ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

JANA STARÁ

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REGULARITY RESULTS FOR NON-LINEAR ELLIPTIC SYSTEMS IN TWO DIMENSIONS

JANA STARÁ, Praha

The purpose of this paper is to prove the regularity of the weak solution of Dirichlet problem for non linear elliptic systems in two dimensions. This problem is considered in the following form:

Let Ω be a bounded domain in E_N, u be a weak solution of the system

$$\sum_{|i| \leq \varkappa_r} (-1)^i D^i a_{ir}(x, Du(x)) = f_r(x); \; r = 1, \dots m, \; Du = \{D^\gamma u_s\}_{\substack{s=1,\dots,m \\ |\gamma| \leq \varkappa_s}}$$

with a boundary condition u_0 i. e.

1) $$u = (u_1, \dots, u_m), u_0 = (u_1^0, \dots, u_m^0); \; u_r - u_r^0 \in \overset{\circ}{W}{}_k^{\varkappa_r}(\Omega); \; r = 1, \dots, m$$

2) $$\int_\Omega \sum_{r=1}^m \Big(\sum_{|i| \leq \varkappa_r} a_{ir}(x, Du(x)) D^i \varphi_r(x) \Big) - f_r(x) \varphi_r(x) \, dx = 0$$

for every $\varphi_r \in \overset{\circ}{W}{}_k^{\varkappa_r}(\Omega)$.

The regularity means that u_r belongs to $C_{\varkappa_r, \mu}(\overline{\Omega})$ for $r = 1, \dots, m$.

This result was proved by

1. Ch. B. Morrey (1937) for $N = 2$, $m = 1$, $\varkappa_r = 1$, $k = 2$,
2. E. De Giorgi (1957) for $N \geq 2$, $m = 1$, $\varkappa_r = 1$, $k = 2$,
3. O. A. Ladyženskaja - N. N. Uralceva (1959) for $N \geq 2, m = 1, \varkappa_r = 1, 1 < k < \infty$,
4. Ch. B. Morrey (1960) $N \geq 2$, $m = 1$, $\varkappa_r = 1$, $1 < k < \infty$,
5. J. Nečas (1966) $N = 2$, $m = 1$, $\varkappa_r \geq 1$, $k = 2$,
6. J. Nečas (1967) $N = 2$, $m = 1$, $\varkappa_r \geq 1$, $1 < k < \infty$.

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In this paper, the regularity is proved for $N = 2$, $m \geqq 1$, $\varkappa_r \geq 1$, $k \geq 2$.

For $N \geq 2$ there was proved a partial regularity (see Morrey, [12]) as follows: for every $\Omega_0, \overline{\Omega}_0 \subset \Omega$ there exists an Ω_1 so that u is regular on Ω_1 and the Lebesgue's measure of $\Omega_0 - \Omega_1$ is equal to zero. A stronger result concerning the Hausdorff measure of $\Omega_0 - \Omega_1$ and under weaker conditions, was proved by E. Giusti, M. Miranda (see [7]). The regularity in this case ($N > 2$) cannot be proved; there exist counter-examples (De Giorgi [4], Giusti-Miranda [6]) of non-regular solutions of the equations with coefficients analytical in u. For the present we do not know a counter-example satisfying the stronger Morrey's conditions of the growth of coefficients.

Let us put the problem considered here in the following way:

Ω is a bounded domain in E_N with infinitely smooth boundary $\partial\Omega$; $\overline{\Omega} = \Omega \cup \partial\Omega$. N_i are linear defferential operators with constant coefficients.

$$(0) \qquad N_i u = \sum_{r=1}^{m} N_{ir} u_r = \sum_{r=1}^{m} \sum_{|\alpha| \leq \varkappa_r} a_{ir\alpha} D^\alpha u_r ; \; i = 1, \dots, h.$$

Let us denote $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_N^{\alpha_N}$; $N_{ir}\xi = \sum_{|\alpha| = \varkappa_r} a_{ir\alpha} \xi^\alpha$ and suppose

$$\text{rang } N\xi = \text{rang } (N_{ir}\xi)_{\substack{i=1,\dots,h \\ r=1,\dots,m}} = m$$

for every $\xi \in E_N$; $\xi \neq (0, \dots, 0)$.

As a special case we may take

$$N_i u = N_{r\alpha} u = D^\alpha u_r$$

for every $r = 1, \dots, m$; $|\alpha| \leq \varkappa_r$.

The functions $F_i(x, \xi)$ (for $i = 1, \dots, h$) are defined and continuous with all their first derivatives on $\overline{\Omega} \times E_h$ (*) and are nonlinear with polynomial growth (of the order $k - 1$) in ξ.

Let us denote $\theta\xi = \left(1 + \sum_{i=1}^{h} |\xi_i|^2\right)^{1/2}$;

$$\theta u(x) = \left(1 + \sum_{i=1}^{h} |N_i u(x)|^2\right)^{1/2}$$

(*) They are differentiable on $\Omega \times E_h$ and the derivatives may be continously extended on $\overline{\Omega} \times E_h$

for

$$u \in \prod_{r=1}^{m} W_k^{\varkappa_r}(\Omega)\,; \ \frac{\partial F_i}{\partial \xi_j} = F_{ij}\,.$$

Let us suppose that there exists $C > 0$ such that for every $x \in \bar{\Omega}$, $\xi \in E_h$

$$\sum_{i=1}^{h} \left(|F_i(x, \xi)| + \sum_{l=1}^{N} \left| \frac{\partial F_i}{\partial x_l}(x, \xi) \right| \right) \leq C \cdot \theta_\xi^{k-1}, \tag{1}$$

$$\sum_{i,j=1}^{h} |F_{ij}(x, \xi)| \leq C \cdot \theta_\xi^{k-2}, \tag{2}$$

$$F_{ij}(x, \xi) = F_{ji}(x, \xi). \tag{3}$$

We shall consider a weak solution of the equation

$$\sum_{|\alpha| \leq \varkappa_r} (-1)^{|\alpha|} \left(\sum_{i=1}^{h} a_{ir\alpha} D^\alpha F_i(x, \{N_j u(x)\}_{j=1}^{h}) \right) = \sum_{|\alpha| \leq \varkappa_r} (-1)^{|\alpha|} \left(\sum_{i=1}^{h} a_{ir\alpha} D^\alpha f_i(x) \right)$$

for $r = 1, \dots, m$, which may be written in a divergent form

$$\int_\Omega \sum_{i=1}^{h} [F_i(x, \{N_j u(x)\}_{j=1}^{h}) - f_i(x)]\, N_i \varphi(x)\, dx = 0. \tag{1.1}$$

We shall suppose that the operator on the left represents a monotone operator from $\prod_{r=1}^{m} \mathring{W}_k^{\varkappa_r}(\Omega)$ into $\left(\prod_{r=1}^{m} \mathring{W}_k^{\varkappa_r}(\Omega) \right)'$.

In Case A operators N_i which consist only of their main parts, i.e.

$$N_i u = \sum_{r=1}^{m} \sum_{|\alpha| = \varkappa_r} a_{ir\alpha} D^\alpha u_r;\ i = 1, , \dots, h$$

will be considered. In this case it is sufficient to suppose that

(4) there exist two positive constants γ_1, γ_2 so that

$$\gamma_1 \theta_\xi^{k-2} |\eta|^2 \leq \sum_{i,j=1}^{h} F_{ij}(x, \xi) \eta_i \eta_j \leq \gamma_2 \theta_\xi^{k-2} |\eta|^2$$

for every $\eta \in E_h$, $x \in \bar{\Omega}$, $\xi \in E_h \cdot |\eta|^2 = \sum_{i=1}^{h} |\eta_i|^2$.

Case B: Let us decompose N_i in the main part N_i' and

$$N_i'' = N_i - N_i'$$

(the corresponding notation $v' = (v_1', \dots, v_h')$; $v'' = (v_1'', \dots v_h'')$; $v = (v', v'') \in E_{2h}$).

$F_i(x, v)$ are defined and continuous with all their first derivatives on $\bar{\Omega} \times E_{2h}$. The conditions of growth are the same as in A. Instead (4) let us suppose

(4′) there exist C_1, C_2 positive so that

$$\sum_{i=1}^{h} F_i(x, v)(v_i' + v_i'') \geqq C_1 \sum_{i=1}^{2h} | v_i |^k - C_2 \qquad \sum_{i=1}^{h} \sum_{j=1}^{2h} F_{ij}(x, v) \mu_j (\mu_i' + \mu_i'') > 0$$

$$\forall \mu \in E_{2h};\ \mu \neq 0$$

(4″) there exist two positive constants γ_1, γ_2 so that

$$\gamma_1 \theta_v^{k-2} | \eta |^2 \leq \sum_{i,j=1}^{h} F_{ij}(x, v) \eta_i \eta_j \leq \gamma_2 \theta_v^{k-2} | \eta |^2$$

for every $x \in \bar{\Omega}$; $v \in E_{2h}$; $\eta \in E_h$.

(5) Suppose that for the regularity conditions (0) — (4) (in case A) or (0) — (4″) (in case B) are satisfied uniformly with regard to an orthonormal transformation of a coordinate system in E_N.

(6) The right part $f_i \in W_p^1(\Omega)$ for $i = 1, \dots, h, p > 2$, the boundary condition $u_r^0 \in W_{\tilde{p}}^{\varkappa_r + 1}(\Omega)$ for $r = 1, \dots, m$, $\tilde{p} > \max(p, k)$.

§ 1 consists of some lemmas on L_p-estimates of solutions of the linear equations. Lemma 1.4 gives such estimates for an equation with measurable coefficients, whose bilinear form is the following

$$\int_{\Omega} \sum_{i,j=1}^{h} A_{ij} N_i u N_j \varphi.$$

Here $A_{ij} \in L_\infty(\Omega)$; $u_r, \varphi_r \in \overset{\circ}{W}{}_2^{\varkappa_r}(\Omega)$; $r = 1, \dots, m$; for $\gamma_1, \gamma_2 > 0$ and every $\eta \in E_h$ is

i) $$\gamma_1 | \eta |^2 \leq \sum_{i,j=1}^{h} A_{ij} \eta_i \eta_j \leq \gamma_2 | \eta |^2.$$

This condition is weaker than the usually required condition of ellipticity which, in this case, has the form

ii) $$\sum_{i,j=1}^{h} A_{ij} \left(\sum_{r=1}^{m} \sum_{|\alpha| = \varkappa_r} a_{ir\alpha} \xi_r^\alpha \right) \left(\sum_{s=1}^{h} \sum_{|\beta| = \varkappa_s} a_{js\beta} \xi_s^\beta \right) \geq C \sum_{r=1}^{m} \sum_{|\alpha| = \varkappa_r} | \xi_r^\alpha |^2.$$

For example, for

$$N_1 u = \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}, \; N_2 u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}; \; A_{ij} = \partial_{ij}$$

is i) evidently satisfied,

$$N\xi = \begin{pmatrix} \xi_1, \xi_2 \\ -\xi_2, \xi_1 \end{pmatrix}$$

is regular for non-vanishing ξ. But ii) has the form

$$(\xi_2^1 - \xi_1^2)^2 + (\xi_1^1 + \xi_2^2)^2 \geq C \sum_{i,j=1}^{2} |\xi_j^i|^2$$

and such constant C does not exist.

§ 2 contains some remarks on existence of solution and continuous dependence on f and u_0 and a proof of the main theorem. A homotopy is used there between a linear equation with constant coefficients with well-known properties and the investigated non linear equation.

The proof is based on a priori estimate denoted as « property $\mathcal{A}$ » of the equation and having this form:

Let us suppose the solution u belongs to

$$\prod_{r=1}^{m} [C_{\varkappa_r}(\bar{\Omega}) \cap W_2^{\varkappa_r+1}(\Omega)], \quad f \in \prod_{i=1}^{h} W_p^1(\Omega).$$

Then u belongs to $\prod_{r=1}^{m} [W_p^{\varkappa_r+1}(\Omega)]$ and its norm is bounded by a constant which depends only on f, u_0.

Several cases of operators which possess the above property, are investigated in § 3.

The author is indebted to Professor J. Nečas for much valuable advice concerning the paper.

NOTATIONS. D^α denotes the partial derivative $\dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}}$ where $\alpha = (\alpha_1, .., \alpha_N)$; all α_i are integers, non-negative numbers, $|\alpha| = \sum_{i=1}^{N} \alpha_i$.

The functional spaces $D(\Omega)$, $\mathcal{E}(\bar{\Omega})$, $W_p^k(\Omega)$, $\overset{\circ}{W}{}_p^k(\Omega)$, $C_k(\bar{\Omega})$, $C_{k,\mu}(\bar{\Omega})$ (with k integer, nonnegative, $p \geq 1$, $0 \leq \mu \leq 1$) are denoted as usually (See for example [15]).

Let u be a vector-function $u = (u_1, \dots, u_m)$, $\varkappa = (\varkappa_1, \dots, \varkappa_m)$ with $\varkappa_i \geq 0$, integer for $i = 1, \dots, m$. Then $u \in W^{\varkappa}{}_p(\Omega)$ $[\overset{\circ}{W}{}_p^{\varkappa}(\Omega)$, $C_{\varkappa}(\bar{\Omega})$, $C_{\varkappa, \mu}(\bar{\Omega})]$ means that each $u_i \in W_p^{\varkappa_i}(\Omega)$ $[\overset{\circ}{W}{}_p^{\varkappa_i}(\Omega)$, $C_{\varkappa_i}(\bar{\Omega})$, $C_{\varkappa_i, \mu}(\bar{\Omega})]$ for $i = 1, \dots, m$.

$\varkappa + 1$ denotes the vector $(\varkappa_1 + 1, \dots, \varkappa_m + 1)$, i. e. $\varkappa + (1, \dots, 1) \cdot M = W_2^{\varkappa+1}(\Omega) \cap C_{\varkappa}(\bar{\Omega})$.

1. Properties of the operators N_i.

We shall be concerned with linear differential equation which may be written as follows:

$$A(u, \varphi) = \int_{\Omega} \sum_{i=1}^{h} N_i u \, N_i \varphi = \int_{\Omega} \sum_{r=1}^{m} \sum_{|\alpha| = \varkappa_r} f_{r\alpha} D^{\alpha} \varphi_r \, . \tag{1.2}$$

N_i consist only of their principal parts N_i', i. e.

$$N_i u = \sum_{r=1}^{m} \sum_{|\alpha| = \varkappa_r} a_{ir\alpha} D^{\alpha} u_r \tag{1.3}$$

and satisfy condition (0), φ, $u \in \overset{\circ}{W}{}_2^{\varkappa}(\Omega)$, $f_{r\alpha} \in L_2(\Omega)$.

LEMMA 1.1 : The linear differential operator given by (1.2) is uniformly elliptic and strongly elliptic.

PROOF : (1, 2) can be written in the form

$$A(u, \varphi) = \int_{\Omega} \sum_{s=1}^{m} \varphi_s \sum_{r=1}^{m} l_{rs}(u_r) = \int_{\Omega} \sum_{s=1}^{m} \varphi_s \sum_{|\alpha| = \varkappa_s} D^{\alpha} f_{s\alpha}$$

where u, $\varphi \in [D(\Omega)]^m$ and $f_{s\alpha} \in W_2^{\varkappa_s}(\Omega)$,

$$l_{rs}(u_r) = \sum_{\substack{|\alpha| = \varkappa_r \\ |\beta| = \varkappa_s}} \left(\sum_{i=1}^{h} a_{ir\alpha} a_{is\beta} \right) D^{\alpha+\beta} u_r \, .$$

Let us denote for $\xi \in E_N$:

$$l_{rs}(\xi) = \sum_{\substack{|\alpha| = \varkappa_r \\ |\beta| = \varkappa_s}} \left(\sum_{i=1}^{h} a_{ir\alpha} a_{is\beta} \right) \xi^{\alpha+\beta} \, .$$

Now we shall prove the uniform ellipticity, i. e.

$$\det | l_{rs}(\xi) | \geq C | \xi |^{2\sum_{i=1}^{m} \varkappa_i} \tag{1.4}$$

and strong ellipticity, i. e.

$$\sum_{i,j=1}^{m} l_{ij}(\xi)\, \eta_i \eta_j \geq C \sum_{i=1}^{m} | \eta_i |^2 \cdot | \xi |^{2\varkappa_i} \quad \text{for a positive } C. \tag{1.5}$$

Let us denote $N^*\xi$ the adjoint matrix to $N\xi$. Then $\det | l_{rs}(\xi) | = \det | N^*\xi \cdot N\xi |$ and so it is the Gramm's determinant of column vectors of $N^*\xi$. Therefore it is equal to zero only in the case of linear dependence of column vectors of $N^*\xi$, i. e. for $\xi = (0, \dots, 0)$ and it is positive for non vanishing ξ.

The quadratic form in (1.5) is positively defined if and only if all the main subdeterminants of its coefficients are positive (according to Silvestr's theorem). But they have the same form as $\det | l_{rs}(\xi) |$.

Let us suppose that for every n there exists a real vector $\xi^n \in E_N$; $\xi^n \neq (0, \dots, 0)$ such that

$$\det | l_{rs}(\xi^n) | < \frac{1}{n} | \xi^n |^{2\sum_{i=1}^{m} \varkappa_i}.$$

Let us consider the sequence $\{\eta^n\}_{n=1}^{\infty}$; $\eta^n = \frac{\xi^n}{|\xi^n|}$. We may choose a convergent subsequence (let us denote also η^n) such that

1) $| \eta^n | = 1$; $n = 1, 2, \dots$

2) $\eta^n \longrightarrow \eta$ for $n \longrightarrow \infty$

3) $0 < \det | l_{rs}(\eta^n) | < \frac{1}{n}$.

Then $\det | l_{rs}(\eta) | = 0$ for non vanishing vector η and that is a contradiction with condition (0). In the same way (1.5) may be proved.

The equation (1.2) has a solution $u \in \overset{\circ}{W}{}_2^{\varkappa}$ for $f_{r\alpha} \in L_2$. Using the estimates of Agmon, Douglis, Nirenberg (see [1]) and continuous dependence on the right part, we see that $u \in \overset{\circ}{W}{}_p^{\varkappa}$ for $f_{r\alpha} \in L_p$ and there exists $C > 0$ so that

$$\| u \|_{W_p^{\varkappa}} \leq C \cdot \sum_{r=1}^{m} \sum_{|\alpha| = \varkappa_r} \| f_{r\alpha} \|_{L_p}.$$

The functions $g_i = N_i u$ satisfy the equation

$$\int_\Omega \sum_{i=1}^h g_i N_i \varphi = \int_\Omega \sum_{r=1}^m \sum_{|\alpha| = \varkappa_r} f_{r\alpha} D^\alpha \varphi_r \tag{1.6}$$

for every $\varphi \in [D(\Omega)]^m$ and there exists $C > 0$ so that

$$\| g_i \|_{L_p} \leq C \cdot \sum_{r=1}^m \sum_{|\alpha| = \varkappa_r} \| f_{r\alpha} \|_{L_p} .$$

That means that the right part of any equation may be written in the form $\int_\Omega \sum_{i=1}^h g_i \cdot N_i \varphi$ and the L_p-norms of f and g are equivalent.

Next, let us write (1.2) in the form

$$\int_\Omega \sum_{i=1}^h N_i u \, N_i \varphi = \int_\Omega \sum_{i=1}^h g_i N_i \varphi \tag{1.7}$$

and let us interest in the dependence of the estimates of u on p.

LEMMA 1.2 : Let $u \in \overset{\circ}{W}{}_2^\varkappa(\Omega)$ be a solution of (1.7), $2 \leq p \leq 2 + \varrho$. Then there exists a positive constant $C_1(\varrho)$ such that

$$\left(\sum_{i=1}^h \| N_i u \|_{L_p}^p \right)^{1/p} \leq C_1(\varrho)^{1 - \frac{2}{p}} \cdot \left(\sum_{i=1}^h \| g_i \|_{L_p}^p \right)^{1/p} . \tag{1.8}$$

PROOF : According to the foregoing remarks

$$\left(\sum_{i=1}^h \| N_i u \|_{L_{2+\varrho}}^{2+\varrho} \right)^{1/2+\varrho} \leq C_1(\varrho) \cdot \left(\sum_{i=1}^h \| g_i \|_{L_{2+\varrho}}^{2+\varrho} \right)^{1/2+\varrho} .$$

From (1.7) we obtain immediately

$$\left(\sum_{i=1}^h \| N_i u \|_{L_2}^2 \right)^{1/2} \leq \left(\sum_{i=1}^h \| g_i \|_{L_2}^2 \right)^{1/2} .$$

The result follows according to the interpolation theorem of Riesz-Thorin (see [22]).

LEMMA 1.3 : Let $1 < p < \infty$; N_i satisfy condition (0). Then

$$\left(\sum_{i=1}^h \| N_i u \|_{L_p(\Omega)}^p \right)^{1/p} \quad \text{is an equivalent norm in} \quad \overset{\circ}{W}{}_p^\varkappa(\Omega).$$

PROOF: For $p = 2$ the result is an immediate consequence of Lemma 1.1. For $p \neq 2$ we may use the method of J. Nečas (see [14]), consisting of applying the Lizorkin's theorem on multiplicators (see [10]) to this special case. $F(f)$ denotes the Fourier transformation of the function $f \in L_p$ (in the sense of distributions).

THEOREM 1 (Lizorkin): Let $\Phi(\xi)$ be a function defined and continuous with all its derivatives $D^\alpha \Phi$ $(\alpha = (\alpha_1, \dots, \alpha_N),\ \alpha_i = 0$ or $1)$ for every $\xi = (\xi_1, \dots, \xi_N)$; $\xi_j \neq 0$ for $j = 1, \dots, N$.

Let all such derivatives satisfy condition

$$(1.9) \qquad |\xi^\alpha D^\alpha \Phi(\xi)| \leq M < \infty \quad \text{on} \quad \{\xi;\ \xi_j \neq 0,\ j = 1, \dots, N\}.$$

Then $Tf = F^{-1}(\Phi(\xi) \cdot F(f)(\xi))$ is a linear and bounded mapping from $L_p(E_N)$ into $L_p(E_N)$ for $1 < p < \infty$.

We shall use the equivalent norm in $\overset{\circ}{W}{}_p^{\varkappa}(\Omega)$;

$$\|u\|_{\overset{\circ}{W}{}_p^{\varkappa}(\Omega)} = \left(\sum_{r=1}^{m} \sum_{|\alpha| = \varkappa_r} \|D^\alpha u_r\|_{L_p}^p\right)^{1/p}.$$

We obtain immediately

$$(1.10) \qquad \left(\sum_{i=1}^{h} \|N_i u\|_{L_p}^p\right)^{1/p} \leq C \|u\|_{\overset{\circ}{W}{}_p^{\varkappa}}.$$

On the contrary, we may suppose $u \in [D(\Omega)]^m$. Using Fourier transform we may write

$$f_i(\xi) = F(N_i u)(\xi) = \sum_{r=1}^{m} N_{ir}\, \xi\, (-i)^{\varkappa_r} F(u_r)(\xi).$$

Let us denote $f = (f_1, \dots, f_h)$, $\varphi = (\varphi_1, \dots, \varphi_m)$ with $\varphi_j = (-i)^{\varkappa_j} F(u_j)$. Then $f = N \cdot \varphi$. Moreover, let $\{\Delta_j\}$ be set of all the determinants $(m \times m)$ of $N\xi$.

For arbitrary $\xi \neq (0, \dots, 0)$ there exists (at least one) $\Delta_j(\xi) \neq 0$. Let $\{\Delta_{jre}\}$ be its subdeterminants of the orders $(m-1) \times (m-1)$. (We may define $\Delta_{jre} = 0$ if N_{re} does not belong to Δ_j). Let us write (from Cramer's rule)

$$\varphi_r = \frac{\sum_{e=1}^{h} \Delta_{jre} f_e}{\Delta_j}$$

and also

$$\sum_{j=1}^{h} \Delta_j^2 \cdot \varphi_r = \sum_{j=1}^{h} \sum_{e=1}^{h} \Delta_j \cdot \Delta_{jre} f_e.$$

But $\sum_{j=1}^{h} \Delta_j^2 \neq 0$ for every $\xi \neq (0, \dots, 0)$, therefore

$$\varphi_r = \frac{\sum_{j=1}^{h} \sum_{e=1}^{h} \Delta_j \Delta_{jre} f_e}{\sum_{j=1}^{h} \Delta_j^2}. \tag{1.11}$$

It remains to prove the estimate (1.5) for

$$\Phi(\xi) = \frac{\Delta_j(\xi) \cdot \Delta_{jre}(\xi) \cdot \xi^\alpha}{\sum_{j=1}^{h} \Delta_j^2(\xi)}, \quad \text{where} \quad |\alpha| = \varkappa_r. \tag{1.12}$$

In the same way as in the proof of Lemma (1.1) it may be shown that

$$\sum_{j=1}^{h} \Delta_j^2(\xi) \geq C \cdot |\xi|^{2 \sum_{i=1}^{m} \varkappa_i}.$$

Thus the assumptions of Theorem 1 are satisfied and hence

$$\left(\sum_{i=1}^{m} \sum_{|\alpha| = \varkappa_i} \| D^\alpha u_i \|_{L_p}^p\right)^{1/p} \leq C \cdot \left(\sum_{i=1}^{h} \| N_i u \|_{L_p}^p\right)^{1/p}. \tag{1.13}$$

Let us now consider the equation

$$\int_\Omega \sum_{i,j=1}^{h} A_{ij}(x) N_i u(x) N_j \varphi(x) \, dx = \int_\Omega \sum_{j=1}^{h} g_j(x) N_j \varphi(x) \, dx, \tag{1.14}$$

where $\varphi, u \in \overset{\circ}{W}{}_2^{\varkappa}(\Omega)$, $g_j \in L_p(\Omega)$ for $j = 1, \dots, h$, $p \geq 2$ and $A_{ij} \in L_\infty(\Omega)$ for $i, j = 1, \dots, h$. Let us suppose

$$A_{ij} = A_{ji}; \quad i, j = 1, \dots, h \tag{1.15}$$

and

$$\gamma_1 |\eta|^2 \leq \sum_{i,j=1}^{h} A_{ij} \eta_i \eta_j \leq \gamma_2 |\eta|^2 \quad \text{for some } \gamma_1, \gamma_2 > 0$$

and every $\eta \in E_h$.

LEMMA 1.4: Let $u \in \overset{\circ}{W}{}_2^{\varkappa}$ be a solution of (1.14) with A_{ij} satisfying (1.15), $2 \leq p \leq 2 + \varrho$. Then there exist two positive constants $\gamma_3(\varrho) > 1$

and $\gamma_4(\varrho) > 1$ such that

$$\| u \|_{\overset{\circ}{W}{}_p^{\varkappa}} \leq \gamma_5 \cdot \gamma_4(\varrho)^{1-\frac{2}{p}} \cdot \left(\sum_{i=1}^{h} \| g_i \|_{L_p}^{p} \right)^{1/p} \tag{1.16}$$

for p satisfying

$$p \leq 2 \left(1 - \log \left[\frac{1 - \frac{1}{2} \cdot \frac{\gamma_1}{\gamma_2}}{1 - \frac{\gamma_1}{\gamma_2}} \right] / \log \gamma_3 \right)^{-1}. \tag{1.17}$$

PROOF: It is sufficient to prove (1.16) for $A_{ij} \in \mathcal{E}(\overline{\Omega})$ $g_i \in \mathcal{E}(\overline{\Omega})$ $(i, j = = 1, \dots, h)$ as Lemma 1.4 is the easy corollary of continuous dependence on A_{ij}, g_i.

Such a solution $u \in \overset{\circ}{W}{}_2^{\varkappa}$ is also a solution of

$$\int_{\Omega} \sum_{i=1}^{h} N_i u N_i \varphi = \int_{\Omega} \sum_{i,j=1}^{h} \left(\delta_{ij} - \frac{1}{\gamma_2} A_{ij} \right) N_j u N_i \varphi + \frac{1}{\gamma_2} \sum_{i=1}^{h} g_i \cdot N_i \varphi. \tag{1.18}$$

We shall estimate the L_p-norm of the first term on the right.

$$\left(\sum_{i=1}^{h} \left\| \sum_{j=1}^{h} \left(\delta_{ij} - \frac{1}{\gamma_2} A_{ij} \right) N_j u \right\|_{L_p}^{p} \right)^{\frac{1}{p}} \leq \sup_{\left(\sum_{j=1}^{h} \| \psi_j \|_{L_{p'}}^{p'} \right)^{\frac{1}{p'}} = 1} \int_{\Omega} \sum_{i,j=1}^{h} \left(\delta_{ij} - \frac{1}{\gamma_2} A_{ij} \right) N_j u \cdot \psi_i \leq$$

$$\leq \sup_{\left(\sum_{j=1}^{h} \| \psi_j \|_{L_{p'}}^{p'} \right)^{1/p'} = 1} \left(1 - \frac{\gamma_1}{\gamma_2} \right) \cdot \int_{\Omega} \left(\sum_{j=1}^{h} | N_j u |^2 \right)^{1/2} \cdot \left(\sum_{j=1}^{h} | \psi_j |^2 \right)^{1/2} \leq$$

$$\leq \left(1 - \frac{\gamma_1}{\gamma_2} \right) \cdot C^{1-\frac{2}{p}} \cdot \left(\sum_{j=1}^{h} \| N_j u \|_{L_p}^{p} \right)^{1/p}.$$

Now, according to Lemma 1.2 we obtain

$$\left(\sum_{i=1}^{h} \| N_i u \|_{L_p}^{p} \right)^{1/p} \leq \tag{1.19}$$

$$\leq (C \cdot C_1(\varrho))^{1-\frac{2}{p}} \left(1 - \frac{\gamma_1}{\gamma_2} \right) \left(\sum_{i=1}^{h} \| N_i u \|_{L_p}^{p} \right)^{1/p} + C_1^{1-\frac{2}{p}}(\varrho) \cdot \frac{1}{\gamma_2} \left(\sum_{i=1}^{h} \| g_i \|_{L_p}^{p} \right)^{1/p}.$$

Let us denote $\gamma_3(\varrho) = C \cdot C_1(\varrho)$, $\gamma_4(\varrho) = C_1(\varrho)$.

The p satisfying (1.17) satisfies also

$$\gamma_3^{1-\frac{2}{p}}\left(1-\frac{\gamma_1}{\gamma_2}\right) \leq 1-\frac{1}{2}\frac{\gamma_1}{\gamma_2}. \tag{1.20}$$

Therefore it follows from (1.19)

$$\left(\sum_{i=1}^{h} \| N_i u \|_{L_p}^{p}\right)^{1/p} \leq \gamma_4^{1-\frac{2}{p}} \frac{2}{\gamma_1} \left(\sum_{i=1}^{h} \| g_i \|_{L_p}^{p}\right)^{1/p}.$$

The equivalence of the norms in $\overset{\circ}{W}{}_p^{\varkappa}(\Omega)$ (Lemma 1.3) implies the result.

§ 2.

The existence and unicity of the solution of (1.1) follows immediately from a special case of the Lerray-Lions Theorem:

THEOREM 2. Let V be a reflexive Banach-space, $A(v)$ a bounded operator from V to V' which is weakly continuous from all finitely dimensional subspaces of V to V'. (Let us denote (F, φ) the value of the functional F in the point φ). Let the following assumptions be satisfied:

1) $$\lim_{\|\varphi\| \to \infty} \frac{(A(\varphi), \varphi)}{\|\varphi\|} = \infty,$$

2) $A(\varphi)$ is strictly monotone, i.e.

$$(A(\varphi) - A(\psi), \varphi - \psi) > 0 \quad \text{for} \quad \varphi \neq \psi; \ \varphi, \psi \in V.$$

Then A is a one-to-one mapping on V' and A^{-1} is a bounded mapping from V' to V.

In our notation, there is $V = \overset{\circ}{W}{}_k^{\varkappa}(\Omega)$ and

$$(A(\psi), \varphi) = \int_{\Omega} \sum_{i=1}^{h} F_i, (x, \{N_j(\psi + u_0)(x)\}_{j=1}^{h}) \, N_i \varphi(x) \, dx.$$

The boundedness and continuity is proved in [2], [20], [21] in the theorem of Nemyckij's operators. Conditions 1,2 follow immediately from assumptions A, B (see [21], [16]). Moreover, we obtain 2) in the form

$$(A(\varphi) - A(\psi), \varphi - \psi) \geq C \cdot \| \varphi - \psi \|_{\overset{\circ}{W}{}_k^{\varkappa}}^{k}$$

in case A and therefore A^{-1} is a continuous mapping; the solution depends continuously on the right part and the boundary condition.

In the case B there is A^{-1} only demi continuous, i.e. it is continuous from the strong topology in $(\overset{\circ}{W}{}_k^{\varkappa})'$ into the weak topology in $\overset{\circ}{W}{}_k^{\varkappa}$ (see [2], [15], [16]).

Let us denote $\mathcal{B}$ a bounded mapping from $W = \{u \in W_k^{\varkappa},\ u - u_0 \in \overset{\circ}{W}{}_k^{\varkappa}\}$ into $(\overset{\circ}{W}{}_k^{\varkappa})'$ such that $\mathcal{B}(u) = A(u - u_0)$ and consider the following equation

$$(\mathcal{B}(u), \varphi) = \int_\Omega \sum_{i=1}^{h} f_i \cdot N_i \varphi,$$

where $f_i \in L_{k'}$, $\varphi \in \overset{\circ}{W}{}_k^{\varkappa}$, $u \in W$.

Let us say $\mathcal{B}$ has the property $\mathcal{A}$ if and only if $f = (f_1, \dots, f_h) \in [W_p^1]^h$, $u = \mathcal{B}^{-1}(f) \in M$ implies $u \in W_p^{\varkappa+1}$ and

$$\| u \|_{W_p^{\varkappa+1}} \leq C(\| f \|_{[W_p^1]^h}),$$

where C is bounded uniformely for $k \in \langle 2, k_0 \rangle$.

Let us denote

$$F_i(\xi, s) = \xi_i \cdot \theta_\xi^{s-2}; \qquad i = 1, \dots, h,\ 2 \leq s \leq k$$

and

$$F_i(x, \xi, t) = t \cdot F_i(\xi, k) + (1 - t) \cdot F_i(x, \xi); \qquad i = 1, \dots, h,\ 0 \leq t \leq 1.$$

Let us define

$$(\mathcal{B}_s(u), \varphi) = \int_\Omega \sum_{i=1}^{h} F_i(\{N_j u\}_{j=1}^h, s) \cdot N_i \varphi$$

and

$$(\mathcal{B}_t(u), \varphi) = \int_\Omega \sum_{i=1}^{h} F_i(x, \{N_j u\}_{j=1}^h, t) \cdot N_i \varphi$$

analogously to $\mathcal{B}$.

THEOREM 3 (ON REGULARITY): Let $\mathcal{B}$ satisfy A or B and let $\mathcal{B}, \mathcal{B}_s$, $\mathcal{B}_t$ have property $\mathcal{A}$.

Then there exists $\mathcal{B}^{-1}$ and it is a bounded mapping from $[W_p^1(\Omega)]^h$ into $W_p^{\varkappa+1}(\Omega)$.

Using Sobolev embedding theorem, it follows immediately:

COROLLARY: Let u be a solution of (1.1), where $\mathcal{B}$ satisfies A or B and $\mathcal{B}, \mathcal{B}_s, \mathcal{B}_t$ have property $\mathcal{A}$. Then $u \in C_{\varkappa, \mu}(\bar{\Omega})$ with $\mu = 1 - \dfrac{2}{p}$ and $\| u \|_{C_{\varkappa, \mu}} \leq C(\| f \|_{[W_p^1]^h})$.

PROOF: $\mathcal{B}_s$ satisfy conditions (0) — (4) or (0) — (4″) with s instead k. Let us denote $\mathcal{P}$ the subset of $s \in \langle 2, k \rangle$ such that for $u \in \mathcal{B}_s^{-1}(f)$

$$\| u \|_{W_p^{\varkappa+1}} \leq C(\| f \|_{[W_p^1]^h}).$$

holds with C independent on s. $\mathcal{P} \neq \emptyset$ for $2 \in \mathcal{P}$. (see results of Agmon, Douglis, Nirenberg [1]). $\mathcal{P}$ is closed:

Let $s_n \in \mathcal{P}$ converge to s; $\mathcal{B}_{s_n}(u_{s_n}) = f$ then

$$\| u_{s_n} \|_{W_p^{\varkappa+1}} \leq C(\| f \|_{[W_p^1]^h}) \tag{2.1}$$

and there exists a subsequence (let us denote it also u_{sn}) such that $u_{s_n} \to u_s$ in $W_p^\varkappa$. But such u_s belongs to $W_p^{\varkappa+1}$ (use (2.1)), therefore it solves $\mathcal{B}_s(u_s) = f$ and according to $\mathcal{A}$

$$\| u_s \|_{W_p^{\varkappa+1}} \leq C(\| f \|_{[W_p^1]^h})$$

$\mathcal{P}$ is open: Let $s_0 \in \mathcal{P}$; $\mathcal{B}^{-1}$ be an inverse operator to $\mathcal{B}_{s_0}$ and

$$C_s(u) = \mathcal{B}_{s_0}(u) - \mathcal{B}_s(u) + \mathcal{B}_{s_0}(u_{s_0})$$

then $\mathcal{B}^{-1} \cdot C_s$ is defined on

$$\mathcal{V} = \{ u \in W_p^{\varkappa+1} ; \| u - u_{s_0} \|_{W_p^{\varkappa+1}} \leq 1 ; u - u_0 \in \mathring{W}_2^\varkappa \},$$

is weakly continuous on $\mathcal{V}$ (see remarks before Theorem 3) and $\mathcal{B}^{-1}(C_s(\mathcal{V})) \subset \subset \mathcal{V}$ for sufficiently small $s - s_0$. According to Schauder's fixed point theorem (see [19]) there exists $u \in \mathcal{V}$, $u = \mathcal{B}^{-1} C_s(u)$. Then $\mathcal{B}_s(u) = \mathcal{B}_{s_0}(u_{s_0})$ and according to $\mathcal{A}$ $\| u \|_{W_p^{\varkappa+1}} \leq C$. We may conclude $\mathcal{P} = \langle 2, k \rangle$. All the proceedings may be repeated for $\mathcal{B}_t$, which completes the proof.

Operators which satisfy $\mathcal{A}$.

THEOREM 4: Let N_i be all the highest derivatives, i. e. $N_i = N_{r\alpha} = D^\alpha u_r$ for $r = 1, \dots, m$, $|\alpha| = \varkappa_r$, let $\mathcal{B}$ satisfy A. Then $\mathcal{B}$ has property $\mathcal{A}$.

The proof is based on the following two estimates

$$\| \theta_u^{k-1} \|_{W_2^1(\Omega)} \leq C(f) \cdot \| \theta u \|_{C(\bar{\Omega})}^{\frac{k}{2}-1}$$

and

$$\| \theta_u^{k-1} \|_{W_p^1(\Omega)} \leq C(f) \cdot \| \theta u \|_{C(\bar{\Omega})}^{\frac{3}{2}(k-2)}, \quad p > 2.$$

They may be obtained in this way:

All first derivatives of u solve a linear equation with measurable coefficients. In the interior of Ω or in the directions « parallel with boundary $\partial\Omega$ « it is sufficient to use the theorems about linear equations (see § 1) or to choose suitable test-functions. In the normal directions, more precise theorems about dual norms must be used.

To this purpose, the following description of boundary will be considered: [see [15]] a neighborhood of every point of $\partial\Omega$ is described by an infinitely differentiable function a which is defined on the cube

$$K_r = \{x' ; | x' | < r\} ; x' = (x_1, \dots, x_{N-1}) ; a(x') = x_N$$

and $\sum_{i=1}^{N-1} \left| \frac{\partial a}{\partial x_i}(0) \right| = 0$ in a corresponding coordinate system. The boundary is covered by a finite number P of such systems.

Let us suppose

$$V_r^i = \{x ; | x' | < r ; a^i(x') < x_N < a^i(x') + r\} \subset \Omega$$

$$U_r^i = \{x ; | x' | < r ; a^i(x') - r < x_N < a^i(x')\} \; U_r^i \cap \Omega = \emptyset$$

for every sufficiently small r and $i = 1, \dots, P$. Let us denote V_r^0 the domain with infinitely smooth boundary

$$\bar{V}_r^0 \subset \Omega ; \qquad \bigcup_{i=0}^{P} V_r^i \supset \Omega \qquad \text{for every } r > 0.$$

In [13], the existence of the functions γ_r^i ; $i = 0, \dots, P$ $\gamma_r^i \in \mathcal{E}(V_r^i) \cap \cap C(\bar{V}_r^i)$ is proved, having the following properties:

1) $\gamma_r^i = 0$ on $\Omega - V_r^i$,

2) γ_r^0 is equivalent to $\sigma_0(x) = \operatorname{dist}(x, \partial V_r^0)$,

γ_r^i is equivalent to $\sigma_i(x) = \operatorname{dist}(x, \partial(\Omega - V_r^i))$; $i = 1, \dots, P$,

3) $| D^\alpha \gamma_r^i | \leq C \cdot | \gamma_r^i |^{1-|\alpha|}$; $i = 0, 1, \dots, P$.

In the next lemmas the right part f is supposed in $[W_p^1(\Omega)]^h$,

$$K \text{ denotes } \max_{i=1,\dots,m} x\, x_i .$$

LEMMA 3.1 : Let $u = \mathcal{B}^{-1}(f) \in M$. Then

$$\int_\Omega \theta_u^k \leq C(f), \tag{3.1}$$

$$\int_\Omega \gamma_0^{2K} \theta_u^{k-2} \sum_{i=1}^{h} \sum_{l=1}^{N} \left| N_i\left(\frac{\partial u}{\partial x_l}\right)\right|^2 \leq C(f). \tag{3.2}$$

The constant C depends on $\|f\|_{[W_p^1]^h}$ and does not depend on $k \in \in \langle 2, k_0 \rangle$.

PROOF :

$$1) \quad \int_\Omega \theta_u^k \leq C \cdot \left\{ 1 + \|u_0\|_{W_k^\varkappa}^k + \int_\Omega \sum_{i=1}^{h} |N_i(u-u_0)|^k \right\} \leq$$

$$\leq C \cdot \left\{ 1 + \|u_0\|_{W_k^\varkappa}^k + \int_\Omega \sum_{i=1}^{h} N_i(u-u_0) \cdot [F_i(x, \{N_j u\}) - \right.$$

$$\left. - F_i(x, \{N_j u_0\})] \right\}.$$

The last term on the left is equal to

$$\int_\Omega \sum_{i=1}^{h} N_i(u-u_0) \cdot (f_i - F_i(x, \{N_j u_0\}) \leq$$

$$\leq C \cdot \{[\|f\|_{[L_2(\Omega)]^h} + \|u_0\|_{W_k^\varkappa}^{k-1}] \cdot [\|u\|_{W_k^\varkappa} + \|u_0\|_{W_k^\varkappa}]\}.$$

Then

$$\int_\Omega \theta_u^k \leq C_1 + C_2 \cdot \left(\int_\Omega \theta_u^k \right)^{1/k}$$

and

$$\int_\Omega \theta_u^k \leq C.$$

2) Let us take $\varphi = \dfrac{\partial \psi}{\partial x_l} \in [D(V^0)]^m$ for $\psi \in [D(V^0)]^m$. Integrating in parts (1.1), we find

$$\int_\Omega \sum_{i=1}^{h} N_i \psi \cdot \left\{ \frac{\partial F_i}{\partial x_l} + \sum_{j=1}^{h} F_{ij} \cdot N_j \left(\frac{\partial u}{\partial x_l}\right)\right\} = \int_\Omega \sum_{i=1}^{h} \frac{\partial f_i}{\partial x_l} \cdot N_i \psi.$$

According to the assumptions on F_i and N_i, f_i; this equation is satisfied for all $\psi \in \overset{\circ}{W}{}_2^{\varkappa}$, thereby also for $\psi = \gamma_0^{2K} \cdot \frac{\partial u}{\partial x_l}$

$$\int\limits_{\Omega} \gamma_0^{2K} \sum_{i,j=1}^{h} F_{ij} \cdot N_i \frac{\partial u}{\partial x_l} \cdot N_j \frac{\partial u}{\partial x_l} = \int\limits_{\Omega} \sum_{i=1}^{h} N_i \left(\gamma_0^{2K} \frac{\partial u}{\partial x_l}\right) \left[\frac{\partial f_i}{\partial x_l} - \frac{\partial F_i}{\partial x_l}\right] + R, \tag{3.4}$$

where R consists of the terms $a_{i\alpha} F_{ij} \cdot D^\alpha u_k \, D^\beta u_l \gamma_0^K$ with smooth $a_{\alpha i}$; $|\alpha| = \varkappa_k + 1$; $|\beta| \leq \varkappa_l$.

Let us denote

$$j = \int\limits_{\Omega} \gamma_0^{2K} \cdot \theta^{k-2} u \sum_{i=1}^{h} N_i^2 \left(\frac{\partial u}{\partial x_l}\right).$$

From the ellipticity

$$\gamma_1 \cdot j \leq \int\limits_{\Omega} \gamma_0^{2K} \sum_{i,j=1}^{h} F_{ij} \cdot N_i \frac{\partial u}{\partial x_l} \cdot N_j \frac{\partial u}{\partial x_l}.$$

Let us estimate the right part of (3.4).

Now,

i) $$\left| \int\limits_{\Omega} N_i \left(\gamma_0^{2K} \frac{\partial u}{\partial x_l}\right) \cdot \frac{\partial f_i}{\partial x_l} \right| \leq C \cdot \| f \|_{[W_2^1]^h} \cdot \left(\int\limits_{\Omega} \sum_{\alpha, r} \left| D^\alpha \left(\gamma_0^{2K} \frac{\partial u_r}{\partial x_l}\right) \right|^2 \right)^{1/2} \leq$$

$$\leq C \| f \|_{[W_2^1]^h} \cdot \left\{ \| u \|_{W_2^{\varkappa}} + \left(\int\limits_{\Omega} \gamma_0^{2K} \sum_{\alpha, r} \left| D^\alpha \left(\frac{\partial u_r}{\partial x_l}\right) \right|^2 \right) \right\}^{1/2} \leq$$

$$\leq C_1 j^{1/2} + C_2 .$$

ii) $$\left| \int\limits_{\Omega} N_i \left(\gamma_0^{2K} \frac{\partial u}{\partial x_l}\right) \cdot \frac{\partial F_i}{\partial x_l} \right| \leq C \cdot \left\{ \int\limits_{\Omega} \theta^{k-1} u \cdot \left(\gamma_0^{2K} \left| N_i \left(\frac{\partial u}{\partial x_l}\right) \right| \right) + \theta^k u \right\} \leq$$

$$\leq C_1 \cdot j^{1/2} \cdot \left(\int\limits_{\Omega} \theta^k u \right)^{1/2} + C_2 .$$

iii) $$\left| \int\limits_{\Omega} \gamma_0^K a_{i\alpha} \cdot F_{ij} \, D^\alpha u_k \cdot D^\beta u_l \right| \leq C \cdot \int\limits_{\Omega} \theta^{k-2} u \cdot \gamma_0^K \cdot | D^\alpha u_k | \cdot | D^\beta u_l | \leq$$

$$\leq C \cdot j^{1/2} \cdot \| u \|_{W_k^{\varkappa}}^{k/2} .$$

Therefore, from i), ii) and iii) it follows

$$j \leq C_1 j^{1/2} + C_2$$

which implies the boundedness of j.

Let us define on V_r^i the derivatives « parallel with the boundary », i. e. the derivatives in the plane orthogonal to the direction

$$\left(-\frac{\partial a^i}{\partial x_1}, -\frac{\partial a^i}{\partial x_2}, \ldots, -\frac{\partial a^i}{\partial x_{N-1}}, 1\right).$$

The index of the coordinate system will be omitted.

If $\partial^l = \frac{\partial}{\partial x_l} + \frac{\partial a}{\partial x_l} \cdot \frac{\partial}{\partial x_N}$ denotes such a derivative for $l = 1, .., N-1$, then the following conclusion is true:

$$u \in \overset{\circ}{W}{}_k^{\varkappa} \cap W_k^{\varkappa+1} \Longrightarrow \partial^l u \in \overset{\circ}{W}{}_k^{\varkappa}. \tag{3.5}$$

It allows us to prove the analogue of the foregoing lemma for ∂^l. Let us denote

$$H(u) = \sum_{i=1}^{h} N_i^2 (\partial^l u).$$

LEMMA 3.2 : Let $u = \mathcal{B}^{-1}(f) \in M$, then

$$\int_\Omega \gamma_r^{2K} \theta^{k-2u} H(u) \leq C(f). \tag{3.6}$$

PROOF : Let us take a test-function φ in (1.1) in the form $\varphi = \partial^l \psi$; $\psi \in [D(V_r^i)]^m$. Then

$$\int_\Omega \sum_{i=1}^{h} F_i(x, \{N_j(u)\}) N_i \partial^l \psi = \int_\Omega \sum_{i=1}^{h} f_i \cdot N_i \partial^l \psi$$

and

$$\int_\Omega \sum_{i=1}^{h} F_i \cdot \partial^l (N_i \psi) = \int_\Omega \sum_{i=1}^{h} f_i \partial^l (N_i \psi) + (f_i - F_i) \cdot R_1,$$

where R_1 involves the derivatives of ψ_i up to the order $\varkappa_i$. After integrating in parts, there holds

$$\int_\Omega \sum_{i,j=1}^{h} F_{ij}\cdot N_j\partial^l u\cdot N_i\psi = \int_\Omega \sum_{i=1}^{h} \partial^l f_i N_i\psi + (f_i - F_i)\,R_1 + N_i(\psi)\cdot R_2\,,$$

where R_2 involves the derivatives of u_i up to the order $\varkappa_i$.

According to remark (3.5) we may take

$$\psi = \gamma_r^{2K}\cdot\partial^l(u-u_0)\in \overset{\circ}{W}{}_2^{\varkappa}$$

and conclude

$$j = \int_\Omega \gamma_r^{2K}\,\theta^{k-2}u\cdot H(u) \leq \frac{1}{\gamma_1}\cdot\int_\Omega \gamma_r^{2K}\sum_{i,j=1}^{h} F_{ij}\cdot N_i\partial^l u\cdot N_j\partial^l u =$$

$$= \frac{1}{\gamma_1}\left\{\int_\Omega \sum_{i=1}^{h} N_i(\gamma_r^{2K}\partial^l(u-u_0))\,(\partial^l f_i + R_2) + R_1(f_i - F_i) + \right.$$

$$\left. + \sum_{j=1}^{h} F_{ij}N_i\partial^l u\,[R_3 - N_j(\gamma_r^{2K}\cdot\partial^l u_0)]\right\},$$

where R_3 involves the derivatives of u_i up to the order $\varkappa_i$. The right part may be estimated as in Lemma 3.1 and it implies $j\leq C(f)$.

LEMMA 3.3: Let $u = \mathcal{B}^{-1}(f)\in M$ then

$$I = \int_\Omega \gamma_s^{2K}\cdot\theta^{2(k-2)}u\cdot\sum_{i=1}^{h} N_i^2\left(\frac{\partial u}{\partial x_N}\right) \leq C(s)\cdot V^{k-2} \tag{3.7}$$

for sufficiently small s; $V = \left\| 1 + \sum_{i=1}^{h} |N_i u| \right\|_{C(\bar\Omega)}$.

PROOF: Let us denote $\alpha_r = (0, 0, \ldots, \varkappa_r)$, $N_{\beta_r}u = D^{\alpha_r}u_r$. We shall estimate the L_2-norm of the functions

$$g_r = \frac{\partial}{\partial x_N}\cdot\{\gamma_s^{2K}\cdot F_{\beta_r}(x, \{N_j u(x)\})\}, \tag{3.8}$$

using the following theorem (see [14], [15]).

Let $f\in W_p^l(\Omega)$; l entire, v entire, non-negative. Then

$$\|f\|_{W_p^l(\Omega)} \leq C\{\sum_{|\alpha|=v}\|D^\alpha f\|_{W_p^{l-v}(\Omega)} + \|f\|_{W_p^{l-v}(\Omega)}\}. \tag{3.9}$$

Let us set

$$p = 2, \quad l = 0, \quad v = \varkappa_r - 1 \qquad \text{for} \qquad \varkappa_r > 1$$

and $v = 1$, otherwise. The second case is quite analogous.

(3.10) $$\| g_r \|_{L_2} \leq C \{ \sum_{|\alpha| = \varkappa_r - 1} \| D^\alpha g_r \|_{W_2^{1-\varkappa_r}} + \| g_r \|_{W_2^{1-\varkappa_r}} \}.$$

1) $$\| g_r \|_{W_2^{1-\varkappa_r}} = \sup_{\varphi \in \overset{\circ}{W}{}_2^{\varkappa_r - 1};\ \| \varphi \| = 1} \left| \int_\Omega g_r \cdot \varphi \right| =$$

$$= \sup \left| \int_\Omega \frac{\partial \varphi}{\partial x_N} \cdot \gamma_s^{2K} \cdot F_{\beta_r} \right| \leq C \cdot \| F_{\beta_r} \|_{L_2} \leq$$

$$\leq C \| \theta^{k-1} u \|_{L_2} \leq C V^{\frac{k}{2} - 1}.$$

Let us denote $\bar{\alpha}_r = (0, 0, \dots, \varkappa_r - 1)$. Let $\alpha \neq \bar{\alpha}_r$; $|\alpha| = \varkappa_r - 1$.

2) $$\| D^\alpha g_r \|_{W_2^{1-\varkappa_r}} = \sup_{\varphi \in \overset{\circ}{W}{}_2^{\varkappa_r - 1},\ \| \varphi \| = 1} \left| \int_\Omega D^\alpha g_r \cdot \varphi \right| =$$

$$= \sup \left| \int_\Omega D^{\alpha'} \varphi \cdot \frac{\partial}{\partial x_j} [\gamma_s^{2K} \cdot F_{\beta_r}] \right|,$$

where $|\alpha'| = \varkappa_r - 1$; $j \neq N$. Then

$$\| D^\alpha g_r \|_{W_2^{1-\varkappa_r}} \leq C \cdot \left\| \frac{\partial}{\partial x_j} (\gamma_s^{2K} \cdot F_{\beta_r}) \right\|_{L_2} =$$

$$= \left\| \gamma_s^{2K} \left(\frac{\partial F_{\beta_r}}{\partial x_j} + \sum_{l=1}^{h} F_{\beta_r l} \cdot N_l \frac{\partial u}{\partial x_j} \right) + F_{\beta_r} \cdot \frac{\partial \gamma_s^{2K}}{\partial x_j} \right\|_{L_2} \leq$$

$$\leq C \cdot \left\{ V^{\frac{k}{2} - 1} + \left\| \gamma_s^{2K} \theta^{k-2} u \sum_{l=1}^{h} \left| N_l \left(\frac{\partial u}{\partial x_j} + \frac{\partial a}{\partial x_j} \frac{\partial u}{\partial x_N} \right) \right| \right\|_{L_2} + \right.$$

$$+ \sup_{x' \in K_s} \left| \frac{\partial a}{\partial x_j} (x') \right| \cdot \left\| \gamma_s^{2K} \theta^{k-2} u \sum_{l=1}^{h} \left| N_l \left(\frac{\partial u}{\partial x_N} \right) \right| \right\|_{L_2} \leq$$

$$\leq C \cdot V^{\frac{k}{2} - 1} + C(s) \cdot I^{1/2},$$

where $C(s) \to 0$ for $s \to 0$.

3) $$\left(\sum_{r=1}^{m}\| D^{\bar{\alpha}_r} g_r \|_{w_2^{1-\varkappa_r}}\right)^{1/2} = \sup_{\varphi_r \in \overset{\circ}{W}_r^{\varkappa_r - 1};\ \Sigma \|\varphi_r\|^2 = 1} \left|\int_{\Omega}\sum_{r=1}^{m}\varphi_r D^{\bar{\alpha}_r} g_r\right| =$$

$$= \sup\left|\int_{\Omega}\sum_{r=1}^{m} D^{\alpha_r}\varphi_r\cdot\gamma_s^{2K}F_{\beta_r}\right| \leq \sup\left|\int_{\Omega}\sum_{r=1}^{m} F_{\beta_r}\cdot N_{\beta_r}(\varphi\cdot\gamma_s^{2K})\right| +$$

$$+ CV^{\frac{k}{2}-1} \leq \sup\left\{\left|\int_{\Omega}\sum_{i=1}^{h} f_i\cdot N_i(\varphi\cdot\gamma_s^{2K})\right| + \left|\int_{\Omega}\sum_{\substack{i=1\\ i\neq\beta_r}}^{h} F_i\cdot N_i(\varphi\cdot\gamma_s^{2K})\right| + CV^{\frac{k}{2}-1}.\right.$$

The first term is estimated by $C\|f\|_{[W_2^1]^h}$ the second one has the same form as 2).

Finally, from (3.10) and 1), 2), 3)

$$\left(\sum_{r=1}^{m}\| g_r\|_{L_2}^2\right)^{1/2} \leq CV^{\frac{k}{2}-1} + C(s)\, I^{1/2}. \tag{3.11}$$

As in 2) all the terms

$$\left(\int_{\Omega}\gamma_s^{2K}\theta^{2(k-2)} N_l^2\left(\frac{\partial u}{\partial x_N}\right)\right)^{1/2} \quad \text{for} \quad l \neq \beta_r;\ r = 1, \dots, m$$

may be estimated by $CV^{\frac{k}{2}-1} + C(s)\, I^{1/2}$

$$j = \int_{\Omega}\gamma_s^{2K}\theta^{2(k-2)}u \sum_{r=1}^{m} N_{\beta_r}^2\left(\frac{\partial u}{\partial x_N}\right) \leq \int_{\Omega}\gamma_s^{2K}\theta^{k-2}u\cdot\sum_{r,s=1}^{m} F_{\beta_r\beta_s} N_{\beta_r}\frac{\partial u}{\partial x_N}\cdot N_{\beta_s}\frac{\partial u}{\partial x_N} \leq$$

$$\leq \int_{\Omega}\left|\sum_{r=1}^{m}\theta^{k-2}u\, N_{\beta_r}\frac{\partial u}{\partial x_N}\cdot\left\{g_r - F_{\beta_r}\frac{\partial\gamma_s^{2K}}{\partial x_N} - \gamma_s^{2K}\left[\sum_{l\neq\beta_s} F_{\beta_r l} N_l\frac{\partial u}{\partial x_N} +\right.\right.\right.$$

$$\left.\left.\left. + \frac{\partial F_{\beta_r}}{\partial x_N}\right]\right\}\right| \leq j^{1/2}\cdot\{CV^{\frac{k}{2}-1} + C(s)\, I^{1/2}\}.$$

Therefore $I \leq C(s)\cdot V^{k-2}$ for sufficiently small s.

From these lemmas it follows immediately:

COROLLARY 3 4:

$$\|\theta^{k-1}u\|_{W_2^1} \leq CV^{\frac{k}{2}-1}. \tag{3.12}$$

L_p-estimates are based on the generalization of Meyer's theorem.

LEMMA 3.5 : Let $u = \mathcal{B}^{-1}(f) \in M$. Then $w = \frac{\partial u}{\partial x_l} \cdot \gamma_0^K$ is a weak solution of

$$\int_\Omega \sum_{i,j=1}^h F_{ij} \cdot N_i w \, N_j \varphi = \int_\Omega \sum_{i=1}^h g_i N_i \varphi, \tag{3.13}$$

i. e. this equation is satisfied for every $\varphi \in [D(\Omega)]^m$. The right part $g_j \in L_p(\Omega)$ with

$$\| g \|_{[L_p]^h} \leq C V^{k/2-1}.$$

PROOF : Analogously to the proof of (3,1) it may be written

$$\int_\Omega \sum_{i,j=1}^h F_{ij} \cdot N_i \left(\frac{\partial u}{\partial x_l}\right) \cdot N_j \psi = \int_\Omega \sum_{j=1}^h \frac{\partial}{\partial x_l}(f_j - F_j) \cdot N_j \psi.$$

For w, $\psi = \gamma_0^K \cdot \varphi$ we obtain

$$\int_\Omega \sum_{i,j=1}^h F_{ij} N_i w \, N_j \varphi =$$

$$= \int_\Omega \sum_{j=1}^h N_j \varphi \cdot \left\{ \gamma_0^K \frac{\partial}{\partial x_l}(f_j - F_j) + \sum_{i=1}^h F_{ij} \sum_{r, |\alpha'| < \varkappa_r} C_{\alpha' j} D^{\alpha'} u_r \right\} + R$$

where $C_{\alpha' j}$ are smooth and R involves the terms

$$C D^\alpha \varphi_r \cdot D^\beta (\gamma_0^K) \cdot \left\{ F_{ij} N_i \frac{\partial u}{\partial x_l} + \frac{\partial}{\partial x_l}(f_i - F_i) \right\}$$

with $|\alpha| < \varkappa_r$; $|\beta| \leq \varkappa_r$.

The expression at $N_j \varphi$ (let us denote it $\overline{g_j}$) may be estimated by $C \cdot \left\{ \theta^{k-1} u + \left| \frac{\partial f_j}{\partial x_l} \right| \right\}$. According to corollary (3.4) $\theta^{k-1} u \in L_q(\Omega)$ for every $1 \leq q < \infty$ and $\overline{g_j} \in L_p$ with the corresponding norm.

R may be written in the form

$$R = \sum_{r=1}^m \sum_{|\alpha| = \varkappa_r} g_{r\alpha} D^\alpha \varphi_r$$

with $\| g_{r\alpha} \|_{L_p} \leq C V^{\frac{k}{2}-1}$ according to the result of Nečas (see [17], [18]):

Let

$$g \in L_q(\Omega), \quad 1 < q < \infty; \quad 0 \leq |l| \leq k-1,$$

hence there exist the functions g_j such that

$$\int_\Omega D^l \varphi \cdot g = \int_\Omega \sum_{|j|=k} g_j D^j \varphi \quad \text{for every} \quad \varphi \in D(\Omega).$$

Moreover,

$$\| g_j \|_{L_p} \leq C \cdot \| g \|_{L_q}$$

where

$$\frac{1}{p} \geq \frac{1}{q} - \frac{k-|l|}{N} \quad \text{for} \quad (k-|l|)\, q < N$$

and

$$1 \leq p < \infty \quad \text{for} \quad (k-|l|) \cdot q \geq N.$$

Applying Lemma 1.4 with $\tilde{\gamma}_1 = \gamma_1$, $\tilde{\gamma}_2 = \gamma_2 \cdot V^{k-2}$ to (3.13), we obtain immediately

LEMMA 3.6: Let $u \in \mathcal{B}^{-1}(f) \in M$ then there exist a constant $\gamma_5 > 0$ and $q = 2 + \gamma_5 V^{2-k}$ such that $\gamma_0^K \cdot u \in W_q^{\varkappa+1}(\Omega)$ and

$$\| \gamma_0^K u \|_{W_q^{\varkappa+1}} \leq C \cdot V^{\frac{1}{2} k-1} \tag{3.14}$$

LEMMA 3.7: Let $u = \mathcal{B}^{-1}(f) \in M$ then there exist a constant $\gamma_5 > 0$ and $q = 2 + \gamma_5 V^{2-k}$ such that $\gamma_r^K u \in W_q^{\varkappa+1}(\Omega)$ and

$$\| \gamma_r^K u \|_{W_q^{\varkappa+1}} \leq C(r) \cdot V^{\frac{3}{2}(k-2)}. \tag{3.15}$$

PROOF: For $w = \gamma_r^K \partial^l (u - u_0)$ we may use Lemma 1.4 analogously to Lemmas 3.5, 3.6.

For the normal derivatives we obtain by repeating the estimates in the proof of 3.3 for $q = 2 + \gamma_5 V^{2-k}$ instead 2

$$\left\| \gamma_r^K \cdot \theta^{k-2} u \cdot \sum_{i=1}^{h} \left| N_i \left(\frac{\partial u}{\partial x_N} \right) \right| \right\|_{L_q} \leq C(r) \cdot V^{\frac{3}{2}(k-2)}.$$

From (3.14), (3.15) it follows:

COROLLARY 3.8 :

$$\| \theta^{k-1} u \|_{W_q^1} \leq C V^{\frac{3}{2}(k-2)} \quad \text{with} \quad q = 2 + \gamma_5 V^{2-k}. \tag{3.16}$$

LEMMA 3.9 : Let $u = \mathcal{B}^{-1}(f) \in M$. Then there exist $p_0 > 2$ such that $u \in W_{p_0}^{\varkappa+1}(\Omega)$ with

$$\| u \|_{W_{p_0}^{\varkappa+1}} \leq C (\| f \|_{[W_p^1]^h}), \tag{3.17}$$

which implies

$$\| u \|_{C_\varkappa(\bar{\Omega})} + \| u \|_{W_2^{\varkappa+1}(\Omega)} \leq C.$$

PROOF : From (3.12), (3.16)

$$\| \theta^{k-1} u \|_{W_2^1} \leq C V^{\frac{k}{2}-1}$$

$$\| \theta^{k-1} u \|_{W_q^1} \leq C V^{\frac{3}{2}(k-2)}$$

hold for $q = 2 + \gamma_5 V^{2-k}$.

Using the Riesz-Thorin interpolation theorem, we have

$$\| \theta^{k-1} u \|_{W_{p_0}^1} \leq C \cdot V^{a \frac{3}{2}(k-2) + (1-a)\left(\frac{k}{2}-1\right)}$$

for

$$\frac{1}{p_0} = \frac{a}{q} + \frac{1-a}{2} ; \quad a \in \langle 0, 1 \rangle .$$

Using embedding theorem (see [17], [18]) for $N = 2$:

$$\| \theta^{k-1} u \|_{C(\bar{\Omega})} \leq C \cdot \left(\frac{p_0 - 1}{p_0 - 2} \right)^{1 - \frac{1}{p_0}} \cdot \| \theta^{k-1} u \|_{W_{p_0}^1} \leq \tag{3.18}$$

$$\leq C_1 \left(\frac{p_0 - 1}{p_0 - 2} \right)^{1 - \frac{1}{p_0}} \cdot V^{(2a+1)\left(\frac{k}{2}-1\right)} + C_2 \cdot V^{\frac{k}{2}-1}$$

But

$$1 - \frac{1}{p_0} \leq \frac{1}{2}(1 + a \gamma_5 V^{2-k}) \quad \text{and} \quad \frac{p_0 - 1}{p_0 - 2} \leq C \cdot \frac{V^{k-2}}{a} .$$

Then

$$\| \theta u \|_{C(\bar{\Omega})}^{k-1} \leq C_1 V^{\left(\frac{k}{2}-1\right)(2 + a(2+\gamma_5))} + C_2 V^{\frac{k}{2}-1} .$$

For

$$a \cdot (2 + \gamma_5) \cdot \left(\frac{k}{2} - 1 \right) < \frac{1}{2}$$

the inequality may be satisfied only for bounded V.

The proceedings of Lemmas (3.5)-(3.7) may be repeated with this result. The coefficients in the equations (3.13) are continuous, the right parts belong to L_p. According to the result of Agmon, Douglis, Nirenberg (see [1]) or Schechter (see [23]) $u \in W_p^{\varkappa+1}$ and its norm depends only on $\| f \|$, $\| u_0 \|$.

This completes the proof of Theorem 4.

Only the dependence of the highest derivatives of u is important in this proof and it allows us to prove, quite analogously and without changes, the following

THEOREM 5. Let N_i be all the derivatives, i. e.

$$N_i u = N_{r_\alpha} u = D^\alpha u_r \quad \text{for} \quad r = 1, \dots, m\,; \quad | \alpha | \leq \varkappa_r\,;$$

let $\mathcal{B}$ satisfies B. Then $\mathcal{B}$ has property $\mathcal{A}$.

THEOREM 6. Let $\mathcal{B}$ satisfy A or B with $h = m$. Then $\mathcal{B}$ has property $\mathcal{A}$.

PROOF. The proof in case B is a slight modification of case A.

Lemmas (3.1), (3.2) may be proved without changes. Let us set

$$g_r = \frac{\partial}{\partial x_N} \left\{ \sum_{i=1}^{h} a_{ir\,a_r}\, \gamma_s^{2K}\, F_i \,(x_1 \{N_j u\,(x)\}) \right\}.$$

The estimates of L_2-norms of g_r are the same as in (3.3). The matrix

$$N\,((0, \dots, 0, 1)) = (a_{ir\,a_r})_{\substack{i = 1, \dots, m \\ r = 1, \dots, m}}$$

is regular and

$$h_i = \frac{\partial}{\partial x_N} (\gamma_s^{2K} F_i) = \sum_{r=1}^{m} C_{r_i}\, g_r \in L_2$$

with

$$\| h_i \|_{L_2} \leq C V^{k/2-1} + C\,(s)\, I^{1/2}.$$

But

$$I \leq \frac{1}{\gamma_1} \cdot \int_\Omega \gamma_s^{2K}\, \theta^{k-2u} \cdot \sum_{i,j=1}^{h} F_{ij} N_i \frac{\partial u}{\partial x_N} \cdot N_j \frac{\partial u}{\partial x_N} =$$

$$= \frac{1}{\gamma_1} \int_\Omega \theta^{k-2u} \cdot N_i \frac{\partial u}{\partial x_N} \cdot \gamma_s^{2K} \cdot \left\{ h_i - \frac{\partial F_i}{\partial x_N} \cdot \gamma_s^{2K} - F_i \cdot \frac{\partial \gamma_s^{2K}}{\partial x_N} \right\}$$

and it is bounded as in (3.3). Analogously Lemma (3.7) is proved and this completes the proof.

THEOREM 7. Let $\mathcal{B}$ satisfy A or B, $k < 4$. Then $\mathcal{B}$ has property $\mathcal{A}$.

PROOF. (3.1), (3.2) are proved in the same way. However, we are not able, under these conditions, to obtain from (3.1), (3.2) a better estimate for

$$\int_{\Omega} \theta^{k-2} u \,|\, D^{\alpha} u_r \,|^2$$

with

$$|\,\alpha\,| = \varkappa_r + 1, \quad \alpha \neq (0, 0, \dots, \varkappa_r + 1)$$

than $C(f) \cdot V^{k-2}$. Therefore,

$$\| \theta^{k-1} u \|_{W_2^1} \leq C V^{k-2},$$

analogously

$$\| \theta^{k-1} u \|_{W_p^1} \leq C V^{2k-4}.$$

In Lemma (3.4) the boundedness of V may be obtained only for $k < 4$.

The difficulties lie in the fact that for more precise estimate of $\int_{\Omega} \theta^{k-2} u \, D^{\alpha} u_r$ a theorem on a very general class of multiplicators of the form $\theta^{k-2} u$ would be necessary — which is not known to us at present.

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