Annali della
Scuola Normale Superiore di Pisa
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 25, nº 2 (1971), p. 221-228

<http://www.numdam.org/item?id=ASNSP_1971_3_25_2_221_0>

NON-NEGATIVE SOLUTIONS OF LINEAR PARABOLIC EQUATIONS: AN ADDENDUM

by D. G. ARONSON

Let D be a domain such that $\bar{D} \setminus D \neq \emptyset$, and suppose that in D we have a solution (in some sense) of a certain partial differential equation. Depending upon the particular context, there are many problems which arise naturally concerning the behavior of the given solution. One such problem is always that of determining if the given solution has a trace on $\bar{D} \setminus D$. Recently E. Magenes [2] has solved this problem and its converse in certain domains for distribution and ultradistribution solutions of linear elliptic and parabolic equations with analytic coefficients. In the elliptic case D is a bounded open set $\Omega \subset E^n$ with analytic boundary $\partial\Omega$, and the trace on $\partial\Omega$ of a distribution or ultradistribution solution is always an analytic functional. Conversely, given any analytic functional on $\partial\Omega$ there corresponds a unique distribution solution defined in Ω whose trace on $\partial\Omega$ is the given functional. For the parabolic case, D is the cylinder $\Omega \times (0, T]$ and the results are analogous but not as simple to describe.

In April 1969 the author presented some of the results from reference [1] in a lecture at the University of Pavia. After the lecture, Professor Magenes observed that the Widder Representation Theorem and its generalizations are related to the results obtained in [2], and asked if it is possible to obtain a complete characterization of the trace on $t = 0$ of a non-negative solution in $E^n \times (0, T]$ of an equation of the type considered in [1]. The purpose of this note is to provide such a characterization. We are indebted to Professor Magenes for suggesting this problem and for his interest in this work. The arguments which we give here are based entirely on reference [1] and are quite different from those of reference [2]. Since this note is an addendum to [1] we will make free use of the definitions, nota-

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Pervenuto alla Redazione l'11 Marzo 1970.

tions and results of [1] without further explanation. We will also use this occasion to correct some typographical errors in [1].

THEOREM. *Suppose that L satisfies (H) in $E^n \times (0, T]$. Then $u = u(x, t)$ is a non-negative weak solution of $Lu = 0$ in $S_1 = E^n \times (0, T_1]$ for some $T_1 \in (0, T]$ if and only if*

$$u(x, t) = \int_{E^n} \Gamma(x, t; \xi, 0)\, \varrho(d\xi), \tag{1}$$

where ϱ is a non-negative Borel measure on E^n such that

$$\int_{E^n} e^{-\sigma |x|^2} \varrho(dx) < \infty \tag{2}$$

for some $\sigma > 0$, and Γ is the weak fundamental solution of $Lu = 0$.

In the sequel the word solution will always mean a weak solution, and the word measure will always mean a non-negative Borel measure on E^n.

PROOF. Suppose first that u is a non-negative solution of $Lu = 0$ in S_1. Then according to Theorem 12 there exists a unique measure ϱ such that

$$u(x, t) = \int_{E^n} \Gamma(x, t; \xi, 0)\, \varrho(d\xi).$$

In view of Theorem 10 (ii) there exist constants $c_1, c_2 > 0$ such that

$$\Gamma(x, t; \xi, \tau) \geq c_1 (t - \tau)^{-n/2} \exp\{- c_2 | x - \xi |^2/(t - \tau)\}.$$

Note that the constants c_1, c_2 depend on T and the structure of L, but are independent of T_1. Since every solution is locally bounded and continuous in its domain of definition we have

$$u(0, T_1) \geq c_1 T_1^{-n/2} \int_{E^n} \exp\{- c_2 | \xi |^2/T_1\}\, \varrho(d\xi).$$

Therefore

$$\int_{E^n} e^{-\sigma |\xi|^2} \varrho(d\xi) \leq u(0, T_1)\, T_1^{n/2}/c_1 < \infty$$

with $\sigma = c_2/T_1$.

Now suppose that ϱ is a measure which satisfies (2) for some $\sigma > 0$. For integers $k \geq 1$, let $\gamma^k(x, t; \xi, \tau)$ denote the weak Green function for L in the cylinder $(|x| < k) \times (0, T]$. By Lemma 7, we have $\gamma^k \nearrow \Gamma$ as $k \to \infty$. Moreover, according to Theorem 9 (ii), if we hold x, t and τ fixed with $\tau < t$ then, for all k, such that $|x| < k$, the function $\gamma^k(x, t; \xi, \tau)$ is continuous for $|\xi| \leq k$ with $\gamma^k(x, t; \xi, \tau) = 0$ for $|\xi| = k$. Extend the domain of definition of γ^k by setting $\gamma^k = 0$ for $|\xi| > k$. Then it is clear that, for fixed (x, t) and k sufficiently large, $\gamma^k(x, t; \xi, 0)$ is ϱ-integrable and we have

$$\int_{E^n} \gamma^k(x, t; \xi, 0)\, \varrho(d\xi) \nearrow \int_{E^n} \Gamma(x, t; \xi, 0)\, \varrho(d\xi)$$

as $k \to \infty$. In view of Theorem 9 (iii) there exist constants $c_3, c_4 > 0$ independent of k such that

$$\gamma^k(x, t; \xi, 0) \leq c_3\, t^{-n/2} \exp\{-c_4 |x - \xi|^2/t\}.$$

It is easily verified that if $0 < t < c_4/\sigma$ then

$$-\frac{c_4}{t} |x - \xi|^2 + \sigma |\xi|^2 = -$$

$$-\frac{1}{t} \left| (c_4 - \sigma t)^{1/2} \xi - \frac{c_4}{(c_4 - \sigma t)^{1/2}} x \right|^2 + \frac{c_4 \sigma}{c_4 - \sigma t} |x|^2.$$

Therefore

$$\int_{E^n} \gamma^k(x, t; \xi, 0)\, \varrho(d\xi) \leq \left\{ c_3 t^{-n/2} \int_{E^n} e^{-\sigma|\xi|^2} \varrho(d\xi) \right\} \exp\{c_4 \sigma |x|^2/(c_4 - \sigma t)\},$$

and the same inequality holds in the limit as $k \to \infty$. In particular, it follows that

$$\int_{E^n} \Gamma(x, t; \xi, 0)\, \varrho(d\xi) \in L^2[\delta, T_1; L^2_{\text{loc}}(E^n)]$$

for every $\delta \in (0, T_1)$, where T_1 is any number in the set $(0, c_4/\sigma) \cap (0, T]$. Thus

$$\int_{E^n} \Gamma(x, t; \xi, 0)\, \varrho(d\xi)$$

satisfies the hypothesis of Corollary 12.1 and is a non-negative solution of $Lu = 0$ in S_1 for $T_1 \in (0, c_4/\sigma) \cap (0, T]$.

We have shown that there is a one to-one correspondence between non-negative solutions of $Lu = 0$ and measures which satisfy (2). Roughly speaking, given a measure ϱ which satisfies (2) we can regard the function u given by the representation formula (1) as the solution of the Cauchy problem

$$Lu = 0 \text{ in } S_1, \quad u = \varrho \text{ on } E^n \times \{0\}.$$

To make this precise we must, however, determine the manner in which u assumes its initial data ϱ.

Let u be a non-negative solution of $Lu = 0$ in S_1. To u there corresponds a unique measure ϱ which satisfies (2) for some $\sigma > 0$. Define

$$\sigma[u] = \inf \left\{ \sigma : \sigma > 0, \int_{E^n} e^{-\sigma |x|^2} \varrho(dx) < \infty \right\}.$$

Clearly $\sigma[u] \geq 0$. Moreover, from the first part of the proof of the theorem we have $\sigma[u] \leq c_2/T_1$, where c_2 is the constant which occurs in the lower bound for the fundamental solution Γ. Recall that c_2 depends only on T and the structure of L. In the example on page 638 of [1] we have $T_1 < < 1/4\lambda$, $\varrho(dx) = e^{\lambda |x|^2} dx$ and, since we are dealing with the equation of heat conduction, $c_2 = 1/4$. It is easily verified that in this case $\sigma[u] = \lambda = \inf\{c_2/T_1 : 0 < T_1 < 1/4\lambda\}$.

COROLLARY. *Let $u = u(x, t)$ be a non-negative solution of $Lu = 0$ in S_1 and ϱ the corresponding measure. Then*

$$\lim_{t \to 0+} \int_{E^n} u(x, t)\, \psi(x)\, dx = \int_{E^n} \psi(x)\, \varrho(dx) \tag{3}$$

for all $\psi \in C(E^n)$ such that $|\psi(x)| \leq K e^{-\delta |x|^2}$ for some constants $K > 0$ and $\delta > \sigma[u]$.

Note the similarity between (3) and the definition of initial values for a general weak solution given on page 619 of [1].

PROOF. Assume for the moment that $\psi \geq 0$. Then, by (1) and the Fubini-Tonelli theorem,

$$\int_{E^n} u(x, t)\, \psi(x)\, dx = \int_{E^n} \left\{ \int_{E^n} \Gamma(x, t; \xi, 0)\, \psi(x)\, dx \right\} \varrho(d\xi). \tag{4}$$

We will show first that the integral on the right hand side is finite for all sufficiently small t. Note that for fixed $t > 0$, $\int_{E^n} \Gamma(x, t; \xi, 0)\, \psi(x)\, dx$ is a continuous function of $\xi \in E^n$. Moreover,

$$\int_{E^n} \Gamma(x, t; \xi, 0)\, \psi(x)\, dx \leq K c_3 t^{-n/2} \int_{E^n} \exp\left\{-\frac{c_4}{t} |x-\xi|^2 - \delta |x|^2\right\} dx.$$

It is easily verified that

$$-\frac{c_4}{t} |x-\xi|^2 - \delta |x|^2 = -|\zeta|^2 - \frac{c_4 \delta}{c_4 + \delta t} |\xi|^2$$

and

$$\int_{E^n} \Gamma(x, t; \xi, 0)\, \psi(x)\, dx \leq K c_3 (c_4 + \delta t)^{-n/2} \left\{\int_{E^n} e^{-|\zeta|^2} d\zeta\right\} \exp\{-c_4 \delta |\xi|^2/(c_4 + \delta t)\},$$

where

$$\zeta = t^{-1/2} \{(c_4 + \delta t)^{1/2} x - c_4 (c_4 + \delta t)^{-1/2} \xi\}.$$

Set $\sigma = (\delta + \sigma[u])/2$. In view of the definition of $\sigma[u]$, we have

$$\int_{E^n} e^{-\sigma |\xi|^2} \varrho(d\xi) < \infty.$$

Thus if

$$t \leq c_4 \left(\frac{1}{\sigma} - \frac{1}{\delta}\right)$$

it follows that

$$\int_{E^n} \Gamma(x, t; \xi, 0)\, \psi(x)\, dx \leq K' e^{-\sigma |\xi|^2}$$

and

$$\int_{E^n} \left\{\int_{E^n} \Gamma(x, t; \xi, 0)\, \psi(x)\, dx\right\} \varrho(d\xi) \leq K' \int_{E^n} e^{-\sigma |\xi|^2} \varrho(d\xi) < \infty.$$

Since $\psi \in C(E^n)$ we conclude from Lemma 8 that

$$\lim_{t \to 0+} \int_{E^n} \Gamma(x, t; \xi, 0)\, \psi(x)\, dx = \psi(\xi)$$

for all $\xi \in E^n$. Therefore, by the dominated convergence theorem,

$$\lim_{t \to 0+} \int_{E^n} \left\{\int_{E^n} \Gamma(x, t; \xi, 0)\, \psi(x)\, dx\right\} \varrho(d\xi) = \int_{E^n} \psi(\xi)\, \varrho(d\xi).$$

In view of (4), this completes the proof in the special case $\psi \geq 0$. For general ψ we write $\psi = \psi^+ - \psi^-$ and apply the above argument to $\psi^\pm$ in the usual manner.

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ERRATA TO [1]

Page	*Line*	*Printed*	*Correct*
607	10 —	rapresentation	representation
608	7 —	Δ	Δu
609	10	Bj	B_j
611	7	section 2	section 3
612	5 —	problem	problem,
614	12	$\int$	$\int_{E^n}$
615	14 —	constant	constants
618	12 —	covex	convex
621	10	convertion	convention
624	1 —	$\| w \|_{2,2}^{2q'/p}$	$\| w_x \|_{2,2}^{2q'/p}$
626	1	couvolution	convolution
627	11 —	exponenents	exponents
627	6 —	$\iint_{\Omega_\tau}$	$\iint_{Q_\tau}$
633	1 —	$H_0^{1,}(\Omega)$	$H_0^{1,2}(\Omega)$
634	8 —	$H^{1,2}(\Omega)$	$H_0^{1,2}(\Omega)$
634 637	3 — 14	m_1	min $(0, m_1)$
634 637	3 — 14	m_2	max $(0, m_2)$
635	9	$\| F_j \|_{2,2,Q}^2$	$\| F_j \|_{2,2,Q}$
635	11	$\| \Omega \|^{(p-1)/2pT(q-1)/2q}$	$\| \Omega \|^{(p-1)/2p} T^{(q-1)/2q}$
635	14	$\| G \|_{p,q,Q}$	$\| G \|_{p,q,Q}^2$
639 640	5 — 8 —	$\| e^{-\gamma \|x\|^2} u_0 \|_{L^2(E^n)}$	$\| e^{-\gamma \|x\|^2} u_0 \|_{L^2(E^n)}^2$

Page	*Line*	*Printed*	*Correct*
639	4 —	$\| e^{-\gamma \|x\|^2} G \|_{p,q}$	$\| e^{-\gamma \|x\|^2} G \|^2_{p,q}$
640	7 —	(same as above)	(same as above)
640	4 —	$H^{1,2}_{\text{loc}}(E)$	$H^{1,2}_{\text{loc}}(E^n)$
641	7 —	fnnction	function
641	5 —	$\| \zeta u^k \|_{2p', 2q'}$	$\| \zeta u^k \|^2_{2p', 2q'}$
641	4 —	coniugates	conjugates
641	3 —	$\| u^k \|_{2p', 2q', Q}$	$\| u^k \|^2_{2p', 2q', Q}$
643	12	consides	consider
653	2	$\inf \| \widetilde{F} \|_{p,q}$	$\mathcal{C} \inf \| \widetilde{F} \|_{p,q}$
656	11	$\Gamma(x, t) ; \cdot , \cdot)$	$\Gamma(x, t ; \cdot , \cdot)$
656	9 —	$(\Gamma x, t ; \cdot , \cdot)$	$\Gamma(x, t ; \cdot , \cdot)$
665	6	Cauhy	Cauchy
666	11	$\int_{\|y-\zeta \geq \sigma}$	$\int_{\|y-\zeta\| \geq \sigma}$
673	13	satisfyng	satisfying
677	3 —	Theorem 1 (ii)	Theorem A
682	7 —	$J_2 \|$	$\| J_2 \|$
685	14	$\int_{E_n}$	$\int_{E^n}$
686	2	(same as above)	(same as above)
686	5	(same as above)	(same as above)
686	6 —	(same as above)	(same as above)
687	6	(same as above)	(same as above)
687	10	(same as above)	(same as above)
689	10 —	$\int_{EK}$	$\int_{E \setminus K}$
690	6	$\Gamma(x, t ; 0)$	$\Gamma(x, t ; \xi, 0)$
690	8	$L^2_{\text{loc}}(E)$	$L^2_{\text{loc}}(E^n)$

REFERENCES

[1] D. G. ARONSON: *Non-negative solutions of linear parabolic equations,* Annali della Scuola Normale Superiore di Pisa Classe di Scienze, **22** (1968), 607-694.

[2] ENRICO MAGENES: *Alcuni aspetti della teoria delle ultradistribuzioni e delle equazioni a derivate parziali,* Istituto Nazionale di Alta Matematica, Symposia Mathematica, **2** (1968), 235-254.

[3] M. KATO: *On positive solutions of the heat equation,* Nagoya Math. Journal, **30** (1967) 203-207.

Added in Proof. I am indebted to Professor B. Frank Jones for the observation that another corollary to our theorem is a pointwise result of Fatou type. Specifically, let u be a non negative solution of $Lu = 0$ and let ϱ be the corresponding measure. Then, for almost every $x \in E^n$, we have

$$\lim_{t \to 0+} u(x, t) = f(x),$$

where f is the density of the absolutely continuous part of the Lebesgue decomposition of ϱ. The proof of this assertion is based on the representation formula (1) and the bounds for the fundamental solution. We omit further details since the proof is essentially the same as the proof of the corresponding result for the equation of heat conduction given in [3].