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# AN EXISTENCE THEOREM FOR BOUNDED VECTOR-VALUED FUNCTIONS

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## Introduction.

In professor's L. Amerio paper [1], supposing existence of bounded solutions for  $t \geq 0$  ( $t$ -time), of non-linear almost-periodic differential equations, one proves existence of bounded solutions which are defined on the whole time axis,  $-\infty < t < \infty$ .

In our paper [2] we proved a very similar result for solutions of the heat equation, with almost-periodic known term. We shall see below that this situation can be extended to a certain class of Banach-space valued functions admitting a certain representation through a given semi-group of class  $O^0$ .

§ 1. Let us consider first a reflexive Banach space  $X$ ; then, a one-parameter semi-group of operators in  $L(X, X): T_t, t \geq 0$ ; such that  $T_0 = I$ ,  $T_{t+\tau} = T_t T_\tau$ ;  $T_t \in L(X, X) \forall t \geq 0$  and  $T_t x$  is continuous from  $0 \leq t < \infty$  to  $X$ .

Consider also a continuous function  $-\infty < t < \infty$  to  $X$ , which is almost-periodic in Bochner's sense, that is:

*Each sequence  $(f(t + a_n))_{n=1}^\infty$  contains a subsequence  $(f(t + a_{n_p}))_{p=1}^\infty$  which is uniformly convergent on  $-\infty < t < \infty$ , in strong topology of  $X$ .*

Let now  $u(t)$  be a continuous function:  $0 \leq t < \infty$  to  $X$ , admitting representation

$$(1.1) \quad u(t) = T_t u(0) + \int_0^t T_{t-\zeta} f(\zeta) d\zeta, \quad \forall t \geq 0$$

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and let us assume

$$(1.2) \quad M = \sup_{t \geq 0} \|u(t)\|_X < \infty.$$

Then we have

**THEOREM.** *There exists a continuous function  $W(t)$ ,  $-\infty < t < \infty$  to  $X$ , such that*

$$(1.3) \quad W(t) = T_{t-t_0} W(t_0) + \int_{t_0}^t T_{t-\zeta} f(\zeta) d\zeta, \quad \forall t \geq t_0$$

$$(1.4) \quad \sup_{-\infty < t < \infty} \|W(t)\| < \infty.$$

**PROOF.** Let us consider the sequence of translates

$$u_n(t) = u(t + n)$$

They are defined for  $t \geq -n$ , and we have

$$(1.5) \quad \sup_{t \geq -n} \|u_n(t)\| = M = \sup_{t \geq 0} \|u(t)\|$$

As well known, reflexivity implies weak sequential compactness of bounded sets in  $X$ . Then, using almost-periodicity of  $f(t)$  and the diagonal procedure, we obtain a sequence of positive integers  $(n_k)_1^\infty$  with following properties:

$$(1.6) \quad \lim_{k \rightarrow \infty} f(t + n_k) = g(t), \text{ uniformly on } -\infty < t < \infty \text{ (and consequently } g(t) \text{ is an almost-periodic function)}$$

$$(1.7) \quad \text{For each } N = 0, 1, 2, \quad u_{n_k}(-N) \text{ is defined for } k > N,$$

and

$$(1.8) \quad (w) \lim_{k \rightarrow \infty} u_{n_k}(-N) = W_N \text{ exists and belongs to } X \text{ (here } (w) \text{ — means weak topology in } X; \text{ remember that a reflexive space is weakly sequentially complete).}$$

Remark now, that for each real  $t$ ,  $u_{n_k}(t)$  is defined for  $k \geq k_t$ . Then we shall see that,  $\forall t \in (-\infty, \infty)$

$$(1.9) \quad (w) \lim_{k \rightarrow \infty} u_{n_k}(t) = V(t) \text{ exists.}$$

$$(1.10) \quad \sup_{-\infty < t < \infty} \|V(t)\| < \infty$$

$$(1.11) \quad V(t) = T_{t-t_0} V(t_0) + \int_{t_0}^t T_{t-\sigma} g(\sigma) d\sigma, \quad \forall t \geq t_0, \quad \forall t_0 \in \mathbb{R}^1.$$

In fact (1.10) is a consequence of (1.9) and (1.5). To prove (1.9) we use following

LEMMA 1. Let  $t \in (-\infty, \infty)$  be given, and  $N$  a positive integer such that  $t + N > 0$ . Then,  $\forall k > N$ , we have

$$(1.12) \quad u_{n_k}(t) = T_{t+N} u_{n_k}(-N) + \int_{-N}^t T_{t-\tau} f(\tau + n_k) d\tau.$$

This Lemma is a Corollary of a slightly more general result

LEMMA 2. Let  $u(t)$ ,  $t \geq 0 \rightarrow \mathcal{X}$  (arbitrary Banach space), be a continuous function;  $T_t$ ;  $t \geq 0 \rightarrow L(\mathcal{X}, \mathcal{X})$  be a strongly continuous one parameter semi-group of linear bounded operators in  $\mathcal{X}$ ;  $f(t)$ ,  $-\infty < t < \infty \rightarrow \mathcal{X}$  be a continuous function.

Suppose

$$u(t) = T_t u(0) + \int_0^t T_{t-\tau} f(\tau) d\tau, \quad \forall t \geq 0.$$

Then, if  $t \in (-\infty, \infty)$ , is given and  $b > a > 0$ ,  $a + t > 0$ , we have

$$(1.13) \quad u(t+b) = T_{t+a} u(b-a) + \int_{-a}^t T_{t-\zeta} f(\zeta + b) d\zeta$$

REMARK. Lemma 1 follows from Lemma 2 if we take  $b = n_k$ ,  $a = N$ .

PROOF OF LEMMA 2.

As  $t + b > t + a > 0$ , we have using (1.1)

$$(1.14) \quad u(t+b) = T_{t+b} u(0) + \int_0^{t+b} T_{t+b-\zeta} f(\zeta) d\zeta = T_{t+a} T_{b-a} u(0) + \int_0^{t+b} T_{t+b-\sigma} f(\sigma) d\sigma.$$

Next remark, again by (1.1), the representation

$$u(b-a) = T_{b-a} u(0) + \int_0^{b-a} T_{b-a-\sigma} f(\sigma) d\sigma.$$

Introducing in (1.14) the value of  $T_{b-a} u(0)$  we get

$$(1.15) \quad u(t+b) = T_{t+a} \left( u(b-a) - \int_0^{b-a} T_{b-a-\sigma} f(\sigma) d\sigma \right) + \int_0^{t+b} T_{t+b-\sigma} f(\sigma) d\sigma = \\ T_{t+a} u(b-a) + \int_{b-a}^{t+b} T_{t+b-\sigma} f(\sigma) d\sigma.$$

Now, set  $\sigma = \zeta + b$ ; it follows  $\int_{b-a}^{t+b} T_{t+b-\sigma} f(\sigma) d\sigma = \int_{-a}^t T_{t-\zeta} f(\zeta + b) d\zeta$  which proves our Lemma, and consequently Lemma 1 too.

Actually we see that (1.9) is true in the following way: Fix an arbitrary real  $t$ ; then take  $N$  a positive integer, such that  $t + N > 0$ , and take  $k > N$ . We use then (1.12); as  $f(t + n_k) \rightarrow g(t)$  uniformly on  $(-\infty, \infty)$  and in  $X$  strong, we have obviously

$$\lim_{k \rightarrow \infty} \int_{-N}^t T_{t-\zeta} f(\zeta + n_k) d\zeta = \int_{-N}^t T_{t-\zeta} g(\zeta) d\zeta.$$

Then we have also

$$(w) \lim_{k \rightarrow \infty} T_{t+N} u_{n_k}(-N) = T_{t+N} W_N.$$

because a linear continuous operator in a  $B$ -space is continuous also in respect to the weak convergence.

Now, we shall see that for function  $V(t)$ ,  $-\infty < t < \infty \rightarrow X$  defined by (1.9), the representation formula (1.11) holds for each semi axis  $t \geq t_0$ .

Take in fact two reals  $t \geq t_0$ , and choose an integer  $N$  such that  $-N < t_0$ . Apply Lemma 1 to  $t, t_0, N$ . We have

$$u_{n_k}(t) = T_{t+N} u_{n_k}(-N) + \int_{-N}^t T_{t-\zeta} f(\zeta + n_k) d\zeta \\ u_{n_k}(t_0) = T_{t_0+N} u_{n_k}(-N) + \int_{-N}^{t_0} T_{t_0-\zeta} f(\zeta + n_k) d\zeta.$$

Then, reasoning as above, we obtain

$$V(t) = T_{t+N} W_N + \int_{-N}^t T_{t-\zeta} g(\zeta) d\zeta$$

$$V(t_0) = T_{t_0+N} W_N + \int_{-N}^{t_0} T_{t_0-\zeta} g(\zeta) d\zeta$$

Trying now to get (1.11) we write

$$T_{t-t_0} V(t_0) = T_{t-t_0} \left( T_{t_0+N} W_N + \int_{-N}^{t_0} T_{t_0-\zeta} g(\zeta) d\zeta \right) = T_{t+N} W_N +$$

$$+ \int_{-N}^{t_0} T_{t-\zeta} g(\zeta) d\zeta.$$

Hence

$$T_{t-t_0} V(t_0) + \int_{t_0}^t T_{t-\zeta} g(\zeta) d\zeta = T_{t+N} W_N + \int_{-N}^t T_{t-\zeta} g(\zeta) d\zeta = V(t)$$

that is (1.11).

Now we give the idea of the final step in the proof. Using uniform (on real axis) convergence of sequence  $f(t + n_k)$  to  $g(t)$ , we obtain uniform convergence of sequence  $g(t - n_k)$  to  $f(t)$ . Starting now with  $V(t)$  and repeating above procedure, we obtain function  $W(t)$ ,  $-\infty < t < \infty$  which is continuous, bounded, and admits representation (1.3)  $\forall t \geq t_0$ .

## REFERENCES

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