Annali della
Scuola Normale Superiore di Pisa
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 24, nº 1 (1970), p. 159-163

<http://www.numdam.org/item?id=ASNSP_1970_3_24_1_159_0>

NONEQUIVALENCE OF REGULAR BOUNDARY POINTS FOR THE LAPLACE AND NONDIVERGENCE EQUATIONS, EVEN WITH CONTINUOUS COEFFICIENTS (*)

by KEITH MILLER

In [7] the author showed that the regular boundary points for the uniformly elliptic equation in nondivergence form

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} = 0 \tag{1}$$

when the coefficients are only required to be measurable, with the eigenvalues of the symmetric $(a_{ij}(x))$ in $[\alpha, 1]$, $0 < \alpha < 1$, are not necessarily the same as those for Laplace's equation, even though this equivalence of regular points does hold for the equation in divergence form,

$$Mu = \sum_{i,j=1}^{n} (a_{ij}(x)\, u_{x_i})_{x_j} = 0, \tag{2}$$

with the same class of coefficients, as proved by Littman, Stampacchia, and Weinberger [5]. In fact, [7] gives examples of « both ways nonequivalence » for α arbitrarily close to 1 when $n = 3$, and examples of « one way nonequivalence » for α arbitrarily close to 1 when $n = 2$ and for α sufficiently small when $n \geq 4$.

Several of the author's fellow workers (being of little faith in the future of the nondivergence equation with discontinuous coefficients) have raised the question whether continuity of the coefficients in (1) is sufficient to

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Pervenuto alla Redazione il 16 Ottobre 1969.

(*) The research for this paper was partially supported by Air Force Contract Number AF-AFOSR 553-64.

restore equivalence of regular points. We show in this note that it is not; in fact, we exhibit a nondivergence equation in the plane with continuous coefficients (analytic except at 0 and tending to those of the Laplacian at 0) for which the origin is an isolated regular point.

To be precise, a boundary point x^0 of the bounded domain Ω is a *regular point for the operator* L (or for the equation $Lu = 0$) if there exists a barrier for L at x^0. Otherwise x^0 is an *exceptional point for* L. A *barrier for* L *at* x^0 is a function w, defined in some relative neighborhood $N = U \cap \Omega$ (U an open neighborhood of x^0), satisfying

$$\begin{gathered} w \in C^2(N) \cap C^0(\bar{N}), \\ w(x^0) = 0, \; w > 0 \text{ on } \bar{N} - \{x^0\}, \\ Lw \leq 0 \text{ in } N. \end{gathered} \tag{3}$$

It suffices to deal with C^2 barriers because we will be considering only operators with smooth coefficients in the interior. If L has nonsmooth coefficients then solvability of the Dirichlet problem is not known even for the sphere. However, we proved in [6, p. 98] that if L has Hölder continuous coefficients on compact subsets of Ω then solvability of the Dirichlet problem for every continuous boundary fnnction is equivalent to existence of a barrier for L at every boundary point.

THEOREM. *There exist, for $n = 2$, nondivergence equations for which the origin is an isolated regular boundary point, even though the coefficients are continuous (analytic except at the origin) and tend at the origin to the coefficients of the Laplacian.*

PROOF. We investigate the case $n = 2$ because there we have from Theorem 3 of [7] a particularly simple example (in fact radially symmetric) of « one way nonequivalence » even when α is arbitrarily close to 1.

Consider radial functions $w(x) = g(r)$. Now at each point x_0 chose the following orthogonal coordinate system: let the y_1 axis be in the radial direction and let the $y_2, y_3 \dots y_n$ axes be in tangential directions. These are directions of principal curvature i. e., the cross derivatives vanish. Therefore L, with ellipticity constant α, applied to radial functions w, have the representation

$$Lw = aw_{y_1y_1} + b\,(w_{y_2y_2} + \dots + w_{y_ny_n}) = ag_{rr} + b\,(n-1)\,\frac{g_r}{r} \tag{4}$$

where $a(x)$, $b(x)$ are measurable coefficients in $[\alpha, 1]$. When $n = 2$ and $\alpha = 1 - \beta$ is any ellipticity constant < 1, the equation

$$L_0 w = g_{rr} + [1 - \beta] g_r / r = 0 \tag{5}$$

has the solution

$$w(x) = g(r) = r^\beta \tag{6}$$

which is a barrier at the origin for this equation.

Let us see if we can still get a barrier as a solution when we replace the ellipticity « constant » $1 - \beta$ by $1 - \beta(r)$ where β is continuous and positive and $\beta(r) \to 0$. We want a solution for some positive δ of:

$$\begin{gathered} g_{rr} + [1 - \beta(r)] g_r / r + 0, \\ g \in C^2(0, \delta] \cap C^0[0, \delta], \\ g > 0 \text{ on } (0, \delta], \quad g(0) = 0. \end{gathered} \tag{7}$$

Two integrations then show that this is possible iffi β is such that

$$\frac{1}{t} e^{-\left(\int_t^\delta \frac{\beta(r)}{r} dr\right)} \tag{8}$$

is integrable on $[0, \delta]$.

Well, a necessary condition is that $\int_t^\delta \frac{\beta(r)}{r} dr \to \infty$ as $t \to 0$. Thus (7) has no solutions if $\beta(r)$ tends Dini continuously to zero. However, a solution of (7) does exist if $\beta(r)$ tends only logarithmically continuously to zero, for

$$g(r) = -(\log r)^{-1} \tag{9}$$

satisfies (7) with $\beta(r) = -2(\log r)^{-1}$. This completes the proof.

REMARK. Landis [4] has recently introduced a sufficient criterion for regularity in terms of « s-capacity » and has also discovered examples of nonequivalence (with discontinuous coefficients) quite similar to those of [7]. The equivalence of regular boundary points for the Laplace and nondivergence equations was established by Oleinik [8] with $C^{3+\alpha}$ coefficients and by Hervé [2] with Lipschitz continuous coefficients. Recently Krylov [3] has extended this equivalence to Dini continuous coefficients, i. e. coefficients with modulus of continuity $\beta(r)$ such that the integral $\int_t^\delta \frac{\beta(r)}{r}\,dr$ converges as $t \to 0$. On the other hand, our construction yields an example of non-equivalence if the continuity is only slightly worse, i. e., if this integral merely diverges sufficiently rapidly that (8) is integrable.

NOTES ADDED IN PROOF. We are indebted to D. Strook of *NYU* for calling attention to an early paper by Gilbarg and Serrin [1]. We find in it the same radially symmetric solution $u = a + b/\log r$ and equation with continuous coefficients which has been constructed here; thereby converting the present note into a largely expository exercise. This example therefore also predates those (for nonequivalence in one of the two directions) in [7] and [4]. The application in [1] was not specifically to regular and exceptional points, but to the closely related topic of removable bounded singularities.

The present example shows only « one way nonequivalence » with continuous coefficients. We have just received a new paper by O. N. Zograf [9] contructing a difficult example in 3-dimensions of « other way non-equivalence » with continuous coefficients; this is closely related to the previously mentioned example of Landis and uses an extension of his s-capacity approach.

We point out that the present example is easily extended to n dimensions, $n \geq 3$. In fact, let L_0 be the operator

$$L_0 u(x_1, x_2, x_3, \dots, x_n) = u_{y_1y_1} + [1 - \beta(r)]\, u_{y_2y_2} + u_{x_3x_3} + \dots u_{x_nx_n}$$

where $r = \sqrt{x_1^2 + x_2^2}$ and where the y_1 and y_2 axes are chosen in the radial and tangential directions in the (x_1, x_2) plane as before. Consider the function

$$w(x_1, x_2, x_3, \dots, x_n) = -(\log r)^{-1} + (x_3)^2 + \dots (x_n)^2.$$

Since $w_{y_2y_2} = w_r/r = [r \log r]^{-2}$ becomes infinite as $r \to 0$, we can take care of the added terms $w_{x_3x_3} = 2$, etc., by changing $\beta(r)$ slightly, $\beta(r)$ now being given by $-2(\log r)^{-1} + 2(n-2)[r \log r]^2$. The resulting operator L_0 with continuous coefficients then has $w(x_1, x_2, x_3 - a_3, \dots, x_n - a_n)$ as a barrier at each boundary point $(0, 0, a_3 \dots a_n)$ of the domain $\Omega = \{0 < \sqrt{x_1^2 + x_2^2} < \delta\}$. Such points are of course exceptional for Laplace's equation.

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