI, H. - *Analytic functions of several complex variables* - Prentice-Hall, Englewood Cliffs N. J, (1965).

[14] HARTSHORNE, R. - *Residues and Duality* - Springer Verlag, Berlin (1966).

[15] HARVEY, R. - *The theory of hyperfunctions on totally real subset of a complex manifold with applications to extension problems* - Journal of Math. XCI, 4 october (1969).

[16] JURCHESCU, M. - *On the canonical topology of an analytic algebra and of an analytic module* - Bull. Soc. Math. France 93, 129-153 (1965).

[17] MALGRANGE, B. - *System differentiels à coefficients constants* - Sem. Bourbaki t. 15, 246, Paris (1962-63).

[18] REA, C. - *Un teorema di rigidità* - Annali della Sc. Norm. Sup. Pisa - 23, 185-203 (1969).

[19] RUGET, G. - *Un théorème de dualité* - Sem. de Geometrie Analytique, Orsay (1968-69).

[20] SERRE, J. P. - *Quelques problèmes globaux relatifs aux variétés de Stein* - Coll. Bruxelles, 56-68 (1953).

[21] SERRE, J. P. - *Un théorème de dualité* - Comm. Math. Helv. 29, 9-26 (1955).

[22] SERRE, J. P. - *Algèbre locale - Multiplicités* - Lecture Notes in Math, 11, Springer-Berlin (1965).

[23] SUOMINEN, K. - *Duality for coherent sheaves on analytic manifotds* - Ann. Acad. Sc. Fennicae I Mathematica 424, Helsinki (1968).

[24] TRAUTMANN, G. - *Ein Kontinuitätssatz für die Forsetzung kohäreuten analytiscen Garben* - Arch. der Math. 18, fasc. 2, 188-196 (1967).

[25] TRAUTMANN, G. - *Cohérence de faisceaux analytiques de la cohomologie locale* - C. R. Acad. Sc. Paris 267, 694-695 (1968).

[26] WIEGMANN, K. W. - *Einbettungen komplexer Räume in Zahlenräume* - Inv. Math. 1, 229-242 (1966).