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A PSEUDOCONCAVE GENERALIZATION OF GRAUERT'S DIRECT IMAGE THEOREM: II

by YUM-TONG SIU (*)

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§ 9. Cartan's Theorem B with Bounds.

Suppose $(X, \mathcal{O})$ is a complex space of reduction order $\leqq p < \infty$ and $\mathcal{F}$ is a coherent analytic sheaf on $(X, \mathcal{O})$. Suppose $\varrho^0 \in \mathbf{R}_+^n$ and $\pi: (X, \mathcal{O}) \rightarrow K(\varrho^0)$ is a holomorphic map.

For the remaining of this paper we need the following notations. Suppose A and B are two open subsets of X. We say that $A \underset{\pi}{\subset} B$ if $A(\varrho) \subset B$ for some ϱ. We say that $A \underset{\pi}{\subset\subset} B$ if $A(\varrho) \subset\subset B$ for some ϱ.

Suppose $\mathfrak{U}$ and $\mathfrak{V}$ are two collections of open subsets of X. We say that $\mathfrak{U} \underset{\pi}{<} \mathfrak{V}$ if $\mathfrak{U}(\varrho) < \mathfrak{V}$ for some ϱ. We say that $\mathfrak{U} \underset{\pi}{<<} \mathfrak{V}$ if $\mathfrak{U}(\varrho) << \mathfrak{V}$ for some ϱ.

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PROPOSITION 9.1. Suppose $H_1 \subset\subset H_2$ are Stein open subsets of X and $\mathfrak{U}_i$ is a finite Stein open covering of H_i $(i = 1, 2)$ such that $\mathfrak{U}_1 << \mathfrak{U}_2$ and every member of $\mathfrak{U}_2$ is relatively compact in some Stein open subset of X. Then for $l \geqq 1$ there exists $\omega \in \Omega^{(n)}$ satisfying the following. If $\varrho < \omega$ and $f \in Z^l(\mathfrak{U}_2(\varrho), \mathcal{F})$ with $\|f\|_{\mathfrak{U}_2, \varrho} < e$, then there exists $g \in C^{l-1}(\mathfrak{U}_1(\varrho), \mathcal{F})$ with $\|g\|_{\mathfrak{U}_1, \varrho} < C_\varrho e$ such that $\delta g = f$ on $\mathfrak{U}_1(\varrho)$, where C_ϱ is a constant depending only on ϱ. Moreover, if t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$, then C_ϱ can be chosen to be independent of ϱ_n.

PROOF. Take a Stein open subset H_2' of X such that $H_1 \subset\subset H_2' \subset\subset H_2$. By replacing H_2 by H_2' and by replacing $\mathfrak{U}_2$ by $\mathfrak{U}_2 \cap H_2'$, we can assume without loss of generality that H_2 is relatively compact in a relatively compact Stein open subset $\widetilde{H}$ of X.

By replacing X by $\widetilde{H}$, we can assume without loss of generality that X is a complex subspace of a Stein open subset G of $\mathbf{C}^N$. We use the notations of § 8*B*.

By Proposition 7.2 we can choose Stein open subsets E_i of G, $0 \leqq i \leqq 3$, such that $E_{i+1} \subset\subset E_i$ and $H_1 \subset\subset E_i \cap X \subset\subset H_2$. By Proposition 7.3 we can choose finite collections $\mathfrak{D}_i$ of Stein open subsets of G $(1 \leqq i \leqq 3)$ such that $\mathfrak{U}_1 << \mathfrak{D}_i \cap X < \mathfrak{U}_2$, $\mathfrak{D}_{i+1} << \mathfrak{D}_i$, and $|\mathfrak{D}_i| = E_i$. Let $\mathfrak{W}_i = \mathfrak{D}_i \cap X$.

Take $\omega \in \Omega^{(n)}$ and conditions will be imposed on ω later. Take $\varrho < \omega$ and $f \in Z^l(\mathfrak{U}_2(\varrho), \mathcal{F})$ with $\|f\|_{\mathfrak{U}_2, \varrho} < e$. Let $f' = f \mid \mathfrak{W}_1(\varrho)$. $\|f'\|_{\mathfrak{W}_1, \varrho} < e$. By Proposition 8.4 $|\theta_{\mathfrak{D}_1}(f')|_{\mathfrak{D}_2, \varrho} < C^{(1)} e$, where $C^{(1)}$ is a constant.

$$\theta_{\mathfrak{D}_1}(f') \in Z^l(K(\varrho) \times \mathfrak{D}_1, \mathcal{F}^*).$$

By Proposition 5.2, if $\omega \leqq \omega^1$ for a suitable $\omega^1 \in \Omega^{(n)}$ (and we assume this to be the case), then there exists $g' \in C^{l-1}(K(\varrho) \times \mathfrak{D}_3, \mathcal{F}^*)$ such that $\delta g' = \theta_{\mathfrak{V}_1}(f')$ on $K(\varrho) \times \mathfrak{D}_3$ and $|g'|_{\mathfrak{D}_3, \varrho} < C_\varrho^{(2)} C^{(1)} e$, where $C_\varrho^{(2)}$ is a constant depending only on ϱ. Let $g = \theta_{\mathfrak{D}_3}^{-1}(g') \in C^{l-1}(\mathfrak{W}_3(\varrho), \mathcal{F})$. Then $\delta g = f$ on $\mathfrak{W}_3(\varrho)$. By Proposition 8.2, $\|g\|_{\mathfrak{U}_1, \varrho} < C^{(3)} C_\varrho^{(2)} C^{(1)} e$, where $C^{(3)}$ is a constant.

If t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$, then t_n is not a zero-divisor for $\mathcal{F}_x^*$ for $K(\varrho^0) \times G$. In this case, C_ϱ can be chosen to independent of ϱ_n. q. e. d.

PROPOSITION 9.2. Suppose $H_1 \subset\subset H_2$ are Stein open subsets of X and $\mathfrak{U}_2$ is a finite Stein open covering of H_2 such that every member of $\mathfrak{U}_2$ is relatively compact is some Stein open subset of X. Then there exists $\omega \in \Omega^{(n)}$ satisfying the following. If $\varrho < \omega$ and $f \in \Gamma(H_2(\varrho), \mathcal{F})$ with

$\|\hat{f}\|_{\mathfrak{U}_2, \varrho} < e$, where $\hat{f} \in Z^0(\mathfrak{U}_2(\varrho), \mathcal{F})$ is induced by f, then $\|f\|_{H_1, \varrho} < C_\varrho e$, where C_ϱ is a constant depending only on ϱ. Moreover, if t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$, then C_ϱ can be chosen to be independent of ϱ_n.

PROOF. As in the proof of Proposition 9.1 we can assume that X is a complex subspace of a Stein open subset G of $\mathbf{C}^N$. We use the notations of § 8*B*.

By Proposition 7.2 we can find Stein open subsets G_i of G $(0 \leqq i \leqq 3)$ sdch that $G_{i+1} \subset\subset G_i$ and $H_1 \subset\subset G_i \cap X \subset H_2$. Let $E_i = G_i \cap X$. By Proposition 7.3 we can find finite collections $\mathfrak{D}_i$ of G, $1 \leqq i \leqq 3$, such that $\mathfrak{D}_{i+1} << \mathfrak{D}_i$, $\mathfrak{D}_i \cap X < \mathfrak{U}_2$, and $|\mathfrak{D}_i| = G_i$. Let $\mathfrak{W}_i = \mathfrak{D}_i \cap X$.

Take $\omega \in \Omega^{(n)}$ and we shall impose conditions on ω later. Take $\varrho < \omega$ and $f \in \Gamma(H_2(\varrho), \mathcal{F})$ with $\|\hat{f}\|_{\mathfrak{U}_2, \varrho} < e$, where $\hat{f} \in Z^0(\mathfrak{U}_2(\varrho), \mathcal{F})$ is induced by f. Let $g = \theta_{G_1}(f \mid E_1(\varrho))$ and $\hat{g} = \theta_{\mathfrak{D}_1}(\hat{f} \mid \mathfrak{W}_1(\varrho))$. $\hat{g} \in Z^0(K(\varrho) \times \mathfrak{D}_1, \mathcal{F}^*)$ is induced by $g \in \Gamma(K(\varrho) \times G_1, \mathcal{F}^*)$.

By Proposition 8.4, $|\hat{g}|_{\mathfrak{D}_2, \varrho} < C^{(1)} e$, where $C^{(1)}$ is a constant. By Proposition 5.3, if $\omega \leqq \omega^1$ for a suitable $\omega^1 \in \Omega^{(n)}$ (and we assume this to be the case), then $|g|_{G_3, \varrho} < C_\varrho^{(2)} C^{(1)} e$, where $C_\varrho^{(2)}$ is a constant depending only on ϱ. By Proposition 8.1, $\|f\|_{H_1, \varrho} < C^{(3)} C_\varrho^{(2)} C^{(1)} e$, where $C^{(3)}$ is a constant.

If t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$, then t_n is not a zero-divisor for $\mathcal{F}_x^*$ for $x \in K(\varrho^0) \times G$ and hence $C_\varrho^{(2)}$ can be chosen to be independent of ϱ_n. q. e. d.

PROPOSITION 9.3. Suppose $\mathcal{G}$ is a coherent analytic sheaf on X and φ: $\mathcal{F} \to \mathcal{G}$ is a sheaf-homomorphism. Suppose $H_1 \subset\subset H_2 \subset\subset \tilde{H}$ are Stein open subsets of X. Then there exists $\omega \in \Omega^{(n)}$ satisfying the following. If $\varrho < \omega$ and $g \in \Gamma(H_2(\varrho), \operatorname{Im} \varphi)$ with $\|g\|_{H_2, \varrho} < e$, then for some $f \in \Gamma(H_1(\varrho), \mathcal{F})$, $\varphi(f) = g$ on $H_1(\varrho)$, and $\|f\|_{H_1, \varrho} < C_\varrho e$, where C_ϱ is a constant depending only on ϱ.

PROOF. By replacing X by $\tilde{H}$ and by shrinking $\tilde{H}$, we can assume that X is a complex subspace of a Stein open subset G of $\mathbf{C}^N$. We use the notations of § 8*B*.

By Proposition 7.2, we can find Stein open subsets G_i of G $(1 \leqq i \leqq 3)$ such that $G_{i+1} \subset\subset G_i$ and $H_1 \subset\subset G_i \cap X \subset H_2$. Let $E_i = G_i \cap X$.

Take $\omega \in \Omega^{(n)}$ and we shall impose conditions on ω later. Take $\varrho < \omega$ and $g \in \Gamma(H_2(\varrho), \operatorname{Im} \varphi)$ with $\|g\|_{H_2, \varrho} < e$. Let $g' = g \mid E_1(\varrho)$. $\|g'\|_{E_1, \varrho} < e$. By Proposition 8.3, $|\theta_{G_1}(g')|_{G_2, \varrho} < C^{(1)} e$, where $C^{(1)}$ is a constant.

Let $\mathcal{G}^* = \theta_0(\mathcal{G})$. We have a unique sheaf-homomorphism $\varphi^* : \mathcal{F}^* \longrightarrow \mathcal{G}^*$ on $K(\varrho^0) \times G$ corresponding to φ. $\theta_{G_1}(g') \in \Gamma(K(\varrho) \times G_1, \operatorname{Im} \varphi^*)$. By Proposition 5.4, if $\omega \leqq \omega^1$ for a suitabe $\omega^1 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $f' \in \Gamma(K(\varrho) \times G_3, \mathcal{F}^*)$ such that $|f'|_{G_3, \varrho} < C_\varrho^{(2)} C^{(1)}$ and $\varphi^*(f') = \theta_{G_1}(g')$ on $K(\varrho) \times G_3$, where $C_\varrho^{(2)}$ is a constant depending only on ϱ.

Let $f = \theta_{G_3}^{-1}(f') \in \Gamma(E_3(\varrho), \mathcal{F})$. $\varphi(f) = g$ on $E_3(\varrho)$. By Proposition 8.2, $\|f\|_{H_1, \varrho} < C^{(3)} C_\varrho^{(2)} C^{(1)} e$, where $C^{(3)}$ is a constant. q. e. d.

The following Proposition is a consequence of Proposition 9.3.

Proposition 9.4. Suppose $\mathcal{G}$ is a coherent analytic sheaf on X and $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a sheaf-homomorphism. Suppose $\mathfrak{U}_1 << \mathfrak{U}_2$ are finite collections of Stein open subsets of X such that every member of $\mathfrak{U}_2$ is relatively compact in some Stein open subset of X. Then for $l \geqq 0$ there exists $\omega \in \Omega^{(n)}$ such that, if $\varrho < \omega$ and $g \in C^l(\mathfrak{U}_2(\varrho), \operatorname{Im} \varphi)$ with $\| g \|_{\mathfrak{U}_2, \varrho} < e$, then for some $f \in C^l(\mathfrak{U}_1(\varrho), \mathcal{F})$, $\varphi(f) = g$ and $\|f\|_{\mathfrak{U}_1, \varrho} < C_\varrho e$, where C_ϱ is a constant depending only on ϱ.

Proposition 9.5. If, in Proposition 9.4, the closure of every member of $\mathfrak{U}_1$ admits a neighborhood basis consisting of Stein open subsets of X, then the conclusion of Proposition 9.4 remains valid when the condition $\mathfrak{U}_1 << \mathfrak{U}_2$ is weakened to $\mathfrak{U}_1 \underset{\pi}{<<} \mathfrak{U}_2$.

Proof. Assume the weaker condition $\mathfrak{U}_1 \underset{\pi}{<<} \mathfrak{U}_2$. Since the closure of every member of $\mathfrak{U}_1$ admits a Stein open neighborhood basis, we can find finite collections $\mathfrak{U}'$, $\mathfrak{U}''$ of Stein open subsets of X such that $\mathfrak{U}_1 << \mathfrak{U}' << \mathfrak{U}''$ and $\mathfrak{U}' \underset{\pi}{<<} \mathfrak{U}_2$.

There exists $\varrho^1 \leqq \varrho^0$ in $\mathbf{R}_+^n$ such that $\mathfrak{U}'(\varrho^1) << \mathfrak{U}_2$. Let $\mathfrak{U}_1' = \mathfrak{U}'(\varrho^1)$. Apply Proposition 9.4 to $\mathfrak{U}_1'$ and $\mathfrak{U}_2$ instead of $\mathfrak{U}_1$ and $\mathfrak{U}_2$. Then use Proposition 8.5 with $\mathfrak{W} = \mathfrak{U}_1$, $\widetilde{\mathfrak{W}} = \mathfrak{U}'$, $\mathfrak{U} = \mathfrak{U}'(\varrho^1)$, and $\varrho' = \varrho^1$. q. e. d.

Proposition 9.6. Suppose $H_1 \subset\subset H_2 \subset\subset \widetilde{H}$ are Stein open subsets of X. Suppose t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$. Then there exists $\omega \in \Omega^{(n)}$ satisfying the following. If $\varrho < \omega$ and $f \in \Gamma(H_2(\varrho), \mathcal{F})$ with $\left\| \left(\frac{t_n}{\varrho_n} \right)^l f \right\|_{H_2, \varrho} < e$ for some $l \in \mathbf{N}_*$, then $\|f\|_{H_1, \varrho} < C_\varrho e$, where C_ϱ is a constant depending only on ϱ and is independent of l.

PROOF. We can assume without loss of generality that X is a complex subspace of a Stein open subset G of $\mathbf{C}^N$. We use the notations of § 8B.

By Proposition 7.2 we can choose Stein open subsets G_i of G $(1 \leqq i \leqq 3)$ such that $G_{i+1} \subset\subset G_i$ and $H_1 \subset\subset G_i \cap X \subset H_2$. Let $E_i = G_i \cap X$.

Take $\omega \in \Omega^{(n)}$ and conditions will be imposed on ω later. Take $\varrho < \omega$ and $f \in \Gamma(H_2(\varrho), \mathcal{F})$ such that $\left\| \left(\frac{t_n}{\varrho_n}\right)^l f \right\|_{H_2, \varrho} < e$ for some $l \in \mathbf{N}_*$.

Let $g = \theta_{G_1}(f | E_1(\varrho)) \in \Gamma(K(\varrho) \times G_1, \mathcal{F}^*)$. Then $\left(\frac{t_n}{\varrho_n}\right)^l g = \theta_{G_1}\left(\left(\frac{t_n}{\varrho_n}\right)^l f | E_1(\varrho)\right)$. By Proposition 8.3, $\left| \left(\frac{t_n}{\varrho_n}\right)^l g \right|_{G_2, \varrho} < C^{(1)} e$, where $C^{(1)}$ is a constant.

Since t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$, t_n is not a zero-divisor for $\mathcal{F}_x^*$ for $x \in K(\varrho^0) \times G$. By Proposition 5.6, if $\omega \leqq \omega^1$ for a suitable $\omega^1 \in \Omega^{(n)}$ (and we assume this to be the case), then $| g |_{G_3, \varrho} < C_\varrho^{(2)} C^{(1)} e$, where $C_\varrho^{(2)}$ is a constant depending only on ϱ. Note that ω^1 and $C_\varrho^{(2)}$ are independent of l. By Proposition 8.1, $\| f \|_{H_1, \varrho} < C^{(3)} C_\varrho^{(2)} C^{(1)} e$, where $C^{(3)}$ is a constant. q. e. d.

The following is a consequence of Proposition 9.6.

PROPOSITION 9.7. Suppose $\mathfrak{U}_1 << \mathfrak{U}_2$ are finite collections of Stein open subsets of X such that every member of $\mathfrak{U}_2$ is relatively compact in some Stein open subset of X. Suppose t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$. Then for $r \geqq 0$ there exists $\omega \in \Omega^{(n)}$ satisfying the following. If $\varrho < \omega$ and $f \in C^r(\mathfrak{U}_2(\varrho), \mathcal{F})$ with $\left\| \left(\frac{t_n}{\varrho_n}\right)^l f \right\|_{\mathfrak{U}_2, \varrho} < e$ for some $l \in \mathbf{N}_*$, then $\| f \|_{\mathfrak{U}_1, \varrho} < C_\varrho e$, where C_ϱ is a constant depending only on ϱ and is independent of l.

The following Proposition is derived from Proposition 9.7 in the same way as Proposition 9.5 is derived from Proposition 9.4.

PROPOSITION 9.8. If, in Proposition 9.7, the closure of every member of $\mathfrak{U}_1$ admits a neighborhood basis consisting of Stein open subsets of X, then the conclusion of Proposition 9.7 remains valid when the condition $\mathfrak{U}_1 << \mathfrak{U}_2$ is weakened to $\mathfrak{U}_1 \underset{\pi}{<<} \mathfrak{U}_2$

§ 10. Leray's Isomorphism Theorem with Bounds.

A. Before we go on, we have to introduce some notations which are to be used throughout the remaining of this paper.

Suppose X is a complex space. $S(X)$ denotes the collection of subsets of X defined as follows. $U \in S(X)$ if and only if (i) U is a relatively compact Stein open subset of X and (ii) U^- admits a neighborhood basis consisting of Stein open subsets of X.

LEMMA 10.1. Suppose W is a Stein open subset of X and L is a compact subset of W. Then there exists $U \in S(X)$ such that $L \subset U \subset W$.

PROOF. We can assume that there exists a strictly 1-convex function φ on W such that $B_c := \{\varphi < c\}$ is relatively compact in W for every real number c. By applying Sard's Theorem to the restrictions of φ to the regular points of W, $\sigma(W)$, $\sigma(\sigma(W))$, ... (where $\sigma(Y)$ denotes the singular set of Y), we conclude that the image A under φ of the subset of W where φ attains a local minimum is of linear measure 0. Take a real number c not belonging to A such that B_c contains L. Since the closure of B_c is $\{\varphi \leq c\}$, it suffices to set $U = B_c$. q. e. d.

As a consequence of Lemma 10.1, we have

LEMMA. 10.2. If $U \in S(X)$, then U^- admits a neighborhood basis consisting of elements of $S(X)$.

We define the notation $\mathbf{S}(X)$ as follows. $\mathfrak{U} = \{U_i\}_{i \in I} \in \mathbf{S}(X)$ if and only if (i) I is finite and (ii) $U_i \in S(X)$

The following follows from Lemmas 10.1 and 10.2.

LEMMA. 10.3. (*a*) If $\mathfrak{U}_1 << \mathfrak{U}_2$ are finite collections of open subsets of X and every member of $\mathfrak{U}_2$ is Stein, then there exists $\mathfrak{U} \in \mathbf{S}(X)$ such that $\mathfrak{U}_1 << \mathfrak{U} < \mathfrak{U}_2$.

(*b*) If $\mathfrak{U}_1 << \mathfrak{U}_2$ are finite collections of open subsets of X and $\mathfrak{U}_1 \in \mathbf{S}(X)$, then there exists $\mathfrak{U} \in \mathbf{S}(X)$ such that $\mathfrak{U}_1 << \mathfrak{U} < \mathfrak{U}_2$.

B. Suppose X is a complex space and $\mathcal{F}$ is a coherent analytic sheaf on X. Suppose $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathfrak{V} = \{V_i\}_{i \in I}$ are collections of Stein open subsets of X such that $|\mathfrak{U}| = |\mathfrak{V}|$.

We introduce the following notations.

$$C^\mu(\mathfrak{U}) = C^\mu(\mathfrak{U}, \mathcal{F})$$
$$C^\nu(\mathfrak{V}) = C^\nu(\mathfrak{V}, \mathcal{F})$$

$C^{\mu,\nu}(\mathfrak{U}, \mathfrak{V})$ consists of all $\xi = \{\xi^{\alpha_0 \dots \alpha_\mu}_{i_0 \dots i_\nu}\}$, where $\xi^{\alpha_0 \dots \alpha_\mu}_{i_0 \dots i_\nu} \in \Gamma(U_{\alpha_0 \dots \alpha_\mu} \cap V_{i_0 \dots i_\nu}, \mathcal{F})$ is skew-symmetric in $\alpha_0, \dots, \alpha_\mu$ and skew-symmetric in $i_0, \dots, i_\nu$.

Define $\delta_1 : C^{\mu,\nu}(\mathfrak{U}, \mathfrak{V}) \to C^{\mu+1,\nu}(\mathfrak{U}, \mathfrak{V})$ and $\delta_2 : C^{\mu,\nu}(\mathfrak{U}, \mathfrak{V}) \to C^{\mu,\nu+1}(\mathfrak{U}, \mathfrak{V})$ as follows:

$$(\delta_1 \xi)^{\alpha_0 \dots \alpha_{\mu+1}}_{i_0 \dots i_\nu} = \Sigma_{\lambda=0}^{\mu+1} (-1)^\lambda \xi^{\alpha_0 \dots \hat{\alpha}_\lambda \dots \alpha_{\mu+1}}_{i_0 \dots i_\nu}$$

$$(\delta_2 \xi)^{\alpha_0 \dots \alpha_\mu}_{i_0 \dots i_{\nu+1}} = \Sigma_{\lambda=0}^{\nu+1} (-1)^\lambda \xi^{\alpha_0 \dots \alpha_\mu}_{i_0 \dots \hat{i}_\lambda \dots i_{\nu+1}} .$$

Define $\theta_1 : C^\nu(\mathfrak{U}) \to C^{0,\nu}(\mathfrak{U}, \mathfrak{V})$ and $\theta_2 : C^\mu(\mathfrak{U}) \to C^{\mu,0}(\mathfrak{U}, \mathfrak{V})$ as follows.

$$(\theta_1 \xi)^{\alpha_0}_{i_0 \cdots i_\nu} = \xi_{i_0 \cdots i_\nu} \mid U_{\alpha_0} \cap V_{i_0 \cdots i_\nu}.$$

$$(\theta_2 \xi)^{\alpha_0 \cdots \alpha_\mu}_{i_0} = \xi^{\alpha_0 \cdots \alpha_\mu} \mid U_{\alpha_0 \cdots \alpha_\mu} \cap V_{i_0}.$$

Consider the following diagram:

$$(10.1)_{\mathfrak{U}, \mathfrak{V}} \qquad \begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \to \Gamma(X, \mathcal{F}) & \to & C^0(\mathfrak{U}) & \xrightarrow{\delta} & C^1(\mathfrak{U}) & \xrightarrow{\delta} & \cdots \\
\downarrow & & \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \theta_2} & & \\
0 \to C^0(\mathfrak{V}) & \xrightarrow{\theta_1} & C^{0,0}(\mathfrak{U}, \mathfrak{V}) & \xrightarrow{\delta_1} & C^{1,0}(\mathfrak{U}, \mathfrak{V}) & \xrightarrow{\delta_1} & \cdots \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \delta_2} & & \\
0 \to C^1(\mathfrak{V}) & \xrightarrow{\theta_1} & C^{0,1}(\mathfrak{U}, \mathfrak{V}) & \xrightarrow{\delta_1} & C^{1,1}(\mathfrak{U}, \mathfrak{V}) & \xrightarrow{\delta_1} & \cdots \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \delta_2} & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

The diagram is commutative. Since every member of $\mathfrak{U}$ and $\mathfrak{V}$ is Stein, all rows except the first and all columns except the first are exact.

We say that $f_{*l}, f_{0,l-1}, f_{1,l-2}, \ldots, f_{l-2,1}, f_{l-1,0}, f_{l*}$ form a *zigzag l-sequence* in $(10.1)_{\mathfrak{U},\mathfrak{V}}$ if the following four conditions are satisfied.

(i) $f_{*l} \in Z^l(\mathfrak{U}, \mathcal{F}), f_{\nu, l-\nu-1} \in C^{\nu, l-\nu-1}(\mathfrak{U}, \mathfrak{V})$, and $f_{l*} \in Z^l(\mathfrak{U}, \mathcal{F})$.

(ii) $$\theta_1 f_{*l} = \delta_2 f_{0,l-1}.$$

(iii) $$\delta_1 f_{\nu-1,l-\nu} = \delta_2 f_{\nu,l-\nu-1}$$

(iv) $$\theta_2 f_{l*} = \delta_1 f_{l-1,0}.$$

The following two statements are well-known and can easily be proved by diagram-chasing.

(i) For every $f_{*l} \in Z^l(\mathfrak{V}, \mathcal{F})$ there exists a zigzag l-sequence in $(10.1)_{\mathfrak{U}, \mathfrak{V}}$ with f_{*l} as the first member.

(ii) the correspondence which relates the first member of a zigzag l-sequence in $(10.1)_{\mathfrak{U}, \mathfrak{V}}$ to its last member induces an isomorphism from $H^l(\mathfrak{V}, \mathcal{F})$ to $H^l(\mathfrak{U}, \mathcal{F})$. We call this isomorphism the ***zigzag isomorphism.***

PROPOSITION 10.1. When $\mathfrak{U} = \mathfrak{V}$, the zigzag isomorphism from $H^l(\mathfrak{V}, \mathcal{F})$ to $H^l(\mathfrak{V}, \mathcal{F})$ is equal to the isomorphism mapping every element ξ of $H^l(\mathfrak{V}, \mathcal{F})$ to $(-1)^{\frac{l(l+1)}{2}} \xi$.

PROOF. Take $\xi \in Z^l(\mathfrak{V}, \mathcal{F})$. Let $\xi = \{\xi_{\alpha_0 \dots \alpha_l}\}$. Define a zigzag l-sequence in $(10.1)_{\mathfrak{V}, \mathfrak{V}}$ as follows:

$$f_{*l} = \xi.$$

$$(f_{\nu, l-\nu-1})^{\alpha_0 \dots \alpha_\nu}_{i_0 \dots i_{l-\nu-1}} = (-1)^{\frac{\nu(\nu+1)}{2}} \xi_{\alpha_0 \dots \alpha_\nu i_0 \dots i_{l-\nu-1}}.$$

$$f_{l*} = (-1)^{\frac{l(l+1)}{2}} \xi.$$

To prove that this forms a zigzag l-sequence in $(10.1)_{\mathfrak{V}, \mathfrak{V}}$, we need only check the following:

$$\theta_1 f_{*l} = \delta_2 f_{0, l-1}.$$

$$\delta_1 f_{\nu-1, l-\nu} = \delta_2 f_{\nu, l-\nu-1}.$$

$$\theta_2 f_{l*} = \delta_1 f_{l-1, 0}.$$

(i) Since $\delta \xi = 0$, $(\delta \xi)_{\alpha_0 i_0 \dots i_l} = 0$ on $V_{\alpha_0} \cap V_{i_0 \dots i_l}$.

$$\xi_{i_0 \dots i_l} - \Sigma_{\lambda=0}^{l} (-1)^\lambda \xi_{\alpha_0 i_0 \dots \hat{i}_\lambda \dots i_l} = 0.$$

Hence $\theta_1 f_{*l} = \delta_2 f_{0, l-1}$.

(ii) Since $\delta \xi = 0$, $(\delta \xi)_{\alpha_0 \dots \alpha_\nu i_0 \dots i_{l-\nu}} = 0$ on

$$V_{\alpha_0 \dots \alpha_\nu} \cap V_{i_0 \dots i_{l-\nu}}.$$

$$\Sigma_{\lambda=0}^{\nu} (-1)^\lambda \xi_{\alpha_0 \dots \hat{\alpha}_\lambda \dots \alpha_\nu i_0 \dots i_{l-\nu}} + (-1)^{\nu+1} \Sigma_{\lambda=0}^{l-\nu} (-1)^\lambda \xi_{\alpha_0 \dots \alpha_\nu i_0 \dots \hat{i}_\lambda \dots i_{l-\nu}} = 0.$$

$$(-1)^{\frac{(\nu-1)\nu}{2}} (\delta_1 f_{\nu-1, l-\nu})^{\alpha_0 \dots \alpha_\nu}_{i_0 \dots i_{l-\nu}} + (-1)^{\nu+1} (-1)^{\frac{\nu(\nu+1)}{2}} (\delta_2 f_{\nu, l-\nu-1})^{\alpha_0 \dots \alpha_\nu}_{i_0 \dots i_{l-\nu}} = 0.$$

Hence $\delta_1 f_{\nu-1, l-\nu} = \delta_2 f_{\nu, l-\nu-1}$.

(iii) Since $\delta\xi = 0$, $(\delta\xi)_{\alpha_0 \dots \alpha_l i_0} = 0$ on $V_{\alpha_0 \dots \alpha_l} \cap V_{i_0}$.

$$\Sigma_{\lambda=0}^{l} (-1)^{\lambda} \xi_{\alpha_0 \dots \hat{\alpha}_\lambda \dots \alpha_l i_0} + (-1)^{l+1} \xi_{\alpha_0 \dots \alpha_l} = 0.$$

$$(-1)^{\frac{(l-1)l}{2}} (\delta_1 f_{l-1,0})_{i_0}^{\alpha_0 \dots \alpha_l} + (-1)^{l+1} \xi_{\alpha_0 \dots \alpha_l} = 0.$$

Hence $\theta_2 f_{l^*} = \delta_1 f_{l-1,0}$. q. e. d.

B. Suppose X is a complex space of reduction order $\leqq p < \infty$ and $\mathcal{F}$ is a coherent analytic sheaf on X. Suppose $\varrho^0 \in \mathbf{R}_+^n$ and $\pi : X \to K(\varrho^0)$ is a holomorphic map.

If $\mathfrak{U}, \mathfrak{V} \in \mathbf{S}(X)$ with $|\mathfrak{U}| = |\mathfrak{V}|$ and $\xi = \{\xi_{i_0 \dots i_\nu}^{\alpha_0 \dots \alpha_\mu}\} \in C^{\mu,\nu}(\mathfrak{U}, \mathfrak{V})$, then we denote by $\xi^{\alpha_0 \dots \alpha_\mu}$ the element $\{\xi_{i_0 \dots i_\nu}^{\alpha_0 \dots \alpha_\mu}\}_{i_0, \dots, i_\nu}$ of $C^\nu(U_{\alpha_0 \dots \alpha_\mu} \cap \mathfrak{V}, \mathcal{F})$.

PROPOSITION 10.2. Suppose $\mathfrak{U}, \mathfrak{V}, \mathfrak{W} \in \mathbf{S}(X)$ such that $|\mathfrak{U}| \subset\subset |\mathfrak{V}|$, $\mathfrak{W} << \mathfrak{U}$, and $\mathfrak{W} << \mathfrak{V}$. Then for $l \geqq 1$ there exists $\omega \in \Omega^{(n)}$ satisfying the following. If $\varrho < \omega$ and $\xi \in Z^l(\mathfrak{V}(\varrho), \mathcal{F})$ with $\|\xi\|_{\mathfrak{U},\varrho} < e$, then there exists $\zeta \in Z^l(\mathfrak{U}(\varrho), \mathcal{F})$ and $\eta \in C^{l-1}(\mathfrak{W}(\varrho), \mathcal{F})$ such that $\xi - \delta\eta = \zeta$ on $\mathfrak{W}(\varrho)$, $\|\zeta\|_{\mathfrak{U},\varrho} < C_\varrho\, e$ and $\|\eta\|_{\mathfrak{W},\varrho} < C_\varrho\, e$, where C_ϱ is a constant depending only on ϱ. Moreover, if t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$, then C_ϱ can be chosen to be independent of ϱ_n.

PROOF. (*a*) Construct $\mathfrak{U}_i \in \mathbf{S}(X)$ with the same index set as $\mathfrak{U}$, $0 \leqq i \leqq 2l+1$, such that $\mathfrak{U}_{i+1} << \mathfrak{U}_i$, $\mathfrak{U} = \mathfrak{U}_{2l+1}$, and $|\mathfrak{U}_0| \subset\subset |\mathfrak{V}|$. Constant $\mathfrak{V}'_i \in \mathbf{S}(X)$ with the same index set as $\mathfrak{V}$, $0 \leqq i \leqq 2l+1$, such that $\mathfrak{W} << \mathfrak{V}'_{i+1} << \mathfrak{V}'_i$, $\mathfrak{V} = \mathfrak{V}'_0$, and $|\mathfrak{U}_0| \subset |\mathfrak{V}'_{2l+1}|$.

Define $\mathfrak{V}_i$ as follows. $V \in \mathfrak{V}_i$ if and only if V is the intersection of a member of $\mathfrak{V}'_i$ and a member of $\mathfrak{U}_i$. We have the following diagram:

$$\begin{array}{ccccccccccccc} & & \mathfrak{U} = \mathfrak{U}_{2l+1} & << & \mathfrak{U}_{2l} & << & \dots & << & \mathfrak{U}_1 & << & \mathfrak{U}_0 & & \\ & & \vee & & \vee & & & & \vee & & \vee & & \\ \mathfrak{W} & << & \mathfrak{V}_{2l+1} & << & \mathfrak{V}_{2l} & << & \dots & << & \mathfrak{V}_1 & << & \mathfrak{V}_0 & < & \mathfrak{V} \end{array}$$

and $|\mathfrak{U}_i| = |\mathfrak{V}_i|$. Write $\mathfrak{U}_r = \{U_\alpha^{(r)}\}$ and $\mathfrak{V}_r = \{V_i^{(r)}\}$.

(*b*) Take $\omega \in \Omega^{(n)}$ and we shall impose conditions on ω later. Take $\varrho < \omega$ and $\xi \in Z^l(\mathfrak{V}(\varrho), \mathcal{F})$ with $\|\xi\|_{\mathfrak{V},\varrho} < e$.

We are going to construct, by induction on ν, $\xi_{\nu,l-\nu-1} \in C^{\nu,l-\nu-1}(\mathfrak{U}_{\nu+1}(\varrho), \mathfrak{V}_{\nu+1}(\varrho))$, $0 \leqq \nu \leqq l-1$, and $\xi_{l^*} \in Z^l(\mathfrak{U}_l(\varrho), \mathcal{F})$ such that

(i) $\xi^*, \xi^*_{0,l-1}, \xi^*_{1,l-2}, \dots, \xi^*_{l-2,1}, \xi^*_{l-1,0}, \xi_{l^*}$ form a zigzag l-sequence in $(10.1)_{\mathfrak{U}_l(\varrho), \mathfrak{V}_l(\varrho)}$, where ξ^* is the restriction of ξ to $Z^l(\mathfrak{V}_l(\varrho), \mathcal{F})$ and

$\xi^*_{\nu,\, l-\nu-1}$ is the restriction of $\xi_{\nu,\, l-\nu-1}$ to $C^{\nu,\, l-\nu-1}(\mathfrak{U}_l(\varrho), \mathfrak{V}_l(\varrho))$,

(ii) $\| (\xi_{\nu,\, l-\nu-1})^{\alpha_0 \dots \alpha_\nu} \|_{U^{(\nu+1)}_{\alpha_0 \dots \alpha_\nu} \cap \mathfrak{V}_{\nu+1},\, \varrho} < C_{\nu,\varrho}\, e$, and

(iii) $\| \xi_{l^*} \mid \mathfrak{U}_{l+1}(\varrho) \|_{\mathfrak{U}_{l+1},\, \varrho} < C_{l,\varrho}\, e$,

where $C_{l,\varrho}$, $0 \leqq \nu \leqq l$, are constants depending only on ϱ.

Suppose we have constructed $\xi_{0,\, l-1}, \dots, \xi_{\nu-1,\, l-\nu}$ for some $\nu < l$. (The case $\nu = 0$ means that we have only the empty set to start with).

Define $a \in C^{\nu,\, l-\nu}(\mathfrak{U}_\nu(\varrho), \mathfrak{V}_\nu(\varrho))$ to be

$$\begin{cases} \theta_1(\xi) & \text{for } \nu = 0 \\ \delta_1(\xi_{\nu-1,\, l-\nu}) & \text{for } \nu > 0. \end{cases}$$

Then it is easily seen that $\delta_2\, a = 0$.

$$a^{\alpha_0 \dots \alpha_\nu} \in Z^{l-\nu}(U^{(\nu)}_{\alpha_0 \dots \alpha_\nu}(\varrho) \cap \mathfrak{V}_\nu(\varrho), \mathcal{F}).$$

$$\| a^{\alpha_0 \dots \alpha_\nu} \|_{U^{(\nu)}_{\alpha_0 \dots \alpha_\nu} \cap \mathfrak{V}_\nu,\, \varrho} < C^{(1)}\, C_{\nu-1,\varrho}\, e,$$

where $C^{(1)}$ is a constant and $C_{-1,\varrho} = 1$.

Apply Proposition 9.1 to $U^{(\nu+1)}_{\alpha_0 \dots \alpha_\nu} \cap \mathfrak{V}_{\nu+1} << U^{(\nu)}_{\alpha_0 \dots \alpha_\nu} \cap \mathfrak{V}_\nu$. If $\omega \leqq \omega^1$ for a suitable $\omega^1 \in \Omega^{(n)}$ (and we assume this to be the case), then we can find $b^{\alpha_0 \dots \alpha_\nu} \in C^{l-\nu-1}(U^{(\nu+1)}_{\alpha_0 \dots \alpha_\nu}(\varrho) \cap \mathfrak{V}_{\nu+1}(\varrho), \mathcal{F})$ such that $\delta(b^{\alpha_0 \dots \alpha_\nu}) = a^{\alpha_0 \dots \alpha_\nu}$ on $U^{(\nu+1)}_{\alpha_0 \dots \alpha_\nu}(\varrho) \cap \mathfrak{V}_{\nu+1}(\varrho)$ and $\| b^{\alpha_0 \dots \alpha_\nu} \|_{U^{(\nu+1)}_{\alpha_0 \dots \alpha_\nu} \cap \mathfrak{V}_{\nu+1},\, \varrho} < C^{(2)}_\varrho\, C^{(1)}\, C_{\nu-1,\varrho}\, e$, where $C^{(2)}_\varrho$ is a constant depending only on ϱ. By going to the skew-symmetrization, we can assume that $b^{\alpha_0 \dots \alpha_\nu}$ is skew-symmetric in $\alpha_0, \dots, \alpha_\nu$.

There exists a unique $\xi_{\nu,\, l-\nu-1} \in C^{\nu,\, l-\nu-1}(\mathfrak{U}_{\nu+1}(\varrho), \mathfrak{V}_{\nu+1}(\varrho))$ such that $b^{\alpha_0 \dots \alpha_\nu} = (\xi_{\nu,\, l-\nu-1})^{\alpha_0 \dots \alpha_\nu}$. Set $C_{\nu,\varrho} = C^{(2)}_\varrho\, C^{(1)}\, C_{\nu-1,\varrho}$. The construction of $\xi_{\nu,\, l-\nu-1}$, $0 \leqq \nu \leqq l-1$, by induction on ν is complete.

We are ready to construct ξ_{l^*}. Since

$$\delta_2\, \delta_1\, \xi_{l-1,\, 0} = \delta_1\, \delta_2\, \xi_{l-1,\, 0} = \left\{ \begin{matrix} \delta_1\, \theta_1\, \xi & \text{for} \quad l = 1 \\ \delta_1\, \delta_1\, \xi_{l-2,\, 1} & \text{for} \quad l > 1 \end{matrix} \right\} = 0,$$

there exists a unique $\xi_{l^*} \in Z^l(\mathfrak{U}_l(\varrho), \mathcal{F})$ such that $\theta_2\, \xi_{l^*} = \delta_1\, \xi_{l-1,\, 0}$.

$$\| (\delta_1\, \xi_{l-1,\, 0})^{\alpha_0 \dots \alpha_l} \|_{U^{(l)}_{\alpha_0 \dots \alpha_l} \cap \mathfrak{V}_{l-1},\, \varrho} < C^{(3)}\, C_{l-1,\varrho}\, e,$$

where $C^{(3)}$ is a constant. By Proposition 9.2, if $\omega \leqq \omega^2$ for a suitable

$\omega^2 \in \Omega^{(n)}$ (and we assume this to be the case), then

$$\| (\xi_{l^*})^{\alpha_0 \cdots \alpha_l} \|_{U^{(l+1)}_{\alpha_0 \dots \alpha_l}, \varrho} < C^{(4)}_\varrho C^{(3)} C_{l-1, \varrho} e,$$

where $C^{(4)}_\varrho$ is a constant depending only on ϱ and $(\xi_{l^*})^{\alpha_0 \cdots \alpha_l}$ is the value of ξ_{l^*} at the simplex $(\alpha_0, \dots, \alpha_l)$ of the nerve of $\mathfrak{U}_l(\varrho)$. Hence $\| \xi_{l^*} \|_{\mathfrak{U}_{l+1}, \varrho} < C_{l, \varrho} e$, where $C_{l, \varrho} = C^{(4)}_\varrho C^{(3)} C_{l-1, \varrho}$.

(c) Let ξ', $\xi'_{\nu, l-\nu-1}$, ξ'_{l^*} be the restrictions of ξ, $\xi_{\nu, l-\nu-1}$, ξ_{l^*} to $Z^l(\mathfrak{V}_{l+1}(\varrho), \mathcal{F})$, $C^{\nu, l-\nu-1}(\mathfrak{V}_{l+1}(\varrho), \mathfrak{V}_{l+1}(\varrho))$, and $Z^l(\mathfrak{V}_{l+1}(\varrho), \mathcal{F})$ respectively.

By the proof of Proposition 10.1 we can find a zigzag l-sequence $\hat{\xi}, \hat{\xi}_{0, l-1}, \dots, \hat{\xi}_{l-1, 0}, \hat{\xi}_{l^*}$ in $(10.1)_{\mathfrak{V}_{l+1}(\varrho), \mathfrak{V}_{l+1}(\varrho)}$ such that

(i) $\hat{\xi} = \xi'$,

(ii) $\hat{\xi}_{l^*} = (-1)^{\frac{l(l+1)}{2}} \xi'$, and

(iii) $\| (\hat{\xi}_{\nu, l-\nu-1})^{\alpha_0 \dots \alpha_\nu}_{i_0 \dots i_{l-\nu-1}} \|_{V^{(l+1)}_{\alpha_0 \dots \alpha_\nu} \cap V^{(l+1)}_{i_0 \dots i_{l-\nu-1}}, \varrho} < e$.

We are going to construct, by induction on ν,

$$\gamma_{\nu, l-\nu-2} \in C^{\nu, l-\nu-2}(\mathfrak{V}_{l+\nu+2}(\varrho), \mathfrak{V}_{l+\nu+2}(\varrho)), \quad 0 \leqq \nu \leqq l-2,$$

such that

(i) $\delta_2 \gamma_{0, l-2} = \xi'_{0, l-1} - \hat{\xi}_{0, l-1}$ in $C^{0, l-1}(\mathfrak{V}_{l+2}(\varrho), \mathfrak{V}_{l+2}(\varrho))$,

(ii) $\delta_2 \gamma_{\nu, l-\nu-2} = \xi'_{\nu, l-\nu-1} - \hat{\xi}_{\nu, l-\nu-1} - \delta_1 \gamma_{\nu-1, l-\nu-1}$ in

$$C^{\nu, l-\nu-1}(\mathfrak{V}_{l+\nu+2}(\varrho), \mathfrak{V}_{l+\nu+2}(\varrho)) \quad \text{for} \quad \nu > 0,$$

and

(iii) $\| (\gamma_{\nu, l-\nu-2})^{\alpha_0 \dots \alpha_\nu}_{i_0 \dots i_{l-\nu-2}} \|_{V^{(l+\nu+2)}_{\alpha_0 \dots \alpha_\nu} \cap V^{(l+\nu+2)}_{i_0 \dots i_{l-\nu-2}}, \varrho} < D_{\nu, \varrho} e$, where $D_{\nu, \varrho}$ is a constant depending only on ϱ. Note that, when $l = 1$, we do not have any $\gamma_{\nu, l-\nu-2}$.

Suppose we have found $\gamma_{0, l-2}, \dots, \gamma_{\nu-1, l-\nu-1}$ for some $0 \leqq \nu < l-2$. For $\nu = 0$, $\delta_2(\xi'_{\nu, l-\nu-1} - \hat{\xi}_{\nu, l-\nu-1}) = \theta_1 \xi' - \theta_1 \hat{\xi} = 0$ in $C^{0, l}(\mathfrak{V}_{l+1}(\varrho), \mathfrak{V}_{l+1}(\varrho))$. For $\nu > 0$,

$$\delta_2(\xi'_{\nu, l-\nu-1} - \hat{\xi}_{\nu, l-\nu-1} - \delta_1 \gamma_{\nu-1, l-\nu-1}) = \delta_1(\xi'_{\nu-1, l-\nu} - \hat{\xi}_{\nu-1, l-\nu} - \delta_2 \gamma_{\nu-1, l-\nu-1})$$

$$= \begin{cases} 0 \text{ for } \nu = 1 \\ \delta_1 \delta_1 \gamma_{\nu-2, l-\nu} \end{cases} = 0 \text{ for } \nu > 1 \text{ in } C^{\nu, l-\nu}(\mathfrak{V}_{l+\nu+1}(\varrho), \mathfrak{V}_{l+\nu+1}(\varrho)).$$

Hence $(\xi'_{\nu, l-\nu-1} - \hat{\xi}_{\nu, l-\nu-1} - \delta_1 \gamma_{\nu-1, l-\nu-1})^{\alpha_0 \dots \alpha_\nu} \in Z^{l-\nu-1}(\mathfrak{V}_{l+\nu+1}(\varrho), \mathcal{F})$, where

$\delta_1 \gamma_{-1, l-1} = 0.$

$$\| (\xi'_{\nu, l-\nu-1} - \hat{\xi}_{\nu, l-\nu-1} - \delta_1 \gamma_{\nu-1, l-\nu-1})^{\alpha_0 \ldots \alpha_\nu} \|_{V^{(l+\nu+1)}_{\alpha_0 \ldots \alpha_\nu} \cap \mathfrak{V}_{l+\nu+1}, \varrho} <$$

$$< (C_{\nu, \varrho} + 1 + C^{(5)} D_{\nu-1, \varrho}) e,$$

where $C^{(5)}$ is a constant and $D_{-1, \varrho} = 0$.

Apply Proposition 9.1 to $V^{(l+\nu-2)}_{\alpha_0 \ldots \alpha_\nu} \cap \mathfrak{V}_{l+\nu+2} << V^{(l+\nu+1)}_{\alpha_0 \ldots \alpha_\nu} \cap \mathfrak{V}_{l+\nu+2}$. If $\omega \leqq \omega^3$ for some suitable $\omega^3 \in \Omega^{(n)}$ (and we assume this to be the case), then there exists $c^{\alpha_0 \ldots \alpha_\nu} \in C^{l-\nu-2} (V^{(l+\nu+2)}_{\alpha_0 \ldots \alpha_\nu} (\varrho) \cap \mathfrak{V}_{l+\nu+2} (\varrho), \mathcal{F})$ such that

$$\delta (c^{\alpha_0 \ldots \alpha_\nu}) = (\xi'_{\nu, l-\nu-1} - \hat{\xi}_{\nu, l-\nu-1} - \delta_1 \gamma_{\nu-1, l-\nu-1})^{\alpha_0 \ldots \alpha_\nu}$$

and

$$\| c^{\alpha_0 \ldots \alpha_\nu} \|_{V^{(l+\nu+2)}_{\alpha_0 \ldots \alpha_\nu} \cap \mathfrak{V}_{l+\nu+2}, \varrho} < C^{(6)}_\varrho (C_{\nu, \varrho} + 1 + C^{(5)} D_{\nu-1, \varrho}) e,$$

where $C^{(6)}_\varrho$ is a constant depending only on ϱ. By going to the skew-symmetrization, we can assume that $c^{\alpha_0 \ldots \alpha_\nu}$ is skew-symmetric in $\alpha_0, \ldots, \alpha_\nu$.

There exists a unique element $\gamma_{\nu, l-\nu-2} \in C^{\nu, l-\nu-2} (\mathfrak{V}_{l+\nu+2} (\varrho), \mathfrak{V}_{l+\nu+2} (\varrho))$ such that $(\gamma_{\nu, l-\nu-2})^{\alpha_0 \ldots \alpha_\nu} = c^{\alpha_0 \ldots \alpha_\nu}$.

$$\| (\gamma_{\nu, l-\nu-2})^{\alpha_0 \ldots \alpha_\nu}_{i_0 \ldots i_{l-\nu-2}} \|_{V^{(l+\nu+2)}_{\alpha_0 \ldots \alpha_\nu} \cap V^{(l+\nu+2)}_{i_0 \ldots i_{l-\nu-2}}, \varrho}$$

$$< D_{\nu, \varrho} e, \text{ where } D_{\nu, \varrho} = C^{(6)} (C_{\nu, \varrho} + 1 + C^{(5)} D_{\nu-1, \varrho}).$$

The construction by induction is complete.

(*d*) For $l = 1$, $\delta_2 (\xi'_{l-1, 0} - \hat{\xi}_{l-1, 0}) = \theta_1 \xi' - \theta_1 \hat{\xi} = 0$ in $C^{l-1, 1} (\mathfrak{V}_{2l} (\varrho), \mathfrak{V}_{2l} (\varrho))$. For $l > 1$,

$$\delta_2 (\xi'_{l-1, 0} - \hat{\xi}_{l-1, 0} - \delta_1 \gamma_{l-2, 0}) = \delta_1 (\xi'_{l-2, 1} - \hat{\xi}_{l-2, 1} - \delta_2 \gamma_{l-2, 0})$$

$$= \begin{cases} 0 \text{ for } l = 2 \\ \delta_1 \delta_1 \gamma_{l-3, 1} = 0 \text{ for } l > 2 \end{cases} \quad \text{in } C^{l-1, 1} (\mathfrak{V}_{2l} (\varrho), \mathfrak{V}_{2l} (\varrho)).$$

Hence there exists a unique element $\eta' \in C^{l-1} (\mathfrak{V}_{2l} (\varrho), \mathcal{F})$ such that $\theta_2 \eta' =$

$= \xi'_{l-1,0} - \hat{\xi}_{l-1,0} - \delta_1 \gamma_{l-2,0}$, where $\delta_1 \gamma_{-1,0} = 0$.

$$\| (\xi'_{l-1,0} - \hat{\xi}_{l-1,0} - \delta_1 \gamma_{l-2,0})^{\alpha_0 \cdots \alpha_l} \|_{V^{(2l)}_{\alpha_0 \cdots \alpha_{l-1}} \cap \mathfrak{V}_{2l}, \varrho} < (C_{l-1,\varrho} + 1 + C^{(6)} D_{l-2,\varrho}) e,$$

where $C^{(6)}$ is a constant.

By Proposition 9.2, if $\omega \leqq \omega^4$ for a suitable $\omega^4 \in \Omega^{(n)}$ (and we assume this to be the case), then $\| (\eta')^{\alpha_0 \cdots \alpha_{l-1}} \|_{V^{(2l+1)}_{\alpha_0 \cdots \alpha_{l-1}}, \varrho} < C^{(7)}_\varrho (C_{l-1,\varrho} + 1 + C^{(6)} D_{l-2,\varrho}) e$, where $C^{(7)}_\varrho$ is a constant depending only on ϱ and $(\eta')^{\alpha_0 \cdots \alpha_{l-1}}$ is the value of η' at the simplex $(\alpha_0, \dots, \alpha_{l-1})$ of the nerve of $\mathfrak{V}_{2l}(\varrho)$. Hence $\| \eta' \|_{\mathfrak{V}_{2l}, \varrho} < C^{(7)}_\varrho (C_{l-1,\varrho} + 1 + C^{(6)} D_{l-2,\varrho}) e$.

(*e*) Let ζ be the restriction of $-(-1)^{\frac{l(l+1)}{2}} \xi_{l^*}$ to $Z^l(\mathfrak{U}(\varrho), \mathcal{F})$ and let η be the restriction of $(-1)^{\frac{l(l+1)}{2}} \eta'$ to $C^{l-1}(\mathfrak{W}(\varrho), \mathcal{F})$. $\| \zeta \|_{\mathfrak{V}, \varrho} < C_\varrho e$ and $\| \eta \|_{\mathfrak{W}, \varrho} < C_\varrho e$, where C_ϱ is the maximum of $C^{(4)}_\varrho C^{(3)} C_{l-1,\varrho}$ and $C^{(7)}_\varrho (C_{l-1,\varrho} + 1 + C^{(6)} D_{l-2,\varrho})$.

In $C^{l,0}(\mathfrak{V}_{2l+1}(\varrho), \mathfrak{V}_{2l+1}(\varrho))$, $\theta_2(\delta \eta') = \delta_1(\theta_2 \eta') = \delta_1(\xi'_{l-1,0} - \hat{\xi}_{l-1,0} - \delta_1 \gamma_{l-2,0}) = \theta_2(\xi_{l^*} - \hat{\xi}_{l^*})$. Hence $\delta\eta' = \xi_{l^*} + (-1)^{\frac{l(l+1)}{2}} \xi$ in $C^l(\mathfrak{V}_{2l+1}(\varrho), \mathcal{F})$. On $\mathfrak{W}(\varrho)$ we have $\xi - \delta\eta = \zeta$.

(*f*) If t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$, then the constants $C_{\nu,\varrho}$, $D_{\nu,\varrho}$, $C^{(i)}_\varrho$ can be chosen to be independent of ϱ_n. Hence C_ϱ can be chosen to independent of ϱ_n. q. e. d.

PROPOSITION 10.3. Suppose $\mathfrak{U}, \mathfrak{U}' \mathfrak{V} \in \mathbf{S}(X)$ such that $\mathfrak{U} << \mathfrak{U}'$, $\mathfrak{V} < \mathfrak{U}'$, and $|\mathfrak{U}| \subset\subset |\mathfrak{V}|$. Then for $l \geqq 1$ there exists $\omega \in \Omega^{(n)}$ satisfying the following. If $\varrho < \omega$ and $\xi \in Z^l(\mathfrak{U}'(\varrho), \mathcal{F})$ and $\eta \in C^{l-1}(\mathfrak{V}(\varrho), \mathcal{F})$ such that $\delta\eta = \xi$ on $\mathfrak{V}(\varrho)$ and $\| \xi \|_{\mathfrak{U}', \varrho} < e$ and $\| \eta \|_{\mathfrak{V}, \varrho} < e$, then there exists $\zeta \in C^{l-1}(\mathfrak{U}(\varrho), \mathcal{F})$ such that $\delta\zeta = \xi$ on $\mathfrak{U}(\varrho)$ and $\| \zeta \|_{\mathfrak{U}, \varrho} < C_\varrho e$, where C_ϱ is a constant depending only on ϱ. Moreover, if t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$, then C_ϱ can be chosen to be independent of ϱ_n.

PROOF. (*a*) Choose $\mathfrak{U}_i \in \mathbf{S}(X)$ with the same index set as $\mathfrak{U}$, $0 \leqq i \leqq l+1$, such that $\mathfrak{U} = \mathfrak{U}_{l+1}$, $\mathfrak{U}_{i+1} << \mathfrak{U}_i << \mathfrak{U}'$, and $|\mathfrak{U}_0| \subset\subset |\mathfrak{V}|$. Choose $\mathfrak{V}'_i \in \mathbf{S}(X)$ with the same index set as $\mathfrak{V}$, $0 \leqq i \leqq l+1$, such that $\mathfrak{V} = \mathfrak{V}'_0$, $\mathfrak{V}'_{i+1} << \mathfrak{V}'_i$, and $|\mathfrak{U}_0| \subset\subset |\mathfrak{V}'_{l+1}|$. Define $\mathfrak{V}_i$ as follows. $V \in \mathfrak{V}_i$ if and only if V is the intersection of a member of $\mathfrak{V}'_i$ and a member of $\mathfrak{U}_i$. We have the following diagram:

$$\begin{array}{ccccccccccc} \mathfrak{U} = \mathfrak{U}_{l+1} & << & \mathfrak{U}_l & << & \dots & << & \mathfrak{U}_1 & << & \mathfrak{U}_0 & << & \mathfrak{U}' \\ \vee & & \vee & & & & \vee & & \vee & & \\ \mathfrak{V}_{l+1} & << & \mathfrak{V}_l & << & \dots & << & \mathfrak{V}_1 & << & \mathfrak{V}_0 & < & \mathfrak{V} \end{array}$$

and $|\mathfrak{U}_i| = |\mathfrak{V}_i|$. Write $\mathfrak{U}_r = \{U_\alpha^{(r)}\}$ and $\mathfrak{V}_r = \{V_i^{(r)}\}$.

(*b*) Take $\omega \in \Omega^{(n)}$ and we shall impose conditions on ω later. Suppose $\varrho < \omega$ and $\xi \in Z^l(\mathfrak{U}'(\varrho), \mathcal{F})$ and $\eta \in C^{l-1}(\mathfrak{V}(\varrho), \mathcal{F})$ such that $\delta\eta = \xi$ on $\mathfrak{V}(\varrho)$ and $\|\xi\|_{\mathfrak{U}', \varrho} < e$ and $\|\eta\|_{\mathfrak{V}, \varrho} < e$.

Let ξ' be the restriction of ξ to $Z^l(\mathfrak{U}_1(\varrho), \mathcal{F})$. From the proof of Proposition 10.1, we can construct a zigzag l-sequence $\xi', \xi_{0, l-1}, \xi_{1, l-2}, \dots$ $\dots, \xi_{l, -1, 0}, (-1)^{\frac{l(l+1)}{2}} \xi'$ in $(10.1)_{\mathfrak{U}_1(\varrho), \mathfrak{U}_1(\varrho)}$ such that

$$\|(\xi_{\nu, l-\nu-1})^{\alpha_0 \dots \alpha_\nu}\|_{U^{(1)}_{\alpha_0 \dots \alpha_\nu} \cap \mathfrak{U}_{1, \varrho}} < e.$$

(*e*) We are going to construct, by induction on ν,

$$\eta_{\nu, l-\nu-2} \in C^{\nu, l-\nu-2}(\mathfrak{U}_{\nu+2}(\varrho), \mathfrak{V}_{\nu+2}(\varrho)), \quad 0 \leqq \nu \leqq l-2,$$

such that

(i) $$\xi_{0, l-1} - \theta_1 \eta = \delta_2 \eta_{0, l-2} \text{ in } C^{0, l-1}(\mathfrak{U}_2(\varrho), \mathfrak{V}_2(\varrho)),$$

(ii) $$\xi_{\nu, l-\nu} - \delta_1 \eta_{\nu-1, l-\nu-1} = \delta_2 \eta_{\nu, l-\nu-2} \text{ in } C^{\nu, l-\nu}(\mathfrak{U}_{\nu+2}(\varrho), \mathfrak{V}_{\nu+2}(\varrho))$$

for $\nu \geqq 1$, and

(iii) $$\|(\eta_{\nu, l-\nu-2})^{\alpha_0 \dots \alpha_\nu}\|_{U^{(\nu+2)}_{\alpha_0 \dots \alpha_\nu} \cap \mathfrak{V}_{\nu+2}, \varrho} < D_{\nu, \varrho}\, e,$$

where $D_{\nu, \varrho}$ is a constant depending only on ϱ.

Suppose we have constructed $\eta_{0, l-2}, \eta_{1, l-3}, \dots, \eta_{\nu-1, l-\nu-1}$ for some $0 \leqq \nu < l-2$.

When $\nu = 0$, we have $\delta_2(\xi_{0, l-1} - \theta_1 \eta) = \theta_1(\delta\xi - \eta) = 0$ in $C^{0, l}(\mathfrak{U}_1(\varrho), \mathfrak{V}_1(\varrho))$ and $\|(\xi_{0, l-1} - \theta_1 \eta)^{\alpha_0}\|_{U^{(1)}_{\alpha_0} \cap \mathfrak{V}_1, \varrho} < 2e$.

When $\nu > 0$, we have

$$\delta_2(\xi_{\nu, l-\nu-1} - \delta_1 \eta_{\nu-1, l-\nu-1}) = \delta_1(\xi_{\nu-1, l-\nu} - \delta_2 \eta_{\nu-1, l-\nu-1})$$

$$= \left\{ \begin{array}{ll} \delta_1 \theta_1 \eta & \text{for } \nu = 1 \\ \delta_1 \delta_1 \eta_{\nu-2, l-\nu} & \text{for } \nu > 1 \end{array} \right\} = 0 \quad \text{in} \quad C^{\nu, l-\nu}(\mathfrak{U}_{\nu+1}(\varrho), \mathfrak{V}_{\nu+1}(\varrho))$$

and

$$\| (\xi_{\nu, l-\nu-1} - \delta_1 \eta_{\nu-1, l-\nu-1})^{\alpha_0 \cdots \alpha_\nu} \|_{U^{(\nu+1)}_{\alpha_0 \cdots \alpha_\nu} \cap \mathfrak{V}_{\nu+1}, \varrho} < (1 + C^{(1)} D_{\nu, \varrho}) e,$$

where $C^{(1)}$ is a constant.

Apply Proposition 9.1 to $U^{(\nu+2)}_{\alpha_0 \cdots \alpha_\nu} \cap \mathfrak{V}_{\nu+2} << U^{(\nu+1)}_{\alpha_0 \cdots \alpha_\nu} \cap \mathfrak{V}_{\nu+1}$. If $\omega \leqq \omega^1$ for a suitable $\omega^1 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $b^{\alpha_0 \cdots \alpha_\nu} \in C^{l-\nu-2}(\mathfrak{V}_{\nu+2}(\varrho), \mathcal{F})$ we have

$$\delta(b^{\alpha_0 \cdots \alpha_\nu}) = \begin{cases} \xi_{0, l-1} - \theta_1 \eta & \text{for } \nu = 0 \\ \xi_{\nu, l-\nu} - \delta_1 \eta_{\nu-1, l-\nu-1} & \text{for } \nu > 0 \end{cases}$$

and

$$\| b^{\alpha_0 \cdots \alpha_\nu} \|_{U^{(\nu+2)}_{\alpha_0 \cdots \alpha_\nu} \cap \mathfrak{V}_{\nu+2}, \varrho} < C^{(2)}_\varrho (1 + C^{(1)} D_{\nu-1, \varrho}) e,$$

where $C^{(2)}_\varrho$ is a constant depending only on ϱ and $C^{(1)} D_{-1, \varrho} = 1$.

By going to the skew-symmetrization, we can assume that $b^{\alpha_0 \cdots \alpha_\nu}$ is skew-symmetric in $\alpha_0, \dots, \alpha_\nu$. Let $\eta_{\nu, l-\nu-2} \in C^{\nu, l-\nu-2}(\mathfrak{U}_{\nu+2}(\varrho), \mathfrak{V}_{\nu+2}(\varrho))$ be the unique element satisfying $(\eta_{\nu, l-\nu-2})^{\alpha_0 \cdots \alpha_\nu} = b^{\alpha_0 \cdots \alpha_\nu}$. The construction by induction is complete if we set $D_{\nu, \varrho} = C^{(2)}_\varrho (1 + C^{(1)} D_{\nu-1, \varrho})$.

(*d*) For

$$l = 1, \quad \delta_2 \xi_{l-1, 0} = \delta_2 \theta_1 \xi = \theta_1 \delta\xi = \theta_1 \delta\delta\eta = 0$$

in

$$C^{l-1, 1}(\mathfrak{U}_l(\varrho), \mathfrak{V}_l(\varrho)).$$

For

$$l > 1, \quad \delta_2 (\xi_{l-1, 0} - \delta_1 \eta_{l-2, 0}) = \delta_1 (\xi_{l-2, 1} - \delta_2 \eta_{l-2, 0})$$

$$= \left. \begin{cases} \delta_1 \theta_1 \eta & \text{for } l = 2 \\ \delta_1 \delta_1 \eta_{l-3, 1} & \text{for } l > 2 \end{cases} \right\} = 0 \quad \text{in} \quad C^{l-1, 1}(\mathfrak{V}_l(\varrho), \mathfrak{V}_l(\varrho)).$$

Hence there exists a unique element ζ' of $C^{l-1}(\mathfrak{U}_l(\varrho), \mathcal{F})$ such that $\theta_2 \zeta' = \xi_{l-1, 0} - \delta_1 \eta_{l-1, 0}$, where $\delta_1 \eta_{-1, 0} = 0$.

$$\| (\xi_{l-1,0} - \delta_1 \eta_{l-2, 0})^{\alpha_0 \cdots \alpha_{l-1}} \|_{U^{(l)}_{\alpha_0 \cdots \alpha_{l-1}} \cap \mathfrak{V}_l, \varrho} < (1 + C^{(3)} D_{l-2, \varrho}) e,$$

where $C^{(3)}$ is a constant. By Proposition 9.2, if $\omega \leqq \omega^2$ for a suitable $\omega^2 \in \Omega^{(n)}$ (and we assume this to be the case), then

$$\| (\zeta')^{\alpha_0 \cdots \alpha_{l-1}} \|_{\mathfrak{U}^{(l+1)}_{\alpha_0 \cdots \alpha_{l-1}}, \varrho} < C^{(4)}_\varrho (1 + C^{(3)} D_{l-2, \varrho}) e,$$

where $C_{\varrho}^{(4)}$ is a constant depending only on ϱ and $(\zeta')^{\alpha_0 \cdots \alpha_{l-1}}$ is the value of ζ at the simplex $(\alpha_0, \ldots, \alpha_{l-1})$ of the nerve of $\mathfrak{U}_{l+1}(\varrho)$.

(*e*) Let ζ be the restriction of $(-1)^{\frac{l(l+1)}{2}} \zeta'$ to $C^{l-1}(\mathfrak{U}_{l+1}(\varrho), \mathcal{F})$. Then $\| \zeta \|_{\mathfrak{U}, \varrho} < C_{\varrho}^{(4)} (1 + C^{(3)} D_{l-2, \varrho})\, e$. In

$$C^{l,0}(\mathfrak{U}_{l+1}(\varrho), \quad \mathfrak{V}_{l+1}(\varrho)), \quad \theta_2(\delta\zeta') = \delta_1(\theta_2 \zeta') =$$

$$\delta_1(\xi_{l-1,0} - \delta_1 \eta_{l-2,0}) = \delta_1 \xi_{l-1,0} = \theta_2((-1)^{\frac{l(l+1)}{2}} \xi).$$

Hence $\delta\zeta = \xi$ on $\mathfrak{U}(\varrho)$.

(*f*) If t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$, then the constants $C_{\varrho}^{(i)}$, $D_{\nu, \varrho}$ can be chosen to be independent of ϱ_n. q. e. d.

PROPOSITION 10.4. The conclusion of Proposition 10.2 remains valid when the conditions $|\mathfrak{U}| \subset\subset |\mathfrak{V}|$, $\mathfrak{W} << \mathfrak{U}$, and $\mathfrak{W} << \mathfrak{V}$ are weakened respectively to $|\mathfrak{U}| \underset{\pi}{\subset\subset} |\mathfrak{V}|$, $\mathfrak{W} \underset{\pi}{<<} \mathfrak{U}$, and $\mathfrak{W} \underset{\pi}{<<} \mathfrak{V}$.

PROOF. Assume the weaker conditions. Choose $\tilde{\mathfrak{U}}, \tilde{\mathfrak{W}} \in \mathrm{S}(X)$ and $\varrho^1 \leqq \varrho^0$ in $\mathbf{R}_+^n$ such that $\mathfrak{U} << \tilde{\mathfrak{U}}$, $\mathfrak{W} << \tilde{\mathfrak{W}}$, $|\tilde{\mathfrak{U}}(\varrho^1)| \subset\subset |\mathfrak{V}|$, $\tilde{\mathfrak{W}}(\varrho^1) << \mathfrak{U}$, and $\tilde{\mathfrak{W}}(\varrho^1) << \mathfrak{V}$. Let $\mathfrak{U}' = \tilde{\mathfrak{U}}(\varrho^1)$ and $\mathfrak{W}' = \mathfrak{W}(\varrho^1)$.

Take $\omega \in \Omega^{(n)}$ with $\omega < \varrho^1$ and we shall impose more conditions on ω later. Take $\varrho < \omega$ and $\xi \in Z^l(\mathfrak{V}(\varrho), \mathcal{F})$ with $\| \xi \|_{\mathfrak{V}, \varrho} < e$.

By Proposition 10.2, if $\omega \leqq \omega^1$ for a suitable $\omega^1 \in \Omega^{(n)}$ (and we assume this to be the case), then there exist $\zeta' \in Z^l(\mathfrak{U}'(\varrho), \mathcal{F})$ and $\eta' \in C^{l-1}(\mathfrak{W}'(\varrho), \mathcal{F})$ such that $\xi - \delta\eta' = \zeta'$ on $\mathfrak{W}'(\varrho)$, $\| \zeta' \|_{\mathfrak{U}', \varrho} < C_{\varrho}^{(1)} e$, and $\| \eta' \|_{\mathfrak{W}', \varrho} < C_{\varrho}^{(1)} e$, where $C_{\varrho}^{(1)}$ is a constant depending only on ϱ (and is independent of ϱ_n when t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$).

Let $\zeta = \zeta' \mid \mathfrak{U}(\varrho)$ and $\eta = \eta' \mid \mathfrak{W}(\varrho)$. By Proposition 8.5, $\| \zeta \|_{\mathfrak{U}, \varrho} < C^{(2)} C_{\varrho}^{(1)}$ e and $\| \eta \|_{\mathfrak{W}, \varrho} < C^{(2)} C_{\varrho}^{(1)}$ e, where $C^{(2)}$ is a constant. $\xi - \delta\eta = \zeta$ on $\mathfrak{W}(\varrho)$. q. e. d.

The following Proposition is derived from Proposition 10.3 in the same way as Proposition 10.4 is derived from Proposition 10.2

PROPOSITION 10.5. The conclusion of Proposition 10.3 remains valid when the conditions $\mathfrak{U} << \mathfrak{U}'$, $\mathfrak{V} < \mathfrak{U}'$, and $|\mathfrak{U}| \subset\subset |\mathfrak{V}|$ are weakened respectively to $\mathfrak{U} \underset{\pi}{<<} \mathfrak{U}'$, $\mathfrak{V} \underset{\pi}{<} \mathfrak{U}'$, and $|\mathfrak{U}| \underset{\pi}{\subset\subset} |\mathfrak{V}|$.

D. Suppose $\varrho^0 \in \mathbf{R}_+^n$, G is an open subset of $\mathbf{C}^N$, and $\mathcal{F}$ is a coherent analytic sheaf on $K(\varrho^0) \times G$. The following two Propositions are proved in the same way as Propositions 10.2 and 10.3. (The last statement of each Proposition follows trivially when we make use of Taylor series expansions.)

PROPOSITION 10.6. Suppose $\mathfrak{U}, \mathfrak{V}, \mathfrak{W} \in \mathbf{S}(G)$ such that $|\mathfrak{U}| \subset\subset |\mathfrak{V}|$, $\mathfrak{W} << \mathfrak{U}$, and $\mathfrak{W} << \mathfrak{V}$. Then for $l \geqq 1$ there exists $\omega \in \Omega^{(n)}$ satisfying the following. If $\varrho < \omega$ and $\xi \in Z^l(K(\varrho) \times \mathfrak{V}, \mathcal{F})$ with $|\xi|_{\mathfrak{V}, \varrho} < e$, then there exists $\zeta \in Z^l(K(\varrho) \times \mathfrak{U}, \mathcal{F})$ and $\eta \in C^{l-1}(K(\varrho) \times \mathfrak{W}, \mathcal{F})$ such that $\xi - \delta\eta = \zeta$ on $K(\varrho) \times \mathfrak{W}$, $|\zeta|_{\mathfrak{U}, \varrho} < C_\varrho e$, and $|\eta|_{\mathfrak{W}, \varrho} < C_\varrho e$, where C_ϱ is a constant depending only on ϱ. Moreover, if t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in K(\varrho^0) \times G$, then C_ϱ can be chosen to be independent of ϱ_n. In the special case $\mathcal{F} = {}_{n+N}\mathcal{O}^q$, ω can be chosen to be ϱ^0 and C_ϱ can be chosen to be independent of ϱ.

PROPOSITION 10.7. Suppose $\mathfrak{U}, \mathfrak{U}', \mathfrak{V} \in \mathbf{S}(G)$ such that $\mathfrak{U} << \mathfrak{U}'$, $\mathfrak{V} < < \mathfrak{U}'$, and $|\mathfrak{U}| \subset\subset |\mathfrak{V}|$. Then for $l \geqq 1$ there exists $\omega \in \Omega^{(n)}$ satisfying the following. If $\varrho < \omega$ and $\xi \in Z^l(K(\varrho) \times \mathfrak{U}', \mathcal{F})$ and $\eta \in C^{l-1}(K(\varrho) \times \mathfrak{V}, \mathcal{F})$ such that $\delta\eta = \xi$ on $K(\varrho) \times \mathfrak{V}$ and $|\xi|_{\mathfrak{U}', \varrho} < e$ and $|\eta|_{\mathfrak{V}, \varrho} < e$, then there exists $\zeta \in C^{l-1}(K(\varrho) \times \mathfrak{U}, \mathcal{F})$ such that $\delta\zeta = \xi$ on $K(\varrho) \times \mathfrak{U}$ and $|\zeta|_{\mathfrak{U}, \varrho} < C_\varrho e$, where C_ϱ is a constant depending only on ϱ. Moreover, if t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in K(\varrho^0) \times G$, then C_ϱ can be chosen to be independent of ϱ_n. In the special case $\mathcal{F} = {}_{n+N}\mathcal{O}^q$, ω can be chosen to be ϱ^0 and C_ϱ can be chosen to be independent of ϱ.

§. 11 Bounded Sheaf Cocycles on Pseudoconcave Spaces.

This section deals with the extension of sheaf cohomology classes on a family of pseudoconcave spaces. The techniques used here are essentially those developed in [1]. The situation we have here is far more complicated than that of [1], because we have parameters and also we have to take care of the bounds. Results developed in § 10 concerning Leray's isomorphism theorem with bounds will be used to cope with this complicated situation.

In the remaining of this paper, we use the following notation. If $\varrho = (\varrho_1, \dots, \varrho_n) \in \mathbf{R}_+^n$, then we denote $(\varrho_1, \dots, \varrho_{n-1}) \in \mathbf{R}_+^{n-1}$ by $\overline{\varrho}$.

A.

DEFINITION. Suppose D is an open subset of $\mathbf{C}^N$ and $q \in \mathbf{N}$.

(*a*) D is called $(H)_q$ if $H^\nu(D, {}_N\mathcal{O}) = 0$ for $1 \leqq \nu < N - q$.

(b) D is called $(H^*)_q$ if for every compact subset L of D there exists a relatively compact $(H)_q$ open neighborhood of L in D.

DEFINITION. Suppose $D, \tilde{D}$ are open subsets of $\mathbf{C}^N$ and $q \in \mathbf{N}$.

(a) $(D, \tilde{D})$ is called an *extension couple* if $D \subset \tilde{D}$ and the restriction map $\Gamma(\tilde{D}, {}_N\mathcal{O}) \to \Gamma(D, {}_N\mathcal{O})$ is bijective.

(b) $(D, \tilde{D})$ is called $(E)_q$ if D is $(H)_q$ and $(D, \tilde{D})$ is an extension couple.

(c) $(D, \tilde{D})$ is called $(E^*)_q$ if for every compact subset L of D and every compact subset $\tilde{L}$ of $\tilde{D}$ there exist a relatively compact open neighborhood D_1 of L in D and a relatively compact open neighborhood $\tilde{D}_1$ of $\tilde{L}$ in $\tilde{D}$ such that $(D_1, \tilde{D}_1)$ is $(E)_q$.

Suppose $q \in \mathbf{N}$, $m \in \mathbf{N}_*$, and $\varrho^0 \in \mathbf{R}^n_+$. Suppose D_i is an $(H)_q$ open subset of $\mathbf{C}^N$, $0 \leqq i \leqq m$, and $\mathcal{F}$ is a coherent analytic sheaf on $K(\varrho^0) \times D_0$ such that

(i) $D_{i+1} \subset\subset D_i$,

(ii) t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in K(\varrho^0) \times D_0$, and

(iii) $\mathcal{F}$ admits a finite free resolution of length m on $K(\varrho^0) \times D_0$.

PROPOSITION 11.1. Suppose $\mathfrak{U}, \mathfrak{V} \in \mathbf{S}(\mathbf{C}^N)$ such that $|\mathfrak{V}| \subset\subset D_m$, $D_0 \subset\subset |\mathfrak{U}|$, and $\mathfrak{V} << \mathfrak{U}$. Then for $1 \leqq \nu < N - q - m$ there exists $\omega \in \Omega^{(n)}$ such that, if $\varrho < \omega$ and $f \in Z^\nu(K(\varrho) \times \mathfrak{U}, \mathcal{F})$ and $|f|_{\mathfrak{U}, \varrho} < e$, then there exists $g \in C^{\nu-1}(K(\varrho) \times \mathfrak{V}, \mathcal{F})$ with $\delta g = f$ on $K(\varrho) \times \mathfrak{V}$ and $|g|_{\mathfrak{V}, \varrho} < C_{\bar{\varrho}}\, e$, where $C_{\bar{\varrho}}$ is a constant depending only on $\bar{\varrho}$. In the special case $m = 0$, $C_{\bar{\varrho}}$ can be chosen to be independent of $\bar{\varrho}$.

PROOF. Use induction on m. Take $\omega \in \Omega^n$ and we shall impose conditions on ω later. Fix $1 \leqq \nu < N - q - m$. Take $\varrho < \omega$ and $f \in Z^\nu(K(\varrho) \times \mathfrak{U}, \mathcal{F})$ with $|f|_{\mathfrak{U}, \varrho} < e$.

(a) $m = 0$. $\mathcal{F} = {}_{n+N}\mathcal{O}^p$ on $K(\varrho^0) \times D_0$ for some $p \in \mathbf{N}$.

Choose a countable Stein open covering $\mathfrak{W}$ of D_0 such that $\mathfrak{W} < \mathfrak{U}$. Choose $\mathfrak{W}^* \in \mathbf{S}(\mathbf{C}^N)$ such that $|\mathfrak{V}| \subset\subset |\mathfrak{W}^*|$ and $\mathfrak{W}^* << \mathfrak{W}$.

By considering Taylor series expansions in $t_1, \dots, t_n$ and applying the open mapping theorem to the continuous linear surjection of Fréchet spaces $\delta : C^{\nu-1}(\mathfrak{W}, {}_N\mathcal{O}^p) \to Z^\nu(\mathfrak{W}, {}_N\mathcal{O}^p)$, we can find $g' \in C^{\nu-1}(K(\varrho) \times \mathfrak{W}^*, {}_{n+N}\mathcal{O}^p)$ such that $\delta g' = f$ on $K(\varrho) \times \mathfrak{W}^*$ and $|g'|_{\mathfrak{W}^*, \varrho} < C^{(1)} e$, where $C^{(1)}$ is a constant.

By Proposition 10.7 there exists $g \in C^{\nu-1}(K(\varrho) \times \mathfrak{V}, {}_{n+N}\mathcal{O}^p)$ such that $\delta g = f$ on $K(\varrho) \times \mathfrak{V}$ and $|g|_{\mathfrak{V}, \varrho} < C^{(2)} C^{(1)} e$, where $C^{(2)}$ is a constant.

(*b*) For the general case assume $m > 0$. Let $\varphi : {}_{n+N}\mathcal{O}^p \to \mathcal{F}$ be the sheaf-epimorphism on $K(\varrho^0) \times D_0$ which is part of the finite free resolution of $\mathcal{F}$ of length m on $K(\varrho^0) \times D_0$. Let $\mathcal{G} = \operatorname{Ker} \varphi$.

Choose $\mathfrak{U}' \in \mathbf{S}(\mathbf{C}^N)$ such that $D_0 \subset\subset |\mathfrak{U}'|$ and $\mathfrak{U}' << \mathfrak{U}$. Choose $\mathfrak{W} \in \mathbf{S}(\mathbf{C}^N)$ such that $D_m \subset\subset |\mathfrak{W}| \subset\subset D_{m-1}$ and $\mathfrak{W} << \mathfrak{U}'$. Choose $\mathfrak{V}' \in \mathbf{S}(\mathbf{C}^N)$ such that $|\mathfrak{V}| \subset\subset |\mathfrak{V}'| \subset\subset D_m$ and $\mathfrak{V}' << \mathfrak{W}$.

By definition of $|\cdot|_{\mathfrak{U}, \varrho}$, we can find $f' \in C^\nu(K(\varrho) \times \mathfrak{U}, {}_{n+N}\mathcal{O}^p)$ such that $\varphi(f') = f$ and $|f'|_{\mathfrak{U}, \varrho} < e$. $\delta f' \in Z^{\nu+1}(K(\varrho) \times \mathfrak{U}, \mathcal{G})$. $|\delta f'|_{\mathfrak{U}, \varrho} < C^{(1)} e$, where $C^{(1)}$ is a constant.

By Proposition 5.1, if $\omega \leqq \omega^1$ for a suitable $\omega^1 \in \Omega^{(n)}$ (and we assume this to be the case), then $|\delta f'|_{\mathfrak{U}', \varrho} < C_{\bar{\varrho}}^{(2)} C^{(1)} e$, where $C_{\bar{\varrho}}^{(2)}$ is a constant depending only on $\bar{\varrho}$.

By induction hypothesis, if $\omega \leqq \omega^2$ for a suitable $\omega^2 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $f'' \in C^\nu(K(\varrho) \times \mathfrak{W}, \mathcal{G})$, $\delta f'' = \delta f'$ on $K(\varrho) \times \mathfrak{W}$ and $|f''|_{\mathfrak{W}, \varrho} < C_{\bar{\varrho}}^{(3)} C_{\bar{\varrho}}^{(2)} C^{(1)} e$, where $C_{\bar{\varrho}}^{(3)}$ is a constant depending only on $\bar{\varrho}$.

$f' - f'' \in Z^\nu(K(\varrho) \times \mathfrak{W}, {}_{n+N}\mathcal{O}^p)$ and $|f' - f''|_{\mathfrak{W}, \varrho} < (1 + C^{(4)} C_{\bar{\varrho}}^{(3)} C_{\bar{\varrho}}^{(2)}) C^{(1)} e$, where $C^{(4)}$ is a constant. By (*a*) there exists $g' \in C^{\nu-1}(K(\varrho) \times \mathfrak{V}', {}_{n+N}\mathcal{O}^p)$ such that $\delta g' = f' - f''$ on $K(\varrho) \times \mathfrak{V}'$ and

$$|g'|_{\mathfrak{V}', \varrho} < C^{(5)} (1 + C^{(4)} C_{\bar{\varrho}}^{(3)} C_{\bar{\varrho}}^{(2)}) C^{(1)} e,$$

where $C^{(5)}$ is a constant $\geqq 1$.

Let $g'' = \varphi(g') \in C^{\nu-1}(K(\varrho) \times \mathfrak{V}', \mathcal{F})$. Then $f = \delta g''$ on $K(\varrho) \times \mathfrak{V}'$ and $|g''|_{\mathfrak{V}', \varrho} < C^{(5)} (1 + C^{(4)} C_{\bar{\varrho}}^{(3)} C_{\bar{\varrho}}^{(2)}) C^{(1)} e$.

By Proposition 10.7, if $\omega \leqq \omega^3$ for a suitable $\omega^3 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $g \in C^{\nu-1}(K(\varrho) \times \mathfrak{V}, \mathcal{F})$, $f = \delta g$ on $K(\varrho) \times \mathfrak{V}$ and $|g|_{\mathfrak{V}, \varrho} < C_{\bar{\varrho}}^{(6)} C^{(5)} (1 + C^{(4)} C_{\bar{\varrho}}^{(3)} C_{\bar{\varrho}}^{(2)}) C^{(1)} e$, where $C_{\bar{\varrho}}^{(6)}$ is a constant depending only on $\bar{\varrho}$. q. e. d..

PROPOSITION 11.2. Suppose $N > q + m$. Suppose $\tilde{D}$ is an open subset of $\mathbf{C}^N$ such that $(D_m, \tilde{D})$ is an extension couple. Suppose $\mathfrak{U}, \mathfrak{V}, \mathfrak{W} \in \mathbf{S}(\mathbf{C}^N)$ such that $|\mathfrak{W}| \subset\subset D_m$, $D_0 \subset\subset |\mathfrak{V}|$, $|\mathfrak{U}| \subset\subset \tilde{D}$, $\mathfrak{W} < \mathfrak{V}$, and $\mathfrak{W} < \mathfrak{U}$. Then there exists $\omega \in \Omega^{(n)}$ such that, if $\varrho < \omega$ and $f \in Z^0(K(\varrho) \times \mathfrak{V}, \mathcal{F})$ and $|f|_{\mathfrak{V}, \varrho} < e$, then there exists $g \in Z^0(K(\varrho) \times \mathfrak{U}, \mathcal{F})$ such that $g = f$ on $K(\varrho) \times \mathfrak{W}$ and $|g|_{\mathfrak{U}, \varrho} < C_{\bar{\varrho}} e$, where $C_{\bar{\varrho}}$ is a constant depending only on $\bar{\varrho}$. In the special case $m = 0$, ω can be chosen to be ϱ^0 and $C_{\bar{\varrho}}$ can be chosen to be independent of $\bar{\varrho}$.

PROOF. Take $\omega \in \Omega^{(n)}$ and we shall impose conditions on $\dot{\omega}$ later. Take $\varrho < \dot{\omega}$ and $f \in Z^0(K(\varrho) \times \mathfrak{V}, \mathcal{F})$ with $|f|_{\mathfrak{V}, \varrho} < e$.

(*a*) Consider first the special case $m = 0$. $\mathcal{F} = {}_{n+N}\bar{O}^p$ for some $p \in \mathbf{N}$.

Choose a countable Stein open covering $\mathfrak{U}'$ of $\widetilde{D}$ such that $\mathfrak{U} << \mathfrak{U}'$. Choose a countable Stein open covering $\mathfrak{V}'$ of D_0 such that $\mathfrak{V}' < \mathfrak{V}$ and $\mathfrak{V}' < \mathfrak{U}'$.

By considering Taylor series expansions in $t_1, \dots, t_n$ and applying the open mapping theorem to the continuous linear bijection of Fréchet spaces $\tau: Z^0(\mathfrak{U}', {}_N\bar{O}^p) \longrightarrow Z^0(\mathfrak{V}', {}_N\bar{O}^p)$ induced by restriction, we conclude that $|\tau^{-1}(f)|_{\mathfrak{U}, \varrho} < Ce$, where C is a constant. Hence $g = \tau^{-1}(f)$ satisfies the requirement.

(*b*) For the general case assume $m > 0$. Let $\varphi: {}_{n+N}\bar{O}^p \longrightarrow \mathcal{F}$ be the epimorphism on $K(\varrho^0) \times D_0$ which is part of the finite free resolution of $\mathcal{F}$ of length m on $K(\varrho^0) \times D_0$. Let $\mathcal{G} = \operatorname{Ker} \varphi$.

Choose $\mathfrak{V}' \in \mathbf{S}(\mathbf{C}^N)$ such that $\mathfrak{V}' << \mathfrak{V}$ and $D_0 \subset\subset |\mathfrak{V}'|$. Choose $\mathfrak{V}'' \in \mathbf{S}(\mathbf{C}^N)$ such that $D_m \subset\subset |\mathfrak{V}''| \subset\subset D_{m-1}$ and $\mathfrak{V}'' << \mathfrak{V}'$. Choose $\mathfrak{W}' \in \mathbf{S}(\mathbf{C}^N)$ such that $|\mathfrak{W}| \subset |\mathfrak{W}'| \subset\subset D_m$, $\mathfrak{W}' < \mathfrak{V}''$, and $\mathfrak{W}' < \mathfrak{U}$.

By definition of $|\cdot|_{\mathfrak{V}, \varrho}$, we can find $f' \in C^0(K(\varrho) \times \mathfrak{V}, {}_{n+N}\bar{O}^p)$ such that $\varphi(f') = f$ on $K(\varrho) \times \mathfrak{V}$ and $|f'|_{\mathfrak{V}, \varrho} < e$. $\delta f' \in Z^1(K(\varrho) \times \mathfrak{V}, \mathcal{G})$ and $|\delta f'|'_{\mathfrak{V}, \varrho} < 2e$.

By Proposition 5.1, if $\omega \leqq \omega^1$ for a suitable $\omega^1 \in \Omega^{(n)}$ (and we assume this to be the case), then $|\delta f'|_{\mathfrak{V}', \varrho} < C^{(1)}_{\bar{\varrho}}\, 2e$, where $C^{(1)}_{\bar{\varrho}}$ is a constant depending only on $\bar{\varrho}$.

Since $N > q + m$, by Proposition 11.1, if $\omega \leqq \omega^2$ for a suitable $\omega^2 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $f' \in C^0(K(\varrho) \times \mathfrak{V}'', \mathcal{G})$, $\delta f' = \delta f''$ and $|f''|_{\mathfrak{V}'', \varrho} < C^{(2)}_{\bar{\varrho}}\, C^{(1)}_{\bar{\varrho}}\, 2e$, where $C^{(2)}_{\bar{\varrho}}$ is a constant depending only on $\bar{\varrho}$.

$f' - f'' \in Z^0(K(\varrho) \times \mathfrak{V}'', {}_{n+N}\bar{O}^p)$ and $|f' - f''| < (1 + C^{(3)}\, C^{(2)}_{\bar{\varrho}}\, C^{(1)}_{\bar{\varrho}})\, 2e$, By (*a*) there exists $g' \in Z^0(K(\varrho) \times \mathfrak{U}, {}_{n+N}\bar{O}^p)$ such that $f' - f'' = g'$ on $K(\varrho) \times \mathfrak{W}'$ and $|g'|_{\mathfrak{U}, \varrho} < C^{(4)}(1 + C^{(3)}\, C^{(2)}_{\bar{\varrho}}\, C^{(1)}_{\bar{\varrho}})\, 2e$, where $C^{(4)}$ is a constant.

Let $g = \varphi(g')$. Then $f = g$ on $K(\varrho) \times \mathfrak{W}$ and $|g|_{\mathfrak{U}, \varrho} < C^{(4)}(1 + C^{(3)}\, C^{(2)}_{\bar{\varrho}}\, C^{(1)}_{\bar{\varrho}})\, 2e$. q. e. d..

B. Suppose X is a complex subspace of a polydisc G of $\mathbf{C}^N$ and the reduction order of X is $\leqq p < \infty$. Suppose $\varrho^0 \in \mathbf{R}^n_+$ and $\pi: X \longrightarrow K(\varrho^0)$ is a holomorphic map. Suppose $\mathcal{F}$ is a coherent analytic sheaf on X such that $\operatorname{codh} \mathcal{F} = r$ and t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$.

Suppose $q \in \mathbf{N}$ and D is a relatively compact open subset of G such that D is $(H^*)_q$. Let $E = D \cap X$.

PROPOSITION 11.3. Suppose $\mathfrak{U}, \mathfrak{V} \in \mathrm{S}(X)$ such that $|\mathfrak{V}| \subset\subset E \subset\subset |\mathfrak{U}|$ and $\mathfrak{V} << \mathfrak{U}$. Then for $1 \leqq \nu < r - q - n$ there exists $\omega \in \Omega^{(n)}$ such that, if $\varrho < \omega$ and $f \in Z^{\nu}(\mathfrak{U}(\varrho), \mathcal{F})$ and $\|f\|_{\mathfrak{U}, \varrho} < e$, then there exists $g \in C^{\nu-1}(\mathfrak{V}(\varrho), \mathcal{F})$ with $\delta g = f$ on $\mathfrak{V}(\varrho)$ and $\|g\|_{\mathfrak{V}, \varrho} < C_{\bar{\varrho}}\, e$, where $C_{\bar{\varrho}}$ is a constant depending only on $\bar{\varrho}$.

PROOF. We use the notations of § 8 *B*.

Since codh $\mathcal{F}^* = r$ on the polydisc $K(\varrho^0) \times G$, by shrinking ϱ^0 and G, we can assume that $\mathcal{F}^*$ admits a finite free resolution of length $m \leqq n + N - r$ on $K(\varrho^0) \times G$. Since t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$, t_n is not a zero-divisor for $\mathcal{F}^*_x$ for $x \in K(\varrho^0) \times G$.

Since D is $(H^*)_q$, we can choose $(H)_q$ open subsets D_i of D, $0 \leqq i \leqq m$, such that $|\mathfrak{V}| \subset\subset D_m$ and $D_{i+1} \subset\subset D_i \subset\subset D$.

Choose $\mathfrak{D}_3 << \mathfrak{D}_2 << \mathfrak{D}_1$ in $\mathrm{S}(G)$ such that $|\mathfrak{V}| \subset\subset |\mathfrak{D}_3| \subset\subset D_m$, $D_0 \subset\subset |\mathfrak{D}_2|$, and $\mathfrak{D}_1 \cap X < \mathfrak{U}$. Let $\mathfrak{U}_i = \mathfrak{D}_i \cap X$, $1 \leqq i \leqq 3$. Choose $\mathfrak{U}_4 \in \mathrm{S}(X)$ such that $\mathfrak{U}_4 << \mathfrak{U}_3$ and $|\mathfrak{V}| \subset\subset |\mathfrak{U}_4|$.

Take $\omega \in \Omega^{(n)}$ and we shall impose conditions on ω later. Fix $1 \leqq \nu < r - q - n$. Take $\varrho < \omega$ and $f \in Z^{\nu}(\mathfrak{U}(\varrho), \mathcal{F})$ with $\|f\|_{\mathfrak{U}, \varrho} < e$.

Let $f^* = \theta_{\mathfrak{D}_1}(f \mid \mathfrak{U}_1(\varrho)) \in Z^{\nu}(K(\varrho) \times \mathfrak{D}_1, \mathcal{F}^*)$. By Proposition 8.4, $|f^*|_{\mathfrak{D}_2, \varrho} < C^{(1)} e$, where $C^{(1)}$ is a constant $\geqq 1$.

By Proposition 11.1, if $\omega \leqq \omega^1$ for a suitable $\omega^1 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $g^* \in C^{\nu-1}(K(\varrho) \times \mathfrak{D}_3, \mathcal{F}^*)$, $\delta g^* = f^*$ on $K(\varrho) \times \mathfrak{D}_3$ and $|g^*|_{\mathfrak{D}_3, \varrho} < C^{(2)}_{\bar{\varrho}}\, C^{(1)} e$, where $C^{(2)}_{\bar{\varrho}}$ is a constant $\geqq 1$ depending only on $\bar{\varrho}$.

Let $g' = \theta^{-1}_{\mathfrak{D}_3}(g^*) \in C^{\nu-1}(\mathfrak{U}_3(\varrho), \mathcal{F})$. By Proposition 8.2, $\|g'\|_{\mathfrak{U}_4, \varrho} < C^{(3)} C^{(2)}_{\bar{\varrho}}\, C^{(1)} e$, where $C^{(3)}$ is a constant $\geqq 1$. $\delta g' = f$ on $\mathfrak{U}_4(\varrho)$.

By Proposition 10.3, if $\omega \leqq \omega^2$ for a suitable $\omega^2 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $g \in C^{\nu-1}(\mathfrak{V}(\varrho), \mathcal{F})$, $\delta g = f$ on $\mathfrak{V}(\varrho)$ and $\|g\|_{\mathfrak{V}, \varrho} < C^{(4)}_{\bar{\varrho}}\, C^{(3)} C^{(2)}_{\bar{\varrho}}\, C^{(1)} e$, where $C^{(4)}_{\bar{\varrho}}$ is a constant depending only on $\bar{\varrho}$.

q. e. d.

Suppose $\widetilde{D}$ is a relatively compact open subset of G such that $(D, \widetilde{D})$ is $(E^*)_q$. Let $\widetilde{E} = \widetilde{D} \cap X$.

PROPOSITION 11.4. Suppose $r > q + n$. Suppose $\mathfrak{U}, \mathfrak{V}, \mathfrak{W} \in \mathrm{S}(X)$ such that $|\mathfrak{W}| \subset\subset E \subset\subset |\mathfrak{V}|$, $|\mathfrak{U}| \subset\subset \widetilde{E}$, $\mathfrak{W} < \mathfrak{V}$ and $\mathfrak{W} < \mathfrak{U}$. Then there exists $\omega \in \Omega^{(n)}$ such that, if $\varrho < \omega$ and $f \in Z^0(\mathfrak{V}(\varrho), \mathcal{F})$ and $\|f\|_{\mathfrak{V}, \varrho} < e$, then for some $g \in Z^0(\mathfrak{U}(\varrho), \mathcal{F})$, $g = f$ on $\mathfrak{W}(\varrho)$ and $\|g\|_{\mathfrak{U}, \varrho} < C_{\bar{\varrho}}\, e$, where $C_{\bar{\varrho}}$ is a constant depending only on $\bar{\varrho}$.

PROOF. We use the notations of § 8*B*.

As in the proof of Proposition 11.3, t_n is not a zero-divisor for $\mathcal{F}^*_x$ for $x \in K(\varrho^0) \times G$ and we can assume that $\mathcal{F}^*$ admits a finite free resolution of length $m \leqq n + N - r$ on $K(\varrho^0) \times G$.

Since $(D, \widetilde{D})$ is $(E^*)_q$, we can choose a relatively compact open subset D' of $\widetilde{D}$ and $(H)_q$ open subsets D_i of D, $0 \leqq i \leqq m$, such that $|\mathfrak{W}| \subset\subset D_m$, $D_{i+1} \subset\subset D_i \subset\subset D$, $|\mathfrak{U}| \subset\subset D'$, and (D_m, D') is an extension couple. Choose $\mathfrak{D}_2 << \mathfrak{D}_1$ in $\mathbf{S}(G)$ such that $D_0 \subset\subset |\mathfrak{D}_2|$ and $\mathfrak{D}_1 \cap X << \mathfrak{V}$. Choose $\widetilde{\mathfrak{D}}_2 << \widetilde{\mathfrak{D}}_1$ in $\mathbf{S}(G)$ such that $|\mathfrak{U}| \subset\subset |\widetilde{\mathfrak{V}}_i| \subset\subset D'$. Choose $\mathfrak{D}' \in \mathbf{S}(G)$ such that $|\mathfrak{W}| \subset |\mathfrak{D}'| \subset\subset D_{n+N-1}$, $\mathfrak{D}' < \mathfrak{D}_2$, and $\mathfrak{D}' < \widetilde{\mathfrak{D}}_2$. Let $\mathfrak{U}_1 = \mathfrak{D}_1 \cap X$ and $\widetilde{\mathfrak{U}}_2 = \widetilde{\mathfrak{D}}_2 \cap X$.

Take $\omega \in \Omega^{(n)}$ and we shall impose conditions on ω later. Take $\varrho < \omega$ and $f \in Z^0(\mathfrak{V}(\varrho), \mathcal{F})$ with $\|f\|_{\mathfrak{V}, \varrho} < e$. Let $f^* = \theta_{\mathfrak{D}_1}(f | \mathfrak{U}_1(\varrho)) \in Z^0(K(\varrho) \times \mathfrak{D}_1, \mathcal{F}^*)$. By Proposition 8.4, $|f^*|_{\mathfrak{D}_2, \varrho} < C^{(1)} e$, where $C^{(1)}$ is a constant.

By Proposition 11.2 if $\omega \leqq \omega^1$ for a suitable $\omega^1 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $g^* \in Z^0(K(\varrho) \times \widetilde{\mathfrak{D}}_1, \mathcal{F}^*)$, $g^* = f^*$ on $K(\varrho) \times \mathfrak{D}'$ and $|g^*|_{\widetilde{\mathfrak{D}}_1, \varrho} < C^{(2)}_{\bar{\varrho}} C^{(1)} e$, where $C^{(2)}_{\bar{\varrho}}$ is a constant depending only on $\bar{\varrho}$.

Let $g' = \theta^{-1}_{\widetilde{\mathfrak{D}}_1}(g^*)$. By Proposition 8.2, $\|g'\|_{\widetilde{\mathfrak{U}}_2, \varrho} < C^{(3)} C^{(2)}_{\bar{\varrho}} C^{(1)} e$, where $C^{(3)}$ is a constant. Let $g = g' | \mathfrak{U}(\varrho)$.

By Proposition 9.2, if $\omega \leqq \omega^2$ for a suitable $\omega^2 \in \Omega^{(n)}$ (and we assume this to be the case), then $\|g\|_{\mathfrak{U}, \varrho} < C^{(4)}_{\bar{\varrho}} C^{(3)} C^{(2)}_{\bar{\varrho}} C^{(1)} e$, where $C^{(4)}_{\bar{\varrho}}$ is a constant depending only on $\bar{\varrho}$. $g = f$ on $\mathfrak{W}(\varrho)$. q. e. d.

C. Suppose X is a complex space of reduction order $\leqq p < \infty$ and $\mathcal{F}$ is a coherent analytic sheaf on X. Suppose $\varrho^0 \in \mathbf{R}^n_+$ and $\pi : X \to K(\varrho^0)$ is a holomorphic map.

Suppose H, H_1, H_2 are relatively compact open subsets of X such that $H = H_1 \cup H_2$ and $(H_1 - H_2)^- \cap (H_2 - H_1)^- = \varnothing$. Let $H_{12} = H_1 \cap H_2$. Suppose some open neighborhood U of $\overline{H_{12}}$ is biholomorphic to a complex subspace of a polydisc G in $\mathbf{C}^N$ under a map τ such that $H_{12} = \tau^{-1}(D)$ for some relatively compact $(H^*)_q$ open subset D of G.

Suppose t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$ and codh $\mathcal{F} \geqq r$ on U.

PROPOSITION 11.5. Suppose $\mathfrak{U}, \mathfrak{V}, \mathfrak{W} \in \mathbf{S}(X)$ such that $|\mathfrak{W}| \subset\subset H_1 \subset\subset |\mathfrak{V}|$, $|\mathfrak{U}| \subset\subset H$, $\mathfrak{W} << \mathfrak{U}$, and $\mathfrak{W} << \mathfrak{V}$. Then for $1 \leqq \nu < r - q - n$ there exists $\omega \in \Omega^{(n)}$ satisfying the following. If $\varrho < \omega$ and $f \in Z^\nu(\mathfrak{V}(\varrho), \mathcal{F})$ and $\|f\|_{\mathfrak{V}, \varrho} < e$, then there exist $g \in Z^\nu(\mathfrak{U}(\varrho), \mathcal{F})$ and $h \in C^{\nu-1}(\mathfrak{W}(\varrho), \mathcal{F})$ such that $f - \delta h = g$ on $\mathfrak{W}(\varrho)$, $\|g\|_{\mathfrak{U}, \varrho} < C_{\bar{\varrho}} e$ and $\|h\|_{\mathfrak{W}, \varrho} < C_{\bar{\varrho}} e$, where $C_{\bar{\varrho}}$ is a constant depending only on $\bar{\varrho}$.

PROOF. Choose $\mathfrak{D}' \in \mathbf{S}(X)$ such that $H_{12} \subset\subset |\mathfrak{D}'| \subset\subset U$ and $\mathfrak{D}' < \mathfrak{V}$.

Since $(H_1 - H_2)^- \cap (H_2 - H_1)^- = \emptyset$, we can choose $\mathfrak{D}, \mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_{12} \in \mathbf{S}(X)$ such that

(i) $\mathfrak{D} = \mathfrak{D}_1 \cup \mathfrak{D}_2, \mathfrak{D}_{12} = \mathfrak{D}_1 \cap \mathfrak{D}_2$,

(ii) $\mathfrak{D}_1 < \mathfrak{V}, \mathfrak{D}_{12} << \mathfrak{D}'$,

(iii) $|\mathfrak{U}| \subset\subset |\mathfrak{D}|, |\mathfrak{W}| \subset\subset |\mathfrak{D}_1|, |\mathfrak{D}_{12}| \subset\subset H_{12}$, and

(iv) If $V \in \mathfrak{D}_1$ and $V' \in \mathfrak{D} - \mathfrak{D}_1$ and $V \cap V' \neq \emptyset$, then $V \in \mathfrak{D}_{12}$.

Choose $\mathfrak{W}' \in \mathbf{S}(X)$ such that $|\mathfrak{W}| \subset\subset |\mathfrak{W}'|, \mathfrak{W}' << \mathfrak{U}, \mathfrak{W}' < \mathfrak{D}_1$, and $\mathfrak{W}' << \mathfrak{D}$.

Take $\omega \in \Omega^{(n)}$ and we shall impose conditions on ω later. Fix $1 \leqq \nu < r - q - n$. Take $\varrho < \omega$ and $f \in Z^\nu(\mathfrak{V}(\varrho), \mathcal{F})$ with $\|f\|_{\mathfrak{V}, \varrho} < e$.

Consider the restriction of f to $\mathfrak{D}'(\varrho)$. By Proposition 11.3, if $\omega \leqq \omega^1$ for a suitable $\omega^1 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $h' \in C^{\nu-1}(\mathfrak{D}_{12}(\varrho), \mathcal{F})$, $f = \delta h'$ on $\mathfrak{D}_{12}(\varrho)$ and $\|h'\|_{\mathfrak{D}_{12}, \varrho} < C^{(1)}_{\bar{\varrho}}\, e$, where $C^{(1)}_{\bar{\varrho}}$ is a constant $\geqq 1$ depending only on $\bar{\varrho}$.

Extend h' trivially to $h'' \in C^{\nu-1}(\mathfrak{D}_1(\varrho), \mathcal{F})$. By condition (iv), $(f - \delta h'') | \mathfrak{D}_1(\varrho)$ can be extended trivially to $g' \in Z^\nu(\mathfrak{D}(\varrho), \mathcal{F})$. $\|g'\|_{\mathfrak{D}, \varrho} < (1 + C^{(2)} C^{(1)}_{\bar{\varrho}})\, e$, where $C^{(2)}$ is a constant.

By Proposition 10.2, if $\omega \leqq \omega^2$ for a suitable $\omega^2 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $g \in Z^\nu(\mathfrak{U}(\varrho), \mathcal{F})$ and $h''' \in C^{\nu-1}(\mathfrak{W}'(\varrho), \mathcal{F})$, $g' - \delta h''' = g$ on $\mathfrak{W}'(\varrho)$, $\|g\|_{\mathfrak{U}, \varrho} < C^{(3)}_{\bar{\varrho}} (1 + C^{(2)} C^{(1)}_{\bar{\varrho}})\, e$, and $\|h'''\|_{\mathfrak{W}', \varrho} < C^{(3)}_{\bar{\varrho}} (1 + C^{(2)} C^{(1)}_{\bar{\varrho}})\, e$, where $C^{(3)}_{\bar{\varrho}}$ is a constant depending only on $\bar{\varrho}$.

$f - g = \delta(h'' + h''')$ on $\mathfrak{W}'(\varrho)$. $\|h'' + h'''\|_{\mathfrak{W}', \varrho} < (C^{(1)}_{\bar{\varrho}} + C^{(3)}_{\bar{\varrho}} (1 + C^{(2)} C^{(1)}_{\bar{\varrho}}))\, e$.

Choose $\mathfrak{W}^* \in \mathbf{S}(X)$ such that $\mathfrak{W} << \mathfrak{W}^*$, $\mathfrak{W}^* < \mathfrak{U}$, and $\mathfrak{W}^* < \mathfrak{V}$. Then $\|f - g\|_{\mathfrak{W}^*, \varrho} < (1 + C^{(3)}_{\bar{\varrho}} (1 + C^{(2)} C^{(1)}_{\bar{\varrho}}))\, e$.

By Proposition 10.3, if $\omega \leqq \omega^3$ for a suitable $\omega^3 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $h \in C^{\nu-1}(\mathfrak{W}(\varrho), \mathcal{F})$, $f - g = \delta h$ on $\mathfrak{W}(\varrho)$ and $\|h\|_{\mathfrak{W}, \varrho} < C^{(4)}_{\bar{\varrho}} (C^{(1)}_{\bar{\varrho}} + C^{(3)}_{\bar{\varrho}} (1 + C^{(2)} C^{(1)}_{\bar{\varrho}}))\, e$, where $C^{(4)}_{\bar{\varrho}}$ is a constant depending only on $\bar{\varrho}$. q. e. d.

Suppose in addition that $H_2^- \subset U$ and $H_2 = \tau^{-1}(\tilde{D})$ for some open subset $\tilde{D}$ of G such that $(D, \tilde{D})$ is $(E^*)_q$.

PROPOSITION 11.6. Suppose $r > q + n$. Suppose $\mathfrak{U}, \mathfrak{V}, \mathfrak{W} \in \mathbf{S}(X)$ such that $|\mathfrak{W}| \subset\subset H_1 \subset\subset |\mathfrak{V}|$, $|\mathfrak{U}| \subset\subset H$, $\mathfrak{W} < \mathfrak{U}$, and $\mathfrak{W} < \mathfrak{V}$. Then there exists $\omega \in \Omega^{(n)}$ satisfying the following. If $\varrho < \omega$ and $f \in Z^0(\mathfrak{V}(\varrho), \mathcal{F})$ and $\|f\|_{\mathfrak{V}, \varrho} < e$, then there exists $g \in Z^0(\mathfrak{U}(\varrho), \mathcal{F})$ such that $f = g$ on $\mathfrak{W}(\varrho)$ and $\|g\|_{\mathfrak{U}, \varrho} < C_{\bar{\varrho}}\, e$, where $C_{\bar{\varrho}}$ is a constant depending only on $\bar{\varrho}$.

PROOF. Choose $\mathfrak{D}', \mathfrak{D}, \mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_{12}$ in $\mathbf{S}(X)$ in precisely the same way as in the proof of Proposition 11.5, except that we require the condition $|\mathfrak{D}_2| \subset\subset H_2$.

Take $\omega \in \Omega^{(n)}$ and we shall impose conditions on ω later. Take $\varrho < \omega$ and $f \in Z^0(\mathfrak{V}(\varrho), \mathcal{F})$ with $\|f\|_{\mathfrak{V}, \varrho} < e$.

Consider the restriction of f to $\mathfrak{D}'(\varrho)$. By Proposition 11.4, if $\omega \leqq \omega^1$ for a suitable $\omega^1 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $g' \in Z^0(\mathfrak{D}_2(\varrho), \mathcal{F})$ we have $f = g'$ on $\mathfrak{D}_{12}(\varrho)$ and $\|g\|_{\mathfrak{D}_2, \varrho} < C^{(1)}_{\bar{\varrho}}\, e$, where $C^{(1)}_{\bar{\varrho}}$ is a constant $\geqq 1$ depending only on $\bar{\varrho}$.

Conditions (iv) in the proof of Proposition 11.5 implies that there exists a unique $g'' \in Z^0(\mathfrak{D}(\varrho), \mathcal{F})$ such that $g'' = f$ on $\mathfrak{D}_1(\varrho)$ and $g'' = g'$ on $\mathfrak{D}_2(\varrho)$. $\|g''\|_{\mathfrak{D}, \varrho} < C^{(1)}_{\bar{\varrho}}\, e$.

Since $|\mathfrak{U}| \subset\subset |\mathfrak{D}|$, we can choose $\mathfrak{U}' \in \mathbf{S}(X)$ such that $\mathfrak{U} << \mathfrak{U}'$ and $|\mathfrak{U}'| \subset |\mathfrak{D}|$. g'' corresponds uniquely to an element g^* of $\Gamma(|\mathfrak{D}(\varrho)|, \mathcal{F})$. g^* induces uniquely an element g of $Z^0(\mathfrak{U}'(\varrho), \mathcal{F})$.

By Proposition 9.2, if $\omega \leqq \omega^2$ for a suitable $\omega^2 \in \Omega^{(n)}$ (and we assume this to be the case), then $\|g\|_{\mathfrak{W}, \varrho} < C^{(2)}_{\bar{\varrho}}\, C^{(1)}_{\bar{\varrho}}\, e$, where $C^{(2)}_{\bar{\varrho}}$ is a constant depending only on $\bar{\varrho}$.

$f = g$ on $\mathfrak{W}(\varrho)$, because $|\mathfrak{W}| \subset |\mathfrak{D}_1|$ and $f = g''$ on $\mathfrak{D}_1(\varrho)$. q. e. d.

In the remaining of this paper we adopt the following convention: If $\mathfrak{U} \in \mathbf{S}(X)$, then $C^{-1}(\mathfrak{U}(\varrho), \mathcal{F}) = 0$ and $\delta : C^{-1}(\mathfrak{U}(\varrho), \mathcal{F}) \to C^0(\mathfrak{U}(\varrho), \mathcal{F})$ is the zero-map.

PROPOSITION 11.7. Suppose $\mathfrak{U}, \mathfrak{V}, \mathfrak{W} \in \mathbf{S}(X)$ such that $|\mathfrak{W}| \underset{\pi}{\subset\subset} H_1 \underset{\pi}{\subset\subset} |\mathfrak{V}|$, $|\mathfrak{U}| \underset{\pi}{\subset\subset} H$, $\mathfrak{W} \underset{\pi}{<<} \mathfrak{U}$, and $\mathfrak{W} \underset{\pi}{<<} \mathfrak{V}$. Then for $0 \leqq \nu < r - q - n$ there exists $\omega \in \Omega^{(n)}$ satisfying the following. If $\varrho < \omega$ and $f \in Z^\nu(\mathfrak{V}(\varrho), \mathcal{F})$ and $\|f\|_{\mathfrak{V}, \varrho} < e$, then there exists $g \in Z^\nu(\mathfrak{U}(\varrho), \mathcal{F})$ and $h \in C^{\nu-1}(\mathfrak{W}(\varrho), \mathcal{F})$ such that $f - \delta h = g$ on $\mathfrak{W}(\varrho)$, $\|g\|_{\mathfrak{U}, \varrho} < C_{\bar{\varrho}}\, e$ and $\|h\|_{\mathfrak{W}, \varrho} < C_{\bar{\varrho}}\, e$, where $C_{\bar{\varrho}}$ is a constant dedending only on $\bar{\varrho}$.

PROOF. There exists $\varrho^1 \leqq \varrho^0$ in $\mathbf{R}^n_+$ such that $\mathfrak{W}(\varrho^1) \subset\subset H_1$, $H_1(\varrho^1) \subset\subset |\mathfrak{V}|$, $|\mathfrak{U}(\varrho^1)| \subset\subset H$, $\mathfrak{W}(\varrho^1) << \mathfrak{U}$, and $\mathfrak{W}(\varrho^1) << \mathfrak{V}$.

Take $\varrho^2 < \varrho^1$ in $\mathbf{R}_+^n$. Choose $\mathfrak{V}^* \in \mathbf{S}(X)$ of the form $\mathfrak{V}^* = \mathfrak{V} \cup \mathfrak{V}'$ such that $H_1 \subset\subset |\mathfrak{V}^*|$ and $|\mathfrak{V}'| \cap \pi^{-1}(K(\varrho^2)) = \emptyset$.

For $\varrho \leqq \varrho^2$, since $\mathfrak{V}^*(\varrho) = \mathfrak{V}(\varrho)$, we have $C^\nu(\mathfrak{V}^*(\varrho), \mathcal{F}) = C^\nu(\mathfrak{V}(\varrho), \mathcal{F})$. It is clear that $\|\cdot\|_{\mathfrak{V}^*, \varrho} = \|\cdot\|_{\mathfrak{V}, \varrho}$ on $C^\nu(\mathfrak{V}^*(\varrho), \mathcal{F})$ for $\varrho \leqq \varrho^2$.

Choose $\mathfrak{U}', \mathfrak{W}' \in \mathbf{S}(X)$ such that $\mathfrak{U} << \mathfrak{U}'$, $\mathfrak{W} << \mathfrak{W}'$, $|\mathfrak{U}'(\varrho^2)| \subset\subset H$, $|\mathfrak{W}'(\varrho^2)| \subset\subset H_1$, $\mathfrak{W}'(\varrho^2) << \mathfrak{U}$, and $\mathfrak{W}'(\varrho^2) << \mathfrak{V}$. Let $\mathfrak{U}^* = \mathfrak{U}'(\varrho^2)$ and $\mathfrak{W}^* = \mathfrak{W}'(\varrho^2)$.

Take $\omega \in \Omega^{(n)}$ with $\omega < \varrho^2$ and we shall impose conditions on ω later. Fix $0 \leqq \nu < r - q - n$. Take $\varrho < \omega$ and $f \in Z^\nu(\mathfrak{V}(\varrho), \mathcal{F})$ with $\|f\|_{\mathfrak{V}, \varrho} < e$. We can regard f as an element of $Z^\nu(\mathfrak{V}^*(\varrho), \mathcal{F}) \cdot \|f\|_{\mathfrak{V}^*, \varrho} < e$.

By Proposition 11.5 and 11.6, if $\omega \leqq \omega^1$ for a suitable $\omega^1 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $g' \in Z^\nu(\mathfrak{U}^*(\varrho), \mathcal{F})$ and $h' \in C^{\nu-1}(\mathfrak{W}^*(\varrho), \mathcal{F})$, we have $g' = f - \delta h'$ on $\mathfrak{W}^*(\varrho)$, $\|g'\|_{\mathfrak{W}^*, \varrho} < C_{\bar{\varrho}}^{(1)} e$, and $\|h'\|_{\mathfrak{W}^*, \varrho} < C_{\bar{\varrho}}^{(1)} e$, where $C_{\bar{\varrho}}^{(1)}$ is a constant depending only on $\bar{\varrho}$.

Let g be the restriction of g' to $\mathfrak{U}(\varrho)$ and let h be the restriction of h' to $\mathfrak{W}(\varrho)$. By Proposition 8.5, $\|g\|_{\mathfrak{U}, \varrho} < C^{(2)} C_{\bar{\varrho}}^{(1)} e$ and $\|h\|_{\mathfrak{W}, \varrho} < C^{(2)} C_{\bar{\varrho}}^{(1)} e$, where $C^{(2)}$ is a constant. q. e. d.

D. For $0 \leqq a < b$ in $\mathbf{R}^N$ we denote $\{(z_1, \ldots, z_N) \in K^N(b) \mid |z_i| > a_i$, for some $1 \leqq i \leqq N\}$ by $G^N(a, b)$.

DEFINITION. Suppose φ is a strictly q-convex function and f is a holomorphic function on some open neighborhood U of $K^N(a)^-$, where $a = (a_*, a_{**}) \in \mathbf{R}_+^{q-1} \times \mathbf{R}_+^{N-q+1}$. Suppose $c \in \mathbf{R}$ and $\alpha \in \mathbf{R}_+$. $K^N(a)$ is said to be *well-situated* with respect to $(q, \varphi, c; f, \alpha)$ if there exist $a'_{**} < a_{**}$ in $\mathbf{R}_+^{N-q+1}$ and $\alpha' < \alpha$ in $\mathbf{R}_+$ such that $\{\varphi \leqq c\}$ is disjoint from both $\{|f| \leqq \alpha\} \cap (K^{q-1}(a_*)^- \times G^{N-q+1}(a'_{**}, a_{**})^-)$ and $\{|f| \leqq \alpha'\} \cap K^N(a)^-$.

The following two Lemmas are clear.

LEMMA 11.1. Suppose φ_1 is a C^∞ function on U such that the restriction of φ_1 to $U \cap (\{y\} \times \mathbf{C}^{N-q+1})$ is strictly q-convex for $y \in \mathbf{C}^{q-1}$. Suppose $c_1 \leqq c$ and $0 < \alpha_1 \leqq \alpha$. If $\varphi_1 \leqq \varphi$ and $K^N(a)$ is well-situated with respect ot $(q, \varphi, c; f, \alpha)$, then $K^N(a)$ is well-situated with respect to $(q, \varphi_1, c_1; f, \alpha_1)$.

LEMMA 11.2. Suppose $K^N(a)$ is well-situated with respect to $(q, \varphi, c; f, \alpha)$. If a^1 is sufficiently close to a, c_1 is sufficiently close to c, α_1 is sufficiently close to α, and $D^\gamma \varphi_1$ is sufficiently close to $D^\gamma \varphi$ for $\gamma \in \mathbf{N}_*^N$ and $|\gamma| \leqq 2$, then $K^N(a^1)$ is well-situated with respect to $(q, \varphi_1, c_1; f, \alpha_1)$.

The following Proposition follows from the arguments on pp. 223-224 of [1].

PROPOSITION 11.8. If $K^N(a)$ is well-situated with respect to $(a, \varphi, c; f, \alpha)$, then $(K^N(a) \cap \{|f| < \alpha\} \cap \{\varphi > c\},\ K^N(a) \cap \{|f| < \alpha\})$ is $(E)_q$.

PROPOSITION 11.9. Suppose $K^N(a)$ is well-situated with respect to $(q, \varphi_i, c; f, \alpha)$, $i = 1, 2$. Let $D_i = K^N(a) \cap \{|f| < \alpha\} \cap \{\varphi_i > c\}$. If $\varphi_2 \geqq \varphi_1$, then (D_1, D_2) is $(E^*)_q$.

PROOF. Let L_i be a compact subset of D_i, $i = 1, 2$. By Lemma 11.2 we can choose $a' < a$ in $\mathbf{R}_+^N$, $0 < \alpha' < \alpha$ and $c' > c$ such that

(i) $K^N(a')$ is well-situated with respect to $(q, \varphi_1, c', f, \alpha')$, and

(ii) $L_i \subset D_i'$, where $D_i' = K^N(a') \cap \{|f| < \alpha'\} \cap \{\varphi_i > c'\}$.

Let $\widetilde{D} = K^N(a') \cap \{|f| < \alpha'\}$. By Proposition 11.8, $(D_i', \widetilde{D})$ is $(E)_q$ for $i = 1,2$. We have $H^\nu(D_1', {}_N\mathcal{O}) = 0$ for $1 \leqq \nu < N - q$. Since $\Gamma(\widetilde{D}, {}_N\mathcal{O}) \rightarrow \Gamma(D_i', {}_N\mathcal{O})$ is bijective for $i = 1, 2$, the restriction map $\Gamma(D_2', {}_N\mathcal{O}) \rightarrow \Gamma(D_1', {}_N\mathcal{O})$ is bijective. Hence (D_1', D_2') is $(E)_q$. q. e. d.

The following Proposition follows from the arguments on pp. 219-223 of [1].

PROPOSITION 11.10. Suppose U is an open neighborhood of 0 in $\mathbf{C}^N$ and φ is a strictly q-convex function on U. If $\varphi(0) = c$, then, after a linear coordinates transformation and a shrinking of U, there exist $a \in \mathbf{R}_+^N$, a holomorphic function f on U, and $\alpha \in \mathbf{R}_+$ such that $K^N(a)^- \subset U$, $|f(0)| < \alpha$, and $K^N(a)$ is well-situated with respect to $(q, \varphi, c; f, \alpha)$.

E. Suppose X is a complex space, $\varrho^0 \in \mathbf{R}_+^n$ and $\pi: X \rightarrow K(\varrho^0)$ is a q-concave holomorphic map with exhaustion function φ and concavity bounds c_*, $c_\#$.

In the remaining of this paper we use the following notations, some of which have been introduced earlier. For $c \in (c_*, \infty)$, $X_c = \{\varphi > c\}$. X_c^0 denotes $X_c \cap \pi^{-1}(0)$. $X_c(\varrho)$ denotes $X_c \cap \pi^{-1}(K(\varrho))$ for $\varrho \in \mathbf{R}_+^n$. $X_{c^*}(\varrho)$ is also denoted simply by $X(\varrho)$.

The following notation is used for this section only. Suppose T is a coordinates system of $\mathbf{C}^N$. For $x \in \mathbf{C}^N$ and $a \in \mathbf{R}_+^N$ we denote by $P_T(x; a)$ the open set which with respect to the coordinates system T is the polydisc centered at x with polyradii a.

Fix $c \in (c_*, c_\#)$. Take arbitrarily $x \in \partial X_c^0$. We have $\varphi(x) = c$.

There exists a holomorphic embedding Φ from an open neighborhood U of x in $\{\varphi < c_\#\}$ onto a complex subspace V of a polydisc G of $\mathbf{C}^N$ such that $\varphi \mid U = \tilde{\varphi}_0 \circ \Phi$ for some strictly q-convex function $\tilde{\varphi}_0$ of G.

By Proposition 11.10, after a shrinking of U there exist $a \in \mathbf{R}_+^N$, $x' \in G$, a holomorphic function f on G, $\alpha \in \mathbf{R}_+$, and a coordinates system T of $\mathbf{C}^N$ such that $\Phi(x) \in P_T(x'; a) \subset\subset G$, $|f(\Phi(x))| < \alpha$, and $P_T(x'; a)$ is well-situated with respect to $(q, \tilde{\varphi}_0, c; f, \alpha)$. Let $D = \Phi^{-1}(P_T(x'; a) \cap \{|f| < \alpha\})$. D is an open neighborhood of x.

Since ∂X_c^0 is compact, we can choose $x_1, \dots, x_k \in \partial X_c^0$ such that $\partial X_c^0 \subset \bigcup_{i=1}^k D_i$, where D_i has the same relation to x_i as D to x. Let the symbols Φ_i, U_i, G_i, N_i, $\tilde{\varphi}_{0i}$, a^i, x_i', f_i, α_i, T_i have meanings similar to Φ, U, G, N, $\tilde{\varphi}_0$, a, x', f, α, T.

Choose a relatively compact open subset L_j of D_j such that $\partial X_c^0 \subset \bigcup_{i=1}^k L_i$.

Choose non-negative functions σ_j on X, $1 \leqq j \leqq k$, such that

(i) $L_j \subset\subset \operatorname{Supp} \sigma_j \subset\subset D_j$ for $1 \leqq j \leqq k$, and

(ii) for every $1 \leqq i, j \leqq k$ there exists a non-negative C^∞ function $\tilde{\sigma}_{ji}$ on G_i with compact supports satisfying $\sigma_j = \tilde{\sigma}_{ji} \circ \Phi_i$ on D_i.

Choose $\lambda_l \in \mathbf{R}_+$, $1 \leqq l \leqq k$, so small that the restriction of $\tilde{\varphi}_{0i} + \Sigma_{l=1}^j \lambda_l \tilde{\sigma}_{li}$ to $G_i \cap (\{y\} \times \mathbf{C}^{N-q+1})$ is strictly q-convex for $1 \leqq j \leqq l$ and $y \in \mathbf{C}^{q-1}$, where the product $\{y\} \times \mathbf{C}^{N-q+1}$ is taken with respect to the coordinates system T_i.

Let $\varphi_0 = \varphi$ and $\varphi_j = \varphi + \Sigma_{l=1}^j \lambda_l \sigma_l$. Let $\tilde{\varphi}_{ji} = \tilde{\varphi}_{0i} + \Sigma_{l=1}^j \lambda_l \tilde{\sigma}_{li}$. Then $\varphi_j = \tilde{\varphi}_{ji} \circ \Phi_i$. Since $\tilde{\varphi}_{ji} \geqq \tilde{\varphi}_{0i}$, $P_{T_i}(x_i'; a^i)$ is well-situated with respect to $(q, \tilde{\varphi}_{ji}, c; f_i, \alpha_i)$ for $1 \leqq i, j \leqq k$.

For some, $\tilde{c} < c$ $X_{\tilde{c}}^0 \subset\subset \{\varphi_k > c\}$, because $\partial X_c^0 \subset \bigcup_{i=1}^k L_i \subset\subset \bigcup_{i=1}^k \operatorname{Supp} \sigma_i$.

DEFINITION. $\tilde{c}$ is said to be *reachable* from c with respect to (q, φ) if $\tilde{c}$ can be obtained from such a construction as described above.

The following Lemma is clear.

LEMMA 11.3. (*a*) For every $c \in (c_*, c_\#)$ there exists $\tilde{c} \in (c_*, c)$ such that $\tilde{c}$ is reachable from c with respect to (q, φ).

(*b*) If $c \in (c_*, c_\#)$ and $\tilde{c} \in (c_*, c)$ such that $\tilde{c}$ is reachable from c with respect to (q, φ), then

(i) c' is reachable from c with respect to (q, φ) for $c' \in (\tilde{c}, c)$,

(ii) $\tilde{c}$ is reachable from c' with respect to (q, φ) for $c' \in (\tilde{c}, c)$, and

(iii) there exists $c' \in (c, c_\#)$ such that $\tilde{c}$ is reachable from c' with respect to (q, φ).

Suppose $\mathcal{F}$ is a coherent analytic sheaf on X such that codh $\mathcal{F} \geqq r$ on $\{\varphi < c_{\#}\}$ and t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$. Suppose X has reduction order $\leqq p < \infty$.

PROPOSITION 11.11 Suppose $c_{*} < \tilde{c} < c < c_{\#}$ and $\tilde{c}$ is reachable from c with respect to (q, φ). Suppose $\mathfrak{U}, \mathfrak{V}, \mathfrak{W} \in \mathbf{S}(X)$ such that $|\mathfrak{W}| \underset{\pi}{\subset\subset} X_c \underset{\pi}{\subset\subset} |\mathfrak{V}|$, $|\mathfrak{U}| \underset{\pi}{\subset\subset} X_{\tilde{c}}$, $\mathfrak{W} \underset{\pi}{<<} \mathfrak{U}$, and $\mathfrak{W} \underset{\pi}{<<} \mathfrak{V}$. Then for $0 \leqq \nu < r - q - n$ there exists $\omega \in \Omega^{(n)}$ satisfying the following. If $\varrho < \omega$ and $f \in Z^{\nu}(\mathfrak{V}(\varrho), \mathcal{F})$ with $\|f\|_{\mathfrak{W}, \varrho} < e$, then for some $g \in Z^{\nu}(\mathfrak{U}(\varrho), \mathcal{F})$ and $h \in C^{\nu-1}(\mathfrak{W}(\varrho), \mathcal{F})$, we have $f - \delta h = g$ on $\mathfrak{W}(\varrho)$, $\|g\|_{\mathfrak{W}, \varrho} < C_{\bar{\varrho}}\, e$, and $\|h\|_{\mathfrak{W}, \varrho} < C_{\bar{\varrho}}\, e$, where $C_{\bar{\varrho}}$ is a constant depending only on $\bar{\varrho}$.

PROOF. Choose $c' \in (c, c_{\#})$ so close to c that the following three conditions are satisfied:

(i) $P_{T_i}(x'_i\,; a^i)$ is well-situated with respect to $(q, \tilde{\varphi}_{0i}, e'\,; f_i, \alpha_i)$ for every i,

(ii) $X_{\tilde{c}} \underset{\pi}{\subset\subset} \{\varphi_k > c'\}$ and

(iii) $|\mathfrak{W}| \underset{\pi}{\subset\subset} X_{c'}$.

Choose $c' = c_k > c_{k-1} > \dots > c_1 > c_0 = c$. Let $H_i^{(j)} = \{\varphi_i > c_j\}$. Then we have the following:

(i) $X_c = H_0^{(0)}$,

(ii) $|\mathfrak{W}| \underset{\pi}{\subset\subset} H_k^{(0)}$,

(iii) $X_{\tilde{c}} \underset{\pi}{\subset\subset} H_k^{(k)}$,

(iv) $H_i^{(j)} \underset{\pi}{\subset\subset} H_i^{(j-1)}$, and

(v) $H_i^{(j)} \subset H_{i+1}^{(j)}$.

Since $H_i^{(i)} \underset{\pi}{\subset\subset} H_i^{(i-1)}$, we can choose $\mathfrak{U}_i \in \mathbf{S}(X)$, $1 \leqq i \leqq k$, such that $H_i^{(i)} \underset{\pi}{\subset\subset} |\mathfrak{U}_i| \underset{\pi}{\subset\subset} H_i^{(i-1)}$. Let $\mathfrak{U}_0 = \mathfrak{V}$.

Since $|\mathfrak{W}| \underset{\pi}{\subset\subset} H_0^{(k)}$, we can choose $\mathfrak{W}' \in \mathbf{S}(X)$ such that $\mathfrak{W}' \underset{\pi}{<<} \mathfrak{U}_i$ for $0 \leqq i \leqq k$, $\mathfrak{W}' \underset{\pi}{<<} \mathfrak{U}$, and $|\mathfrak{W}| \underset{\pi}{\subset\subset} |\mathfrak{W}'|$.

We are going to prove $(11.1)_i$ for $0 \leqq i < k$.

$$(11.1)_i \quad \left\{\begin{array}{l}\text{For } 0 \leqq \nu < r-q-n \text{ there exists } \omega_i \in \Omega^{(n)} \text{ such that, if } \\ \varrho < \omega \text{ and } f \in Z^\nu(\mathfrak{U}_i(\varrho), \mathcal{F}) \text{ with } \|f\|_{\mathfrak{U}_i, \varrho} < e, \text{ then for some } \\ g \in Z^\nu(\mathfrak{U}_{i+1}(\varrho), \mathcal{F}) \text{ and } h \in C^{\nu-1}(\mathfrak{W}'(\varrho), \mathcal{F}) \text{ we have } f - \delta h = g \\ \text{on } \mathfrak{W}'(\varrho) \text{ and } \|g\|_{\mathfrak{U}_{i+1}, \varrho} < C_{\bar{\varrho}}^{(i)} e \text{ and } \|h\|_{\mathfrak{W}', \varrho} < C_{\bar{\varrho}}^{(i)} e, \text{ where} \\ C_{\bar{\varrho}}^{(i)} \text{ is a constant } \geqq 1 \text{ depending only on } \bar{\varrho}.\end{array}\right.$$

Fix $0 \leqq i < k$. Let $H = H_{i+1}^{(i)}$, $H_1 = H_i^{(i)}$, $H_2 = D_{i+1} \cap H_{i+1}^{(i)}$, and $H_{12} = D_{i+1} \cap H_i^{(i)}$. Since $H_i^{(i)} \subset H_{i+1}^{(i)}$, $H_{12} = H_1 \cap H_2$. Since Supp $(\varphi_{i+1} - \varphi_i) \subset D_{i+1}$, we have $H_{i+1}^{(i)} - H_i^{(i)} \subset\subset D_{i+1}$. Hence $H = H_1 \cup H_2$ and $(H_1 - H_2)^- \cap \cap (H_2 - H_1)^- = \emptyset$.

Let $D' = P_{T_i}(x_i'; a^i) \cap \{|f| < \alpha_i\} \cap \{\widetilde{\varphi}_{i,i} > c_i\}$ and $D'' = P_{T_i}(x_i'; a^i) \cap \cap \{|f| < \alpha_i\} \cap \{\widetilde{\varphi}_{i+1,i} > c_i\}$. We have $H_{12} = \Phi_i^{-1}(D')$ and $H_2 = \Phi_i^{-1}(D'')$. By Proposition 11.9, (D', D'') is $(E^*)_q$.

Now apply Proposition 11.7 (with $\mathfrak{U}$ substituted by $\mathfrak{U}_{i+1}$, $\mathfrak{V}$ by $\mathfrak{U}_i$, and $\mathfrak{W}$ by $\mathfrak{W}'$) and (11.1) follows.

Take $\omega \in \Omega^{(n)}$ and we shall impose conditions on ω later. Fix $0 \leqq \nu < < r - q - n$. Take $\varrho < \omega$ and $f \in Z^\nu(\mathfrak{V}(\varrho), \mathcal{F})$ with $\|f\|_{\mathfrak{V}, \varrho} < e$.

By using $(11.1)_i$, $0 \leqq i < k$, and using induction on l, we obtain (11.2) for $1 \leqq l \leqq k$.

$$(11.2)_l \quad \left\{\begin{array}{l}\text{If } \omega \leqq \omega^i \text{ for } 0 \leqq i < l, \text{ then for some } g^{(l)} \in Z^\nu(\mathfrak{U}_l(\varrho), \mathcal{F}) \text{ and} \\ h^{(l)} \in C^{\nu-1}(\mathfrak{W}'(\varrho), \mathcal{F}) \text{ we have } f - \delta h^{(l)} = g^{(l)} \text{ on } \mathfrak{W}'(\varrho), \\ \|g^{(l)}\|_{\mathfrak{U}_l, \varrho} < (\Sigma_{i=0}^{l-1} C_{\bar{\varrho}}^{(i)}) e, \text{ and } \|h^{(l)}\|_{\mathfrak{W}', \varrho} < (\Sigma_{i=0}^{l-1} C_{\bar{\varrho}}^{(i)}) e.\end{array}\right.$$

Now assume $\omega \leqq \omega^i$ for $0 \leqq i < k$. Let $C_{\bar{\varrho}}' = \Sigma_{i=0}^{k-1} C_{\bar{\varrho}}^{(i)}$.

By Proposition 10.4, if $\omega \leqq \omega'$ for a suitable $\omega' \in \Omega^{(n)}$ (and we assume this to be the case), then for some $g \in Z^\nu(\mathfrak{U}(\varrho), \mathcal{F})$ and $h^* \in C^{\nu-1}(\mathfrak{W}'(\varrho), \mathcal{F})$ we have $g^{(k)} - \delta h^* = g$ on $\mathfrak{W}'(\varrho)$, $\|g\|_{\mathfrak{U}, \varrho} < C_{\bar{\varrho}}'' C_{\bar{\varrho}}' e$, and $\|h^*\|_{\mathfrak{W}', \varrho} < C_{\bar{\varrho}}'' C_{\bar{\varrho}}' e$, where $C_{\bar{\varrho}}''$ is a constant depending only on $\bar{\varrho}$.

$$f - g = \delta(h^{(k)} + h^*) \text{ on } \mathfrak{W}'(\varrho).$$

Choose $\mathfrak{W}^* \in \mathfrak{S}(X)$ such that $\mathfrak{W} \underset{\pi}{<<} \mathfrak{W}^*$, $|\mathfrak{W}^*| \underset{\pi}{\subset\subset} |\mathfrak{W}'|$, $\mathfrak{W}^* < \mathfrak{V}$, and $\mathfrak{W}^* < \mathfrak{U}$. $\|f - g\|_{\mathfrak{W}^*, \varrho} < (1 + C_{\bar{\varrho}}'' C_{\bar{\varrho}}')e$. $\|h^{(k)} + h^*\|_{\mathfrak{W}', \varrho} < (1 + C_{\bar{\varrho}}'') C_{\bar{\varrho}}' e$.

By Proposition 10.5, if $\omega \leqq \omega''$ for a suitable $\omega'' \in \Omega^{(n)}$ (and we assume this to be the case), then for some $h \in C^{\nu-1}(\mathfrak{W}(\varrho)\, \mathcal{F})$ we have $f - g = \delta h$ on $\mathfrak{W}(\varrho)$ and $\| h \|_{\mathfrak{W}, \underline{\varrho}} < C'''_{\underline{\varrho}}\, (1 + C''_{\underline{\varrho}})\, C'_{\underline{\varrho}}\, e$, where $C'''_{\underline{\varrho}}$ is a constant depending only on $\underline{\varrho}$. q. e. d.

PROPOSITION 11.12 Suppose $c \in (c_*, c_\#)$ and $\tilde{c} \in (c_*, c)$ such that $\tilde{c}$ is reachable from c with respect to (q, φ). Then there exists $\varrho^1 \leqq \varrho^0$ in $\mathbf{R}^n_+$ such that, for $\varrho \leqq \varrho^1$ and $0 \leqq \nu < r - q - n$, the restriction map $H^\nu(X_{\tilde{c}}(\varrho), \mathcal{F}) \to H^\nu(X_c(\varrho), \mathcal{F})$ is surjective.

PROOF. Since $\overset{0}{X_{\tilde{c}}} \subset\subset \{\varphi_k > c\}$, there exists $\varrho^1 < \varrho^0$ in $\mathbf{R}^n_+$ such that $X_{\tilde{c}}(\varrho^1) \subset \{\varphi_k > c\}$.

Take $\varrho \leqq \varrho^1$ in $\mathbf{R}^n_+$. Let $M_i = \{\varphi_i > c\} \cap X(\varrho)$, $0 \leqq i \leqq k$. Then $M_0 = X_c(\varrho)$ and $X_{\tilde{c}}(\varrho) \subset M_k$. To finish the proof, we need only show that, for $0 < i \leqq k$ and $0 \leqq \nu < r - q - n$, the restriction map $H^\nu(M_i, \mathcal{F}) \to H^\nu(M_{i-1}, \mathcal{F})$ is surjective.

Let $E_i = D_i \cap X(\varrho) \cap \{\varphi_i > c\}$ and $F_i = D_i \cap X(\varrho) \cap \{\varphi_{i-1} > c\}$. Since $\varphi_i \geqq \varphi_{i-1}$ and $\operatorname{Supp}(\varphi_i - \varphi_{i-1}) \subset D_i$, we have $M_i = M_{i-1} \cup E_i$ and $M_{i-1} \cap E_i = F_i$. The following portion of the Mayer-Vietoris sequence of $\mathcal{F}$ on $M_i = M_{i-1} \cup E_i$ is exact.

$$H^\nu(M_i, \mathcal{F}) \to H^\nu(M_{i-1}, \mathcal{F}) \oplus H^\nu(E_i, \mathcal{F}) \to H^\nu(F_i, \mathcal{F}). \tag{11.3}$$

There exists a unique holomorphic embedding $\Phi_i^*: U_i \to K(\varrho^0) \times G_i$ such that $\Pi_1 \circ \Phi_i^* = \pi$ and $\Pi_2 \circ \Phi_i^* = \Phi_i$, where $\Pi_1: K(\varrho^0) \times G_i \to K(\varrho^0)$ and $\Pi_2: K(\varrho^0) \times G_i \to G_i$ are the projections. Let $\mathcal{F}^{(i)} = (\Phi_i^*)_0(\mathcal{F})$.

Since $P_{T_i}(x'_i; a^i)$ is well-situated with respect to $(q, \tilde{\varphi}_{i-1, i}, c; f_i, \alpha_i)$ and also with respect to $(q, \tilde{\varphi}_{i, i}, c; f_i, \alpha_i)$, $K(\varrho) \times P_{T_i}(x'_i; a^i)$ is well-situated with respect to $(q, \tilde{\varphi}_{i-1, i} \circ \Pi_2, c; f_i \circ \Pi_2, \alpha_i)$ and also with respect to $(q, \tilde{\varphi}_{i, i} \circ \Pi_2, c; f_i \circ \Pi_2, \alpha_i)$.

Let $Q_i = K(\varrho) \times (P_{T_i}(x'_i; a^i) \cap \{|f_i| < \alpha_i\} \cap \{\tilde{\varphi}_{i-1, i} > c_i\})$ $\tilde{Q}_i = K(\varrho) \times (P_{T_i}(x'_i; a^i) \cap \{|f_i| < \alpha_i\} \cap \{\tilde{\varphi}_{i, i} > c_i\})$. By the proof of Proposition 11.8, $(Q_i, \tilde{Q}_i)$ is $(E)_q$.

Since $\operatorname{codh} \mathcal{F}^{(i)} \geqq r$, there exists a finite free resolution of $\mathcal{F}^{(i)}$ of length $\leqq n + N_i - r$ on $K(\varrho) \times P_{T_i}(x'_i; a^i)$. Hence $H^\nu(Q_i, \mathcal{F}^{(i)}) = 0$ for $1 \leqq \nu < r - q - n$ and the restriction map $\Gamma(\tilde{Q}_i, \mathcal{F}^{(i)}) \to \Gamma(Q_i, \mathcal{F}^{(i)})$ is bijective when $r > q - n$.

It is clear that $E_i = (\Phi_i^*)^{-1}(\tilde{Q}_i)$ and $F_i = (\Phi_i^*)^{-1}(Q_i)$. Hence $H^\nu(F_i, \mathcal{F}) = 0$ for $1 \leqq \nu < r - q - n$ and the restriction map $\Gamma(E_i, \mathcal{F}) \to \Gamma(F_i, \mathcal{F})$ is bijective when $r > q - n$. (11.3) implies that the restriction map $H^{\nu}(M_i, \mathcal{F}) \rightarrow$ $\rightarrow H^{\nu}(M_{i-1}, \mathcal{F})$ is surjective for $0 < i \leqq k$ and $0 \leqq \nu < r - q - n$. q. e. d.

§ 12. Proof of H^l-finiteness.

In § 3 we reduce the proof of the Main Theorem essentially to the task of proving a certain property which we call H^l-finiteness. In this section we shall establish H^l-finiteness by assuming a certain property concerning bounds which we call property $(B)_l^n$. Property $(B)_l^n$ corresponds to the Hauptlemma in [2] and is the most vital point in this paper. Its proof will be presented in § 13 and § 14.

Suppose X is a complex space of reduction order $\leqq p < \infty$, $\varrho^0 \in \mathbf{R}_+^n$, and $\pi : X \rightarrow K(\varrho^0)$ is a q-concave holomorphic map with exhaustion function φ and concavity bounds $c_*, c_\#$. Suppose $\mathcal{F}$ is a coherent analytic sheaf on X and $l \in \mathbf{N}_*$. Recall the following notations. For $c \in (c_*, c_\#)$, $X_c = \{\varphi > c\}$, $\pi^c = \pi \mid X_c$, and $\pi_l^c(\mathcal{F})$ denotes the l th direct image of $\mathcal{F} \mid X_c$ under π^c.

Definition. $\mathcal{F}$ has *property* $(B)_l^n$ at 0 with respect to (π, φ) if, for every $c \in (c_*, c_\#)$ and every $\tilde{c} \in (c_*, c)$ which is reachable from c with respect to (q, φ), the following two conditions are satisfied.

(i) $\pi_l^c(\mathcal{F})_0$ is finitely generated over ${}_n\mathcal{O}_0$.

(ii) Suppose $\mathfrak{V}, \mathfrak{U} \in \mathbf{S}(X)$ with $|\mathfrak{V}| \underset{\pi}{\subset\subset} X_c \underset{\pi}{\subset\subset} |\mathfrak{U}| \underset{\pi}{\subset\subset} X_{\tilde{c}}$ and $\mathfrak{V} \underset{\pi}{<<} \mathfrak{U}$. Suppose $\varrho^1 \leqq \varrho^0$ and $\xi_1, \dots, \xi_k \in Z^l(\mathfrak{U}(\varrho^1), \mathcal{F})$ and A is the ${}_n\mathcal{O}_0$-submodule of $\pi_l^c(\mathcal{F})_0$ generated by the images of $\xi_1, \dots, \xi_k$ in $\pi_l^c(\mathcal{F})_0$. Then there exists $\omega \in \Omega^{(n)}$ such that, if $\varrho < \omega$ and $\xi \in Z^l(\mathfrak{U}(\varrho), \mathcal{F})$ with $\| \xi \|_{\mathfrak{U}, \varrho} < e$ and the image of ξ in $\pi_l^c(\mathcal{F})_0$ belongs to A, then for some $a_1, \dots, a_k \in \Gamma(K(\varrho), {}_n\mathcal{O})$ and $\eta \in C^{l-1}(\mathfrak{V}(\varrho), \mathcal{F})$, $\xi = \Sigma\, a_i \xi_i + \delta\eta$ on $\mathfrak{V}(\varrho)$ and $|a_i|_\varrho < C_\varrho e$ and $\| \eta \|_{\mathfrak{V}, \varrho} < C_\varrho e$, where C_ϱ is a constant depending only on ϱ.

Proposition 12.1. Suppose codh $\mathcal{F} > l + q + n$ on $\{\varphi < c_\#\}$ and t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in \{\varphi < c_\#\}$. Suppose $d \in \mathbf{N}_*$ and t_n is not a zero-divisor for $t_n^d \mathcal{F}_x$ for $x \in X$. If $\mathcal{F}/t_n^{d+1} \mathcal{F}$ has property $(B)_l^n$ at 0 with respect to (π, φ), then for every $c \in (c_*, c_\#)$, $\mathcal{F} \mid X_c$ is H^l-finite at 0 with respect to π^c.

Proof. Fix $c \in (c_*, c_\#)$. By Lemma 11.3 (a) there exists $\tilde{c} \in (c_*, c)$ such that $\tilde{c}$ is reachable from c with respect to (q, φ). Choose $\tilde{c} < c_1 < c_2 < c$. By Lemma 11.3 ($b$), $\tilde{c}$ is reachable from both c_1 and c_2.

Choose $\mathfrak{U}_i \in \mathbf{S}(X)$, $0 \leqq i \leqq 3$, such that $X_c \underset{\pi}{\subset} |\mathfrak{U}_3| \underset{\pi}{\subset\subset} X_{c_2} \underset{\pi}{\subset\subset} |\mathfrak{U}_2|$, $|\mathfrak{U}_1| \underset{\pi}{\subset\subset} X_{c_1} \underset{\pi}{\subset\subset} |\mathfrak{U}_0| \underset{\pi}{\subset\subset} X_{\tilde{c}}$, and $\mathfrak{U}_{i+1} \underset{\pi}{<<} \mathfrak{U}_i$.

Let A be the image of $\pi_l^c(\mathcal{F})_0 \longrightarrow \pi_l^c(\mathcal{F}/t_n^{d+1}\mathcal{F})_0$. By Proposition 11.12, after shrinking ϱ^0, we can assume that, for some $\xi_1, \dots, \xi_k \in Z^l(\mathfrak{U}_0(\varrho^0), \mathcal{F})$, A is generated by the images of $\xi_1, \dots, \xi_k$ in $\pi_l^c(\mathcal{F}/t_n^{d+1}\mathcal{F})_0$. By Proposition 8.6, after further shrinking ϱ^0, we can assume that $\| \xi_i \|_{\mathfrak{U}_1, \varrho^0} < \infty$.

We are going to prove the following: For $\varrho^1 \leqq \varrho^0$ we can find $\varrho^2 \leqq \varrho^1$ such that $\operatorname{Im}(H^l(X_c(\varrho^1), \mathcal{F}) \rightarrow H^l(X_c(\varrho^2), \mathcal{F}))$ is contained in the $\Gamma(K(\varrho^2), {}_n\mathcal{O})$-submodule generated by the images of $\xi_1, \dots, \xi_k$ in $H^l(X_c(\varrho^2), \mathcal{F})$. This will imply that $\mathcal{F} | X_c$ is H^l-finite at 0.

Fix $\varrho^1 \leqq \varrho^0$. Choose $\varrho \leqq \varrho^1$ and we shall impose more conditions on ϱ later. Take $\tilde{\xi} \in H^l(X_c(\varrho^1), \mathcal{F})$.

We claim that, if ϱ is small enough, we can find $\xi \in Z^l(\mathfrak{U}_0(\varrho), \mathcal{F})$ with $\| \xi \|_{\mathfrak{U}_0, \varrho} < \infty$ such that ξ and $\tilde{\xi}$ induce the same element in $H^l(X_c(\varrho), \mathcal{F})$. By Proposition 11.12, for some $\varrho' \leqq \varrho^1$ we can find $\xi' \in H^l(X_{\tilde{c}}(\varrho'), \mathcal{F})$ such that ξ' and $\tilde{\xi}$ induce the same element in $H^l(X_c(\varrho'), \mathcal{F})$. Since $|\mathfrak{U}_0| \subset\subset_\pi X_{\tilde{c}}$, we can find a countable Stein open covering $\mathfrak{U}'$ of $X_c(\varrho')$ such that $\mathfrak{U}_0(\varrho'') << \mathfrak{U}'$ for some $\varrho'' \leqq \varrho'$. Let ξ' be represented by $\xi'' \in Z^l(\mathfrak{U}', \mathcal{F})$. If $\varrho < \varrho''$ and $\xi \in Z^l(\mathfrak{U}_0(\varrho), \mathcal{F})$ is induced by ξ'', then $\| \xi \|_{\mathfrak{U}_0, \varrho} < \infty$ by Proposition 8.6, and ξ induces the same element in $H^l(X_c(\varrho), \mathcal{F})$ as $\tilde{\xi}$. The claim is proved.

Choose $e \in \mathbf{R}_+$ such that $e > \| \xi \|_{\mathfrak{U}_0, \varrho}$. We are going to construct by induction on $r \in \mathbf{N}_*$, $\xi^{(r)} \in Z^l(\mathfrak{U}_0(\varrho), \mathcal{F})$, $\eta^{(r+1)} \in C^{l-1}(\mathfrak{U}_3(\varrho), \mathcal{F})$, and $a_1^{(r+1)}, \dots, a_k^{(r+1)} \in \Gamma(K(\varrho), {}_n\mathcal{O})$ such that

(i) $\xi^{(0)} = \xi$,

(ii) $\xi^{(r)} = \Sigma_i a_i^{(r+1)} \xi_i + \delta\eta^{(r+1)} + \left(\dfrac{t_n}{\varrho_n}\right) \xi^{(r+1)}$ on $\mathfrak{U}_3(\varrho)$,

(iii) $\| \xi^{(r)} \|_{\mathfrak{U}_0, \varrho} < (D_\varrho)^r e$,

(iv) $\| \eta^{(r+1)} \|_{\mathfrak{U}_3, \varrho} < (D_\varrho)^{r+1} e$, and

(v) $| a_i^{(r+1)} |_\varrho < (D_\varrho)^{r+1} e$,

where D_ϱ is a constant depending only on ϱ.

Set $\xi^{(0)} = \xi$. Suppose for some $r_0 \in \mathbf{N}_*$ we have constructed $\xi^{(r)}$, $\eta^{(r)}$, $a_i^{(r)}$ for $r \leqq r_0$.

Since $\mathcal{F}/t_n^{d+1}\mathcal{F}$ has properly $(B)_l^n$ with respect to (π, φ), if $\varrho < \omega^1$ for a suitable $\omega^1 \in \Omega^{(n)}$ with $\omega^1 < \varrho^0$ (and we assume this to be the case), then there exist $a_1^{(r_0+1)}, \dots, a_k^{(r_0+1)} \in \Gamma(K(\varrho), {}_n\mathcal{O})$ and $\eta' \in C^{l-1}(\mathfrak{U}_1(\varrho), \mathcal{F})$ such that $(\xi^{(r_0)} - \Sigma_i a_i^{(r_0+1)} \xi_i - \delta\eta') | \mathfrak{U}_1(\varrho) \in Z^{l-1}(\mathfrak{U}_1(\varrho), t_n^{d+1}\mathcal{F})$ and $| a_i^{(r_0+1)} |_\varrho < C_\varrho^{(1)} (D_\varrho)^{r_0} e$ and $\| \eta' \|_{\mathfrak{U}_1, \varrho} < C_\varrho^{(1)} (D_\varrho)^{r_0} e$, where $C_\varrho^{(1)}$ is a constant depending only on ϱ. (Lemma 8.3 is used in obtaining the preceding statement).

Since each member of $\mathfrak{U}_1(\varrho)$ is Stein, there exists $\zeta \in C^l(\mathfrak{U}_1(\varrho), \mathcal{F})$ such that $\xi^{(r_0)} - \Sigma_i a_i^{(r_0+1)} \xi_i - \delta\eta' = \left(\frac{t_n}{\varrho_n}\right)^{d+1} \zeta$ on $\mathfrak{U}_1(\varrho)$. Since $\|\Sigma_i a_i^{(r_0+1)} \xi_i\|_{\mathfrak{U}_1, \varrho} < C^{(2)} C_\varrho^{(1)} (D_\varrho)^{r_0} e$ and $\|\delta\eta'\|_{\mathfrak{U}_1, \varrho} < C^{(2)} C_\varrho^{(1)} (D_\varrho)^{r_0} e$ for some constant $C^{(2)}$,

$$\left\| \left(\frac{t_n}{\varrho_n}\right)^{d+1} \zeta \right\|_{\mathfrak{U}_1, \varrho} < (1 + 2C^{(2)} C_\varrho^{(1)}) (D_\varrho)^{r_0} e.$$

By Proposition 9.5 (with φ defined by multiplication by t_n^{d+1}), if $\varrho < \omega^2$ for a suitable $\omega^2 \in \Omega^{(n)}$ (and we assume this to be the case), then there exists $\zeta' \in C^l(\mathfrak{U}_2(\varrho), \mathcal{F})$ such that $\left(\frac{t_n}{\varrho_n}\right)^{d+1} \zeta' = \left(\frac{t_n}{\varrho_n}\right)^{d+1} \zeta$ on $\mathfrak{U}_2(\varrho)$ and $\|\zeta'\|_{\mathfrak{U}_2, \varrho} < C_\varrho^{(3)} (1 + 2C^{(2)} C_\varrho^{(1)}) (D_\varrho)^{r_0} e$, where $C_\varrho^{(3)}$ is a constant depending only on ϱ.

Since t_n is not a zero-divisor for $t_n^d \mathcal{F}_x$ for $x \in X$, $\left(\frac{t_n}{\varrho_n}\right)^d \zeta' \in Z^l(\mathfrak{U}_2(\varrho), t_n^d \mathcal{F})$. Since t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in \{\varphi < c_\#\}$, the sheaf-epimorphism $\mathcal{F} \rightarrow t_n^d \mathcal{F}$ defined by multiplication by t_n^d is a sheaf-isomorphism on $\{\varphi < c_\#\}$. Hence codt $t_n^d \mathcal{F} > l + q + n$ on $\{\varphi < c_\#\}$.

By Proposition 11.11 (applied to $t_n^d \mathcal{F}$), if $\varrho < \omega^3$ for a suitable $\omega^3 \in \Omega^{(n}$ (and we assume this to be the case), then there exist $\xi^{(r_0+1)} \in Z^l(\mathfrak{U}_0(\varrho), t_n^d \mathcal{F}) \subset Z^l(\mathfrak{U}_0(\varrho), \mathcal{F})$ and $\eta'' \in C^{l-1}(\mathfrak{U}_3(\varrho), t_n^d \mathcal{F}) \subset C^{l-1}(\mathfrak{U}_3(\varrho), \mathcal{F})$ such that $\left(\frac{t_n}{\varrho_n}\right)^d \zeta' = \xi^{(r_0+1)} + \delta\eta''$ on $\mathfrak{U}_3(\varrho)$ and $\|\xi^{(r_0+1)}\|_{\mathfrak{U}_0, \varrho} < C_\varrho^{(4)} C_\varrho^{(3)} (1 + 2C^{(2)} C_\varrho^{(1)}) (D_\varrho)^{r_0} e$ and $\|\eta''\|_{\mathfrak{U}_3, \varrho} < C_\varrho^{(4)} C_\varrho^{(3)} (1 + 2C^{(2)} C_\varrho^{(1)}) (D_\varrho)^{r_0} e$, where $C_\varrho^{(4)}$ is a constant depending only on ϱ.

Set $\eta^{(r_0+1)} = \eta' + \left(\frac{t_n}{\varrho_n}\right) \eta''$ on $\mathfrak{U}_3(\varrho)$. Then

$$\eta^{(r_0+1)} \in C^{l-1}(\mathfrak{U}_3(\varrho), \mathcal{F}).$$

$$\xi^{(r_0)} = \Sigma_i a_i^{(r_0+1)} \xi_i + \delta\eta^{(r_0+1)} + \left(\frac{t_n}{\varrho_n}\right) \xi^{(r_0+1)} \text{ on } \mathfrak{U}_3(\varrho).$$

$$\|\eta^{(r_0+1)}\|_{\mathfrak{U}_3, \varrho} < [C_\varrho^{(1)} + C_\varrho^{(4)} C_\varrho^{(3)} (1 + 2C^{(2)} C_\varrho^{(1)})] (D_\varrho)^{r_0} e.$$

We need only set $D_\varrho > C_\varrho^{(1)} + C_\varrho^{(4)} C_\varrho^{(3)} (1 + 2C_\varrho^{(2)} C_\varrho^{(1)})$ and the construction by induction is complete.

Let $a_i = \Sigma_r \left(\frac{t_n}{\varrho_n}\right)^r a_i^{(r+1)}$ and $\eta = \Sigma_r \left(\frac{t_n}{\varrho_n}\right)^r \eta^{(r+1)}$. Let $\varrho^2 = (\varrho_1, \dots, \varrho_{n-1}, D_\alpha^{-1} \varrho_n)$. The first series converges on $K(\varrho^2)$ and the second series converges on $\mathfrak{U}_3(\varrho^2)$. It is easily seen that $\xi = \Sigma_i a_i \xi_i + \delta\eta$ on $\mathfrak{U}_3(\varrho^2)$. Therefore $\operatorname{Im}(H^l(X_c(\varrho^1), \mathcal{F}) \rightarrow H^l(X_c(\varrho^2), \mathcal{F}))$ is contained in the $\Gamma(K(\varrho^2), {}_n\mathcal{O})$-submodule generated by the images of $\xi_1, \dots, \xi_k$ in $H^l(X_c(\varrho^2), \mathcal{F})$. q. e. d.

§ 13. Some Preparations for the Proof of (B_n^l).

A. Suppose X is a complex space and $\mathcal{F}$ is a coherent analytic sheat on X. Suppose $\varrho^0 \in \mathbf{R}_+^n$ and $\pi: X \to K(\varrho^0)$ is a holomorphic map. Suppose $l \in \mathbf{N}_*$.

PROPOSITION 13.1. Suppose t_n is not a zero-divisor for $t_n^d \mathcal{F}_x$ for $x \in X$ and t_n is not a zero-divisor for $t_n^d \pi_{l+1}(\mathcal{F})_0$. If $r \in \mathbf{N}_*$, then $\operatorname{Im} \alpha = \operatorname{Im} \beta$ in

$$\pi_l(\mathcal{F})_0 \xrightarrow{\alpha} \pi_l(\mathcal{F}/t_n^r \mathcal{F})_0 \xleftarrow{\beta} \pi_l(\mathcal{F}/t_n^{r+2d} \mathcal{F})_0 ,$$

where α and β are induced by quotient maps.

PROOF. It is clear that $\operatorname{Im} \alpha \subset \operatorname{Im} \beta$. We are going to prove $\operatorname{Im} \beta \subset \operatorname{Im} \alpha$. The commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & t_n^{r+2d} \mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}/t_n^{r+2d} \mathcal{F} & \longrightarrow & 0 \\
 & & \cap\downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & t_n^r \mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}/t_n^r \mathcal{F} & \longrightarrow & 0
\end{array}$$

gives rise to the following diagram with exact rows

$$\begin{array}{ccccccc}
 & & \pi_l(\mathcal{F}/t_n^{r+2d} \mathcal{F})_0 & \xrightarrow{a} & \pi_{l+1}(t_n^{r+2d} \mathcal{F}) & \xrightarrow{c} & \pi_{l+1}(\mathcal{F})_0 \\
 & & \beta \downarrow & & b \downarrow & & \\
\pi_l(\mathcal{F})_0 & \xrightarrow{\alpha} & \pi_l(\mathcal{F}/t_n^r \mathcal{F})_0 & \longrightarrow & \pi_{l+1}(t_n^r \mathcal{F})_0 . & &
\end{array}$$

All we need to prove is that $ba = 0$.

Consider the following commutative diagram:

$$\begin{array}{ccccc}
\pi_{l+1}(\mathcal{F})_0 & \xleftarrow{c} & \pi_{l+1}(t_n^{r+2d} \mathcal{F})_0 & \xrightarrow{b} & \pi_{l+1}(t_n^r \mathcal{F})_0 \\
\| & & \uparrow u & & \\
\| & & \pi_{l+1}(t_n^d \mathcal{F})_0 & & \uparrow q \\
\| & & \downarrow v & & \\
\pi_{l+1}(\mathcal{F})_0 & \xleftarrow{w} & \pi_{l+1}(\mathcal{F})_0 & \xrightarrow{p} & \pi_{l+1}(\mathcal{F})_0 ,
\end{array}$$

where u is induced by the map $\tilde{u}: t_n^d \mathcal{F} \longrightarrow t_n^{d+2r} \mathcal{F}$ defined by multiplication by t_n^{r+d}, v is defined by the inclusion map $t_n^d \mathcal{F} \hookrightarrow \mathcal{F}$, w is defined by multiplication by t_n^{r+d}, p is defined by multiplication by t_n^d, and q is induced by the map $\mathcal{F} \longrightarrow t_n^r \mathcal{F}$ defined by multiplication by t_n^r.

Since t_n is not a zero-divisor for $t_n^d \mathcal{F}_x$ for $x \in X$, $\tilde{u}$ is a sheaf-isomorphism. Hence u is an isomorphism. Since t_n is not a zero-divisor for $t_n^d \pi_{l+1}(\mathcal{F})_0$, $\operatorname{Ker} w \subset \operatorname{Ker} p$.

$$\operatorname{Im} a = \operatorname{Ker} c = \operatorname{Ker} wvu^{-1} = (vu^{-1})^{-1}(\operatorname{Ker} w)$$

$$\subset (vu^{-1})^{-1}(\operatorname{Ker} p) = \operatorname{Ker} pvu^{-1} \subset \operatorname{Ker} qpvu^{-1} = \operatorname{Ker} b.$$

Hence $ba = 0$. q. e. d.

B. Suppose $(X, \mathcal{O})$ is a complex space of reduction order $\leqq p < \infty$. Suppose $\varrho^0 \in \mathbf{R}_+^n$ and $\pi: (X, \mathcal{O}) \to K(\varrho^0)$ is a q-concave map with exhaustion function φ and concavity bounds $c_*, c_\#$. Suppose $l \in \mathbf{N}_*$.

PROPOSITION 13.2. Suppose $0 \longrightarrow \mathcal{G} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{R} \longrightarrow 0$ is an exact sequence of coherent analytic sheaves on X. If $\mathcal{G}$ and $\mathcal{R}$ have property $(B)_l^n$ at 0 with respect to (π, φ), then $\mathcal{F}$ has property $(B)_l^n$ at 0 with respect to (π, φ).

PROOF. Fix $c_* < \tilde{c} < c < c_\#$ such that $\tilde{c}$ is reachable from c with respect to (q, φ).

Since $\pi_l^c(\mathcal{G})_0 \longrightarrow \pi^c(\mathcal{F})_0 \xrightarrow{\beta_c} \pi_l^c(\mathcal{R})_0$ is exact, $\pi_l^c(\mathcal{F})_0$ is finitely generated over ${}_n\mathcal{O}_0$.

Suppose $\mathfrak{V}, \mathfrak{U} \in \mathbf{S}(X)$ such that $|\mathfrak{V}| \underset{\pi}{\subset\subset} X_c \underset{\pi}{\subset\subset} |\mathfrak{U}| \underset{\pi}{\subset\subset} X_{\tilde{c}}$ and $\mathfrak{V} \underset{\pi}{<<} \mathfrak{U}$. Suppose $\varrho^1 \leqq \varrho^0$ and $\xi_1, \dots, \xi_k \in Z^l(\mathfrak{U}(\varrho^1), \mathcal{F})$. Let A be the ${}_n\mathcal{O}_0$-submodule of $\pi_l^c(\mathcal{F})_0$ generated by the images of $\xi_1, \dots, \xi_k$ in $\pi_l^c(\mathcal{F})_0$.

Take $c' \in (c, c_\#)$ such that $\tilde{c}$ is reachable from c' and $|\mathfrak{V}| \underset{\pi}{\subset\subset} X_{c'}$. Choose $\mathfrak{U}_i \in \mathbf{S}(X)$, $1 \leqq i \leqq 3$, such that

$$\mathfrak{U}_{i+1} \underset{\pi}{<<} \mathfrak{U}_i \underset{\pi}{<<} \mathfrak{U}, \quad |\mathfrak{V}| \underset{\pi}{\subset\subset} |\mathfrak{U}_3| \underset{\pi}{\subset\subset} X_{c'} \underset{\pi}{\subset\subset} |\mathfrak{U}_2|,$$

and $|\mathfrak{U}_1| \underset{\pi}{\subset\subset} X_c$.

By Proposition 8.5 and 8.6, for some $\varrho^2 \leqq \varrho^1$ we have $\| \xi_i \|_{\mathfrak{U}_1, \varrho^2} < \infty$.

Take $\omega \in \Omega^{(n)}$ with $\omega < \varrho^2$ and we shall impose more conditions on ω later on. Take $\varrho < \omega$ and $\xi \in Z^l(\mathfrak{U}(\varrho), \mathcal{F})$ with $\| \xi \|_{\mathfrak{U}, \varrho} < \varepsilon$ such that the image of ξ in $\pi_l^c(\mathcal{F})_0$ belongs to A.

Let $A_1 = \beta_c(A)$. The image of $\beta(\xi)$ in $\pi_l^c(\mathcal{R})_0$ belongs to A_1. Since $\mathcal{R}$ has property $(B)_l^n$, if $\omega \leqq \omega^1$ for a suitable $\omega^1 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $a_i' \in \Gamma(K(\varrho), {}_n\mathcal{O})$ and $\eta' \in C^{l-1}(\mathfrak{U}_1(\varrho), \mathcal{R})$, $\beta(\xi) = \Sigma_i\, a_i'\, \beta(\xi_i) + \delta\eta'$ on $\mathfrak{U}_1(\varrho)$ and $|a_i'|_\varrho < C_\varrho^{(1)} e$ and $\|\eta'\|_{\mathfrak{U}_1, \varrho} < C_\varrho^{(1)} e$, where $C_\varrho^{(1)}$ is a constant depending only on ϱ.

By Lemma 8.3, for some $\eta'' \in C^{l-1}(\mathfrak{U}_1(\varrho), \mathcal{F})$, $\beta(\eta'') = \eta'$ on $\mathfrak{U}_1(\varrho)$ and $\|\eta''\|_{\mathfrak{U}_1, \varrho} < C_\varrho^{(1)} e$. $\xi - \Sigma_i\, a_i'\, \xi_i - \delta\eta'' \in Z^l(\mathfrak{U}_1(\varrho), \operatorname{Im} \alpha)$. $\|\xi - \Sigma_i\, a_i'\, \xi_i - - \delta\eta''\|_{\mathfrak{U}_1, \varrho} < (1 + C^{(2)}\, C_\varrho^{(1)})\, e$, where $C^{(2)}$ is a constant.

By Proposition 9.5, if $\omega \leqq \omega^2$ for a suitable $\omega^2 \in \Omega^{(n)}$ (and we assume this to be the case), then $\|\alpha^{-1}(\xi - \Sigma_i\, a_i'\, \xi_i - \delta\eta'')\|_{\mathfrak{U}_2, \varrho} < C_\varrho^{(3)}(1 + C^{(2)}\, C_\varrho^{(1)})\, e$, where $C_\varrho^{(3)}$ is a constant depending only on ϱ.

Let π' be the restriction of π to $|\mathfrak{U}_2|$. Let A' be the ${}_n\mathcal{O}_0$-submodule of $\pi_l'(\mathcal{F})_0\, (= \pi_l'(\mathcal{F}\,\|\, \mathfrak{U}_2|)_0)$ generated by the images of $\xi_1, \ldots, \xi_k$ in $\pi_l'(\mathcal{F})_0$. Let $\alpha' : \pi_l'(\mathcal{G})_0 \to \pi_l'(\mathcal{F})_0$ be induced by α and let $\sigma\colon\ \pi_l'(\mathcal{G})_0 \to \pi_l^{c'}(\mathcal{G})_0$ be induced by restriction map. Let $A_2 = \sigma((\alpha')^{-1}(A'))$.

Since $\pi_l^{c'}(\mathcal{G})_0$ is finitely generated over ${}_n\mathcal{O}_0$, $A_2 = \Sigma_{i=1}^q\, {}_n\mathcal{O}_0 f_i$ for some $f_i \in \pi_l^{c'}(\mathcal{G})_0$. $f_i = \sigma(g_i)$ for some $g_i \in (\alpha')^{-1}(A') \subset \pi_l'(\mathcal{G})_0$. Since $\mathfrak{U}_2(\varrho)$ is a Stein open covering of $|\mathfrak{U}_2(\varrho)|$, we can find $\zeta_i \in Z^l(\mathfrak{U}_2(\varrho^3), \mathcal{G})$ for some $\varrho^3 \leqq \varrho^0$ such that ζ_i induces g_i.

The images of $\zeta_1, \ldots, \zeta_p$ in $\pi_l^{c'}(\mathcal{G})_0$ generate A_2. Since $\alpha'(g_i) \in A'$, by shrinking ϱ^3, we can find $b_j^{(i)} \in \Gamma(K(\varrho^3), {}_n\mathcal{O})$ and $\eta^{(i)} \in C^{l-1}(\mathfrak{U}_2(\varrho^3), \mathcal{F})$ such that $\alpha(\zeta_i) = \Sigma_j\, b_j^{(i)}\, \xi_j + \delta\eta^{(i)}$ on $\mathfrak{U}_2(\varrho^3)$.

Since the image of $\xi - \Sigma_i\, a_i'\, \xi_i$ in $\pi_l^c(\mathcal{F})_0$ belongs to A, the image of $\xi - \Sigma_i\, a_i'\, \xi_i - \delta\eta''$ in $\pi_l'(\mathcal{F})_0$ belongs to A'. The image of $\alpha^{-1}(\xi - \Sigma_i\, a_i'\, \xi_i - \delta\eta'')$ in $\pi_l'(\mathcal{G})_0$ belongs to A_2.

Since $\mathcal{G}$ has property $(B)_l^n$ at 0 with respect to (π, φ), if $\varrho < \omega^3$ for a suitable $\omega^3 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $a_i^* \in \Gamma(K(\varrho), {}_n\mathcal{O})$ and $\eta^* \in C^{l-1}(\mathfrak{U}_3(\varrho), \mathcal{G})$, $\alpha^{-1}(\xi - \Sigma_i\, a_i'\, \xi_i - \delta\eta'') = \Sigma_i\, a_i^* \zeta_i + + \delta\eta^*$ and $|a_i^*|_\varrho < C_\varrho^{(4)}\, C_\varrho^{(3)}(1 + C^{(2)}\, C_\varrho^{(1)})\, e$ and $\|\eta^*\|_{\mathfrak{U}_3, \varrho} < C_\varrho^{(4)}\, C_\varrho^{(3)}(1 + + C^{(2)}\, C_\varrho^{(1)})\, e$, where $C_\varrho^{(4)}$ is a constant depending only on ϱ.

Let $a_i = a_i' + \Sigma_j\, a_j^*\, b_i^{(j)} \in \Gamma(K(\varrho), {}_n\mathcal{O})$ and $\tilde{\eta} = \eta'' + \Sigma_i\, a_i^*\, \eta^{(i)} + \alpha(\eta^*) \in \in C^{l-1}(\mathfrak{U}_3(\varrho), \mathcal{F})$. Then $\xi = \Sigma_i\, a_i\, \xi_i + \delta\tilde{\eta}$ on $\mathfrak{U}_3(\varrho)$. If $\omega < \varrho^3$ (and we assume this to be the case), then $|a_i|_\varrho < (C_\varrho^{(1)} + C^{(5)}\, C_\varrho^{(4)}\, C_\varrho^{(3)}(1 + C^{(2)}\, C_\varrho^{(1)}))\, e$ and $\|\tilde{\eta}\|_{\mathfrak{U}_3, \varrho} < (C_\varrho^{(1)} + C^{(5)}\, C_\varrho^{(4)}\, C_\varrho^{(3)}(1 + C^{(2)}\, C_\varrho^{(1)}))\, e$, where where $C^{(5)}$ is a constant.

$$\|\xi - \Sigma_i\, a_i\, \xi_i\|_{\mathfrak{U}_3, \varrho} < (1 + C^{(6)}(C_\varrho^{(1)} + C^{(5)}\, C_\varrho^{(4)}\, C_\varrho^{(3)}(1 + C^{(2)}\, C_\varrho^{(1)})))\, e,$$

where $C^{(6)}$ is a constant $\geqq 1$. By Proposition 10.5, if $\omega \leqq \omega^{(4)}$ for a suitable $\omega^4 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $\eta \in$

$\in C^{l-1}(\mathfrak{V}(\varrho), \mathcal{F})$, $\xi - \Sigma_i a_i \xi_i = \delta\eta$ on $\mathfrak{V}(\varrho)$ and $\|\eta\|_{\mathfrak{V},\varrho} < C_\varrho^{(7)}(1 + C^{(6)}(C_\varrho^{(1)} + C^{(5)} C_\varrho^{(4)} C_\varrho^{(3)}(1 + C^{(2)} C_\varrho^{(1)})))\, e$, where $C_\varrho^{(7)}$ is a constant depending only on ϱ. Hence $\mathcal{F}$ has properly $(B)_l^n$ at 0 with respect to (π, φ). q. e. d.

Let $X' = X \cap \{t_n = 0\}$ and $\mathcal{O}' = \mathcal{O}/t_n\,\mathcal{O}$. Suppose $(X', \mathcal{O}')$ has reduction order $\leqq p' < \infty$. Let $\pi' : (X', \mathcal{O}') \to K^{n-1}(\bar{\varrho}^0)$ be induced by π. Let $\varphi' = \varphi \mid X'$. Suppose $\mathcal{F}$ is a coherent analytic sheaf on X such that $t_n\, \mathcal{F} = 0$. Let $\mathcal{F}' = \mathcal{F} \mid X'$.

PROPOSITION 13.3. If $\mathcal{F}'$ has properly $(B)_l^{n-1}$ at 0 with respect to (π', φ'), then $\mathcal{F}$ has property $(B)_l^n$ at 0 with respect to (π, φ).

PROOF. We use the notations of § 8.*D*.

Fix $c_* < \tilde{c} < c < c_\#$ such that $\tilde{c}$ is reachable from c with respect to (q, φ). Then $\tilde{c}$ is reachable from c with respect to (q, φ').

Since we have a natural isomorphism $\sigma : \pi_l^c(\mathcal{F})_0 \to (\pi')_l^c(\mathcal{F})_0$, $\pi_l^c(\mathcal{F})_0$ is finitely generated over ${}_n\mathcal{O}_0$.

Suppose $\mathfrak{V}, \mathfrak{U} \in \mathbf{S}(X)$ such that $|\mathfrak{V}| \underset{\pi}{\subset\subset} X_c \underset{\pi}{\subset\subset} |\mathfrak{U}| \underset{\pi}{\subset\subset} X_{\tilde{c}}$ and $\mathfrak{V} \underset{\pi}{<<} \mathfrak{U}$. Suppose $\varrho^1 \leqq \varrho^0$ and $\xi_1, \dots, \xi_k \in Z^l(\mathfrak{U}(\varrho^1), \mathcal{F})$. Let A be the ${}_n\mathcal{O}_0$-submodule of $\pi_l^c(\mathcal{F})$ generated by the images of $\xi_1, \dots, \xi_k$ in $\pi_l^c(\mathcal{F})_0$.

Choose $\mathfrak{V}_1, \mathfrak{U}_1 \in \mathbf{S}(X)$ such that $\mathfrak{V} \underset{\pi}{<<} \mathfrak{V} \underset{\pi}{<<} \mathfrak{U}_1 \underset{\pi}{<<} \mathfrak{U}$, and $|\mathfrak{V}_1| \underset{\pi}{\subset\subset} X_c \underset{\pi}{\subset\subset} |\mathfrak{U}_1|$.

Take $\omega \in \Omega^{(n)}$ and we shall impose more conditions on ω later. Take $\varrho < \omega$ and $\xi \in Z^l(\mathfrak{U}(\varrho), \mathcal{F})$ with $\|\xi\|_{\mathfrak{U},\varrho} < e$ such that the image of ξ in $\pi_l^c(\mathcal{F})_0$ belongs to A.

Let $A' = \sigma(A)$ and $\xi_i' = \sigma_{\mathfrak{U}}(\xi_i) \in Z^l(\mathfrak{U}'(\bar{\varrho}^{-1}), \mathcal{F}')$. A' is generated by the images of ξ_i' in $(\pi')_l^c(\mathcal{F}')_0$. Let $\xi' = \sigma_{\mathfrak{U}}(\xi) \in Z^l(\mathfrak{U}'(\bar{\varrho}), \mathcal{F}')$.

By Proposition 8.8, if $\omega < \varrho^2$ for a suitable $\varrho^2 \in \mathbf{R}_+^n$ (and we assume this to be the case), then $\|\xi'\|_{\mathfrak{U}_1', \bar{\varrho}} < C^{(1)} e$, where $C^{(1)}$ is a constant.

Since $\mathcal{F}'$ has property $(B)_l^{n-1}$ at 0 with respect to (π', φ'), if $(\omega_1, \dots, \omega_{n-1}) \leqq \bar{\omega}'$ for a suitable $\bar{\omega}' \in \Omega^{(n-1)}$ (and we assume this to be the case), then for some $a_i' \in \Gamma(K^{n-1}(\bar{\varrho}), {}_{n-1}\mathcal{O})$ and $\eta' \in C^{l-1}(\mathfrak{V}_1'(\bar{\varrho}), \mathcal{F}')$, $\xi' = \Sigma_i a_i' \xi_i + \delta\eta'$ on $\mathfrak{V}_1'(\bar{\varrho})$ and $|a_i'|_{\bar{\varrho}} < C_{\bar{\varrho}}^{(2)} C^{(1)} e$ and $\|\eta'\|_{\mathfrak{V}_1', \bar{\varrho}} < C_{\bar{\varrho}}^{(2)} C^{(1)} e$, where $C_{\bar{\varrho}}^{(2)}$ is a constant depending only on $\bar{\varrho}$.

Let $P : K(\varrho) \to K^{n-1}(\bar{\varrho})$ be the projection. Let $a_i = a_i' \circ P$. Then $|a_i|_\varrho < C_{\bar{\varrho}}^{(2)} C^{(1)} e$. Let $\eta = \sigma_{\mathfrak{V}_1}^{-1}(\eta') \in C^{l-1}(\mathfrak{V}_1(\varrho), \mathcal{F})$.

By Proposition 8.8, if $\omega < \varrho^3$ for a suitable $\varrho^3 \in \mathbf{R}_+^n$ (and we assume this to be the case), then $\|\eta\|_{\mathfrak{V},\varrho} < C^{(3)} C_{\bar{\varrho}}^{(2)} C^{(1)} e$, where $C^{(3)}$ is a constant. $\xi = \Sigma_i a_i \xi_i + \delta\eta$ on $\mathfrak{V}(\varrho)$. q. e. d.

C. We are going to introduce a property weaker than $(B)_l^n$ which, in the cases we are interested in, is equivalent to property $(B)_l^n$. We call it property pre-$(B)_l^n$.

Suppose X is a complex space of reduction order $\leqq p < \infty$, $\varrho^0 \in \mathbf{R}_+^n$, and $\pi: X \to K(\varrho^0)$ is a q-concave map with exhaustion function φ and concavity bounds $c_*, c_\#$. Suppose $\mathcal{F}$ is a coherent analytic sheaf on X and $l \in \mathbf{N}_*$.

DEFINITION. $\mathcal{F}$ has *property pre-*$(B)_l^n$ at 0 with respect to (π, φ) if, for every $c \in (c_*, c_\#)$ and every $\tilde{c} \in (c_*, c)$ such that $\tilde{c}$ is reachable from c with respect to (q, φ), the following two conditions are satisfied.

(i) $\pi_l^c(\mathcal{F})_0$ is finitely generated over ${}_n\mathcal{O}_0$.

(ii) Suppose $\mathfrak{V}, \mathfrak{U} \in \mathbf{S}(X)$ with $|\mathfrak{V}| \underset{\pi}{\subset\subset} X_c \underset{\pi}{\subset\subset} |\mathfrak{U}| \underset{\pi}{\subset\subset} X_{\tilde{c}}$ and $\mathfrak{V} \underset{\pi}{<<} \mathfrak{U}$. Suppose $\varrho^1 \leqq \varrho^0$ and $\xi_1, \ldots, \xi_k \in Z^l(\mathfrak{U}(\varrho^1), \mathcal{F})$ such that the images of $\xi_1, \ldots, \xi_k$ in $\pi_l^c(\mathcal{F})_0$ generate $\pi_l^c(\mathcal{F})_0$ over ${}_n\mathcal{O}_0$. Then there exists $\omega \in \Omega^{(n)}$ such that, if $\varrho < \omega$ and $\xi \in Z^l(\mathfrak{U}(\varrho), \mathcal{F})$ with $\|\xi\|_{\mathfrak{U}, \varrho} < e$, then for some $a_1, \ldots, a_k \in \Gamma(K(\varrho), {}_n\mathcal{O})$ and $\eta \in C^{l-1}(\mathfrak{V}(\varrho), \mathcal{F})$, $\xi = \Sigma_i a_i \xi_i + \delta\eta$ on $\mathfrak{V}(\varrho)$ and $|a_i|_\varrho < C_\varrho e$ and $\|\eta\|_{\mathfrak{V}, \varrho} < C_\varrho e$, where C_ϱ is a constant depending only on ϱ.

PROPOSITION 13.4. Suppose $\mathcal{F}$ satisfies following: if $c_* < \tilde{c} < c < c_\#$ and $\tilde{c}$ is reachable from c, then $\pi_l^{\tilde{c}}(\mathcal{F})_0 \to \pi_l^c(\mathcal{F})_0$ is surjective. Then $\mathcal{F}$ has property $(B)_n^l$ at 0 with respect to (π, φ) if and only if $\mathcal{F}$ has property pre-$(B)_n^l$ at 0 with respect to (π, φ).

PROOF. The «only if» part is clear.

To prove the «if» part, fix $c_* < \tilde{c} < c < c_\#$ such that $\tilde{c}$ is reachable from c. Take $\mathfrak{V}, \mathfrak{U} \in \mathbf{S}(X)$ such that $|\mathfrak{V}| \underset{\pi}{\subset\subset} X_c \underset{\pi}{\subset\subset} |\mathfrak{U}| \underset{\pi}{\subset\subset} X_{\tilde{c}}$ and $\mathfrak{V} \underset{\pi}{<<} \mathfrak{U}$. Suppose $\varrho^1 \leqq \varrho^0$ and $\xi_1, \ldots, \xi_k \in Z^l(\mathfrak{U}(\varrho^1), \mathcal{F})$ and A is the ${}_n\mathcal{O}_0$-submodule of $\pi_l(\mathcal{F})_0$ generated by the images of $\xi_1, \ldots, \xi_k$ in $\pi_l^c(\mathcal{F})_0$.

Since $\pi_l^{\tilde{c}}(\mathcal{F})_0 \to \pi_l^c(\mathcal{F})_0$ is surjective and $\pi_l^c(\mathcal{F})_0$ is finitely generated over ${}_n\mathcal{O}_0$, there exist $\zeta_1, \ldots, \zeta_m \in Z^l(\mathfrak{U}(\varrho^2), \mathcal{F})$ for some $\varrho^2 \leqq \varrho^1$ such that the images of $\zeta_1, \ldots, \zeta_m$ in $\pi_l^c(\mathcal{F})_0$ generate $\pi_l^c(\mathcal{F})_0$ over ${}_n\mathcal{O}_0$.

Choose $\mathfrak{V}' \in \mathbf{S}(X)$ such that $\mathfrak{V} \underset{\pi}{<<} \mathfrak{V}' \underset{\pi}{<<} \mathfrak{U}$ and $|\mathfrak{V}'| \subset X_c$.

Since the images of $\zeta_1, \ldots, \zeta_m$ in $\pi_l^c(\mathcal{F})_0$ generate $\pi_l^c(\mathcal{F})_0$ over ${}_n\mathcal{O}_0$, for some $\varrho^3 \leqq \varrho^2$ there exist $b_j^{(i)} \in \Gamma(K(\varrho^2), {}_n\mathcal{O})$ and $\eta_i \in C^{l-1}(\mathfrak{U}'(\varrho^2), \mathcal{F})$ such that $\xi_i = \Sigma_j b_j^{(i)} \zeta_j + \delta\eta_i$ on $\mathfrak{V}'(\varrho^2)$. By Propositions 8.5 and 8.6, after shrinking ϱ^3 we have $\|\eta_i\|_{\mathfrak{V}, \varrho^3} < \infty$. Let $b^{(i)} = (b_1^{(i)}, \ldots, b_m^{(i)})$.

Let $\zeta_i' \in \pi_l^c(\mathcal{F})_0$ be the image of ζ_i in $\pi_l^c(\mathcal{F})_0$. Let $T \subset {}_n\mathcal{O}_0^m$ be the relation-module of $\zeta_1', \dots, \zeta_m'$ over ${}_n\mathcal{O}_0$. $T = \Sigma_{i=1}^r {}_n\mathcal{O}_0 g_i$ for some $g_i \in {}_n\mathcal{O}_0$.

By shrinking ϱ^3, we can assume the following:

(i) g_i is the germ at 0 of $f^{(i)} = (f_1^{(i)}, \dots, f_m^{(i)}) \in \Gamma(K(\varrho^3), {}_n\mathcal{O}^m)$, and

(ii) $\Sigma_j f_j^{(i)} \zeta_j = \delta\gamma^{(i)}$ on $\mathfrak{W}'(\varrho^3)$ for some $\gamma^{(i)} \in C^{l-1}(\mathfrak{W}'(\varrho^3), \mathcal{F})$. By further shrinking ϱ^3, we can assume $\| \gamma^{(i)} \|_{\mathfrak{W}, \varrho^3} < \infty$.

Let $\mathcal{M} = \Sigma_{i=1}^k {}_n\mathcal{O}\, b^{(i)} + \Sigma_{i=1}^r {}_n\mathcal{O} f^{(i)} \subset {}_n\mathcal{O}^m$ on $K(\varrho^3)$.

Take $\omega \in \Omega^{(n)}$ with $\omega < \varrho^3$ and we shall impose more conditions on ω later on. Take $\varrho < \omega$ and $\xi \in Z^l(\mathfrak{U}(\varrho), \mathcal{F})$ such that $\| \xi \|_{\mathfrak{U}, \varrho} < e$ and the image of ξ in $\pi_l^c(\mathcal{F})_0$ belongs to A.

Since $\mathcal{F}$ has property pre-$(B)_n^l$ at 0 with respect to (π, φ), if $\omega \leqq \omega^1$ for a suitable $\omega^1 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $a_1', \dots, a_m' \in \Gamma(K(\varrho), {}_n\mathcal{O})$ and $\eta' \in C^{l-1}(\mathfrak{W}(\varrho), \mathcal{F})$, $\xi = \Sigma_i a_i' \zeta_i + \delta\eta'$ on $\mathfrak{W}(\varrho)$ and $| a_i' |_\varrho < C_\varrho^{(1)} e$ and $\| \eta' \|_{\mathfrak{W}, \varrho} < C_\varrho^{(1)} e$, where $C_\varrho^{(1)}$ is a constant depending only on ϱ.

Since the image of ξ in $\pi_l^c(\mathcal{F})_0$ belongs to A, $(a_1', \dots, a_m')_0 \in \mathcal{M}_0$. By Proposition 1 of [7] (with $N = 0$), if $\omega \leqq \omega^2$ for a suitable $\omega^2 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $a_1, \dots, a_k, a_1'', \dots, a_r'' \in \Gamma(K(\varrho), {}_n\mathcal{O})$, $(a_1', \dots, a_m') = \Sigma_{i=1}^k a_i b^{(i)} + \Sigma_{i=1}^r a_i'' f^{(i)}$ on $K(\varrho)$ and $| a_i |_\varrho < C_\varrho^{(2)} C_\varrho^{(1)} e$ and $| a_i'' |_\varrho < C_\varrho^{(2)} C_\varrho^{(1)} e$, where $C_\varrho^{(2)}$ is a constant depending only on ϱ.

$$\xi = \Sigma_{i,j}\, a_i b_j^{(i)} \zeta_j + \Sigma_{i,j}\, a_i'' f_j^{(i)} \zeta_j + \delta\eta' = \Sigma_i a_i \xi_i + \delta\eta,$$

where $\eta = \Sigma_i a_i'' \gamma^{(i)} - \Sigma_i a_i \eta_i + \eta'$. $\| \eta \|_{\mathfrak{W}, \varrho} < (1 + C^{(3)} C_\varrho^{(2)})\, C_\varrho^{(1)} e$, where $C^{(3)}$ is a constant. q. e. d.

§ 14. Proof of Property $(B)_l^n$.

Suppose X is a complex space of reduction order $\leqq p < \infty$, $\varrho^0 \in \mathbf{R}_+^n$, and $\pi: X \to K(\varrho^0)$ is a q-concave map with exhaustion function φ and concavity bounds $c_*, c_\#$.

PROPOSITION $(14.1)_n$. Suppose $\mathcal{F}$ is a coherent analytic sheaf on X such that (i) codh $\mathcal{F} \geqq r$ on $\{\varphi < c_\#\}$ and (ii) $t_1, \dots, t_n$ form an $\mathcal{F}_x$-sequence for $x \in \{\varphi < c_\#\} \cap \pi^{-1}(0)$. Then $\mathcal{F}$ has property $(B)_l^n$ at 0 with respect to (π, φ) for $0 \leqq l < r - q - 2n$.

PROOF. We are going to prove by induction on n.

(*a*) Assume $n=0$. Fix $0 \leqq l < r-q$ and $c \in (c_*, c_\#)$. By Theorem *A-G*, $\pi_l^c(\mathcal{F})_0 = H^l(X_c, \mathcal{F})$ is finite-dimensional over $\mathbf{C}$.

Suppose $\mathfrak{V}, \mathfrak{U} \in \mathbf{S}(X)$ such that $|\mathfrak{V}| \subset\subset X_c \subset\subset |\mathfrak{U}|$ and $\mathfrak{V} << \mathfrak{U}$. Suppose $\xi_1, \dots, \xi_k \in Z^l(\mathfrak{U}, \mathcal{F})$ and A is the vector subspace of $H^l(X_c, \mathcal{F})$ generated by $\xi_1, \dots, \xi_k$.

Take $\xi \in Z^l(\mathfrak{U}, \mathcal{F})$ such that $\|\xi\|_{\mathfrak{U}} < e$ and the image of ξ in $H^l(X_c, \mathcal{F})$ belongs to A.

Choose a countable Stein open covering $\mathfrak{W}$ of X_c such that $\mathfrak{V} << \mathfrak{W} << \mathfrak{U}$. Let $\xi_i' = \xi_i | \mathfrak{W}$ and $\xi' = \xi | \mathfrak{W}$. Define $\psi : \mathbf{C}^k \oplus C^{l-1}(\mathfrak{W}, \mathcal{F}) \to Z^l(\mathfrak{W}, \mathcal{F})$ by $\psi(a_1, \dots, a_k, \eta) = \Sigma_i a_i \xi_i' + \delta\eta$ for $a_i \in \mathbf{C}$ and $\eta \in C^{l-1}(\mathfrak{W}, \mathcal{F})$.

Since $\dim_{\mathbf{C}} \operatorname{Coker} \psi \leqq \dim_{\mathbf{C}} H^l(\mathfrak{W}, \mathcal{F}) = \dim_{\mathbf{C}} H^l(X_c, \mathcal{F}) < \infty$, ψ has closed range when $\mathbf{C}^k$, $C^{l-1}(\mathfrak{W}, \mathcal{F})$, and $Z^l(\mathfrak{W}, \mathcal{F})$ are given the natural Fréchet space structures. $\operatorname{Im} \psi$ is a Fréchet space. Since $\xi \in \operatorname{Im} \psi$, by applying the open mapping theorem to ψ, we conclude that $\xi' = \Sigma_i a_i \xi_i' + \delta\eta$ for some $a_i \in \mathbf{C}$ and $\eta \in C^{l-1}(\mathfrak{W}, \mathcal{F})$ such that $|a_i| < Ce$ and $\|\eta\|_{\mathfrak{V}} < Ce$, where C is a constant. Proposition $(14.1)_0$ is proved.

(*b*) In the rest of the proof we assume $n > 0$ and tuat Proposition $(14.1)_{n-1}$ is proved.

Take arbitrarily $c_* < c_*' < c_\#' < c_\#$.

Since t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in \{\varphi < c_\#\} \cap \pi^{-1}(0)$, the support S of the kernel of the sheaf-homomorphism $\mathcal{F} \to \mathcal{F}$ defined by multiplication by t_n is disjoint from $\{\varphi < c_\#\} \cap \pi^{-1}(0)$. For some $\varrho^2 < \varrho^0$, S is disjoint from $\{c_*' \leqq \varphi \leqq c_\#'\} \cap X(\varrho^2)$. Hence t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in \{c_*' \leqq \varphi \leqq c_\#'\} \cap X(\varrho^2)$.

Since $X_{c_*'}(\varrho^2) \subset\subset X$, by Lemma 3.3 (*b*), there exists $d_* \in \mathbf{N}_*$ such that t_n is not a zero-divisor for $t_n^{d_*} \mathcal{F}_x$ for $x \in X_{c_*'}(\varrho^2)$.

By replacing X by $X_{c_*'}(\varrho^2)$ and $c_\#$ by c_*', we can assume without loss of generality the following.

(14.1) $\qquad t_n$ is not a zero-divisor for $\mathcal{F}_x$ for $x \in \{\varphi < c_\#\}$.

(14.2) $\quad \left\{ \begin{array}{l} \text{There exists } d_* \in \mathbf{N}_* \text{ such that } t_n \text{ is not a zero-divisor for } t_n^{d_*} \mathcal{F}_x \\ \text{for } x \in X. \end{array} \right.$

By Proposition 6.6, we can also assume that $X \cap \{t_n = 0\}$ has reduction order $\leqq p'$ for some $p' \in \mathbf{N}_*$.

(*c*) We are going to reduce the proof of Proposition $(14.1)_n$ to the special case where t_n is not a zero-divisor of $\mathcal{F}_x$ for $x \in X$.

By (14.1), for $d \in \mathbf{N}_*$ the sheaf-homomorphism $\mathcal{F} \to t_n^d \mathcal{F}$ defined by multiplication by t_n^d induces a sheaf-isomorphism $\mathcal{F}/t_n \mathcal{F} \to t_n^d \mathcal{F}/t_n^{d+1} \mathcal{F}$ on $\{\varphi < c_\#\}$. Hence $\operatorname{codh}(t_n^d \mathcal{F}/t_n^{d+1} \mathcal{F}) = \operatorname{codh}(\mathcal{F}/t_n \mathcal{F}) \geqq r-1$ on $\{\varphi < c_\#\}$ and $t_1, \dots, t_{n-1}$ is a $(t_n^d \mathcal{F}/t_n^{d+1} \mathcal{F})_x$-sequence for $x \in \{\varphi < c_\#\} \cap \pi^{-1}(0)$.

Since $t_n(t_n^d \mathcal{F}/t_n^{d+1}\mathcal{F}) = 0$, by applying Proposition $(14.1)_{n-1}$ to the sheaf $t_n^d \mathcal{F}/t_n^{d+1}\mathcal{F}$ restricted to $X \cap \{t_n = 0\}$ and by using Proposition 13.3, we conclude that

$$(14.3)\quad \begin{cases} t_n^d \mathcal{F}/t_n^{d+1}\mathcal{F} \text{ has property } (B)_l^n \text{ at } 0 \text{ with respect to } (\pi, \varphi) \text{ for} \\ d \in \mathbf{N}_* \text{ and } 0 \leqq l < r - q - 2n + 1. \end{cases}$$

By using (14.3) and applying Proposition 13.2 to the exact sequence $0 \to t_n^d \mathcal{F}/t_n^{d+1}\mathcal{F} \to \mathcal{F}/t_n^{d+1}\mathcal{F} \to \mathcal{F}/t_n^d\mathcal{F} \to 0$, we conclude by induction on d that

$$(14.4)\quad \begin{cases} \mathcal{F}/t_n^d \mathcal{F} \text{ has property } (B)_l^n \text{ at } 0 \text{ with respect to } (\pi, \varphi) \text{ for } d \in \mathbf{N} \\ \text{and } 0 \leqq l < r - q - 2n + 1. \end{cases}$$

By using (14.4) and applying Proposition 13.2 to the exact sequence $0 \to t_n^{d_*}\mathcal{F} \to \mathcal{F} \to \mathcal{F}/t_n^{d_*}\mathcal{F} \to 0$, we conclude that, if $t_n^{d_*}\mathcal{F}$ has property $(B)_l^n$ at 0 with rispect to (π, φ), then $\mathcal{F}$ has property $(B)_l^n$ at 0 with respect to (π, φ).

Since $\mathcal{F} \approx t_n^{d_*}\mathcal{F}$ on $\{\varphi < c_\#\}$ by (14.1), by replacing $\mathcal{F}$ by $t_n^{d_*}\mathcal{F}$ we can assume without loss of generality that t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$.

(*d*) Fix $0 \leqq l < r - q - 2n$.

By Proposition 11.12, for any $c_* < \tilde{c} < c < c_\#$ such that $\tilde{c}$ is reachable from c, the map $\pi_l^{\tilde{c}}(\mathcal{F})_0 \to \pi_l^c(\mathcal{F})_0$ is surjective. Hence by Proposition 13.4 we need only prove that $\mathcal{F}$ has property pre-$(B)_l^n$.

Fix $c_* < \tilde{c} < c < c_\#$ such that $\tilde{c}$ is reachable from c. By (14.4) and Proposition 12.1, $\mathcal{F} | X_c$ is H^l-finite at 0 with respect to π^c. In particular, $\pi_l^c(\mathcal{F})_0$ is finitely generated over ${}_n\mathcal{O}_0$.

Choose $\mathfrak{V}, \mathfrak{U} \in \mathbf{S}(X)$ such that $|\mathfrak{V}| \underset{\pi}{\subset\subset} X_c \underset{\pi}{\subset\subset} |\mathfrak{U}| \underset{\pi}{\subset\subset} X_{\tilde{c}}$ and $\mathfrak{V} << \mathfrak{U}$. Suppose $\varrho^1 \leqq \varrho^0$ and $\xi^{(1)}, \dots, \xi^{(k)} \in Z^l(\mathfrak{U}(\varrho^1), \mathcal{F})$ such that the images of $\xi^{(1)}, \dots, \xi^{(k)}$ in $\pi_l^c(\mathcal{F})_0$ generate $\pi_l^c(\mathcal{F})_0$ over ${}_n\mathcal{O}_0$.

Choose $\tilde{c} < c_1 < c < c_2 < c_3 < c_\#$ such that $|\mathfrak{V}| \underset{\pi}{\subset\subset} X_{c_3}$, $X_{c_1} \underset{\pi}{\subset\subset} |\mathfrak{U}|$, and $\tilde{c}$ is reachable from c_3 with respect to (q, φ). Choose $\mathfrak{U}_i \in \mathbf{S}(X)$, $1 \leqq i \leqq 6$, such that $\mathfrak{U}_{i+1} \underset{\pi}{<<} \mathfrak{U}_i \underset{\pi}{<<} \mathfrak{U}$, $|\mathfrak{V}| \underset{\pi}{\subset\subset} |\mathfrak{U}_6| \underset{\pi}{\subset\subset} X_{c_3} \underset{\pi}{\subset\subset} |\mathfrak{U}_5|$, and $|\mathfrak{U}_4| \underset{\pi}{\subset\subset} X_{c_2} \underset{\pi}{\subset\subset} |\mathfrak{U}_3| \underset{\pi}{\subset\subset} X_c \underset{\pi}{\subset\subset} |\mathfrak{U}_2| \underset{\pi}{\subset\subset} X_{c_1} \underset{\pi}{\subset\subset} |\mathfrak{U}_1|$.

By shrinking ϱ^1, we can assume that $\|\xi^{(i)}\|_{\mathfrak{U}_1, \varrho^1} < \infty$.

Consider the following statement.

$$(14.5)\quad \begin{cases} \text{There exists } \omega \in \Omega^{(n)} \text{ such that, if } \varrho < \omega \text{ and } \xi \in Z^l(\mathfrak{U}(\varrho), \mathcal{F}) \text{ with} \\ \|\xi\|_{\mathfrak{U}, \varrho} < e, \text{ then for some } a_i \in \Gamma(K(\varrho), {}_n\mathcal{O}) \text{ and } \chi \in C^{l-1}(\mathfrak{U}_5(\varrho), \mathcal{F}), \\ |a_i|_\varrho < D_{\bar{\varrho}}\, e, \ \|\chi\|_{\mathfrak{U}_5, \varrho} < D_{\bar{\varrho}}\, e, \text{ and } \|\xi - \Sigma_i a_i \xi^{(i)} - \delta\chi\|_{\mathfrak{U}_5, \varrho} < \varrho_n D_{\bar{\varrho}}\, e, \\ \text{where } D_{\bar{\varrho}} \text{ is a constant} \geqq 1 \text{ depending only on } \bar{\varrho}. \end{cases}$$

We are going to assume (14.5) and finish the proof of Proposition $(14.1)_n$. After we finish the proof of Proposition $(14.1)_n$, we will prove (14.5).

Fix $\omega' \in \Omega^{(n)}$ such that ω' satisfies the requirement stated in (14.5).

Take $\omega \in \Omega^{(n)}$ with $\omega \leqq \omega'$ and $\omega < \varrho^1$ and we shall impose more conditions on ω later. Take $\varrho < \omega$ and $\xi \in Z^l(\mathfrak{U}(\varrho), \mathcal{F})$ with $\| \xi \|_{\mathfrak{U}, \varrho} < e$.

We are going to construct by induction on ν,

$$\zeta_\nu \in Z^l(\mathfrak{U}(\varrho), \mathcal{F}),\ b^{(i)}_{\nu+1} \in \Gamma(K(\varrho), {}_n\mathcal{O}), \text{ and } \chi_{\nu+1} \in C^{l-1}(\mathfrak{U}_6)(\varrho), \mathcal{F}),$$

$\nu \in \mathrm{N}_*$, such that

(i) $\zeta_0 = \xi$,

(ii) $\zeta_\nu - \Sigma_i b^{(i)}_{\nu+1} \xi^{(i)} - \delta\chi_{\nu+1} = \zeta_{\nu+1}$ on $\mathfrak{U}_6(\varrho)$,

(iii) $\| \zeta_\nu \|_{\mathfrak{U}, \varrho} < (\varrho_n C'_{\bar\varrho} D_{\bar\varrho})^\nu e$,

(iv) $| b^{(i)}_{\nu+1} |_\varrho < D_{\bar\varrho} (\varrho_n C'_{\bar\varrho} D_{\bar\varrho})^\nu e$ and

(v) $\| \chi_{\nu+1} \|_{\mathfrak{U}_6, \varrho} < D_{\bar\varrho} (1 + \varrho_n C'_n)(\varrho_n C'_{\bar\varrho} D_{\bar\varrho})^\nu e$,

where $D_{\bar\varrho}$ is the constant in (14.5) and $C'_{\bar\varrho}$ is a constant depending only on $\bar\varrho$.

Set $\zeta_0 = \xi$. Suppose for some $\nu_0 \in \mathrm{N}_*$ we have constructed ζ_ν, $b^{(i)}_{\nu+1}$, and $\chi_{\nu+1}$ for $\nu \leqq \nu_0$.

Since $\varrho < \omega'$, we can find $b^{(i)}_{\nu_0+1} \in \Gamma(K(\varrho), {}_n\mathcal{O})$ and $\chi' \in C^{l-1}(\mathfrak{U}_5(\varrho), \mathcal{F})$ such that $| b^{(i)}_{\nu_0+1} |_\varrho < D_{\bar\varrho} (\varrho_n C'_{\bar\varrho} D_{\bar\varrho})^{\nu_0} e$, $\| \chi' \|_{\mathfrak{U}_5, \varrho} < D_{\bar\varrho} (\varrho_n C'_{\bar\varrho} D_{\bar\varrho})^{\nu_0} e$, and $\| \zeta_{\nu_0} - \Sigma_i b^{(i)}_{\nu_0+1} \xi^{(i)} - \delta\chi' \|_{\mathfrak{U}_5, \varrho} < \varrho_n D_{\bar\varrho} (\varrho_n C'_{\bar\varrho} D_{\bar\varrho})^{\nu_0} e$.

By Proposition 11.11, if $\omega \leqq \omega''$ for a suitable $\omega'' \in \Omega^{(n)}$ (and we assume this to be the case), then for some $\zeta_{\nu_0+1} \in Z^l(\mathfrak{U}(\varrho), \mathcal{F})$ and $\chi'' \in C^{l-1}(\mathfrak{U}_6(\varrho), \mathcal{F})$ we have $(\zeta_{\nu_0} - \Sigma_i b^{(i)}_{\nu_0+1} \xi^{(i)} - \delta\chi') - \delta\chi'' = \zeta_{\nu_0+1}$ on $\mathfrak{U}_6(\varrho)$, $\| \zeta_{\nu_0+1} \| < C''_{\bar\varrho}\, \varrho_n D_{\bar\varrho} (\varrho_n C'_{\bar\varrho} D_{\bar\varrho})^{\nu_0} e$, and $\| \chi'' \|_{\mathfrak{U}_6, \varrho} < C''_{\bar\varrho} \varrho_n D_{\bar\varrho} (\varrho_n C'_{\bar\varrho} D_{\bar\varrho})^{\nu_0} e$, where $C'''_{\bar\varrho}$ is a constant depending only on $\bar\varrho$.

Let $\chi_{\nu_0+1} = \chi' + \chi'' \in C^{l-1}(\mathfrak{U}_6(\varrho), \mathcal{F})$. Then $\| \chi_{\nu_0+1} \|_{\mathfrak{U}_6, \varrho} < (1 + C''_{\bar\varrho}\, \varrho_n) \cdot D_{\bar\varrho} (\varrho_n C'_{\bar\varrho} D_{\bar\varrho})^{\nu_0} e$ and $\zeta_{\nu_0} - \Sigma_i b^{(i)}_{\nu_0+1} \xi^{(i)} - \delta\chi_{\nu_0+1} = \zeta_{\nu_0+1}$ on $\mathfrak{U}_6(\varrho)$. The construction by induction is complete when we set $C'_{\bar\varrho} = C''_{\bar\varrho}$.

Impose the following condition on ω: $\omega_n(\bar\varrho) \leqq (2C'_{\bar\varrho} D_{\bar\varrho})^{-1}$.

Since $\varrho_n < (2C'_{\bar\varrho} D_{\bar\varrho})^{-1}$, $| b^{(i)}_{\nu+1} |_\varrho < D_{\bar\varrho}\, 2^{-\nu} e$ and $\| \chi_{\nu+1} \|_{\mathfrak{U}_6, \varrho} < D_{\bar\varrho}\, 2^{-\nu+1} e$. Hence $\Sigma^\infty_{\nu=0} b^{(i)}_{\nu+1}$ converges on $K(\varrho)$ and $\Sigma^\infty_{\nu=0} \chi_{\nu+1}$ converges on $\mathfrak{U}_6(\varrho)$. Let $a_i = \Sigma^\infty_{\nu=0} b^{(i)}_{\nu+1}$ and $\eta' = \Sigma^\infty_{\nu=0} \chi_{\nu+1}$. Then $| a_i |_\varrho < 2D_{\bar\varrho} e$ and $\| \eta' \|_{\mathfrak{U}_6, \varrho} < 4D_{\bar\varrho} e$.

Since $\| \xi - \Sigma_i (\Sigma_{\nu=0}^{m} b_{m+1}^{(i)} \xi^{(i)}) - \delta (\Sigma_{\nu=0}^{m} \chi_{\nu+1}) \|_{\mathfrak{U}_\delta, \varrho} = \| \zeta_{m+1} \|_{\mathfrak{U}_\delta, \varrho} < 2^{-m-1} e$ for $m \in \mathrm{N}_*$, $\xi = \Sigma_i a_i \xi^{(i)} + \delta\eta'$ on $\mathfrak{U}_\delta(\varrho)$.

$\| \xi - \Sigma_i a_i \xi^{(i)} \|_{\mathfrak{U}_\delta, \varrho} < (1 + C''' 2D_{\bar{\varrho}}) e$, where C''' is a constant $\geqq 1$. By Proposition 10.5, if $\omega \leqq \omega'''$ for a suitable $\omega''' \in \Omega^{(n)}$ (and we assume this to be the case), then there exists $\eta \in C^{l-1}(\mathfrak{V}(\varrho), \mathcal{F})$ such that $\xi - \Sigma_i a_i \xi^{(i)} = \delta\eta$ on $\mathfrak{V}(\varrho)$ and $\| \eta \|_{\mathfrak{V}, \varrho} < C''''_{\bar{\varrho}} (1 + C''' 2D_{\bar{\varrho}}) e$, where $C''''_{\bar{\varrho}}$ is a constant depending only on $\bar{\varrho}$.

Hence $\mathcal{F}$ has property pre $(B)_l^n$ at 0 with respect to (π, φ). Proposition $(14.1)_n$ is proved under the assumption of the validity of (14.5).

(*e*) We are going to reduce the proof of (14.5) to the proof of a simpler statement.

Since $l + 1 < r - q - 2n + 1$, by (14.4) $\mathcal{F}/t_n \mathcal{F}$ has property $(B)_{l+1}^n$ at 0 with respect to (π, φ). By Proposition 12.1, $\mathcal{F} | X_c$ is H^{l+1}-finite at 0 with respect to π. In particular, $\pi_{l+1}^c (\mathcal{F})_0$ is finitely generated over ${}_n\mathcal{O}_0$. Since ${}_n\mathcal{O}_0$ is Noetherian, by Lemma 3.3 (*a*) there exists $d \in \mathrm{N}_*$ such that t_n is not a zero-divisor for $t_n^d \pi_{l+1}^c (\mathcal{F})_0$.

Let $m = 2d + 1$. By Proposition 13.1, $\mathrm{Im}(\pi_l^c (\mathcal{F}/t_n^m \mathcal{F})_0 \rightarrow \pi_l^c (\mathcal{F}/t_n \mathcal{F})_0) = \mathrm{Im}(\pi_l^c (\mathcal{F})_0 \rightarrow \pi_l^c (\mathcal{F}/t_n \mathcal{F})_0)$. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\Phi'} & \mathcal{F}/t_n^m \mathcal{F} \\ \| & & \downarrow{\scriptstyle \Phi''} \\ \mathcal{F} & \xrightarrow{\Phi} & \mathcal{F}/t_n \mathcal{F}, \end{array}$$

where Φ, Φ', and Φ'' are quotient maps.

Define ϱ^* as follows: $\varrho_i^* = \varrho_i$ for $1 \leqq i \leqq n - 1$ and $\varrho_n^* = \dfrac{1}{2} \omega_n (\bar{\varrho})$. Then ϱ^* depends only on $\bar{\varrho}$ and $\varrho^* < \omega$. We assume that $\varrho_n < \varrho_n^*$.

By the definition of $\| \xi \|_{\mathfrak{U}, \varrho}$ there exists $\xi'_{\mu_1 \cdots \mu_n} \in C^l(\mathfrak{U}, \mathcal{F})$ such that $\| \xi'_{\mu_1 \cdots \mu_n} \|_{\mathfrak{U}} < e$ and $\Sigma \xi'_{\mu_1 \cdots \mu_n} \left(\dfrac{t_1}{\varrho_1}\right)^{\mu_1} \cdots \left(\dfrac{t_n}{\varrho_n}\right)^{\mu_n} = \xi$ on $\mathfrak{U}$.

For $\nu \in \mathrm{N}_*$ define

$$\xi_\nu = \Sigma_{\mu_1, \ldots, \mu_{n-1}} \xi'_{\mu_1 \cdots \mu_{n-1} \nu} \left(\frac{t_1}{\varrho_1}\right)^{\mu_1} \cdots \left(\frac{t_{n-1}}{\varrho_{n-1}}\right)^{\mu_{n-1}} \in C^l(\mathfrak{U}(\varrho^*), \mathcal{F}),$$

Then $\| \xi_\nu \|_{\mathfrak{U}, \varrho^*} < e$ and $\Sigma_{\nu \in \mathrm{N}_*} \xi_\nu \left(\dfrac{t_n}{\varrho_n}\right)^\nu = \xi$ on $\mathfrak{U}(\varrho)$.

Consider the following statement.

$$(14.6)\quad \begin{cases} \text{If } \omega \leqq \omega^1 \text{ for a suitable } \omega^1 \in \Omega^{(n)}, \text{ then there exists } \sigma_\nu \in C^l(\mathfrak{U}_2(\varrho^*), \mathcal{F}) \\ \text{such that} \\ \quad \text{(i) } \|\sigma_\nu\|_{\mathfrak{U}_2, \varrho^*} < C_{\varrho^*} e, \text{ where } C_{\varrho^*} \text{ is a constant depending only} \\ \text{on } \varrho^*, \\ \quad \text{(ii) } \delta\Phi\left(\xi_\nu - \frac{\varrho_n}{\varrho_n^*}\sigma_\nu\right) = 0 \text{ on } \mathfrak{U}_2(\varrho^*), \text{ and} \\ \quad \text{(iii) the image of } \Phi\left(\xi_\nu - \frac{\varrho_n}{\varrho_n^*}\sigma_\nu\right) \text{ in } \pi_l^c(\mathcal{F}/t_n\mathcal{F})_0 \text{ belongs to} \\ \operatorname{Im}(\pi_l^c(\mathcal{F})_0 \to \pi_l^c(\mathcal{F}/t_n\mathcal{F})_0). \end{cases}$$

We are going to prove (14.5) by assuming (14.6). (14.6) will be proved later.

Assume $\omega \leqq \omega^1$ so that the requirements in (14.6) are satisfied.

By (14.4), $\mathcal{F}/t_n\mathcal{F}$ has property $(B)_l^n$ at 0 with respect to (π, φ). $\left\|\Phi\left(\xi_\nu - \frac{\varrho_n}{\varrho_n^*}\sigma_\nu\right)\right\|_{\mathfrak{U}_2, \varrho^*} < \left(1 + \frac{\varrho_n}{\varrho_n^*} C_{\varrho^*}\right) e$. Since the images of $\Phi(\xi^{(1)}), \ldots, \Phi(\xi^{(k)})$ in $\pi_l^c(\mathcal{F}/t_n\mathcal{F})_0$ generate $\operatorname{Im}(\pi_l^c(\mathcal{F})_0 \to \pi_l^c(\mathcal{F}/t_n\mathcal{F})_0)$ over ${}_n\mathcal{O}_0$, if $\omega \leqq \omega^2$ for a suitable $\omega^2 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $a_\nu^{(1)}, \ldots, a_\nu^{(k)} \in \Gamma(K(\varrho^*), {}_n\mathcal{O})$ and $\eta'_\nu \in C^{l-1}(\mathfrak{U}_3(\varrho^*), \mathcal{F}/t_n\mathcal{F})$, we have

$$\Phi\left(\xi_\nu - \frac{\varrho_n}{\varrho_n^*}\sigma_\nu\right) = \Sigma_i a_\nu^{(i)} \Phi(\xi^{(i)}) + \delta\eta'_\nu \text{ on } \mathfrak{U}_3(\varrho^*),$$

$$|a_\nu^{(i)}|_{\varrho^*} < C_{\varrho^*}^{(1)}\left(1 + \frac{\varrho_n}{\varrho_n^*} C_{\varrho^*}\right) e,$$

and $\|\eta'_\nu\|_{\mathfrak{U}_3, \varrho^*} < C_{\varrho^*}^{(1)}\left(1 + \frac{\varrho_n}{\varrho_n^*} C_{\varrho^*}\right) e$, where $C_{\varrho^*}^{(1)}$ is a constant depending only on ϱ^*.

By Lemma 8.3 there exists $\eta_\nu \in C^{l-1}(\mathfrak{U}_3(\varrho^*), \mathcal{F})$ such that $\|\eta_\nu\|_{\mathfrak{U}_3, \varrho^*} < C_{\varrho^*}^{(1)}\left(1 + \frac{\varrho_n}{\varrho_n^*} C_{\varrho^*}\right) e$ and $\Phi(\eta_\nu) = \eta'_\nu$. Since t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$, there exists a unique $\theta_\nu \in C^l(\mathfrak{U}_3(\varrho^*), \mathcal{F})$ such that $\xi_\nu - \frac{\varrho_n}{\varrho_n^*}\sigma_\nu = \Sigma_i a_\nu^{(i)} \xi^{(i)} + \delta\eta_\nu + \left(\frac{t_n}{\varrho_n}\right)\theta_\nu$ on $\mathfrak{U}_3(\varrho^*)$. $\left\|\frac{t_n}{\varrho_n^*}\theta_\nu\right\|_{\mathfrak{U}_3, \varrho^*} < (1 + C^{(2)} C_{\varrho^*}^{(1)})\left(1 + \frac{\varrho_n}{\varrho_n^*} C_{\varrho^*}\right) e$, where $C^{(2)}$ is a constant. By Proposition 9.5 (applied to the sheaf-homomorphism $\mathcal{F} \to \mathcal{F}$ defined by multiplication by t_n), if $\omega \leqq \omega^3$ for a suitable

$\omega^3 \in \Omega^{(n)}$ (and we assume this to be the case), then

$$\| \theta_\nu \|_{\mathfrak{U}_4, \varrho^*} < C^{(3)}_{\varrho^*}(1 + C^{(2)} C^{(1)}_{\varrho^*}) \left(1 + \frac{\varrho_n}{\varrho_n^*} C_{\varrho^*}\right) e, \text{ where } C^{(3)}_{\varrho^*}$$

is a constant depending only on ϱ^*.

$$\text{Let } a_i = \Sigma_{\nu=0}^{\infty} a_\nu^{(i)} \left(\frac{t_n}{\varrho_n}\right)^\nu \in \Gamma(K(\varrho), {}_n\mathcal{O}),$$

$$\text{let } \chi = \Sigma_{\nu=0}^{\infty} \eta_\nu \left(\frac{t_n}{\varrho_n}\right)^\nu \in C^{l-1}(\mathfrak{U}_3(\varrho), \mathcal{F}),$$

$$\text{and let } \sigma = \Sigma_{\nu=0}^{\infty} \sigma_\nu \left(\frac{t_n}{\varrho_n}\right)^\nu + \Sigma_{\nu=0}^{\infty} \theta_\nu \left(\frac{t_n}{\varrho_n}\right)^{\nu+1} \in C^l(\mathfrak{U}_4(\varrho), \mathcal{F}).$$

By Lemma 8.4, we have

$$| a_i |_\varrho < \left(1 - \frac{\varrho_n}{\varrho_n^*}\right)^{-1} C^{(1)}_{\varrho^*} \left(1 + \frac{\varrho_n}{\varrho_n^*} C_{\varrho^*}\right) e,$$

$$\| \chi \|_{\mathfrak{U}_3, \varrho} < \left(1 - \frac{\varrho_n}{\varrho_n^*}\right)^{-1} C^{(1)}_{\varrho^*} \left(1 + \frac{\varrho_n}{\varrho_n^*} C_{\varrho^*}\right) e,$$

and

$$\| \sigma \|_{\mathfrak{U}_4, \varrho} < \left(1 - \frac{\varrho_n}{\varrho_n^*}\right)^{-1} \left(C_{\varrho^*} + C^{(3)}_{\varrho^*}(1 + C^{(2)} C^{(1)}_{\varrho^*}) \left(1 + \frac{\varrho_n}{\varrho_n^*} C_{\varrho^*}\right)\right) e.$$

It is easily checked that $\xi = \Sigma_i a_i \xi^{(i)} + \delta\chi + \frac{\varrho_n}{\varrho_n^*} \sigma$. Hence

$$\| \xi - \Sigma_i a_i \xi^{(i)} - \delta\chi \|_{\mathfrak{U}_4, \varrho} = \frac{\varrho_n}{\varrho_n^*} \| \sigma \|_{\mathfrak{U}_4, \varrho} <$$

$$< \frac{\varrho_n}{\varrho_n^*} \left(1 - \frac{\varrho_n}{\varrho_n^*}\right)^{-1} \left(C_{\varrho^*} + C^{(3)}_{\varrho^*}(1 + C^{(2)} C^{(1)}_{\varrho^*}) \left(1 + \frac{\varrho_n}{\varrho_\varrho^*} C_{\varrho^*}\right)\right) e.$$

(14.5) is proved if we require $\varrho_n \leqq \frac{\varrho_n^*}{2}$ and choose $D_{\bar{\varrho}} \geqq 1$ so that

$$D_{\bar{\varrho}} > 2 C^{(1)}_{\varrho^*} \left(1 + \frac{1}{2} C_{\varrho^*}\right) \text{ and } D_{\bar{\varrho}} > \frac{2}{\varrho_n^*} \left(C_{\varrho^*} + C^{(3)}_{\varrho^*}(1 + C^{(2)} C^{(1)}_{\varrho^*}) \left(1 + \frac{1}{2} C_{\varrho^*}\right)\right)$$

(*f*) To prove (14.6), let $\gamma_\nu = \left(\frac{\varrho_n}{\varrho_n^*}\right)^{\nu-1} \delta \Sigma_{i<\nu}\, \xi_i \left(\frac{t_n}{\varrho_n}\right)^i$ for $\nu \in \mathbf{N}$. Since

$$\gamma_\nu = \delta \Sigma^{\infty}_{\mu_1,\ldots\mu_{n-1}=0}\, \Sigma^{\nu-1}_{\mu_n=0}\, \xi'_{\mu_1\ldots\mu_n} \left(\frac{t_1}{\varrho_1^*}\right)^{\mu_1} \cdots \left(\frac{t_n}{\varrho_n^*}\right)^{\mu_n} \left(\frac{\varrho_n}{\varrho_n^*}\right)^{\nu-\mu_n-1},$$

$\| \gamma_\nu \|_{\mathfrak{U},\varrho^*} < C^{(4)} e$, where $C^{(4)}$ is a constant.

We claim that $\gamma_\nu \in Z^{l+1}(\mathfrak{U}(\varrho^*), t_n^\nu \mathcal{F})$. To prove the claim, we need only show that $\gamma_\nu \mid \mathfrak{U}(\varrho) \in C^{l+1}(\mathfrak{U}(\varrho), t_n^\nu \mathcal{F})$. Since $\delta\xi = 0$ on $\mathfrak{U}(\varrho)$, $\delta \Sigma_{i<\nu}\, \xi_i \left(\frac{t_n}{\varrho_n}\right)^i = - \delta \Sigma_{i \geqq \nu} \xi_i \left(\frac{t_n}{\varrho_n}\right)^i = \left(\frac{t_n}{\varrho_n}\right)^\nu \left[- \delta \Sigma_{i \geqq \nu} \xi_i \left(\frac{t_n}{\varrho_n}\right)^{i-\nu}\right]$. The claim is proved.

Since t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$, there exists a unique $\tilde{\gamma}_\nu \in Z^{l+1}(\mathfrak{U}(\varrho^*), \mathcal{F})$ such that $\gamma_\nu = \left(\frac{t^n}{\varrho_n^*}\right)^\nu \tilde{\gamma}_\nu$ on $\mathfrak{U}(\varrho^*)$. By Proposition 9.8, if $\omega \leqq \omega^4$ for a suitable $\omega^4 \in \Omega^{(n)}$ (and we assume this to be the case), then $\| \tilde{\gamma}_\nu \|_{\mathfrak{W}_1,\varrho^*} < C^{(5)}_{\varrho^*}\, C^{(4)'} e$, where $C^{(5)}_{\varrho^*}$ is a constant depending only on ϱ^*.

Since $\tilde{\gamma}_\nu = \frac{\varrho_n^*}{\varrho_n} \left[- \delta\, \Sigma_{i \geqq \nu} \xi_i \left(\frac{t_n}{\varrho_n}\right)^{i-\nu}\right]$ on $\mathfrak{U}(\varrho)$, the image of $\tilde{\gamma}_\nu$ in $\pi^{c_1}_{l+1}(\mathcal{F})_0$ is zero. The image of $\Phi'(\tilde{\gamma}_\nu)$ in $\pi^{c_1}_{l+1}(\mathcal{F}/t_n^m \mathcal{F})_0$ is zero.

$\| \Phi'(\tilde{\gamma}_\nu) \|_{\mathfrak{W}_1,\varrho^*} < C^{(5)}_{\varrho^*}\, C^{(4)} e$. By (14.4), $\mathcal{F}/t_n^m \mathcal{F}$ has property $(B)^n_{l+1}$ at 0 with respect to (π, φ) By considering the zero ${}_n\mathcal{O}_0$-submodule of $\pi^{c_1}_{l+1}(\mathcal{F}/t_n^m \mathcal{F})_0$ which is generated by the image of $0 \in Z^{l+1}(\mathfrak{U}_1(\varrho^*), \mathcal{F}/t_n^m \mathcal{F})$ we conclude that, if $\omega \leqq \omega^5$ for a suitable $\omega^5 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $\sigma'_\nu \in C^l(\mathfrak{U}_2(\varrho^*), \mathcal{F}/t_n^m \mathcal{F})$ we have $\Phi(\tilde{\gamma}_\nu) = \delta \sigma'_\nu$ on $\mathfrak{U}_2(\varrho^*)$ and $\| \sigma' \|_{\mathfrak{W}_2,\varrho^*} < C^{(6)}_{\varrho^*}\, C^{(5)}_{\varrho^*}\, C^{(4)} e$, where $C^{(6)}_{\varrho^*}$ is a constant depending only on ϱ^*.

For some $\sigma_\nu \in C^l(\mathfrak{U}_2(\varrho^*), \mathcal{F})$, $\Phi'(\sigma_\nu) = - \sigma'_\nu$ and $\| \sigma_\nu \|_{\mathfrak{W}_2,\varrho^*} < C^{(6)}_{\varrho^*}\, C^{(5)}_{\varrho^*}\, C^{(4} e$. Since t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$, there exists a unique $\tau_\nu \in Z^{l+1}(\mathfrak{U}_2(\varrho^*), \mathcal{F})$ such that $\tilde{\gamma}_\nu + \delta \sigma_\nu = \left(\frac{t_n}{\varrho_n^*}\right)^m \tau_\nu$ on $\mathfrak{U}_2(\varrho^*)$.

$$\delta\, \Sigma^{\nu+m-1}_{i=\nu}\, \xi_i \left(\frac{t_n}{\varrho_n}\right)^i = \delta\, \Sigma_{i<\nu+m} \xi_i \left(\frac{t_n}{\varrho_n}\right)^i - \delta\, \Sigma_{i<\nu} \xi_i \left(\frac{t_n}{\varrho_n}\right)^i =$$

$$= \left(\frac{\varrho_n^*}{\varrho_n}\right)^{\nu+m-1} \gamma_{\nu+m} - \left(\frac{\varrho_n^*}{\varrho_n}\right)^{\nu-1} \gamma_\nu = \left(\frac{\varrho_n^*}{\varrho_n}\right)^{\nu+m-1} \left(\frac{t_n}{\varrho_n^*}\right)^{\nu+m} \tilde{\gamma}_{\nu+m} -$$

$$- \left(\frac{\varrho_n^*}{\varrho_n}\right)^{\nu-1} \left(\frac{t_n}{\varrho_n^*}\right)^{\nu} \tilde{\gamma}_\nu .$$

Since t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in X$, we can take out the factor $\left(\frac{t_n}{\varrho_n}\right)^{\nu}$ from both sides.

$$\delta\, \Sigma_{i=\nu}^{\nu+m-1}\, \xi_i \left(\frac{t^n}{\varrho_n}\right)^{i-\nu} = \left(\frac{\varrho_n^*}{\varrho_n}\right)^{m-1} \left(\frac{t_n}{\varrho_n^*}\right)^{m} \tilde{\gamma}_{\nu+m} - \left(\frac{\varrho_n}{\varrho_n^*}\right) \tilde{\gamma}_{\nu} =$$

$$= \left(\frac{\varrho_n^*}{\varrho_n}\right)^{m-1} \left(\frac{t_n}{\varrho_n^*}\right)^{m} \tilde{\gamma}_{\nu+m} + \frac{\varrho_n}{\varrho_n^*} \left(\delta \sigma_\nu - \left(\frac{t_n}{\varrho_n}\right)^{m} \tau_\nu\right).$$

Hence $\delta\, \Phi' \left(\Sigma_{i=\nu}^{\nu+m-1}\, \xi_i \left(\frac{t_n}{\varrho_n}\right)^{i-\nu} - \frac{\varrho_n}{\varrho_n^*}\, \sigma_\nu\right) = 0$ on $\mathfrak{U}_2(\varrho)$.

$$\Phi' \left(\Sigma_{i=\nu}^{\nu+m-1}\, \xi_i \left(\frac{t_n}{\varrho_n}\right)^{i-\nu} - \frac{\varrho_n}{\varrho_n^*}\, \sigma_\nu\right) \in Z^l(\mathfrak{U}_2(\varrho), \mathcal{F}/t_n^m \mathcal{F}).$$

Let $\alpha : \pi_l^c(\mathcal{F})_0 \longrightarrow \pi_l^c(\mathcal{F}/t_n \mathcal{F})_0$ and $\alpha'' : \pi_l^c(\mathcal{F}/t_n^m \mathcal{F})_0 \longrightarrow \pi_l^c(\mathcal{F}/t_n \mathcal{F})_0$ be induced respectively by Φ and Φ''. Let f_ν be the image of $\Phi' \Big(\Sigma_{i=\nu}^{\nu+m-1} \xi_i \cdot \cdot \left(\frac{t_n}{\varrho_n}\right)^{i-\nu} - \frac{\varrho_n}{\varrho_n^*}\, \sigma_\nu\Big)$ in $\pi_l^c(\mathcal{F}/t_n^m \mathcal{F})_0$ and let g_ν be the image of $\Phi\left(\xi_\nu - \frac{\varrho_n}{\varrho_n^*}\, \sigma_\nu\right)$ in $\pi_l^c(\mathcal{F}/t_n \mathcal{F})_0$. Then $g_\nu = \alpha''(f_\nu)$. Since $\operatorname{Im} \alpha = \operatorname{Im} \alpha''$, $g_\nu \in \operatorname{Im} \alpha$. (14.6) is proved. q. e. d.

COROLLARY. Suppose $n \geqq 1$. Under the assumptions of Proposition $(14.1)_n$, $\mathcal{F}/t_n^d \mathcal{F}$ has property $(B)_l^n$ at 0 with respect to (π, φ) for $d \in \mathbf{N}$ and $0 \leqq l < r - q - 2n + 1$.

§ 15. Proof of Main Theorem.

A. PROPOSITION 15.1. Suppose X is a complex space and $\mathcal{F}$ is a coherent analytic sheaf on X. Suppose $\varrho^0 \in \mathbf{R}_+^n$ and $\pi : X \longrightarrow K(\varrho^0)$ is a proper holomorphic map. Then for $l \in \mathbf{N}_*$ and $t^0 \in K(\varrho^0)$, $\mathcal{F}$ is H^l-finite at t^0 with respect to π.

PROOF. Fix $l \in \mathbf{N}_*$ and $t^0 \in K(\varrho^0)$. Without loss of generality we can assume that $t^0 = 0$.

Take $\varrho^1 < \varrho^0$ in $\mathbf{R}_+^n$. Since $X(\varrho^1) \subset\subset X$, by Proposition 6.6 we can assume (after replacing ϱ^0 by ϱ^1) that X has reduction order $\leqq p$ for some $p \in N_*$. By choosing a bounded real-valued C^∞ function φ on X and $c_* < c_\#$ in $\mathbf{R}$ satisfying $X \subset \{\varphi \geqq c_\#\}$, we can make $\pi : X \longrightarrow K(\varrho^0)$ into 1-concave map.

By proposition $(14.1)_n$, $\mathcal{F}$ has property $(B)_l^n$ at 0 with respect to (π, φ). In particular, $\pi_l(\mathcal{F})_0$ is finitely generated over ${}_nO_0$. For some $\varrho^2 \leqq \varrho^0$ we can find $\mathfrak{V} \underset{\pi}{<<} \mathfrak{U} \underset{\pi}{<<} \mathfrak{W}$ in $\mathbf{S}(X)$ and $\xi_1, \dots, \xi_k \in Z^l(\mathfrak{U}, \mathcal{F})$ such that $|\mathfrak{V}(\varrho^2)| = |\mathfrak{U}(\varrho^2)| = |\mathfrak{W}(\varrho^2)| = X(\varrho^2)$ and the images of $\xi_1, \dots, \xi_k$ in $\pi_l(\mathcal{F})_0$ generate $\pi_l(\mathcal{F})_0$ over ${}_nO_0$.

Since $\mathcal{F}$ has property $(B)_l^n$ at 0 with respect to (π, φ), there exists $\omega \in \Omega^{(n)}$ with $\omega < \varrho^2$ such that, if $\varrho < \omega$ and $\xi \in Z^l(\mathfrak{U}(\varrho), \mathcal{F})$ with $\|\xi\|_{\mathfrak{U}, \varrho} < \infty$, then for some $a_1, \dots, a_k \in \Gamma(K(\varrho), {}_nO)$ and $\eta \in C^l(\mathfrak{V}(\varrho), \mathcal{F})$, $\xi = \Sigma_i a_i \xi_i + \delta\eta$ on $\mathrm{N}(\varrho)$.

Take arbitrarily $\varrho' \leqq \varrho^0$. To finish the proof, we need only find $\varrho'' < \varrho'$ such that the $\Gamma(K(\varrho''), {}_nO)$-submodule $I(\varrho'')$ generated by Im $((H^l(\varrho'), \mathcal{F}) \longrightarrow H^l(X(\varrho''), \mathcal{F}))$ is finitely generated over $\Gamma(K(\varrho''), {}_nO)$.

Choose $\varrho'' < \varrho'$ such that $\varrho'' < \omega$. We claim that ϱ'' satisfies the requirement.

Take $\zeta \in H^l(X(\varrho'), \mathcal{F})$ Since $|\mathfrak{W}(\varrho')| = X(\varrho')$, ζ is induced by some $\zeta^* \in Z^l(\mathfrak{W}(\varrho'), \mathcal{F})$. Since $\varrho'' < \varrho'$, $\|\zeta^*\|_{\mathfrak{U}, \varrho''} < \infty$. Since $\varrho'' < \omega$, for some $a_1, \dots, a_k \in \Gamma(K(\varrho''), {}_nO)$ and $\eta \in C^{l-1}(\mathfrak{V}(\varrho''), \mathcal{F})$, $\zeta^* = \Sigma_i a_i \xi_i + \delta\eta$ on $\mathfrak{V}(\varrho'')$. Hence the images of $\xi_1, \dots, \xi_k$ in $H^l(X(\varrho''), \mathcal{F})$ generate $I(\varrho'')$ over $\Gamma(K(\varrho''), {}_nO)$. q. e. d.

B. Suppose X is a complex space, $\varrho^0 \in \mathbf{R}_+^n$, and $\pi: X \longrightarrow K(\varrho^0)$ is a q-concave holomorphic map with exhaustion function φ and concavity bounds $c_*, c_\#$. Suppose $\mathcal{F}$ is a coherent analytic sheaf on X such that (i) codh $\mathcal{F} \geqq r$ on $\{\varphi < c_\#\}$ and (ii) $t_1 - t_1(\pi(x)), \dots, t_n - t_n(\pi(x))$ form an $\mathcal{F}_x$-sequence for $x \in \{\varphi < c_\#\}$.

PROPOSITION $(15.2)_k$ If $d_{n-k+1}, \dots, d_n \in \mathbf{N}$, $t^0 \in K(\varrho^0)$, $0 \leqq l < r - q - n - \max(0, n-k-1)$, and $c \in (c_*, c_\#)$, then $(\mathcal{F}/\Sigma_{i=n-k+1}^n t_i^{d_i} \mathcal{F}) \mid X_c$ is H^l-finite at t^0 with respect to π^c.

PROOF. We are going to prove by induction on k for $0 \leqq k \leqq n$.

Fix $c \in (c_*, c_\#)$. By Proposition 6.6, after shrinking ϱ^0 and replacing X by $X_{c_*'}$ for some $c_*' \in (c_*, c)$, we can assume without loss of generality that X has reduction order $\leqq p$ for some $p \in \mathbf{N}_*$.

Fix $t^0 \in K(\varrho^0)$. We need only consider the case where $t_i^0 = 0$ for $n - k + 1 \leqq i \leqq n$, because, if $t_i^0 \neq 0$ for some $n - k + 1 \leqq i \leqq n$, then $\mathcal{F}/\Sigma_{i=n-k+1}^n t_i^{d_i} \mathcal{F} = 0$ on $\pi^{-1}(U)$ for some open neighborhood U of t^0 and $(\mathcal{F}/\Sigma_{i=n-k+1}^n t_i^{d_i} \mathcal{F}) \mid X_c$ is trivially H^l-finite at t^0 with respect to π^c. After applying a coordinates transformation of $\mathbf{C}^n$ affecting only $t_1, \dots, t_{n-k}$, we can assume without loss of generality that $t^0 = 0$.

(*a*) Assume $k = 0$. Then $\mathcal{F}/\Sigma_{i=n-k+1}^n t_i^{d_i} \mathcal{F} = \mathcal{F}$.

When $n=0$, $\dim_{\mathbf{C}} H^l(X_c, \mathcal{F}) < \infty$ for $0 \leqq l < r-q$. Hence $\mathcal{F} | X_c$ is H^l-finite at 0 with respect to π^c for $0 \leqq l < r-q-n-\max(0, n-k-1)$.

When $n \geqq 1$, by Corollary to Proposition $(14.1)_n$, $\mathcal{F}/t_n^{\nu}\mathcal{F}$ has property $(B)_l^n$ at 0 with respect to (π, φ) for $\nu \in \mathrm{N}$ and $0 \leqq l < r-q-2n+1$. By Lemma 3.3 (b), after shrinking ϱ^0 and by replacing X by $X_{c_*''}$ for some $c_*'' \in (c_*, c)$, we can assume that there exists $\nu \in \mathrm{N}_*$ such that t_n is not a zero-divisor for $t_n^{\nu}\mathcal{F}_x$ for $x \in X$. By Proposition 12.1, $\mathcal{F} | X_c$ is H^l-finite at 0 with respect to π^c for $0 \leqq l < r-q-2n+1 = r-q-n-\max(0, n-k-1)$.

(b) For the general case, assume $0 < k \leqq n$ and further assume that Proposition $(15.2)_{k-1}$ is true. We are going to prove Proposition $(15.2)_k$ by induction on d_n. Let $\mathcal{G} = \mathcal{F}/t_n\mathcal{F}$.

When $d_n = 1$, $\mathcal{F}/\Sigma_{i=n-k+1}^{n} t_i^{d_i}\mathcal{F} = \mathcal{G}/\Sigma_{i=n-k+1}^{n-1} t_i^{d_i}\mathcal{G}$. codh $\mathcal{G} \geqq r-1$ on $\{\varphi < c_{\#}\}$. By replacing X by $X \cap \{t_n = 0\}$, Preposition $(15.2)_{k-1}$ implies that $(\mathcal{G}/\Sigma_{i=n-k+1}^{n-1} t_i^{d_i}\mathcal{G}) | X_c$ is H^l-finite at 0 with respect to π^c. Proposition $(15.2)_k$ is therefore proved for $d_n = 1$.

Suppose $d_n > 1$. Let $\mathcal{R}^{(\nu)} = \Sigma_{i=n-k+1}^{n-1} t_i^{d_i}\mathcal{F} + t_n^{\nu}\mathcal{F}$. Consider the following exact sequence $0 \to \mathcal{R}^{(d_n-1)}/\mathcal{R}^{(d_n)} \to \mathcal{F}/\mathcal{R}^{(d_n)} \to \mathcal{F}/\mathcal{R}^{(d_n-1)} \to 0$. By induction hypothesis, $(\mathcal{F}/\mathcal{R}^{(d_n-1)}) | X_c$ is H^l-finite at 0 with respect to π^c for $0 \leqq l < r-q-n-\max(0, n-k-1)$. By Lemma 3.1, to complete the induction on d_n, it suffices to show that $(\mathcal{R}^{(d_n-1)}/\mathcal{R}^{(d_n)}) | X_c$ is H^l-finite at 0 with respect to π^c for $0 \leqq l < r-q-n-\max(0, n-k-1)$.

By Lemma 4.4 we have a natural sheaf homomorphism α: $\mathcal{G}/\Sigma_{i=n-k+1}^{n-1} t_i^{d_i}\mathcal{G} \to \mathcal{R}^{(d_n-1)}/\mathcal{R}^{(d_n)}$ and α is a sheaf-isomorphism on $\{\varphi < c_{\#}\}$. Sup Ker $\alpha \subset \{\varphi \geqq c_{\#}\}$ and Supp Coker $\alpha \subset \{\varphi \geqq c_{\#}\}$. The restriction of π to Supp Ker α and Supp Coker α are proper. By Proposition 15.1, $(\mathrm{Ker}\,\alpha) | X_c$ and $(\mathrm{Coker}\,\alpha) | X_c$ are H^l-finite at 0 with respect to π^c for $l \geqq 0$.

Since $(\mathcal{G}/\Sigma_{i=n-k+1}^{n-1} t_i^{d_i}\mathcal{G}) | X_c$ is H^l-finite at 0 with respect to π^c for $0 \leqq l < r-q-n-\max(0, n-k-1)$, from Lemma 3.1 and the following two exact sequences:

$$0 \to \mathrm{Ker}\,\alpha \to \mathcal{G}/\Sigma_{i=n-k+1}^{n-1} t_i^{d_i}\mathcal{G} \to \mathrm{Im}\,\alpha \to 0.$$

$$0 \to \mathrm{Im}\,\alpha \to \mathcal{R}^{(d_n-1)}/\mathcal{R}^{(d_n)} \to \mathrm{Coker}\,\alpha \to 0,$$

we conclude that $(\mathcal{R}^{(d_n-1)}/\mathcal{R}^{(d_n)}) | X_c$ is H^l-finite at 0 with respect to π_c for $0 \leqq l < r-q-n-\max(0, n-k-1)$. The induction on d_n is complete and Proposition $(15.2)_k$ is proved q. e. d.

C. Proof of Main Theorem.

Without loss of generality we can assume that $M = K(\varrho^0)$ for some $\varrho^0 \in \mathbf{R}_+^n$. Since $\mathcal{F} | \{\varphi < c_{\#}\}$ is $(\pi | \{\varphi < c_{\#}\})$ — flat, $t_1 - t_1(\pi(x)), \ldots, t_n - t_n(\pi(x))$ form an $\mathcal{F}_x$-sequence for $x \in \{\varphi < c_{\#}\}$.

First we are going to prove $(15.1)_k$ for $1 \leqq k \leqq n+1$ by induction on k.

$$(15.1)_k \qquad \begin{cases} \text{If } d_k, \dots, d_n \in \mathbf{N},\ 0 \leqq l < r - q - n - k + 1 \text{ and } c \in (c_*, c_\#), \\ \text{then } \pi_l^c(\mathcal{F}/\Sigma_{i=k}^n t_i^{d_i}\mathcal{F}) \text{ is coherent.} \end{cases}$$

When $k = 1$, by Lemma 4.3, $t_1^{d_1}, \dots, t_n^{d_n}$ form an $\mathcal{F}_x$-sequence for $x \in \{\varphi < c_\#\} \cap \pi^{-1}(0)$. On $\{\varphi < c_\#\}$ we have codh $(\mathcal{F}/\Sigma_{i=1}^n t_i^{d_i}) \geqq r - n$. Since Supp $(\mathcal{F}/\Sigma_{i=1}^n \cdot t_i^{d_i}\mathcal{F}) \subset \pi^{-1}(0)$ and $\pi^{-1}(0)$ is a q-concave space, $\dim_{\mathbf{C}} H^l(X_c, \mathcal{F}/\Sigma_{i=1}^n t_i^{d_i}\mathcal{F}) < \infty$ for $0 \leqq l < r - q - n$. $(15.1)_1$ follows.

For the general case, assume $k > 1$. Fix $0 \leqq l < r - q - n - k + 1$. By Proposition 3.2, to prove the coherence of $\pi_l^c(\mathcal{F}/\Sigma_{i=k}^n t_i^{d_i}\mathcal{F})$, we need only verify the following three conditions

(i) $\pi_\mu^c(\mathcal{F}/(\Sigma_{i=k}^n t_i^{d_i}\mathcal{F} + (t_{k-1} - t_{k-1}^0)^\nu \mathcal{F}))$ is coherent for every $\nu \in \mathbf{N}$, $t^0 \in K(\varrho^0)$, and every $0 \leqq \mu \leqq l$.

(ii) $(\mathcal{F}/\Sigma_{i=k}^n t_i^{d_i}\mathcal{F}) \mid X_c$ is H^l-finite and H^{l+1}-finite with respect to π^c.

(iii) For every $t^0 \in K(\varrho^c)$ and every relatively compact open subset U of $K(\varrho^0)$ there exists $\nu \in \mathbf{N}_*$ such that $t_n - t_n^0$ is not a zero-divisor for $(t_n - t_n^0)^\nu \mathcal{F}_x$ for $x \in (\pi^c)^{-1}(U)$.

(i) follows from $(15.1)_{k-1}$ after a coordinates transformation in $\mathbf{C}^n$. (ii) follows from Proposition $(15.2)_{n-k+1}$. (iii) follows from $(\pi^c)^{-1}(U) \subset\subset X$ and Lemma 3.3 (*b*). Hence $(15.1)_k$ is proved.

$(15.1)_{n+1}$ implies that $\pi_l^c(\mathcal{F})$ is coherent for $c \in (c_*, c_\#)$ and $0 \leqq l < r - q - 2n$. The Main Theorem follows from Propositions 4.2 and 11.12.

q. e. d.

REFERENCES

[1] ANDREOTTI, A. and GRAUERT, H. *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962), 193-259.

[2] GRAUERT, H. *Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen*, I. H. E. S. No. 5 (1960). Berichtigung, I. H. E. S. No. 16 (1963), 35-36.

[3] GRAUERT, H. *The coherence of direct images*, L'Enseignement mathématique, **16** (1968), 99-119.

[4] GUNNING, R. C. and ROSSI, H. *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, N. J., 1965.

[5] KNORR, K. *Über den Grauertschen Kohärenzsatz bei eigentlichen holomorphen Abbildungen I, II.* Ann. Scuola Norm. Sup. Pisa **22** (1968), 729-761; **23** (1969), 1-74.

[6] RICHBERG, R. *Stetige streng pseudokonvexe Funktionen.* Math. Ann. **175** (1968), 257-286.

[7] SIU, Y.-T. *Extending coherent analytic sheaves*, Ann. Math. **90** (1969), 108-143.

[8] SIU, Y.-T. *O^N-approximable and holomorphic functions on complex spaces*, Duke Math. J. **36** (1969), 451-454.

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