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ON THE RIESZ MEANS OF THE SOLUTIONS OF THE SCHRÖDINGER EQUATION

by SIGRID SJÖSTRAND

0. Introduction.

Consider the solution $u = G(t)f$ of the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = i\,\Delta u, & \text{if } t > 0, \\ u = f, & \text{if } t = 0. \end{cases} \qquad \left(\Delta = \sum_1^n \frac{\partial^2}{\partial x_1^2}\right),$$

At least formally we have

$$\mathcal{F} u(\xi) = \mathcal{F} G(t) f(\xi) = \exp(\mathrm{it}\,|\,\xi\,|^2)\,\mathcal{F} f(\xi),$$

where $\mathcal{F}$ denotes the Fourier transform. From this it is easily seen that $G(t)$ is a bounded, even unitary, operator in $L^2 = L^2(R^n)$. We also have the group property

$$G(t+s) = G(t)\,G(s),$$

$$G(0) = \text{ identity}.$$

Thus we have a unitary group of operators. In $L^p = L^p(R^n)$, $p \neq 2$, $G(t)$ is not bounded. See Hörmander [2] and Lanconelli [3]. See also Littman-McCarthy-Riviere [4]. A possible substitute for this, motivated by the theory of distribution (semi) groups, is that at least the Riesz means

$$I^k\,G(t)f = kt^{-k}\int_0^t (t-s)^{k-1}\,G(s)f\,ds, \tag{0.1}$$

of sufficiently large order k are bounded in L^p. See Peetre [7].

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More generally we consider the operators $G(t)$, $t \geq 0$, defined by

$$\mathcal{F} G(t) f(\xi) = \exp(\mathrm{it}\, H(\xi)) \mathcal{F} f(\xi), \tag{0.2}$$

where H is a positive homogeneous function of degree $m > 0$, and H is infinitely differentiable for $\xi \neq 0$. We will show that $I^k G(t)$ is bounded in L^p, if $k > n\,|1/p - 1/2|$.

In particular we consider the case $H(\xi) = |\xi|^m$. In this case we show that the bound $n\,|1/p - 1/2|$ is the best possible if $m \neq 1$, but can be improved to $(n-1)\,|1/p - 1/2|$ but not more if $m = 1$. (If $m = 2$, our result easily follows from Lanconelli [3], th. 1, with the aid of lemma 2.1 below.)

The plan of the paper is as follows. Section 1 contains some preliminary theorems, mostly on Fourier multipliers. In section 2 we show that our problem is equivalent to the following one: For which k is the function

$$\Phi(H(\xi))\, H(\xi)^{-k} \exp(iH(\xi))$$

a Fourier multiplier on L^p. Here Φ is infinitely differentiable and $\Phi(t) = 0$ for $t < 1/2$ and $\Phi(t) = 1$ for $t > 1$. In section 3 we prove that $k > n\,|1/p - 1/2|$ implies that $I^k G(t)$ is bounded in L^p. Section 4 and 5, finally, treat the special cases $H(\xi) = |\xi|^m$ for $m \neq 1$ and $m = 1$, respectively.

The problem treated in this paper was suggested to me by professor Jaak Peetre. I thank him for valuable advice and great interest in my work.

1. Preliminairies on Fourier multipliers and asymptotic expansion.

By $\hat{f}$ or $\mathcal{F}f$ we denote the Fourier transform of f and by $\check{f}$ or $\mathcal{F}^{-1}f$ the inverse Fourier transform of f. Thus formally

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{R^n} e^{-i<x,\xi>} f(x)\, dx$$

and

$$\check{g}(x) = \mathcal{F}^{-1} g(x) = (2\pi)^{-n} \int_{R^n} e^{i<x,\xi>} g(\xi)\, d\xi.$$

Let $M_p = M_p(R^n)$ denote the space of Fourier multipliers on $L^p = L^p(R^n)$, i.e. $\hat{a} \in M_p$ if and only if $\hat{a}$ is a tempered distribution and

$$\|\hat{a}\|_{M_p} = \sup_{\|f\|_{L_p}=1} \|(\hat{a}\hat{f})^{\vee}\|_{L_p} < \infty.$$

Then

$$M_p = M_{p'}, \; 1/p + 1/p' = 1.$$

In the sequel we may therefore consider only the case $1 \leq p \leq 2$. We also have

$$\mathcal{F} L^1 \subset M_1 \subset M_p \tag{1.1}$$

for every p, and

$$M_2 = L^\infty. \tag{1.2}$$

If $\hat{a}_1 \in M_p$ and $\hat{a}_2 \in M_p$, then $\hat{a}_1 \hat{a}_2 \in M_p$ and

$$\| \hat{a}_1 \hat{a}_2 \|_{M_p} \leq \| \hat{a}_1 \|_{M_p} \| \hat{a}_2 \|_{M_p} . \tag{1.3}$$

Further M_p is invariant for homotheties, i. e. for a constant t

$$\| \hat{a} (t \xi) \|_{M_p} = \| \hat{a} (\xi) \|_{M_p} . \tag{1.4}$$

For proofs and details see Hörmander [2].

THEOREM 1.1. If $1 < p < 2$ and $\theta = 2(1 - 1/p)$, then

$$\| \hat{a} \|_{M_p} \leq \| \hat{a} \|_{M_1}^{1-\theta} \| \hat{a} \|_{M_2}^{\theta} .$$

PROOF: Apply the Riesz-Thorin convexity theorem.

By $D^N \hat{a}$ we denote the set of all derivatives of $\hat{a}$ of order N and by $\| D^N \hat{a} \|_{L^2}$ the maximum of the L^2-norms of these derivatives.

THEOREM 1.2. (th. of Bernstein; cf. Peetre [6], chap. 1, th. 2.1). If $N > n/2$, then there is a constant C, so that

$$\| \hat{a} \|_{M_1} \leq C \| \hat{a} \|_{L^2}^{1-n/2N} \| D^N \hat{a} \|_{L^2}^{n/2N}.$$

PROOF: Using (1.1), Parseval's formula and the Cauchy-Schwarz inequality we get

$$\| \hat{a} \|_{M_1} \leq \int | a(x) | \, dx = \int_{|x| \leq r} | a(x) | \, dx + \int_{|x| \geq r} | x |^{-N} | x |^{N} | a(x) | \, dx \leq$$

$$\leq C \left(r^{n/2} \| \hat{a} \|_{L^2} + r^{-N+n/2} \| D^N \hat{a} \|_{L^2} \right).$$

If $\|\hat{a}\|_{L^2} \neq$ we choose

$$r = \|\hat{a}\|_{L^2}^{-1/N} \| D^N \hat{a} \|_{L^2}^{1/N}$$

to obtain the desired inequality. If $\|\hat{a}\|_{L^2} = 0$, the inequality is of course trivial.

THEOREM 1.3. Assume that F is infinitely differentiable on $0 < u < \infty$ and that

$$| F(u) - F(0) | \leq C_0 u^{\alpha}, \ 0 < u < 1,$$

$$| F(u) | \leq C_0 u^{-\beta}, \ 1 < u < \infty,$$

$$| D^J F(u) | \leq C_J \min (u^{\alpha - J}, u^{-\beta - J}), \ 0 < u < \infty,$$

where $J \geq 1$ and α, $\beta > 0$. Let H be a positive homogeneous function as in the introduction. Then

$$F(H(\xi)) \in \mathcal{F} L^1.$$

PROOF: Se Löfström [5], lemma 1.4.

Next we want to compute the Fourier transform of a function which has a similar behaviour near the point x_0 as the function f defined by

$$f : x \to (x - x_0)_+^{\alpha} = \begin{cases} (x - x_0)^{\alpha}, & \text{if } x > x_0, \\ 0, & \text{if } x \leq x_0. \end{cases}$$

For $\xi > 0$ and $\alpha \neq -1, -2, \ldots$.

$$\hat{f}(\xi) = K_{\alpha} \xi^{-\alpha - 1} e^{-i x_0 \xi}, \tag{1.5}$$

where

$$K_{\alpha} = - i \Gamma(\alpha + 1) e^{-i\alpha \frac{\pi}{2}}.$$

See Gelfand-Schilow [1], p. 169.

THEOREM 1.4. Suppose that

1^0 f is infinitely differentiable for $x \neq x_0$ and has compact support.

2^0 $D^M f(x) = C_M (x - x_0)_+^{\alpha - M} + 0 (| x - x_0 |^{\beta - M})$, $x \to x_0$ for $0 \leq M \leq N$,

where

$$\alpha \neq -1, \ -2, \ldots, \ \beta > \alpha \quad \text{and} \ N > \beta + 1.$$

Then

$$\hat{f}(\xi) = C\xi^{-\alpha-1}\, e^{-ix_0\xi} + 0\,(\xi^{-\beta-1}),\ \xi \to \infty.$$

PROOF: By (1.5) we may suppose that $C = 0$ and $x_0 = 0$. Here and on several occasions in the sequel we shall make use of the following construction. We let Ψ be a fixed function such that $0 \leq \Psi \in C_0^\infty(R)$ and $\operatorname{supp} \Psi \subset \{x \mid 1/2 < |x| < 2\}$. We write

$$\Psi_\nu(x) = \Psi(2^{-\nu} x)$$

and assume

$$\sum_{-\infty}^{\infty} \Psi_\nu(x) = 1, \quad x \neq 0.$$

(For the existence of such a function, see Hörmander [2].)

To prove the theorem we write $f_\nu = \Psi_\nu f$. Then

$$|f_\nu(x)| \leq C_0' \, 2^{\nu\beta},$$

$$| D^N f_\nu(x) \leq C_N' \, 2^{\nu(\beta-N)}$$

and thus

$$|\hat{f}_\nu(\xi)| \leq C \frac{2^{\nu(\beta+1)}}{1 + (|\xi| \cdot 2^\nu)^N}.$$

This implies

$$|\hat{f}(\xi)| \leq \sum_{-\infty}^{\infty} |\hat{f}_\nu(\xi)| \leq C |\xi|^{-\beta-1} \sum_{-\infty}^{\infty} \frac{(|\xi| \cdot 2^\nu)^{\beta+1}}{1 + (|\xi| \cdot 2^\nu)^N} \leq C' |\xi|^{-\beta-1}.$$

2. A Lemma.

Using (0.1) and (0.2) we get after Fourier transformation

$$\mathcal{F} I^k G(t) f(\xi) = (kt^{-k} \int_0^t (t-s)^{k-1} e^{isH(\xi)}\, ds)\, \mathcal{F} f(\xi).$$

Thus our problem is to find for which k

$$\hat{a}_t(\xi) = kt^{-k} \int_0^t (t-s)^{k-1} e^{isH(\xi)}\, ds$$

is a Fourier multiplier on L^p, i. e. belongs to M_p. By (1.3) it suffices to consider the case $t = 1$, i. e. to find for which k the function

$$\hat{a}(\xi) = k \int_0^1 (1 - s)^{k-1} e^{isH(\xi)} ds$$

belongs to the space M_p.

Now let $\Phi \in C^\infty(R)$ be such that $\Phi(t) = 0$ if $t < 1/2$ and $\Phi(t) = 1$ if $t > 1$.

LEMMA 2.1. $\hat{a} \in M_p$ if and only if $\hat{b} \in M_p$, where

$$\hat{b}(\xi) = \Phi(H(\xi)) H(\xi)^{-k} e^{iH(\xi)}.$$

PROOF: We want to apply theorem 1.3 to the function F, where

$$F(H(\xi)) = \hat{a}(\xi) - C_k \hat{b}(\xi),$$

and the constant C_k is defined by

$$k \int_{-\infty}^1 (1 - s)^{k-1} e^{isu} ds = C_k u^{-k} e^{iu}, \ u > 0.$$

Such a constant C_k exists in view of (1.5). The integral is of course to be understood as a Fourier transform. For $u > 0$ we get

$$F(u) = k \int_0^1 (1 - s)^{k-1} e^{isu} ds - C_k \Phi(u) u^{-k} e^{iu} =$$

$$= C_k u^{-k} e^{iu} - k \int_{-\infty}^0 (1 - s)^{k-1} e^{isu} ds - C_k \Phi(u) u^{-k} e^{iu}.$$

Thus for $u \geq 1$

$$F(u) = - k \int_{-\infty}^0 (1 - s)^{k-1} e^{isu} ds.$$

Put

$$f_k(s) = \begin{cases} (1 - s)^{k-1} & \text{if } s < 0, \\ 0 & \text{if } s \geq 0. \end{cases}$$

Then for $u \geq 1$

$$D^N F(u) = C_{N,k} (s^N f_k(s))^{\vee} (u).$$

But

$$s^N f_k(s) = s_-^N (f_k(s) - 1) + s_-^N,$$

where $s_- = (-s)_+$.

The first term belongs to C^{N+1} in a neighbourhood of $s = 0$ and the derivatives of it of sufficiently high order belong to $L^1 (s < -1)$. This gives the desired estimate

$$| D^N F(u) | \leq C_N u^{-N-1}, \quad u \geq 1.$$

For $0 \leq u \leq 1$ we get

$$| F(u) - F(0) | \leq k \int_0^1 (1 - s)^{k-1} | e^{isu} - 1 | \, ds + | C_k | \, \Phi(u) \, u^{-k} \leq C_0 \, u.$$

We can now apply theorem 1.3 with $\alpha = \beta = 1$ and get

$$F(H(\xi)) \in \mathcal{F} L^1.$$

3. The general case.

THEOREM 3.1. $\hat{a} \in M_p$, if $k > n \, | 1/p - 1/2 |$.

PROOF: According to lemma 2.1 it is sufficient to show the corresponding statement for $\hat{b}$, where

$$\hat{b}(\xi) = \Phi(H(\xi)) \, H(\xi)^{-k} \, e^{iH(\xi)}.$$

We may also assume $1 \leq p \leq 2$. Choose Ψ as in the proof of theorem 1.4 and put

$$\hat{b}_\nu(\xi) = \Psi_\nu(|\xi|) \, \hat{b}(\xi).$$

Then

$$\| \hat{b}_\nu \|_{L^2} \leq C \cdot 2^{-\nu m k} \cdot 2^{\nu n/2}.$$

Since

$$\hat{b}(\xi) = F(H(\xi)),$$

where

$$F(u) = \Phi(u) \, u^{-k} \, e^{iu},$$

and

$$| D^M F(u) | \leq C_M u^{-k}, \quad M \geq 0,$$

we get

$$| D^M \hat{b}(\xi) | \leq C'_M | \xi |^{-mk+M(m-1)}, \quad M \geq 0,$$

and thus

$$\| D^N \hat{b}_\nu \|_{L^2} \leq C''_N 2^{\nu(-mk+N(m-1))} \cdot 2^{\nu n/2}.$$

If we take $N > n/2$, theorem 1.2 gives

$$\| \hat{b}_\nu \|_{M_1} \leq C \cdot (2^{\nu(n/2-mk)})^{1-n/2N} \cdot (2^{\nu(N(m-1)-mk+n/2)})^{n/2N} = C 2^{\nu m(n/2-k)}.$$

Moreover, by (1.2),

$$\| \hat{b}_\nu \|_{M_2} = \| \hat{b}_\nu \|_{L^\infty} \leq C \cdot 2^{-\nu mk}.$$

Theorem 1.1 with $\theta = 2(1 - 1/p)$ gives

$$\| \hat{b}_\nu \|_{M_p} \leq C \cdot 2^{\nu m(n/2-k)(1-\theta)} \cdot 2^{-\nu mk\theta} = C \cdot 2^{\nu m(n(1/p-1/2)-k)}.$$

Remembering that $\hat{b}(\xi)$ vanishes in a neighbourhood of 0, we get for some ν_0

$$\| \hat{b} \|_{M_p} \leq \sum_{\nu_0}^{\infty} \| \hat{b}_\nu \|_{M_p} < \infty,$$

if $k > n(1/p - 1/2)$.

REMARK 3.1. Exactly in the same way we can prove the more general result: Let F be infinitely differentiable on R and vanish in a neighbourhood of 0. Assume that

$$| D^J F(u) | \leq C_J u^{-k}, \quad J = 1, \ldots .$$

Then $F(H(\xi)) \in M_p$, if $k > n | 1/p - 1/2 |$. The corresponding result for the torus T^n is proved in Löfström [5], section 10.

4. The case $H(\xi) = | \xi |^m, m \neq 1$.

In this section we always take $H(\xi) = | \xi |^m$ and $m \neq 1$.

THEOREM 4.1. If $H(\xi) = | \xi |^m$, $m \neq 1$, then $\hat{a} \in M_p$ implies $k \geq n | 1/p - 1/2 |$.

For the proof we need two lemmata.

LEMMA 4.1. If $m \neq 1$ and $n + m'(k - n/2) > 0$, then

$$b(x) = C_1 |x|^{-(n+m'(k-n/2))} \exp(iC_2 |x|^{m'}) [1 + 0(|x|^{-m'/2})] +$$
$$+ b_m(|x|), \quad |x|^{m'} \to \infty,$$

where $m' = m/(m-1)$, C_1 and C_2 are constants $\neq 0$, and b_m is a continuous function if $m < 1$ and identically zero if $m > 1$.

REMARK 4.1. The lemma was proved in the case $m < 1$ by Wainger [8], Part II. We will prove it in the case $m > 1$.

LEMMA 4.2. Let Φ be the function in section 2 and put

$$\hat{f}_\alpha(\xi) = \Phi(|\xi|^m) |\xi|^{-\alpha}, \quad \alpha > 0.$$

Then $f_\alpha \in L^p$, if $\alpha > n/p'$, $1 \leq p \leq \infty$, $1/p + 1/p' = 1$.

PROOF OF THEOREM 4.1.
If $\hat{a} \in M_p$, then $\hat{b} \in M_p$. Consider

$$\hat{f}(\xi) = \Phi(|\xi|^m) |\xi|^{-m(k+\lambda)} \exp(i|\xi|^m), \quad \lambda m > n/p'.$$

Then $f \in L^p$ according to lemma 4.2. On the other hand lemma 4.1 gives

$$f(x) = C_1 |x|^{-(n+m'(k+\lambda-n/2))} \exp(iC_2 |x|^{m'}) [1 + \sigma(1)] + b_m(|x|), \ |x|^{m'} \to \infty,$$

if $n + m'(k + \lambda - n/2) > 0$.
Now choose k so that

$$p(n + m'(k + \lambda - n/2)) = n,$$

i. e. $0 < m\lambda - n/p' = m(n(1/p - 1/2)) - k)$. Thus $\hat{a} \in M_p$ leads to a contradiction if

$$k < n(1/p - 1/2).$$

REMARK 4.2. This method was used by Wainger [8], Part IV, to prove the analogous theorem for Fourier multipliers on the torus T^n.

PROOF OF LEMMA 4.1 IN THE CASE $m > 1$.

We will restrict ourselves to the case $n \geq 2$. The case $n = 1$ can be treated in an analogous way.

Let Ψ_ν be the standard functions introduced in the proof of theorem 1.4. We may assume that

$$\Phi(t) = \sum_0^\infty \Psi_\nu(t) \quad \text{for} \quad t \geq 0.$$

Then $\hat{b}(\xi) = \sum_0^\infty \hat{b}_\nu(\xi)$, where

$$\hat{b}_\nu(\xi) = \Psi_\nu(|\xi|^m)\, |\xi|^{-mk} \exp(i\,|\xi|^m).$$

This implies

$$b_\nu(x) = C \int e^{i\langle x,\xi\rangle} \Psi_\nu(|\xi|^m)\, |\xi|^{-mk} \exp(i\,|\xi|^m)\, d\xi.$$

Clearly b_ν is a function of $|x|$, so we can assume that $x = (|x|, 0, \ldots, 0)$. Then we make the transformation

$$\xi_j \to |x|^{\frac{1}{m-1}} \xi_j, \quad 1 \leq j \leq n,$$

and get

$$b_\nu(x) =$$

$$C\,|x|^{-m'k+n/(m-1)} \int \exp(i\,|x|^{m'}(\xi_1 + |\xi|^m))\, \Psi_\nu(|x|^{m'}\,|\xi|^m)\, |\xi|^{-mk}\, d\xi =$$

$$C\,|x|^{-m'k+n/(m-1)}\,.$$

$$\cdot \iint_{\substack{-\infty<\xi_1<\infty \\ r\geq 0}} \exp[i\,|x|^{m'}(\xi_1 + (\xi_1^2 + r^2)^{m/2})]\, \Psi_\nu(|x|^{m'}(\xi_1^2 + r^2)^{m/2})\, (\xi_1^2 + r^2)^{-mk/2}\, r^{n-2}\, d\xi_1\, dr.$$

Finally we make variable transformation

$$\begin{cases} s = \xi_1 + (\xi_1^2 + r^2)^{m/2} \\ t = \xi_1 \end{cases}$$

or

$$\begin{cases} \xi_1 = t \\ r = ((s-t)^{2/m} - t^2)^{1/2}\,. \end{cases}$$

We get

$$b_\nu(x) = C\,|x|^{-m'k+n/(m-1)}\,.$$

$$\cdot \iint_\Omega \exp(i\,|x|^{m'} s)\, \Psi_\nu(|x|^{m'}(s-t))\, (s-t)^{-k+1-2/m}\, ((s-t)^{2/m} - t^2)^{(n-3)/2}\, ds\,dt,$$

where $\Omega = \{(s, t) \mid s \geq s_0 , t + | t |^m \leq s\}$ and $s_0 = \inf (t + | t |^m)$.

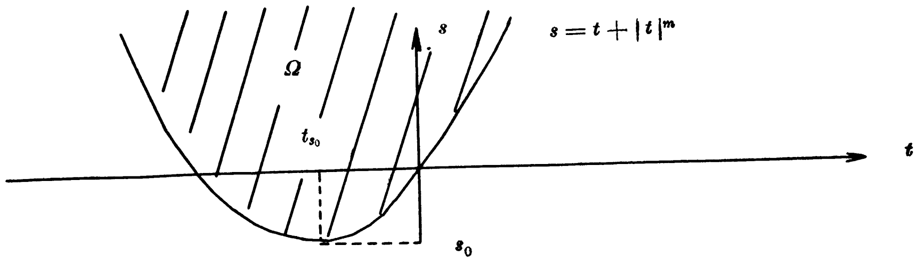

fig. 4.1.

Let t_{s_0} be defined by

$$s_0 = t_{s_0} + | t_{s_0} |^m .$$

We split the integral into two integrals

$$b_\nu (x) = C \cdot | x |^{-m'k+n/(m-1)} \left(\iint_{\Omega \cap \{t > t_{s_0}\}} + \iint_{\Omega \cap \{t < t_{s_0}\}} \right) .$$

Both of them can be handled in the same way. Let us consider only the second one and call it $b_\nu^* (x)$. With t_s defined by

$$s = t_s + | t_s |^m , \; t_s > t_{s_0}$$

we get

$$b_\nu^* (x) = C \, | x |^{-m'k+n/(m-1)} \int_{s_0}^{\infty} \exp (i \, | x |^{m'} s) \, C_\nu (| x |, s) \, ds,$$

where

$$C_\nu (| x |, s) = \int_{t_{s_0}}^{t_s} \Psi_\nu (| x |^{m'} (s - t)) \, (s - t)^{-k+1-2/m} \, ((s - t)^{2/m} - t^2)^{(n-3)/2} \, dt.$$

Since

$$g (| x | , s) = \sum_0^\infty C_\nu (| x | , s) =$$

$$= \int_{t_{s_0}}^{t_s} \Phi (| x |^{m'} (s - t)) \, (s - t)^{-k+1-2/m} \, ((s - t)^{2/m} - t^2)^{(n-3)/2} \, dt$$

is uniformly convergent on compact subsets of $\{s \mid s \geq s_0\}$ we get

$$b^*(x) = \sum_0^\infty b_\nu^*(x) = C \mid x \mid^{-m'k+n/(m-1)} \int \exp(is \mid x \mid^{m'})\, g(\mid x \mid, s)\, ds,$$

where the integral, as in the rest of the proof, is to be understood as an inverse Fourier transform.

Now we write

$$g = g_0 + g_1 + g_2,$$

where

$$g_\nu(\mid x \mid, s) = \chi_\nu(s)\, g(\mid x \mid, s), \qquad \nu = 0, 1, 2,$$

and χ_0, χ_1, χ_2 are as in fig. 4.2.

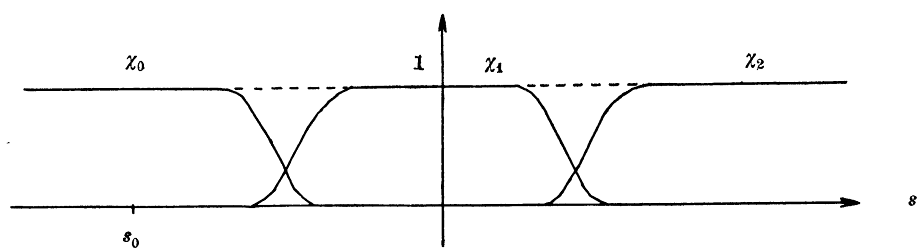

fig. 4.2.

Choosing $\mid x \mid^{m'}$ sufficiently large, we can obtain that $\Phi = 1$ in the integrals defining g_0 and g_2.

We first consider g_0.

$$g_0(\mid x \mid, s) = g_0(s) =$$

$$= \chi_0(s) \int_{t_{s_0}}^{t_s} (s-t)^{-k+1-2/m} ((s-t)^{2/m} - t^2)^{(n-3)/2}\, dt =$$

$$= (t_s - t_{s_0})\, \chi_0(s) \int_0^1 (s - t(y))^{-k+1-2/m} ((s - t(y))^{2/m} - t(y)^2)^{(n-3)/2}\, dy,$$

if

$$t(y) = t_{s_0} + (t_s - t_{s_0})\, y.$$

The integrand is a function f of y and $(t_s - t_{s_0})$. Elementary calculations give

$$f(y, t_s - t_{s_0}) = C(t_s - t_{s_0})^{n-3}(1-y)^{(n-3)/2}\ (1 + \sum_{\substack{\nu \geq 1 \\ \mu \geq 0}} C_{\nu\mu}(t_s - t_{s_0})^{\nu}(1-y)^{\mu})$$

uniformly in $\{0 \leq y \leq 1\} \times \{|t_s - t_{s_0}| < \varepsilon\}$ with $C \neq 0$.

But

$$t_s - t_{s_0} = h((s - s_0)^{1/2}),$$

where h is analytic in a neighbourhood of 0, and $h(0) = 0$, $h'(0) \neq 0$.

Hence $g_0(s)$ fulfils the conditions of theorem 1.4 with $\alpha = (n-2)/2$ and $\beta = (n-1)/2$. We thus get

$$\int \exp(is\,|x|^{m'})\, g_0(s)\, ds =$$

$$= C\,|x|^{-m'((n-2)/2+1)} \exp(-is_0\,|x|^{m'})\ [1 + 0(|x|^{-m'/2})],\ |x|^{m'} \to \infty.$$

We finally turn to g_1 and g_2. One can show that for N sufficiently large

$$\|D_s^N g_1(|x|, s)\|_{L^1} \leq C\,|x|^{N/(m-1)+\text{const.}}$$

and

$$\|D_s^N g_2(s)\|_{L_1} < \infty.$$

This implies

$$\int \exp(is\,|x|^{m'})\, g_\nu(|x|, s)\, ds = 0(|x|^{-m'M}),\ |x|^{m'} \to \infty,$$

for every $M \geq 1$ $\nu = 1,2$.

PROOF OF LEMMA 4.2.

At first we suppose $n > \alpha > n/p'$. Write

$$\hat{f}_\alpha(\xi) = |\xi|^{-\alpha} - (1 - \Phi(|\xi|^m))\,|\xi|^{-\alpha}.$$

Thus

$$f_\alpha(x) = C_{\alpha,n}\,|x|^{\alpha-n} + g_\alpha(x),$$

where $g_\alpha \in L^\infty$.

By differentiation we get

$$f_\alpha(x) = 0(|x|^{-N}),\ |x| \to \infty, \text{ for any } N.$$

Thus $f_\alpha \in L^p$, if

$$p(\alpha - n) > n,$$

i. e. if $\alpha > n/p'$.

Now the proof will be finished, if we show that

$$f_{\alpha_1} \in L^p \text{ and } \alpha_2 > \alpha_1$$

implies

$$f_{\alpha_2} \in L^p.$$

Write

$$f_{\alpha_2}(\xi) = [\Phi(|\xi|^m)^{1/2} |\xi|^{-(\alpha_2-\alpha_1)}] \cdot [\Phi(|\xi|^m)^{1/2} |\xi|^{-\alpha_1}].$$

If $\alpha_2 - \alpha_1 > 0$ the first factor belongs to M^p by theorem 1.3 The proof is finished.

5. The case $H(\xi) = |\xi|$.

Now we turn to the case $m = 1$, i. e. in this section $H(\xi) = |\xi|$.

THEOREM 5.1. $\hat{a} \in M_p$ if $H(\xi) = |\xi|$ and $k > (n-1)|1/p - 1/2|$.

THEOREM 5.2. If $H(\xi) = |\xi|$, then $\hat{a} \in M_p$ implies $k \geq (n-1)|1/p - 1/2|$.

For the proofs we need the following

LEMMA 5.1. If $b(\xi) = \Phi(|\xi|)|\xi|^{-k} e^{i|\xi|}$, then

$$b \in L^\infty_{\text{loc}}(|x| \neq 1), \tag{5.1}$$

$$b(x) = b_1(x) + C(\operatorname{sgn}(1-|x|))\,|1-|x||^{k-(n-1)/2-1}(1+o(1)),\ |x| \to 1, \tag{5.2}$$

where $b_1 \in L^\infty$, if $k-(n-1)/2 < 1$ and $k-(n-1)/2$ is not an integer

$$b(x) = 0(|x|^{-N}),\ |x| \to \infty, \text{ for any } N. \tag{5.3}$$

PROOF OF THEOREM 5.1.

As usual we prove the theorem for $\hat{b}$ instead of $\hat{a}$. By theorem 1.3 it is obvious that if $\hat{b} \in M_1$ for a certain $k = k_0$, then $\hat{b} \in M_1$ for all $k \geq k_0$. By lemma 5.1 we get

$$b \in L^1, \text{ if } 0 < k - (n-1)/2 < 1,$$

and thus

$$\hat{b} \in M_1, \text{ if } 0 < k - (n-1)/2.$$

To prove the theorem for $p > 1$ we use again the standard functions Ψ_ν from the proof of theorem 1.4 and put

$$\hat{b}_\nu(\xi) = \Psi_\nu(|\xi|)\,\hat{b}(\xi).$$

Let k_0 be a number $> (n-1)/2$. Then

$$\hat{b}_\nu(\xi) = \Psi(2^{-\nu}|\xi|)\,\Phi(|\xi|)\,|\xi|^{-k} e^{i|\xi|} =$$

$$= \{\Psi(2^{-\nu}|\xi|)\,|2^{-\nu}\xi|^{k_0-k}\}\cdot\{\Phi(|\xi|)\,|\xi|^{-k_0} e^{i|\xi|}\}\cdot 2^{\nu(k_0-k)},$$

which implies, in view of (1.2), (1.3) and (1.4),

$$\|\hat{b}_\nu\|_{M_1} \leq C\cdot 2^{\nu(k_0-k)}$$

and

$$\|\hat{b}_\nu\|_{M_2} \leq C 2^{-\nu k}.$$

Theorem 1.1 with $\theta = 2(1-1/p)$ gives

$$\|\hat{b}_\nu\|_{Mp} \leq C 2^{\nu(k_0-k)(1-\theta)-\nu k\theta} = C\,2^{\nu(k_0(1-\theta)-k)}.$$

Thus $\hat{b} \in M_p$, if $k > k_0(1-\theta)$. But k_0 is arbitrary $> (n-1)/2$ and hence $\hat{b} \in M_p$, if

$$k > ((n-1)/2)(1-\theta) = (n-1)(1/p - 1/2).$$

PROOF OF THEOREM 5.2.

Let 0 be the annulus $\{x \mid 1/2 \leq |x| \leq 2\}$. Then by (5.2) $b \in L^p(0)$ implies

$$p(k - (n-1)/2 - 1) > -1,$$

i. e.

$$k > (n+1)/2 - 1/p.$$

Suppose now that

$$k = (n-1)(1/p - 1/2) - \varepsilon, \quad \varepsilon > 0.$$

Put

$$\hat{g}(\xi) = \Phi(|\xi|)\,|\xi|^{-\lambda}, \text{ where } \lambda = n/p' + \varepsilon.$$

By lemma 4.2 $g \in L_p$. But

$$\hat{b}(\xi) \cdot \hat{g}(\xi) = \Phi(|\xi|)^2 |\xi|^{-(k+\lambda)} e^{i|\xi|},$$

where

$$k + \lambda = (n-1)(1/p - 1/2) - \varepsilon + n/p' + \varepsilon = (n+1)/2 - 1/p.$$

Thus $(\hat{b}\hat{g})^\vee \notin L_p$, which implies $\hat{b} \notin M_p$.

PROOF OF LEMMA 5.1.

By writing $\hat{b}(\xi) = \sum_0^\infty \hat{b}_\nu(\xi)$ as in the proof of lemma 4.1 we can justify the following calculations:

$$b(x) = C \int e^{i|x|\xi_1} \Phi(|\xi|) |\xi|^{-k} e^{i|\xi|} d\xi =$$

$$= C \int \Phi(s) s^{-k+n-1} e^{is} \left(\int_{-1}^{1} e^{i|x|st} (1-t^2)^{(n-3)/2} dt \right) ds.$$

Write

$$g(|x|s) = (2\pi)^{-1} \int_{-1}^{1} e^{i|x|st} (1-t^2)^{(n-3)/2} dt.$$

Then g is the inverse Fourier transform of the function

$$t \to (1-t^2)_+^{(n-3)/2} = (1-t)_+^{(n-3)/2} (1+t)_+^{(n-3)/2}.$$

Let

$$(1-t)_+^{(n-3)/2} = \sum_0^\infty c_\nu (1+t)^\nu$$

and

$$(1+t)_+^{(n-3)/2} = \sum_0^\infty d_\nu (1-t)^\nu$$

be the corresponding Taylor series.

Put

$$R_N(t) = (1-t^2)_+^{(n-3)/2} - \sum_0^{N-1} c_\nu (1+t)_+^{(n-3)/2+\nu} - \sum_0^{N-1} d_\nu (1-t)_+^{(n-3)/2+\nu}.$$

From

$$D^N R_N \in L^1_{\text{loc}} \text{ and}$$

$$D^M R_N \in L^1(|t| \geq 2) \text{ if } M \text{ is large}$$

we conclude that

$$|\check{R}_N(|x|s)| \leq C_N(|x|s)^{-N}.$$

Thus by (1.5)

$$g(|x|s) = \sum_0^{N-1} c'_\nu (|x|s)^{-(n-3)/2-\nu-1} e^{-i|x|s}$$

$$+ \sum_0^{N-1} d'_\nu (|x|s)^{-(n-3)/2-\nu-1} e^{i|x|s} + \check{R}_N(|x|s)$$

and

$$b(x) = \sum_0^{N-1} c''_\nu |x|^{-(n-3)/2-\nu-1} f_\nu(1-|x|) +$$

$$+ \sum_0^{N-1} d''_\nu |x|^{-(n-3)/2-\nu-1} f_\nu(1+|x|) + G_N(x)|x|^{-N},$$

where $G_N \in L^\infty$ and f_ν is defined by

$$\hat{f}_\nu(s) = \Phi(s)\, s^{-(k-(n-1)/2+\nu)}.$$

Now it is obvious that (5.3) is fulfilled. To obtain (5.2) we consider the function f defined by

(5.4) $$\hat{f}(s) = \Phi(s)\, s^{-\alpha}.$$

If $\alpha > 1$, then $f \in L^\infty$.
If $\alpha < 1$ is not an integer, then

$$\Phi(s)\, s^{-\alpha} = s_+^{-\alpha} - (1-\Phi(s))\, s_+^{-\alpha},$$

The inverse Fourier transformation gives

$$f(y) = C_1 y_+^{\alpha-1} + C_2 y_-^{\alpha-1} + g(y),$$

where $g \in L^\infty$ and $(y_-)^{\alpha-1} = (-y)_+^{\alpha-1}$. (See Gelfand-Schilow [1], p. 169).

Using this with $y = 1 - |x|$ and $y = 1 + |x|$ and $\alpha = k - (n-1)/2 + \nu$ we obtain (5.2). To prove (5.1) it is now sufficient to consider $|x| \leq 1/2$. By changing the order of integration and with f defined by (5.4) and $\alpha = k - n + 1$ we get

$$|b(x)| = \left| C \int_{-1}^{1} (1-t^2)^{(n-3)/2} f(1+|x|t)\, dt \right| \leq |C| \int_{-1}^{1} (1-t^2)^{(n-3)/2} (1+|x|t)^{-N}\, dt$$

for N sufficiently large. But

$$(1 + |x| t)^{-N} \leq (1 - 1/2)^{-N}, \text{ for } |x| \leq 1/2,$$

and thus

$$|b(x)| \leq \text{const. for } |x| \leq 1/2.$$

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