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KILLING FORMS IN A RIEMANNIAN MANIFOLD WITH BOUNDARY(*)

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1. Introduction.

Let M be a compact Riemannian manifold which is the closure of an open submanifold of an n-dimensional orientable Riemannian manifold V. The manifold M has a boundary $\partial M = B$, which is an $(n-1)$-dimensional compact orientable submanifold ([1]). We denote by $K_T^2(M, \mathbb{R})$ and $K_N^2(M, \mathbb{R})$ the Killing 2-forms on the manifold, which are tangential and normal to the boundary, respectively. We assume that the manifold M is negatively k-pinched, then the groups $K_T^2(M, \mathbb{R})$, $K_N^2(M, \mathbb{R})$ have some properties.

The aim of the present paper is to prove that if the number k is greater than a number μ and the second fundamental form on the boundary B satisfies some relations, then the two groups are trivial. These results are an extension of those given in ([7]).

2. A p-form $\alpha = (\alpha_{i_1} \dots i_p)$ is called Killing if it satisfies the relation, ([8], p. 66)

$$\nabla_X \alpha (Y, X_2, \dots, X_p) + \nabla_Y \alpha (Y, X_2, \dots, X_p) = 0, X, Y, X_i \in T(M), i = 2, \dots, p,$$

which implies

$$\delta \alpha = 0. \tag{2.1}$$

For any p-form α, we have the formula, ([5], p. 4)

$$\frac{1}{2} \Delta (|\alpha|^2) = (\alpha, \Delta\alpha) - |\nabla\alpha|^2 + \frac{1}{2[(p-1)!]} Q_p(\alpha), \tag{2.2}$$

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where

$$| \nabla \alpha |^2 = \frac{1}{p!} \nabla^l \alpha^{i_1 \cdots i_p} \nabla_l \alpha_{i_1 \cdots i_p} , \tag{2.3}$$

$$Q_p(\alpha) = (p-1) R_{ijhl} \alpha^{iji_3 \cdots i_p} \alpha^{hl}_{i_3 \cdots i_p} - 2R_{hl} \alpha^{hi_2 \cdots i_p} \alpha^{l}_{i_2 \cdots i_p} . \tag{2.4}$$

If α is a Killing p-form, then (2.2) takes the form ([7])

$$\frac{1}{2} \Delta(|\alpha|^2) = - |\nabla \alpha|^2 - \frac{1}{2(p!)} Q_p(\alpha). \tag{2.5}$$

We consider a point P on the boundary B. Let $(u^1, \dots, u^{n-1})$ be a local coordinate system of a neighborhood of the point P as a point of B and $(v^1, \dots, v^n)$ another local coordinate systen of a neighborhood of the same point considered as a point of V. The local representation of B is given by

$$v^i = f^i(u^1, \dots, u^{n-1}), \; i = 1, \dots, n, \tag{2.6}$$

in $U(P) \cap M$, $U(P)$ being a coordinate neighborhood of V.

We denote by N the normal vector field to the boundary. We choose the local coordinate system $(u^1, \dots, u^{n-1})$ such that the vector fields N, $\partial/\partial u^1, \dots, \partial/\partial u^{n-1}$ form a positive sence of M with respect to vector fields $\partial/\partial v^1, \dots, \partial/\partial v^n$.

We assume that the mapping F of B into M defined by (2.6) is an isometric immersion, therefore the metric $h = (h_{\lambda\nu})$ on the manifold B is given by

$$h_{\lambda\nu} = \partial v^i/\partial u^\lambda \, \partial v^j/\partial u^\nu \, g_{ij} ,$$

where (g_{ij}) is the metric on the manifold M.

We denote by g and h the diterminants of the metrics (g_{ij}) and $(h_{\lambda\nu})$, respectively.

If ω is any $(n-1)$-form on the manifold M, then Stoke's theorem can be stated as follows

$$\int_M d\omega = \int_B \omega,$$

from which we obtain

$$\int_M \delta\gamma \, \eta = - \int_B (N, \gamma) \, \overline{\eta}, \tag{2.7}$$

for any vector field $\gamma = (\gamma_i)$ on M and $\eta, \bar{\eta}$ are the volume elements of M, B, respectively, defined by

$$\eta = \sqrt{g}\, dv^1 \wedge \dots \wedge dv^n, \quad \bar{\eta} = \sqrt{h}\, du^1 \wedge \dots \wedge du^{n-1}.$$

The relation (2.7) is valid, if we define the codifferentation of a p-form $\alpha = (\alpha_{i_1 \dots i_p})$ as follows

$$(2.8) \qquad (\delta\alpha)_{i_2 \dots i_p} = - \nabla_l \, \alpha^{l}_{i_2 \dots i_p} .$$

A p-form $\alpha = (\alpha_{i_1 \dots i_p})$ on the manifold M is tangential to B, if satisfies the relations ([10], p. 431)

$$\alpha^{i_1 \dots i_p} = \partial v^{i_1}/\partial u^{j_1} \dots \partial v^{i_p}/\partial u^{j_p} \, \bar{\alpha}_{j_1 \dots j_p} ,$$

or

$$\alpha^{h i_2 \dots i_p} N_h = 0,$$

where $\bar{\alpha} = (\bar{\alpha}_{j_1 \dots j_p})$ is a p-form defined over B, which imply, ([10], p. 434)

$$(2.9) \qquad (\nabla_h \, \alpha_{j i_2 \dots i_p}) \, (\alpha^{j i_2 \dots i_p}) \, N^h = - H_{ji} \, \bar{\alpha}^{j}_{i_2 \dots i_p} \, \bar{\alpha}^{i i_2 \dots i_p} +$$

$$(\nabla_j \, \alpha_{h i_2 \dots i_p} + \nabla_h \, \alpha_{j i_2 \dots i_p}) \, \alpha^{j i_2 \dots i_p} N^h .$$

We consider a p-form $\alpha = (\alpha_{i_1 \dots i_p})$ on the manifold M. This form is normal to the boundary B, if we have the relation, ([10], p. 432)

$$\alpha_{i_1 \dots i_p} \, \partial v^{i_1}/\partial u^{j_1} \dots \partial v^{i_p}/\partial u^{j_p} = 0,$$

from which, we obtain ([10], p. 435)

$$(2.10) \qquad (\nabla_h \, \alpha_{j i_2 \dots i_p}) \, \alpha^{j i_2 \dots i_p} N^h = p \, (\nabla_h \, \alpha^{h}_{i_2 \dots i_p}) \, \alpha^{j i_2 \dots i_p} N_j +$$

$$p H^{l}_{l} \, \bar{\alpha}_{i_2 \dots i_p} \, \bar{\alpha}^{i_2 \dots i_p} - p\,(p-1)\, H_{ji} \, \bar{\alpha}^{j}_{i_3 \dots i_p} \, \bar{\alpha}^{i i_3 \dots i_p} ,$$

where $\bar{\alpha} = (\bar{\alpha}_{i_2 \dots i_p})$ is a $(p-1)$-form on the manifold B defined by

$$\alpha_{h i_2 \dots i_p} N^h = \bar{\alpha}_{j_2 \dots j_p} \, \partial v^{j_2}/\partial u^{i_2} \dots \partial v^{j_p}/\partial u^{i_p} .$$

3. We assume that the dimension of the manifold M is odd $n = 2m + 1$ and admits a metric which is negatively k-pinched. We also assume that α

is a Killing 2-form and consider the $2m$-form β

$$\beta = \frac{1}{m!} \alpha \wedge \ldots \wedge \alpha, \ (m \text{ times}).$$

Let P be any point of the manifold M. There is a special base of the vector space M_P^*, such that the following inequalities hold at the point P, ([7]),

$$\frac{1}{2} Q_2(\alpha) \geq 2(2m-1) k |\alpha|^2 - \frac{8}{3}(1-k)\delta, \tag{3.1}$$

$$\frac{1}{2[(2m-1)!]} Q_{2m}(\beta) \geq 2mk |\beta|^2, \tag{3.2}$$

where

$$|\alpha|^2 = \alpha_{12}^2 + \alpha_{34}^2 + \ldots + \alpha_{2m-1,\,2m}^2, \ \beta = \alpha_{12}\alpha_{34} \ldots \alpha_{2m-1,\,2m},$$

$$\delta = \alpha_{12}\alpha_{34} + \ldots + \alpha_{12}\alpha_{2m-1,\,2m} + \ldots + \alpha_{2m-3,\,2m-2}\alpha_{2m-1,\,2m},$$

where $\alpha_{12}, \alpha_{34}, \ldots, \alpha_{2m-1,\,2m}$ the components of the Killing 2-form α with respect to the special base of the vector space M_P^*.

The formula (2.5) for $p = 2$ becomes

$$\frac{1}{2} \Delta(|\alpha|^2) = -|\nabla\alpha|^2 - \frac{1}{4} Q_2(\alpha),$$

or

$$\frac{1}{2}\int_M |\alpha|^{2m-2} \Delta(|\alpha|^2)\eta = \int_M [-|\alpha|^{2m-2}|\nabla\alpha|^2 - \frac{1}{4}|\alpha|^{2m-2} Q_2(\alpha)]\eta,$$

which by means of (3.1) becomes

$$\frac{1}{2}\int_M |\alpha|^{2m-2}\Delta(|\alpha|^2)\eta \leq \int_M -[|\alpha|^{2m-2}|\nabla\alpha|^2 - \frac{4}{3}(1-k)|\alpha|^{2m-2}\delta + (2m-1)k|\alpha|^{2m}]\eta. \tag{3.3}$$

It can be easily proved the formula

$$\Delta[(|\alpha|^2)^m] = m|\alpha|^{2m-2}\Delta(|\alpha|^2) - m(m-1)|\alpha|^{2m-4}(d(|\alpha|^2))^2,$$

from which we obtain

$$\int_M \Delta[(|\alpha|^2)^m]\eta \leq \int_M m|\alpha|^{2m-2}\Delta(|\alpha|^2)\eta. \tag{3.4}$$

The relation (3.4) by virtue of (2.7) becomes

(3.5) $$-\int_B (N, |\alpha|^{2m-2} d(|\alpha|^2)\bar{\eta} \leq \int_M |\alpha|^{2m-2} \Delta(|\alpha|^2)\,\eta.$$

From (3.3) and (3.5) we obtain

(3.6) $$\frac{1}{2}\int_B (N, |\alpha|^{2m-2} d(|\alpha|^2))\bar{\eta} \geq \int_M \Big[|\alpha|^{2m-2} |\nabla\alpha|^2 - \frac{4}{3}(1-k)|\alpha|^2\delta + (2m-1)k|\alpha|^{2m}\Big]\eta.$$

The formula (2.2) for the $2m$-form β takes the form

$$\frac{1}{2}\Delta(|\beta|^2) = (\beta, \Delta\beta) - |\nabla\beta|^2 + \frac{1}{2[(2m-1)!]} Q_{2m}(\beta),$$

or

$$\frac{1}{2}\int_M \Delta(|\beta|^2)\,\eta = \int_M \Big[(\beta, \Delta\beta) - |\nabla\beta|^2 + \frac{1}{2[(2m-1)!]} Q_{2m}(\beta)\Big]\eta,$$

which by means of (2.7), (3.2) and the relation, ([4], p. 187)

$$\langle \beta, \Delta\beta \rangle = \| d\beta \|^2 + \| \delta\beta \|^2,$$

becomes

(3.7) $$-\frac{1}{2}\int_B (N, d(|\beta|^2))\bar{\eta} \geq \int_M [\,|\delta\beta|^2 + |d\beta|^2 - |\nabla\beta|^2 + 2mk|\beta|^2]\,\eta.$$

It has been proved the inequality, ([7])

(3.8) $$|\nabla\beta|^2 \leq |\nabla\alpha|^2 |\alpha|^{2m-2}/m^{m-3}.$$

From (3.7) by means of (3.8) we obtain

(3.9) $$-\frac{1}{2}\int_B (N, m^{m-3} d(|\beta|^2))\bar{\eta} \geq \int_M [-|\nabla\alpha|^2 |\alpha|^{2m-2} + 2km^{m-2}|\beta|^2]\,\eta.$$

We add the (3.6), (3.9) and after some calculations we obtain

$$\frac{3}{2}\int_B (N, |\alpha|^{2m-2}\, d(|\alpha|^2) - m^{m-3}\, d(|\beta|^2))\,\overline{\eta} \geq$$

$$\int_M [3(2m-1)k\,|\alpha|^{2m} - 4(1-k)\,|\alpha|^{2m-2}\,\delta + 6km^{m-2}\,|\beta|^2]\,\eta,$$

which can be written

$$(3.10)\qquad 3\int_B [|\alpha|^{2m-2}(\nabla_h\, \alpha_{ji_2})\,\alpha^{ji_2} N^h - m^{m-3}(\nabla_h\, \beta_{ji_2\ldots i_{2m}})\,\beta^{ji_2\ldots i_{2m}} N^h]\,\overline{\eta} \geq$$

$$j < i_2 < \ldots < i_{2m}.$$

$$\int_M [3(2m-1)k\,|\alpha|^{2m} - 4(1-k)\,|\alpha|^{2m-2}\,\delta + 6km^{m-2}\,|\beta|^2]\,\eta.$$

The inequality (3.10), if the Killing 2-form α is tangential to B by means of (2.9) becomes

$$(3.11)\qquad \int_B -3\,[H_{ji}(|\alpha|^{2m-2}\,\overline{\alpha}^j_{i_2}\,\overline{\alpha}^{ii_2} - m^{m-3}\,\overline{\beta}^j_{i_2\ldots i_{2m}}\,\overline{\beta}^{ii_2\ldots i_{2m}}) - (\nabla_j\, \beta_{hi_2\ldots i_{2m}} + \nabla_h\, \beta_{ji_2\ldots i_{2m}})\,\overline{\beta}^{hi_2\ldots i_{2m}} N^j]\,\overline{\eta} \geq$$

$$i_2 < \ldots < i_{2m}.$$

$$\int_M [3(2m-1)k\,|\alpha|^{2m} - 4(1-k)\,|\alpha|^{2m-2}\,\delta + 6\,km^{m-2}\,|\beta|^2]\,\eta.$$

The second member of the (3.11) is positive, if k satisfies the inequality, ([7])

$$(3.12)\qquad k > \mu = 2m^2(m-1)/(8m-5)\,m^2 + 6.$$

We consider the following quadratic form

$$(3.13)\qquad G(\alpha, \alpha) = H_{ij}(|\alpha|^{2m-2}\,\overline{\alpha}^j_{i_2}\,\overline{\alpha}^{ii_2} - m^{m-3}\,\overline{\beta}^j_{i_2\ldots i_{2m}}\,\overline{\beta}^{ii_2\ldots i_{2m}}) -$$

$$(\nabla_j \beta_{hi_2 \dots i_{2m}} + \nabla_h \beta_{ji_2 \dots i_{2m}}) \beta^{hi_2 \dots i_{2m}} N^j .$$

$$i_2 < \dots < i_{2m} .$$

From (3.11), (3.12) and (3.13) we have the theorem

THEOREM (I). *Let M be a compact negatively k-pinched Riemannian manifold of dimension $n = 2m + 1$ with a boundary B. If the number $k > \mu$, given by* (3.12), *and the quadratic form $G(\alpha, \alpha)$ is semipositive, then the group $K_T^2(M, \mathbb{R}) = 0$.*

We assume tha the Killing 2-form α is normal to B, then (3.10) by means of (2,1), (2.10) and the relation, ([10], p. 436)

$$H_l^l \, \overline{\alpha}_{i_2} \, \overline{\alpha}^{i_2} - H_{ji} \, \overline{\alpha}^j \, \overline{\alpha}^i = 0,$$

takes the form

$$(3.14) \qquad \int_B 3m^{m-2} [(2m-1) H_{ji} \overline{\beta}^j_{i_3 \dots i_{2m}} \overline{\beta}^{ii_3 \dots i_{2m}} - H_l^l \, \overline{\beta}_{i_2 \dots i_{2m}} \overline{\beta}^{i_2 \dots i_{2m}} - (\nabla_h \, \beta^h_{i_2 \dots i_{2m}}) \beta^{li_2 \dots i_{2m}} N_l] \, \overline{\eta} \geq$$

$$i_2 < \dots < i_{2m} .$$

$$\int_M [3(2m-1) k |\alpha|^{2m} - 4(1-k) |\alpha|^{2m-2} \delta + 6 k m^{m-2} |\beta|^2] \, \eta .$$

Let $L(\alpha, \alpha)$ be a quadratic form defined by

$$(3.15) \qquad L(\alpha, \alpha) = (2m-1) H_{ji} \overline{\beta}^j_{i_3 \dots i_{2m}} \overline{\beta}^{ii_3 \dots i_{2m}} - H_l^l \, \overline{\beta}_{i_2 \dots i_{2m}} \overline{\beta}^{i_2 \dots i_{2m}} - (\nabla_h \, \beta^h_{i_2 \dots i_{2m}}) \beta^{li_2 \dots i_{2m}} N_l .$$

$$i_2 < \dots < i_{2m} .$$

From (3.12), (3.14) and (3.15) we obtain the theorem :

THEOREM (II). *We consider a compact negatively k-pinched Riemannian manifold M of dimension $n = 2m + 1$ with boundary B. If $k > \mu$, given by* (3.12), *and the quadratic form $L(\alpha, \alpha)$ is semi-negative, then $K_N^2(M, \mathbb{R}) = 0$.*

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