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# SEMI-PRIMAL CLUSTERS

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A primal cluster is essentially a class $\{\mathcal{U}_i\}$ of universal algebras of the same species, where each $\mathcal{U}_i$ is primal ($=$ strictly functionally complete), and such that every finite subset of $\{\mathcal{U}_i\}$ is « independent ». The concept of independence is essentially a generalization to universal algebras of the Chinese Remainder Theorem in number theory. Primal algebras themselves are further subsumed by the broader class of « semi-primal » algebras, and a structure theory for these algebras was recently established by Foster and Pixley [5] and Astromoff [1]. This theory subsumes and substantially generalizes well-known results for Boolean rings, $p$-rings, and Post algebras.

In order to expand the domain of applications of this extended « Boolean » theory, we should attempt to discover semi-primal clusters which, preferably, are as comprehensive as possible. In this paper we prove that certain large classes of semi-primal algebras form semi-primal clusters. Indeed, we show that the class of all two-fold surjective singular subprimal algebras which are pairwise non-isomorphic and in which each finite subset is co coupled forms a semi-primal cluster. A similar result is also shown to hold for regular subprimal algebras with pairwise non-isomorphic cores. Moreover, we prove that the class of all pairwise non-isomorphic $s$-couples, as well as the class of all $r$-frames with pairwise non-isomorphic cores, and even the union of these two classes, forms a semi-primal cluster. Finally, we construct classes of $s$-couples and $r$-frames.

## 1. Fundamental Concepts and Lemmas.

In this section we recall the following basic concepts of [2]-[5]. Let $\mathcal{U} = (A\,;\,\Omega)$ be a universal algebra of species $S = (n_1\,, n_2\,, \ldots)$, where the

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$n_i$ are non-negative integers, and let $\Omega = (\bar{O}_1, \bar{O}_2, \dots)$ denote the primitive operation symbols of $S$. Here, $\bar{O}_i = \bar{O}_i(\xi_1, \dots, \xi_{n_i})$ is of rank $n_i$. By an *$S$-expression* we mean any indeterminate symbol $\xi, \eta, \dots$ or any composition of these indeterminate symbols via the primitive operations $\bar{O}_i$. As usual, we use the same symbols $\bar{O}_i$ to denote the primitive operations of the algebras $\mathcal{U}_1, \mathcal{U}_2, \dots$ when these algebras are of species $S$. We write « $\Psi(\xi, \dots)(\mathcal{U})$ » to mean that the $S$-expression $\Psi$ is interpreted in the $S$-algebra $\mathcal{U}$. This simply means that the primitive operation symbols are identified with the corresponding primitive operations of $\mathcal{U}$, and the indeterminate-symbols $\xi, \dots$ are now viewed as indeterminates over $\mathcal{U}$. Moreover, « $\Psi(\xi, \dots)(\mathcal{U})$ » is called a *strict $\mathcal{U}$-function.* An identity between strict $\mathcal{U}$-functions $\Psi$, $\Phi$ holding throughout $\mathcal{U}$ is called a *strict $\mathcal{U}$-identity,* and is written as $\Psi(\xi, \dots) = \Phi(\xi, \dots)(\mathcal{U})$. We use $Id(\mathcal{U})$ to denote the family of all strict $\mathcal{U}$-identities. A finite algebra $\mathcal{U}$ with more than one element is called *categorical* (respectively, *semi-categorical*) if every algebra, of the same species as $\mathcal{U}$, which satisfies all the strict identities of $\mathcal{U}$ is a subdirect power of $\mathcal{U}$ (respectively, is a subdirect product of subalgebras of $\mathcal{U}$). A map $f(\xi_1, \dots, \xi_k)$ from $A^k$ into $A$ is *$S$-expressible* if there exists an $S$-expression $\Psi(\xi_1, \dots, \xi_k)$ such that $f = \Psi$ for all $\xi_1, \dots, \xi_k$ in $A$. A map $f(\xi_1, \dots, \xi_k)$ is *conservative* if for each subalgebra $\mathcal{B} = (B; \Omega)$ of $\mathcal{U}$ and for all $b_1, \dots, b_k$ in $B$, we have, $f(b_1, \dots, b_k) \in B$. An algebra $\mathcal{U}$ is *primal* (respectively, *semi-primal*) if it is finite, with at least two elements, and every map from $A \times \dots \times A$ into $A$ is $S$-expressible (respectively, every conservative mapping from $A \times \dots \times A$ into $A$ is $S$-expressible). A semi-primal algebra $\mathcal{U}$ which possesses exactly one subalgebra $\mathcal{U}^* = (A^*; \Omega)$ $(\neq \mathcal{U})$ is called a *subprimal* algebra. The subalgebra $\mathcal{U}^*$ is called the *core* of $\mathcal{U}$. If $\mathcal{U}^*$ has exactly one element, $\mathcal{U}$ is called a *singular subprimal*; otherwise it is called a *regular subprimal.* An element a in $A$ is said to be *expressible* if there exists a unary $S$-expression $\Lambda_a(\xi)$ such that $\Lambda_a(\xi) = a$ for each $\xi$ in $A$. An element $a$ in $A \setminus A^*$ is said to be *ex-expressible* provided there exists a unary $S$-expression $\Gamma_a(\xi)$ such that $\Gamma_a(\xi) = a$ for each $\xi$ in $A \setminus A^*$ (here, $A \setminus A^* = \{\xi \mid \xi \in A,\ \xi \notin A^*\}$).

We now proceed to define the concept of independence. Let $\{\mathcal{U}_i\} = \{\mathcal{U}_1, \dots, \mathcal{U}_r\}$ be a finite set of algebras of species $S$. We say that $\{\mathcal{U}_i\}$ is *independent* if corresponding to each set $\Psi_1, \dots, \Psi_r$ of $S$-expressions there exists a single expression $\Psi$ such that $\Psi = \Psi_i(\mathcal{U}_i)$, $i = 1, \dots, r$ (or equivalently, if there exists an $r$-ary $S$-expression $\Psi$ such that $\Psi(\xi_1, \dots, \xi_r) = \xi_i(\mathcal{U}_i)$, $i = 1, \dots, r$). A *primal* (respectively, *subprimal, semi-primal*) *cluster* of species $S$ is defined to be a class $\widetilde{\mathcal{U}} = \{\dots, \mathcal{U}_i, \dots\}$ of primal (respectively, subprimal, semi-primal) algebras of species $S$, any finite subset of which is independent.

We are now in a position to state the following lemmas, the proofs of which have already been given in [2; 5].

LEMMA 1. *A primal algebra is categorical and simple.*

LEMMA 2. *A semi-primal algebra is semi-categorical and simple.*

LEMMA 3. *Let* $\mathcal{B} = (B; \Omega)$ *be a subprimal algebra of species* $S$. *Then,*

(*a*) *the core* $\mathcal{B}^* = (B^*; \Omega)$ *is primal or else is a one-element subalgebra;*
(*b*) *each* $b$ *in* $B^*$ *is* $S$*-expressible;*
(*c*) *each* $b$ *in* $B \setminus B^*$ *is ex-expressible.*

## 2. Semi-primal Clusters.

In this section some semi-primal clusters will be found. The methods of proof are similar to those of Foster [4] and O'Keefe [6]. We will be concerned, mainly, with co-coupled families of subprimal algebras, in the sense of the following

DEFINITION 1. A family $\mathcal{U}_i = (A_i; \Omega)$ $i \in I$, of subprimal algebras of species $S$, with cores $\mathcal{U}_i^* = (A_i^*; \Omega)$, $i \in I$, respectively, is said to be *co-coupled* if there exist two binary $S$-expressions $\xi \times \eta$ $(= \xi \cdot \eta = \xi\eta)$ and $\xi T \eta$, and elements $0_i$, $1_i$ in $A_i (0_i \neq 1_i)$, for each $i \in I$, such that

(*a*) if $\mathcal{U}_i$ is a singular subprimal, then

(1) $$\{0_i\} = A_i^*;$$

(2) $$0_i \times \xi = \xi \times 0_i = 0_i; \quad 1_i \times \xi = \xi \times 1_i = \xi \quad (\text{all } \xi \text{ in } A_i);$$

(3) $$0_i T \xi = \xi T 0_i = \xi \quad (\text{all } \xi \text{ in } A_i);$$

(*b*) if $\mathcal{U}_i$ is a regular subprimal, then (2) and (3) hold in addition to

(4) $$\{0_i, 1_i\} \subseteq A_i^*.$$

DEFINITION 2. A family $\mathcal{U}_i = (A_i; \Omega)$, $i \in I$, of regular subprimal algebras of species $S$ is said to be *co-framal* if there exist $S$-expressions $\xi \times \eta$ $(= \xi \cdot \eta = \xi\eta)$ and $\xi^\cap$, and elements $0_i$, $1_i$ in $A_i$ $(0_i \neq 1_i)$, for each $i \in I$, such that (2) and (4) hold in addition to

(5) $$^\cap \text{ is a permutation of } A_i \text{ with } 0_i^\cap = 1_i.$$

REMARK 1. If $\mathfrak{U}_i$, $i \in I$, is a co-framal family of regular subprimal algebras, then by letting $\xi T \eta = \text{def} = (\xi^{\cap} \times \eta^{\cap})^{\cup}$, where $\xi^{\cup}$ denotes the inverse of $\xi^{\cap}$, it follows that (3) holds and therefore the family is also co-coupled.

THEOREM 1. *Let $\mathfrak{U}_i = (A_i; \Omega)$, $i = 1, \dots, n$, be co-coupled subprimal algebras of species $S$. Then, if the $\mathfrak{U}_i$ are pairwise independent, they are independent.*

PROOF. Assume that $\mathfrak{U}_1, \dots, \mathfrak{U}_n$ are pairwise independent. Then, for any two algebras $\mathfrak{U}_i$, $\mathfrak{U}_j$ (where $i \neq j$), there exists an $S$-expression $\Phi(\xi, \eta)$ such that

$$\Phi(\xi, \eta) = \begin{cases} \xi \ (\mathfrak{U}_i) \\ \eta \ (\mathfrak{U}_j). \end{cases} \tag{6}$$

Let $\mathfrak{U}_i^* = (A_i^*; \Omega)$ denote the core of $\mathfrak{U}_i$. If $\mathfrak{U}_i$ is a regular subprimal (respectively, a singular subprimal), then $1_i \in A_i^*$ (respectively, $1_i \in A_i \setminus A_i^*$), and according to Lemma 3 it is expressible (respectively, ex-expressible). In either case, there exists a unary $S$-expression $\Delta_{1_i}(\xi)$ such that

$$\Delta_{1_i}(\xi) = 1_i \quad (\xi \text{ in } A_i \setminus \{0_i\}). \tag{7}$$

From (6), (7), and the fact that $0_j \in A_j$ is $S$-expressible, say by $\Delta_{0_j}(\xi)$, it follows immediately that

$$\Pi_{ij}(\xi) = \text{def} = \Phi(\Delta_{1_i}(\xi), \quad \Delta_{0_j}(\xi)) = \begin{cases} 1_i \quad (\xi \text{ in } A_i \setminus \{0_i\}) \\ 0_j \quad (\mathfrak{U}_j) \end{cases} \qquad (i \neq j).$$

Define, now, a unary $S$ expression $\Psi_i(\xi)$, $1 \leq i \leq n$, by

$$\Psi_i(\xi) = \Pi_{i1}(\xi) \times \dots \times \widehat{\Pi}_{ii}(\xi) \times \dots \times \Pi_{in}(\xi) = \begin{cases} 1_i \ (\xi \text{ in } A_i \setminus \{0_i\}) \\ 0_j \ (\mathfrak{U}_j) \ (\text{all } j \neq i). \end{cases} \tag{8}$$

where $\widehat{\ }$ denotes deletion and the $\Pi_{ij}(\xi)$ are associated in some fixed manner. Using (8) and the co-coupling binary $S$-expression $\xi T \eta$, define an $n$-ary $S$-expression $\Phi(\xi_1, \dots, \xi_n)$ by

$$\Phi(\xi_1, \dots, \xi_n) = [\Psi_1(\xi_1) \times \xi_1] \, T \dots T \, [\Psi_n(\xi_n) \times \xi_n],$$

the $T$-factors being associated in some fixed manner. It is easily checked that $\Phi(\xi_1, \dots, \xi_n) = \xi_i(\mathfrak{U}_i)$, $1 \leq i \leq n$. This proves the theorem.

Because of Theorem 1, it is important to discuss the pairwise independence of subprimal algebras. To do this we impose a surjectivity property on the primitive operations.

DEFINITION 3. A subprimal algebra $\mathfrak{U}$, with core $\mathfrak{U}^*$, is said to be *two fold surjective* if each primitive operation of $\mathfrak{U}$ is surjective on both $\mathfrak{U}$ and $\mathfrak{U}^*$.

We now show that subprimal algebras which are two-fold surjective satisfy a certain factorization property (compare with [6]).

THEOREM 2. *Let $\mathfrak{U} = (A\,;\,\Omega)$ be a two-fold surjective subprimal algebra of species $S$. Then, for each unary $S$-expression, $\Gamma(\xi)$, and each primitive operation $\bar{O}_i$ (of rank $n_i$) of $\mathfrak{U}$, there exist unary $S$-expressions $\Psi_1(\xi), \dots, \Psi_{n_i}(\xi)$ such that*

$$\bar{O}_i(\Psi_1(\xi), \dots, \Psi_{n_i}(\xi)) = \Gamma(\xi)\,(\mathfrak{U}). \tag{9}$$

PROOF. Let $A = \{a_1, \dots, a_m, a_{m+1}, \dots, a_t\}$ where $A^* = \{a_1, \dots, a_m\}$ is the base set of $\mathfrak{U}^*$ (= core of $\mathfrak{U}$) Clearly. $\Gamma(a_j) \in A^*$ for $1 \leq j \leq m$. Because of two-fold surjectivity, there exist elements $a_{jk}$ $(1 \leq j \leq t,\ 1 \leq k \leq n_i)$ of $A$, with $a_{jk}$ in $A^*$ when $1 \leq j \leq m$, such that

$$\bar{O}_i(a_{j1}, \dots, a_{jn_i}) = \Gamma(a_j).$$

Now let unary functions $g_1(\xi), \dots, g_{n_i}(\xi)$ be defined on $A$ by

$$g_k(a_j) = a_{jk}\ (1 \leq k \leq n_i,\ 1 \leq j \leq t).$$

Since $g_k(a_j) \in A^*$, $1 \leq j \leq m$, each $g_k$ is conservative and hence is $S$-expressible, say by $\Psi_k(\xi)$. It follows that

$$\bar{O}_i(\Psi_1(a_j), \dots, \Psi_{n_i}(a_j)) = \bar{O}_i(g_1(a_j), \dots, g_{n_i}(a_j)) = \bar{O}_i(a_{j1}, \dots, a_{jn_i}) = \Gamma(a_j)$$

for $1 \leq j \leq t$ and (9) is verified.

From [6; Lemma 2.3] and Theorem 2 we immediately obtain the following generalized factorization property.

THEOREM 3. *Let $\mathfrak{U}$ be a two-fold surjective subprimal algebra of species $S$. Then, for each expression $\Sigma(\xi_1, \dots, \xi_q)$ and each expression $\Theta(\xi_1, \dots, \xi_p)$ in which no indeterminate $\xi_i$, $1 \leq i \leq p$, occurs twice in $\Theta$, there exist expressions $\Psi_1, \dots, \Psi_p$ such that*

$$\Theta(\Psi_1, \dots, \Psi_p) = \Sigma\,(\mathfrak{U}). \tag{10}$$

The pairwise independence of any two universal algebras $\mathfrak{A}$, $\mathfrak{B}$ of species $S$ assures that any two subalgebras of $\mathfrak{A}, \mathfrak{B}$, of more than one element each, are non-isomorphic. In establishing independence, therefore, this must be taken as a minimal assumption.

THEOREM 4. *Let $\mathfrak{A} = (A\,;\Omega)$ and $\mathfrak{B} = (B\,;\Omega)$ be subprimal algebras of species $S$ with cores $\mathfrak{A}^* = (A^*\,;\Omega)$ and $\mathfrak{B}^* = (B^*\,;\Omega)$, respectively. Suppose that either of the following holds:*

(i) *$\mathfrak{B}$ is a regular subprimal and $\mathfrak{A}^*, \mathfrak{B}^*$ are non-isomorphic;*

(ii) *$\mathfrak{B}$ is a singular subprimal and $\mathfrak{A}, \mathfrak{B}$ are non-isomorphic.*

*Then, there exist elements $d_1, d_2$ in $B$ $(d_1 \neq d_2)$ and unary expressions $\Gamma_1(\xi), \Gamma_2(\xi)$ such that*

$$(1^0) \qquad \Gamma_1(\xi) = \Gamma_2(\xi)\,(\mathfrak{A});$$

$$(2^0) \qquad \Gamma_1(\xi) = d_1\,(\xi \text{ in } B \setminus B^*);$$

$$(3^0) \qquad \Gamma_2(\xi) = d_2\,(\xi \text{ in } B \setminus B^*).$$

*Moreover, if* (i) *holds, then $d_1, d_2 \in B^*$ and*

$$(4^0) \qquad \Gamma_1(\xi) = d_1\,(\mathfrak{B});$$

$$(5^0) \qquad \Gamma_2(\xi) = d_2\,(\mathfrak{B}).$$

PROOF. First, assume that (i) holds. Then $\mathfrak{B}^*$ is primal (Lemma 3) and hence categorical (Lemma 1). Therefore, if $Id(\mathfrak{A}) \supseteq Id(\mathfrak{B}^*)$ then $\mathfrak{A} \cong \mathfrak{B}^{*(k)}$ ($= k^{th}$ subdirect power of $\mathfrak{B}$) for some $k \geq 1$. Now $k \neq 1$ since $\mathfrak{A}$ has a subalgebra $(\neq \mathfrak{A})$. But if $k \geq 2$, there exists an epimorphism $\mathfrak{A} \rightarrow \mathfrak{B}^*$, contradicting the simplicity of $\mathfrak{A}$. Thus, $Id(\mathfrak{A}) \not\supseteq Id(\mathfrak{B}^*)$. Similarly, if $Id(\mathfrak{B}^*) \supseteq Id(\mathfrak{A})$, then since $\mathfrak{A}$ is semi-categorical (Lemma 2), $\mathfrak{B}^* \cong \mathfrak{A}^{(k_1)} \dot{\times} \mathfrak{A}^{*(k_2)}$ ($=$ subdirect product of subdirect powers of $\mathfrak{A}$ and $\mathfrak{A}^*$) for some $k_1, k_2$. Now $k_1 + k_2 \neq 1$ since $\mathfrak{B}^*, \mathfrak{A}$ are non-isomorphic and by assumption $\mathfrak{B}^*, \mathfrak{A}^*$ are non-isomorphic. Thus, $k_1 + k_2 \geq 2$. But then there exists an epimorphism from $\mathfrak{B}^*$ onto either $\mathfrak{A}$ or $\mathfrak{A}^*$, contradicting the simplicity of $\mathfrak{B}^*$. Thus $Id(\mathfrak{B}^*) \not\supseteq Id(\mathfrak{A})$. These two non-inclusions assure the existence of expressions $\Psi_1(\xi_1, \dots, \xi_p)$ and $\Psi_2(\xi_1, \dots, \xi_p)$ such that

$$(11) \qquad \Psi_1 = \Psi_2\,(\mathfrak{A}) \text{ and } \Psi_1 \neq \Psi_2\,(\mathfrak{B}^*).$$

From (11) it follows that there exist elements $\beta_1, \dots, \beta_p$ of $B^*$ for which

$$(12) \qquad d_1 = \text{def} = \Psi_1(\beta_1, \dots, \beta_p) \neq \Psi_2(\beta_1, \dots, \beta_p) = \text{def} = d_2.$$

Clearly, $d_1, d_2 \in B^*$. Since $\beta_1, \ldots, \beta_p \in B^*$, there exist expressions $\Lambda_1(\xi), \ldots, \Lambda_p(\xi)$ such that (see Lemma 3)

$$\Lambda_i(\xi) = \beta_i \ (\mathfrak{B}),\ 1 \leq i \leq p. \tag{13}$$

If $\Gamma_j(\xi)$ is defined by

$$\Gamma_j(\xi) = \Psi_j(\Lambda_1(\xi), \ldots, \Lambda_p(\xi)),\ 1 \leq j \leq 2, \tag{14}$$

from (12)-(14) it follows that $\Gamma_1(\xi), \Gamma_2(\xi)$ satisfy ($1^0$), ($4^0$), and ($5^0$).

Secondly, assume that (ii) holds. Using arguments similar to those above, it can be established that $Id(\mathfrak{A}) \not\supseteq Id(\mathfrak{B})$ and $Id(\mathfrak{B}) \not\supseteq Id(\mathfrak{A})$. Thus, there exist expressions $\Psi_1(\xi_1, \ldots, \xi_p)$, $\Psi_2(\xi_1, \ldots, \xi_p)$ such that $\Psi_1 = \Psi_2 (\mathfrak{A})$ and $\Psi_1 \neq \Psi_2 (\mathfrak{B})$. Let $\beta_1, \ldots, \beta_p$ be elements of $B$ for which (12) holds. Because of Lemma 3, there exist expressions $\Lambda_1(\xi), \ldots, \Lambda_p(\xi)$ with

$$\Lambda_i(\xi) = \beta_i (\text{in } B \setminus B^*),\ 1 \leq i \leq p.$$

Let $\Gamma_1(\xi), \Gamma_2(\xi)$ be defined as in (14). It is easy to verify that they have the desired properties ($1^0$)-($3^0$).

Next, we prove the following theorems.

THEOREM 5. *Let $\mathfrak{A} = (A; \Omega)$ and $\mathfrak{B} = (B; \Omega)$ be subprimal algebras of species $S$ satisfing either* (i) *or* (ii) *of Theorem* 4. *Then there exist expressions $\Phi_1(\xi), \ldots, \Phi_p(\xi)$ such that $\Phi_1(\xi) = \ldots = \Phi_p(\xi)(\mathfrak{A})$ and such that every conservative unary function on B is identical, in B, to one of $\Phi_1(\xi), \ldots, \Phi_p(\xi)$.*

PROOF. Let the conservative unary functions on B be enumerated as $b_1(\xi), \ldots, b_p(\xi)$ and let $d_1, d_2, \Gamma_1(\xi), \Gamma_2(\xi)$ be as in Theorem 4. Since $\mathfrak{B}$ is semi-primal, each conservative function on $B$ is $S$-expressible. Hence, there exists an expression $\Phi = \Phi(\xi, \xi_1, \ldots, \xi_p, \xi_{p+1})$ for which

$$\Phi(\xi, \underbrace{d_1, \ldots, d_1}_{\text{i terms}}, d_2, \ldots, d_2) = b_i(\xi),\ 1 \leq i \leq p \ \ (\xi \text{ in } B).$$

(This follows since the above equation is a conservative condition). Using $\Phi$ as a skeleton, we now define $\Phi_1(\xi), \ldots, \Phi_p(\xi)$ by

$$\Phi_i(\xi) = \Phi(\xi, \underbrace{\Gamma_1(\xi), \ldots, \Gamma_1(\xi)}_{\text{i terms}}, \Gamma_2(\xi), \ldots, \Gamma_2(\xi)),\ 1 \leq i \leq p.$$

From ($1^0$) of Theorem 4 it follows that $\Phi_i(\xi) = \Phi_j(\xi)(\mathfrak{A})$ for all $1 \leq i, j \leq p$. If (i) holds, then from ($4^0$) and ($5^0$) of Theorem 4, $\Phi_i(\xi) = b_i(\xi)$ ($\xi$ in $B$), $1 \leq i \leq p$. If (ii) holds, then ($2^0$) and ($3^0$) assure that $\Phi_i(\xi) = b_i(\xi)$ ($\xi$ in $B \setminus B^*$).

Moreover, in $\mathfrak{B}$, $\Phi_i(\xi)$ and $b_i(\xi)$ are both conservative. Since $B^*$ consists of exactly one element, say $B^* = \{0\}$, it follows that $\Phi_i(0) = b_i(0) = 0$. Hence, in case (ii) we also have $\Phi_i(\xi) = b_i(\xi)$ ($\xi$ in $B$).

THEOREM 6. *Let $\mathfrak{A} = (A\,; \Omega)$, $\mathfrak{B} = (B\,; \Omega)$ be subprimal algebras of species $S$ (with cores $\mathfrak{A}^* = (A^*; \Omega)$, $\mathfrak{B}^* = (B^*; \Omega)$, respectively) satisfying either* (i) *or* (ii) *of Theorem 4. If $\mathfrak{B}$ is two-fold surjective, then for each $a$ in $A^*$ and each unary expression $\Psi(\xi)$ there exists an expression $\Omega = \Omega(\xi)$ such that*

$$\Omega = \begin{cases} a\,(\mathfrak{A}) \\ \Psi\,(\mathfrak{B}). \end{cases}$$

PROOF. If $a \in A^*$, there exists a unary expression $\Theta(\xi)$ for which $\Theta = a\,(\mathfrak{A})$. Let $\Theta'(\xi_1, \dots, \xi_p)$ be the $S$-expression derived from $\Theta$ by distinguishing each occurrence of $\xi$ in $\Theta$. Thus, by definition, $\Theta'(\xi, \dots, \xi) = \Theta(\xi)$. From Theorem 3, there exist expressions $\Psi_1(\xi), \dots, \Psi_p(\xi)$ such that

$$\Theta'(\Psi_1(\xi), \dots, \Psi_p(\xi)) = \Psi(\xi)\,(\mathfrak{B}).$$

Since $\Psi_1(\xi), \dots, \Psi_p(\xi)$ are conservative in $\mathfrak{B}$, by Theorem 5, there exist expressions $\Phi_1(\xi), \dots, \Phi_p(\xi)$ such that

$$\Phi_i(\xi) = \Phi_j(\xi)\,(\mathfrak{A}), \ 1 \leq i, j \leq p\,;$$

$$\Phi_i(\xi) = \Psi_i(\xi)\,(\mathfrak{B}), \ 1 \leq i \leq p.$$

Let $\Omega(\xi) = \Theta'(\Phi_1(\xi), \dots, \Phi_p(\xi))$. It is easily verified that $\Omega$ has the desired property of the theorem.

If $F$ is a family of subprimal algebras of species $S$ let us use $F_s$ (respectively, $F_r$) to denote the subfamily of all singular subprimal (respectively, regular subprimal) members of $F$.

THEOREM 7. (*Principal Theorem*) *Let $F$ be a family of two-fold surjective subprimal algebras of species $S$, each finite subset of which is co-coupled. If, further,*

(*a*) *the members of $F^s$ are pairwise non-isomorphic,*

(*b*) *the members of $F_r$ have pairwise non-isomorphic cores, then $F$ is a subprimal cluster.*

PROOF. Because of (*a*), (*b*), and Theorem 6, for any two members $\mathcal{U}, \mathcal{B}$ of $F$, there exist expressions $\Omega_1(\xi), \Omega_2(\xi)$ for which

$$\Omega_1(\xi) = \begin{cases} \xi\ (\mathcal{U}) \\ 0\ (\mathcal{B}) \end{cases} ; \quad \Omega_2(\xi) = \begin{cases} 0\ (\mathcal{U}) \\ \xi\ (\mathcal{B}) \end{cases} .$$

Since each finite subset of $F$ is co-coupled, there exists a binary $S$-expression $\xi\, T\, \eta$ satisfying (3). Thus

$$\Omega_1(\xi_1)\, T\, \Omega_2(\xi_2) = \begin{cases} \xi_1\ (\mathcal{U}) \\ \xi_2\ (\mathcal{B}) \end{cases}$$

and therefore $\mathcal{U}$, $\mathcal{B}$ are independent. From Theorem 1 it follows that each finite subset of $F$ is independent, and the theorem is proved.

COROLLARY 1. *Let $F\,(= F_r)$ be a family of two-fold surjective regular subprimal algebras of species $S$ satisfying (b) of Theorem 7. Suppose that each finite subset of $F$ is co-framal. Then $F$ is a regular subprimal cluster.*

This follows from the above theorem, upon applying Remark 1.

We now consider special subclasses of co-coupled and co-framal subprimal algebras.

DEFINITION 4. An *s-couple* is a singular subprimal algebra $\mathcal{U} = (A\,; \times, T)$ of species $S = (2,2)$ containing elements 0,1 $(0 \neq 1)$ such that (1)-(3) hold. An *r-frame* is a regular subprimal algebra $\mathcal{U} = (A\,; \times, {}^{\cap})$ of species $S = (2, 1)$ containing elements 0,1 $(0 \neq 1)$ for which (2), (4), and (5) hold.

Examples of $s$-couples are plentiful. Two such examples are (see [5]):

($1^0$) The « double groups » $\mathcal{C} = (C\,; \times, +)$ of finite order $n \geq 2$ in which $(C\,; +)$ is a cyclic group with identity 0 and generator 1, $(C \setminus \{0\}\,; \times)$ is a group with identity 1, and $0 \times \xi = \xi \times 0 = 0$ ($\xi$ in $C$); and

($2^0$) the algebras $\mathcal{C}_p = (C_p\,; \times +)$ of $p$ elements $0, 1, \dots, p-1$ ($p$ a prime) in which $\xi + \eta =$ addition mod $p$, and $\xi \times \eta = \min(\xi, \eta)$ in the ordering $0, 2, 3, \dots, p-1, 1$.

To establish other classes of $r$-frames and $s$-couples we need the following definitions and lemmas.

DEFINITION 5. A *binary algebra* is an algebra $\mathcal{B} = (B\,; \times)$ of species $S = (2)$ which possesses elements 0,1 $(0 \neq 1)$ satisfying

(15) $\qquad 0 \times \xi = \xi \times 0 = 0\,; \qquad 1 \times \xi = \xi \times 1 = \xi$ (all $\xi$ in $B$).

The element 0 is called the *null* of $\mathcal{B}$; 1 is called the *identity*.

LEMMA 4. (*Foster and Pixley* [5]). *An algebra* $\mathcal{B} = (B\,;\Omega)$ *of species* $S$ *is a regular subprimal if and only if there exist elements* $0,1$ *in* $B$ $(0 \neq 1)$ *and functions* $\times$ *(binary) and* $^{\cap}$ *(unary) defined in* $B$ *such that* (15) *holds, in addition to*

($1^0$) $\mathcal{B}$ *is a finite algebra of at least three elements*;

($2^0$) $\mathcal{B}$ *possesses a unique subalgebra* $(\neq \mathcal{B})$, *denoted by* $\mathcal{B}^* = (B^*; \Omega)$ *and* $B^*$ *contains at least two elements;*

($3^0$) *the elements* $0,1$ *and the functions* $\times, {}^{\cap}$ *are each* $S$*-expressible;*

($4^0$) $^{\cap}$ *is a permutation of* $B$ *in which* $0^{\cap} = 1$;

($5^0$) *for each* $b$ *in* $B$, *the characteristic function* $\delta_b(\xi)$ *(defined below) is* $S$*-expressible :*

$$\delta_b(\xi) = 1 \ \textit{if} \ \xi = b \ \text{and} \ \delta_b(\xi) = 0 \ \textit{if} \ \xi \neq b \ (\textit{all} \ \xi \ \textit{in} \ B);$$

($6^0$) *there exists an element* $b_0$ *in* $B \setminus B^*$ *which is ex-expressible.*

LEMMA 5 (*Foster and Pixley* [5]). *An algebra* $\mathcal{B} = (B\,;\Omega)$ *of species* $S$ *is a singular subprimal if and only if there exist elements* $0,\ 1,\ 1^0$ *in* $B$ $(0 \neq 1)$ *and two binary functions* $\times, T$ *defined in* $B$ *such that* (15) *holds in addition to*

($1^0$) $\mathcal{B}$ *is a finite algebra of at least two elements;*

($2^0$) $\mathcal{B}$ *possesses exactly one one-element subalgebra* $\mathcal{B}^* = (B^*\,; \Omega)$ *and no other subalgebra* $(\neq \mathcal{B})$;

($3^0$) *the element* $0$ *and the functions* $\times, T$ *are each* $S$*-expressible*;

($4^0$) $0\,T\,\xi = \xi\,T\,0 = \xi$ *for each* $\xi$ *in* $B$ *and* $1\,T\,1^0 = 1^0\,T\,1 = 0$;

($5^0$) *for each* $b$ *in* $B \setminus B^*$, *the characteristic function* $\delta_b(\xi)$ *is* $S$*-expressible;*

($6^0$) *there exists an element* $b_0$ *in* $B \setminus B^*$ *which is ex-expressible.*

REMARK 2. If $\xi^{\cap}$ is a permutation on a set, we use $\xi^{\cup}$ to denote its inverse. Moreover, for each positive integer $s$ we define :

$$\xi^{\cap_s} = \mathrm{def} = (\dots(\xi^{\cap})^{\cap}\dots)^{\cap} \ \ (s \ \text{iterations}).$$

We define $\xi^{\cup_s}$ similary. Note that if $\xi^{\cap}$ is a permutation on a finite set, then there exists an integer $s$ such that $\xi^{\cap_s} = \xi^{\cup}$. Hence, any $({}^{\cap}, {}^{\cup})$-expression is just a $({}^{\cap})$-expression.

The following theorems provide large classes of $r$-frames and $s$-couples.

THEOREM 8. *Let* $\mathcal{B} = (P\,; \times, {}^{\cap})$ *be a primal algebra for which*

($1^0$) $(P\,; \times)$ *is a binary algebra (with null* $0$ *and identity* $1$*); and*

($2^0$) $^{\cap}$ *is a cyclic permutation on* $P$ *with* $0^{\cap} = 1$. *If* $P_m = P \cup \{\lambda_1, \dots, \lambda_m\}$

*where $\lambda_i \notin P$, $1 \leq i \leq m$, then the operations $\times$ and $^{\cap}$ can be extended to $P_m$ such that $\mathcal{B}_m = (P_m\,;\, \times, ^{\cap})$ is an r-frame with core $\mathcal{B}$.*

PROOF. Let $^{\cap} = (0, 1, \beta_3, \dots, \beta_n)$ in $P$. Because of primality, there exists a unary $(\times, ^{\cap})$-expression $\Delta(\xi)$ such that

$$\Delta(\xi) = \begin{cases} 1 & \text{if } \xi = 0 \\ 0 & \text{if } \xi \neq 0 \end{cases} \quad (\xi \text{ in } P).$$

We extend the definitions of $\times$ and $^{\cap}$ to $P_m$ as follows:

(i) For $\xi, \eta$ in $P$ define $\xi \times \eta$ and $\xi^{\cap}$ in $P_m$ just as in $P$;

(ii) $\lambda_1^{\cap} = \lambda_2$, $\lambda_2^{\cap} = \lambda_3, \dots, \lambda_m^{\cap} = \lambda_1$;

(iii) $\lambda_i \times \lambda_j = \lambda_{\min(i,j)}$ (if $i \neq j$) and $\lambda_i \times \lambda_i = 1$ (for each $i$);

(iv) $0 \times \lambda_i = \lambda_i \times 0 = 0$, $1 \times \lambda_i = \lambda_i \times 1 = \lambda_i$;

(v) $\xi \times \lambda_i$ and $\lambda_i \times \xi$ ($\xi$ in $P$) are defined arbitrarily for other $\xi$.

Each characteristic function $\delta_\tau(\xi)$, $\tau \in P_m$, is $(\times, ^{\cap})$-expressible since the following identities hold in $P_m$ (for a product of more than two terms, assume that the association is from the left):

$$\delta_0(\xi) = \Delta(\xi)\,\Delta(\xi \times \xi),\ \delta_1(\xi) = \delta_0(\xi^{\cup}), \dots, \delta_{\beta_n}(\xi) = \delta_0(\xi^{\cup^{n-1}});$$

$$\delta_{\lambda_1}(\xi) = (\xi \cdot \xi^{\cap} \dots \xi^{\cap^{n-1}})^2 (((\xi\,\xi^{\cap} \dots \xi^{\cap^{m-1}})(\xi^{\cap}\,\xi^{\cap^2} \dots \xi^{\cap^{m-1}}))^{\cup})^2$$

$$\delta_{\lambda_2}(\xi) = \delta_{\lambda_1}(\xi^{\cup}), \dots, \delta_{\lambda_m}(\xi) = \delta_{\lambda_1}(\xi^{\cup^{m-1}}).$$

In the above, $\xi^{\cup}$ denotes the inverse of $\xi^{\cap}$. Since $P_m$ is finite, $\xi^{\cup}$ is a $(^{\cap})$-expression. Moreover, $0, 1, \beta_3, \dots, \beta_n$ are $(\times, ^{\cap})$-expressible and $\lambda_1, \dots, \lambda_m$ are $(\times, ^{\cap})$-ex-expressible, since

$$0 = \delta_0(\xi) \times \delta_1(\xi),\ 1 = 0^{\cap},\ \beta_3 = 0^{\cap^2}, \dots, \beta_n = 0^{\cap^{n-1}} \ (\xi \text{ in } P_m);$$

$$\lambda_1 = \xi \cdot \xi^{\cap} \dots \xi^{\cap^{m-1}},\ \lambda_2 = \lambda_1^{\cap}, \dots, \lambda_m = \lambda_1^{\cap^{m-1}} \ (\xi \text{ in } P_m \setminus P).$$

Clearly, $\mathcal{B}$ is the unique proper subalgebra of $\mathcal{B}_m$. The conditions $(1^0)$-$(6^0)$ of Lemma 4 are verified. Thus, $\mathcal{B}_m$ is a regular subprimal algebra and, indeed, even an $r$-frame. The theorem is proved.

THEOREM 9. *Let $(B\,;\,\times)$ be a finite binary algebra. Then a binary operation $\xi\,T\,\eta$ can be defined on $B$ such that $(B\,;\,\times, T)$ is an s-couple.*

PROOF. For the two-element binary algebra $(\{0,1\};\times)$ it is easily verified that conditions $(1^0)$-$(6^0)$ of Lemma 5 hold if $\xi T \eta$ is defined by $0 T \xi = \xi T 0 = \xi$ and $1 T 1 = 0$. Let, then $B = \{0, 1, b_1, \dots, b_m\}$ be the base set of a binary algebra of order $m+2$, where $m \geq 1$. Consider the cases (I) $m \geq 2$ and (II) $m = 1$. For (I), define $T$ on $B$ such that

$$0 T \xi = \xi T 0 = \xi \quad (\text{each } \xi \text{ in } B); \tag{16}$$

$$1 T 1 = b_1,\ b_1 T b_1 = b_2, \dots, b_m T b_m = 1; \tag{17}$$

$$1 T b_1 = 1,\ b_1 T b_2 = b_2 T b_3 = \dots = b_{m-1} T b_m = b_m T 1 = 1 T b_m = 0;$$

$$\xi T \eta \text{ is defined arbitrarily for other } \xi, \eta \text{ in } B;$$

hold, while for (II), define $T$ on $B$ such that (16) and (17) hold in addition to

$$1 T b_1 = b_1 T 1 = 0.$$

In either case (I) or (II), let $\xi^{\cap} = \xi T \xi$. In the characteristic function $\delta_1(\xi)$ is $(\times, T)$-expressible then $\delta_{b_1}(\xi), \dots, \delta_{b_m}(\xi)$, $\Gamma_1(\xi)$, and 0 are $(\times, T)$-expressible since

$$\delta_{b_m}(\xi) = \delta_1(\xi^{\cap}),\ \delta_{b_{m-1}}(\xi) = \delta_1(\xi^{\cap_2}), \dots, \delta_{b_1}(\xi) = \delta_1(\xi^{\cap_m})\ (\xi \text{ in } B);$$

$$\Gamma_1(\xi) = \delta_1(\xi)\, T\, \delta_{b_1}(\xi)\, T \dots T\, \delta_{b_m}(\xi) = 1\ (\xi \text{ in } B \setminus \{0\});$$

$$0 = \delta_1(\xi) \times \delta_{b_1}(\xi)\ (\xi \text{ in } B).$$

In case (I), $\delta_1(\xi) = \xi T \xi^{\cap}$, while in case (II), $\delta_1(\xi) = \xi^2$, $(\xi T \xi^2)^2$, or $\xi^{\cap} T (\xi \xi^{\cap})$, according as $b_1^2 = 0, 1$, or $b_1$, respectively. In each case, it is clear that $\{0\}$ is the unique subalgebra of $(B; \times, T)$. The conditions $(1^0)$-$(6^0)$ of Lemma 5 are verified. Thus, $(B; \times, T)$ is a singular subprimal algebra and, in fact, an $s$-couple.

We conclude with the following easily proved corollaries of Theorem 7.

COROLLARY 2. *Any subfamily of the family $F_{s_0}$ of all pairwise non-isomorphic s-couples forms a singular subprimal cluster.*

COROLLARY 3. *Any subfamily of the family $F_{r_0}$ of all r-frames with pairwise non isomorphic cores forms a regular subprimal cluster.*

COROLLARY 4. *Any subfamily of the family $F_{s_0} \cup F_{r_0}$ is a subprimal cluster.*

The algebras given in Theorems 8 and 9 apply, of course, to these corollaries.

*Note Added in Proof.* Theorem 8 was obtained independently by A. L. Foster, *Monatshefte für Mathematik* 72 (1968), 315-324.

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