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THE BOUNDARY REGULARITY OF MINIMAL SURFACES

by DAVID KINDERLEHRER

Suppose that the boundary of a minimal surface contains an open arc Γ whose tangent is $(n-1)$-times differentiable with respect to arc length ($\Gamma \in C^n$), $n \geq 1$. In this paper we determine the differentiability properties of the conformal representation of the surface restricted to that arc γ in the boundary of the parameter domain whose image is Γ. The recent investigations of this question date from M. Tsuji [10] who extended to minimal surfaces the theorem of F. and M. Riesz ([11], p. 318). In 1951, H. Lewy [5] proved that if Γ is analytic, then the conformal representation is analytically extensible across γ. To solve this problem, Lewy employed his method of connecting the conformal representation to its analytic extension with an auxiliary function constructed as the solution to an ordinary differential equation. Although this method does not apply in our case, the differential equation itself does play an important role.

Criteria insuring regularity of the conformal representation have been given by S. Hildebrandt in [2], [3]. In his latest paper, he has proved that if $\Gamma \in C^n$ and the $(n-1)$ *st* derivative of the tangent vector satisfies a Hölder condition with exponent α, $0 < \alpha < 1$, ($\Gamma \in C^{n,\alpha}$), for $n \geq 4$, then the conformal representation is of class $C^{n,\alpha}$ on γ. Hildebrandt's proof proceeds by « straightening » the arc Γ to a line segment by a suitable diffeomorphism of Euclidean space, thereby transforming the problem to a quasilinear elliptic equation with certain boundary conditions. This new problem is then solved with the assistance of methods in the theory of elliptic equations. Unfortunately, this technique seems to require that $\Gamma \in C^{4,\alpha}$, $\alpha > 0$.

The primary object of this paper is to prove that if $\Gamma \in C^{1,\alpha}$, then the conformal representation is also of class $C^{1,\alpha}$, $\alpha > 0$. In an elementary

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manner, it follows that the conformal representation is of class $C^{n,\alpha}$ when $\Gamma \in C^{n,\alpha}$. The precise statements of these results are Theorems 4 and 5.

Motivated by the conformal character of minimal surfaces, the proof given here utilizes methods of function theory. For each point of Γ, we provide a special representation of a portion of the surface where one of the harmonic functions is Re z^m, $z = x + iy$ and m an odd positive integer. The points of Γ for which $m \neq 1$, corresponding in a sense to interior branch points for a minimal surface, pose serious difficulties. None-the-less, such « boundary singular points » comprise a discrete subset of Γ. These considerations only require that Γ has a continuous tangent. The use of a special representation to describe a particular problem in the behavior of minimal surfaces has been found useful by many authors. We mention as examples L. Bers [1], J. C. C. Nitsche [7], and M. Shiffman [9].

The regularity is proven by estimating the special conformal representation and applying Kellogg's Theorem (cf. [11] or [12]). Higher derivatives are discussed in § 6.

Each theorem we prove has an analogue for a minimal surface whose boundary contains a cusp. As an example, we state Theorem 4'. The methods used here provide some information about boundary behavior when Γ is assumed to be differentiable at a single point or when the tangent to Γ has a modulus of continuity $\omega(t)$ satisfying $\int_0^\delta t^{-1}\,\omega(t)\,dt < \infty$, for a $\delta > 0$. We do not discuss these questions here. The local character of the results insure that the conclusions are valid for minimal surfaces bounded by several Jordan curves which may even have self-intersections.

§ 1. Definitions.

Denote by G the upper half $\zeta = \xi + i\eta$ plane. Let S be a minimal surface in three dimensional (x_1, x_2, x_3) space which is the conformal image of G under the triple of harmonic functions

$$\mathfrak{r}(\zeta) = (x_1(\zeta), x_2(\zeta), x_3(\zeta))$$

which are continuous in $\bar{G}$, the closure of G, and admit finite limits at $\zeta = \pm\infty$. Suppose that the curve $\partial S : \mathfrak{r} = \mathfrak{r}(\xi)$, $-\infty < \xi < \infty$, the boundary of S, is rectifiable and that $\mathfrak{r} = \mathfrak{r}(\xi)$ is a strictly monotone map of $(-\infty, \infty)$ onto ∂S. In the conformal representation, $\mathfrak{r}(\zeta)$ satisfies the isothermal relations

$$(1) \qquad \begin{aligned} \mathfrak{r}_\xi(\zeta)^2 &= \mathfrak{r}_\eta(\zeta)^2 \\ \mathfrak{r}_\xi(\zeta)\cdot\mathfrak{r}_\eta(\zeta) &= 0 \end{aligned} \qquad \text{for } \zeta \in G$$

Here, we have adopted the notations $a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$ and $a^2 = a \cdot a$ for vectors $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$. The subscripts ξ, η denote differentation with respect to ξ, η respectively.

Let $F_j(\zeta) = x_j(\zeta) + ix_j^*(\zeta)$, with $x_j^*(\zeta)$ the harmonic conjugate of $x_j(\zeta)$ and $F_j(0) = 0, j = 1, 2, 3$. According to the results of Tsuji [10], the $F_j(\zeta)$ are bounded analytic functions in G assuming absolutely continuous boundary values $F_j(\xi)$ on $\operatorname{Im} \zeta = 0$. Their derivatives $F_j'(\zeta), j = 1, 2, 3$, belong to the Hardy class H^1 of the upper half ζ-plane. This implies there exists an M, $0 < M < \infty$, so that for $1 \leq j \leq 3$,

$$(2) \qquad \begin{aligned} &\int_{-\infty}^{\infty} | F_j'(t + i\eta) | \, dt \leq M \text{ for each } \eta, \qquad 0 \leq \eta < \infty \\ &\int_{0}^{\infty} | F_j'(\xi + it) | \, dt \leq M \text{ for each } \xi, \qquad -\infty < \xi < \infty. \end{aligned}$$

The isothermal relations (1) hold almost everywhere (a. e.) on $\operatorname{Im} \zeta = 0$, null sets on $\operatorname{Im} \zeta = 0$ correspond to null sets on ∂S, and null sets on ∂S correspond to null sets on $\operatorname{Im} \zeta = 0$. In particular,

$$(3) \qquad \begin{aligned} &\mathfrak{r}_\xi(\xi)^2 = \mathfrak{r}_\eta(\xi)^2 \neq 0 \ \text{ a. e. for } -\infty < \xi < \infty \\ &\mathfrak{r}_\xi(\xi)\cdot\mathfrak{r}_\eta(\xi) = 0 \qquad \text{a. e. for } -\infty < \xi < \infty. \end{aligned}$$

The assumption that ∂S be rectifiable is a local one in this sense: if the image C of $-1 < \xi < 1$ under $\mathfrak{r}(\xi)$ is rectifiable, then there is a subset $G' \subset \{ | \zeta | < 1 \} \cap G$ whose image S' under $\mathfrak{r} = \mathfrak{r}(\zeta)$ is bounded by a curve of finite length. See, for example, Lemma 1.1 in [4].

§ 2. Geometric Considerations.

Let S be a minimal surface with rectifiable boundary ∂S. Suppose that ∂S contains the arc

$$I : x_2 = \psi_2(x_1), \quad x_3 = \psi_3(x_1), \quad -B < x_1 < B,$$

where the $\psi_j(x_1)$ are absolutely continuous with $|\psi_j'(x_1)| \leq \omega(|x_1|)$, a. e. in $[-B, B]$, and $\psi_j(0) = 0$, $j = 2, 3$. Here $\omega(t)$ is a modulus of continuity at $t = 0$ (*). Since $\mathfrak{x} = \mathfrak{x}(\xi)$ is a monotone mapping, we may assume that Γ is the topological image of $-1 < \xi < 1$ with $x_1(-1) = -B$ and $x_1(1) = B$. It follows that $\dfrac{ds}{d\xi} \geq 0$, where s is the arc length of Γ in $\bar{S}$ measured from $\xi = -1$. In this section, the behavior of S on Γ will be utilized to provide a special conformal representation for a subset of S. The proofs of Theorem 1 and the lemmas which precede it are only technical modifications of those in § 2 of [4]. They are presented here for completeness.

LEMMA 2.1: There is an absolutely continuous function $H(x_1)$ such that $x_1^*(\xi) = H(x_1(\xi))$, $-B < x_1 < B$, and

(i) $|H'(x_1)| \leq c\,\omega(|x_1|)$, a. e. $-B < x_1 < B$, where c is a constant which depends on Γ,

(ii) $$H'(0) = \lim_{x_1 \to 0} \frac{H(x_1)}{x_1} = 0.$$

PROOF: With the notations introduced above, $\dfrac{ds}{d\xi} > 0$, a. e. in $[-1,1]$ by (3), so that

$$\frac{\partial x_1}{\partial \xi} = (1 + \psi_2'(x_1)^2 + \psi_3'(x_1)^2)^{-\frac{1}{2}} \frac{ds}{d\xi} > 0, \quad \text{a. e.} -1 \leq \xi \leq 1. \tag{5}$$

It follows that the inverse function $\xi = g(x_1)$ to $x_1 = x_1(\xi)$ on $[-1,1]$ is absolutely continuous. Since $g(x_1)$ is also monotone, $H(x_1) = x_1^*(g(x_1))$ is absolutely continuous in $[-B, B]$.

From (3) and (5),

$$\sum_1^3 \left(\frac{x_{j\xi}^*(\xi)}{x_{1\xi}(\xi)}\right)^2 \leq \sup(1 + \psi_2'(x_1)^2 + \psi_3'(x_1)^2) = \frac{1}{2}c^2, \qquad \text{a. e. in } [-1,1]$$

where c depends only on Γ. Using the isothermal relations once more,

$$H'(x_1) + \psi_2'(x_1)\frac{x_{2\xi}^*(\xi)}{x_{1\xi}(\xi)} + \psi_3'(x_1)\frac{x_{3\xi}^*(\xi)}{x_{1\xi}(\xi)} = 0 \text{ a. e.},$$

(*) By modulus of continuity at $t = 0$ we understand a non-decreasing continuous function $\omega(t)$ with $\omega(0) = 0$.

$$|H'(x_1)| \leq \frac{c}{\sqrt{2}}\sqrt{\psi_2'(x_1)^2 + \psi_3'(x_1)^2} \leq c\omega(|x_1|), \text{ a.e for } -1 \leq \xi \leq 1.$$

Statement (ii) follows upon the integration of H' from 0 to a point x_1 and applying the estimate above.

LEMMA 2.2: Recall that $F_1(\zeta) = x_1(\zeta) + ix_1^*(\zeta)$, $F_1(0) = 0$. Then

(i) $\lim\limits_{\xi \to 0^+} \operatorname{Arg} F_1(\xi) = 0$ and $\lim\limits_{\xi \to 0^-} \operatorname{Arg} F_1(\xi) = \pi$, where $\operatorname{Arg}(x + iy)$ denotes the unique number in $\left[-\frac{\pi}{2}, \frac{3}{2}\pi\right)$ satisfying $x + iy = |x + iy| e^{i \operatorname{Arg}(x+iy)}$,

(ii) $|F_1(\xi)|$ is strictly decreasing for $-\xi_0 < \xi \leq 0$ and strictly increasing for $0 \leq \xi < \xi_0$, where $\xi_0 > 0$ is sufficiently small.

PROOF: For $-1 < \xi \leq 0$, $\operatorname{Arg} F_1(\xi) = \operatorname{Arg}(x_1 + iH(x_1)) = \pi + \operatorname{Arg}\left(1 + i\frac{H(x_1)}{x_1}\right)$. Hence $\lim\limits_{\xi \to 0^-} \operatorname{Arg} F_1(\xi) = \pi$ by the previous lemma. The same reasoning applies in the case $\xi \to 0^+$.

For $-1 < \xi < 0$, set $R(x_1) = \sqrt{x_1(\xi)^2 + H(x_1(\xi))^2}$. By an elementary computation

$$\frac{dR}{dx_1} < 0 \text{ when } \omega(|x_1|) < \frac{1}{c}.$$

Since $x_1(\xi)$ is strictly increasing in $[-1,1]$, $|F_1(\xi)|$ is strictly decreasing for $-\xi_0 < \xi \leq 0$, some ξ_0. Analogously, $|F_1(\xi)|$ is strictly increasing for $0 \leq \xi < \xi_0$, some ξ_0.

LEMMA 2.3: The function $F_1(\zeta)$ has an isolated zero at $\zeta = 0$.

The idea of this proof is due to Donald Sarason. It derives from a well known theorem. If a function in the Hardy space H^1 of the disc $|z| < 1$ is real valued almost everywhere on an arc γ of $|z| = 1$, then it can be analytically continued across γ.

PROOF OF LEMMA. Let $z = z(\zeta)$ map G onto the unit $z = x + iy$ disc, $z(0) = 1$, and set $g(z) = F_1(\zeta)^2$. By Lemma 2.2, for some $\theta_0 > 0$ and a, $0 < a < \frac{\pi}{2}$,

$$-\frac{\pi}{2} < -a < \operatorname{Arg} g(e^{i\theta}) < a < \frac{\pi}{2} \text{ for } 0 < |\theta| < \theta_0.$$

Here Arg has the same meaning as in Lemma 2.2. Let $\varphi(z)$ denote the bounded, harmonic function in $|z|<1$ with boundary values

$$\varphi(e^{i\theta}) = \begin{cases} \operatorname{Arg} g(e^{i\theta}) & 0<|\theta|<\theta_0 . \\ 0 & \text{elsewhere.} \end{cases}$$

Again invoking Lemma 2.2 (i), $\varphi(z)$ is continuous at $z=1$. Let $\varphi^*(z)$ denote the harmonic conjugate of $\varphi(z)$, $\varphi^*(0)=0$. The real part of the analytic function $h(z)=e^{\varphi^*(z)-i\varphi(z)}$ is positive in $|z|<1$.

$$h(z) = |h(z)|\, e^{i \arg h(z)},$$

$$\frac{\operatorname{Re} h(z)}{\cos a} > |h(z)|.$$

Integrating,

$$\int_{-\pi}^{\pi} |h(\mathrm{r}e^{i\theta})|\, d\theta \leq \frac{2\pi}{\cos a} \operatorname{Re} h(0).$$

Hence $h(z)$ belongs to the Hardy class H^1 so that the function $g(z)\,h(z)$ belongs to H^1. Moreover, for almost every θ, $|\theta|<\theta_0$, $\lim_{r\to 1} g(re^{i\theta})\,h(re^{i\theta}) = e^{\varphi^*(e^{i\theta})}\, e^{-i \operatorname{Arg} g(e^{i\theta})}\, g(e^{i\theta})$, a real number. By the analytic extension theorem for H^1 functions quoted above, $g(z)\,h(z)$ may be analytically continued across $|\theta|<\theta_0$. Since $h(z)$ is zero free, $z=1$ is an isolated zero of $g(z)$. Therefore $\zeta=0$ is an isolated zero of $F_1(\zeta)$.

THEOREM 1: Let S be a minimal surface with rectifiable boundary ∂S. Suppose that ∂S contains the arc

$$\Gamma: x_2 = \psi_2(x_1), \quad x_3 = \psi_3(x_1), \quad -B < x_1 < B,$$

where the $\psi_j(x_1)$ are absolutely continuous with $|\psi_j'(x_1)| \leq \omega(|x_1|)$, a. e in $-B \leq x_1 \leq B$ and $\psi_j(0)=0$, $j=2,3$.

Then there is a subdomain U of G bounded by a segment $[a,b]$ of $-1<\zeta<1$ and a Jordan arc α joining b to a in G such that $\log F_1$ is a univalent map of U onto a region of bounded argument in the Riemann surface of $\log(x_1+ix_1^*)$.

The notations of the theorem and its proof are explained in § 1. The function $\omega(t)$ is a modulus of continuity at $t=0$.

PROOF. We may assume that $x_1(\xi)$ is strictly increasing in $-1 < \xi < 1$. Choose $r_1 > 0$ so that $\zeta = 0$ is the only zero of F_1 in $\{|\zeta| < r_1\} \cap \bar{G}$, and so that $|F_1(\zeta)|$ is strictly decreasing in $-r_1 \leq \zeta \leq 0$ and strictly increasing in $0 \leq \zeta \leq r_1$. This is possible by Lemmas 2.2, 2.3. Choose a semicircle $re^{i\theta}$, $0 \leq \theta \leq \pi$, with $r \leq r_1$ such that

$$|F_1(re^{i\theta})| \geq L \quad \text{for} \quad 0 \leq \theta \leq \pi, \quad \text{some} \quad L > 0.$$

For $0 < \lambda < L$, it will be shown that the set

$$\alpha = \alpha_\lambda = \{\zeta \in \bar{G}\,;\ |\zeta| < r,\ |F_1(\zeta)| = \lambda\}$$

is a simple arc, analytic in G. By continuity and strict monotonicity of $|F_1(\xi)|$, there are unique points $a < 0$ and $b > 0$ on α. By continuity of F_1, $\alpha \cap G \neq \emptyset$. Since $|F_1(\zeta)|$ is not constant in G, $\alpha \cap G$ is locally one or several branches of analytic curves. No more than one branch passes through each point of α. Suppose that starting at a point of α and traversing α in one direction necessitates passing through a point of α more than once. Then α would separate G into at least two components. On the boundary of at least one of these components, the harmonic function $\log|F_1(\zeta)|$ would be constant. Hence F_1 would be constant in G, a contradiction. Because $F_1(0) = 0$, α does not contain the origin. Since $|F_1(re^{i\theta})| \geq L > \lambda$, α does not intersect the semicircle $re^{i\theta}$, $0 \leq \theta \leq \pi$. Hence $\alpha \subset \{0 < |\zeta| < r\}$. Therefore, starting at the point b and traversing α must lead to the point a. Hence α is a Jordan arc from b to a.

Suppose that $\zeta \in \alpha$ is a point where $F_1'(\zeta) = 0$. Then in a neighborhood of this ζ,

$$F_1(t) - F_1(\zeta) = \sum_{k \geq n} c_k (t - \zeta)^k, \qquad n \geq 2.$$

It follows that more than one branch of α passes through ζ. But this cannot occur. Consequently, α is an analytic arc in G.

For a fixed ε, $0 < \varepsilon < \frac{\pi}{2}$, choose r_0 so small that

$$(6) \qquad \begin{gathered} -\varepsilon < \operatorname{Arg} F_1(\xi) < \varepsilon, \quad 0 < \xi \leq r_0 \\ \pi - \varepsilon < \operatorname{Arg} F_1(\xi) < \pi + \varepsilon, \qquad -r_0 \leq \xi < 0, \end{gathered}$$

and $r_0 < r \leq r_1$. Let $\alpha = \alpha_\lambda$ where $\lambda = \min\{|F_1(r_0)|, |F_1(-r_0)|\}$. Say $\lambda = |F_1(r_0)|$. Put $r_0 = b$, let $a \geq -r_0$ be the terminus of α_λ, and denote by U the simply connected subdomain of G bounded by α and the segment

$[a, b]$. Since $F_1(\zeta)$ does not vanish in U, a single valued branch of $\log F_1(\zeta)$, $\zeta \in U$ may be defined by specifying that $\log F_1(r_0) = \log |F_1(r_0)| + i \operatorname{Arg} F_1(r_0)$.

According to the maximum principle, $\dfrac{d}{dn} \log |F_1(\zeta)| < 0$ for $\zeta \in \alpha_\lambda \cap G$, where n denotes the outward normal derivative on α_λ. Therefore, with s as arc-length on α_λ measured from its right hand end point, the Cauchy-Riemann equations imply that $\dfrac{d}{ds} \arg F_1(\zeta) > 0$ for $\zeta \in \alpha_\lambda$. Hence $\arg F_1(\zeta)$ is strictly increasing on α_λ.

We now show that $\arg F_1(\xi)$ is finite for any $\xi \in (a, 0)$. For a fixed $\xi \in (a, 0)$, choose $\delta > 0$ so that the segment $\left[\xi + i\delta, \dfrac{1}{2} b + i\delta\right] \subset \bar{U}$. Then

$$|\log F_1(\xi)| \leq \left| \int_{\frac{b}{2}}^{\frac{b}{2}+i\delta} \frac{F_1'(\zeta)}{F_1(\zeta)} d\zeta \right| + \left| \int_{\frac{b}{2}+i\delta}^{\xi+i\delta} \frac{F_1'(\zeta)}{F_1(\zeta)} d\zeta \right| +$$

$$+ \left| \int_{\xi+i\delta}^{\xi} \frac{F_1'(\zeta)}{F_1(\zeta)} d\zeta \right| + \left| \log F_1\left(\frac{1}{2} b\right) \right| .$$

Since the total variation of F_1 on segments parallel to the ξ and η axes is bounded (cf. (2)) and since F_1 is not zero in U, $|\log F_1(\xi)| < \infty$. Hence for a given $\xi \in (a, 0)$, there is a positive odd integer m, in view of (6), such that

$$\pi m - \varepsilon < \arg F_1(\xi) < \pi m + \varepsilon. \tag{6'}$$

It also follows that $\log F_1(\xi)$ is continuous on $[a, b]$ for $\xi \neq 0$. Given $\delta > 0$, choose an interval $|\xi - \xi'| < \varrho$ and a box B with vertices $\xi + \varrho$, $\xi + \varrho + id$, $\xi - \varrho + id$, $\xi - \varrho$, for some $d > 0$ so that

$$\int_{-\varrho}^{\varrho} |F_1'(\xi + t)| \, dt < \delta \inf_B |F_1(\zeta)| .$$

Since F_1' is in the Hardy space H^1,

$$\lim_{\eta \to 0} \int_{-\varrho}^{\varrho} |F_1'(\xi + t + i\eta) \, dt = \int_{-\varrho}^{\varrho} |F_1'(\xi + t)| \, dt.$$

Therefore, for $|\xi - \xi'| < \varrho$,

$$| \log F_1(\xi) - \log F_1(\xi') | \leq (\inf_B | F_1(\zeta) |)^{-1} \lim_{\eta \to 0} \int_{-\varrho}^{\varrho} | F_1'(\xi + t + i\eta) | \, dt < \delta.$$

This continuity implies that (6′) holds for all $\xi \in (a, 0)$ with m independent of ξ. Q. E. D.

Maintaining the assumptions and notations of the theorem, determine a branch of $F_1(\zeta)^{1/m}$ by specifying $F_1(b)^{1/m} = |F_1(b)| \exp\left(\frac{i}{m} \operatorname{Arg} F_1(b)\right)$. The image of U under the mapping $z = F_1(\zeta)^{1/m}$ is a domain in the (ordinary) $z = x + iy$ plane.

Corollary 1: With the hypotheses of Theorem 1, there is a subdomain U of G and a positive odd integer m such that $z = F_1(\zeta)^{1/m}$ is a univalent map onto $\Omega = \sqrt[m]{F_1(U)}$. Let γ denote the image of $[-1, 1] \cap \overline{U}$ under this mapping. Then γ is a simple rectifiable arc, null sets on $\overline{U} \cap [-1, 1]$ correspond to null sets on γ, null sets on γ correspond to null sets on $[-1, 1] \cap \overline{U}$, and

$$\text{(7)} \qquad \begin{aligned} &\lim \arg z = 0 \text{ as } z \to 0, \ \operatorname{Re} z > 0, \ z \in \gamma \\ &\lim \arg z = \pi \text{ as } z \to 0, \ \operatorname{Re} z < 0, \ z \in \gamma. \end{aligned}$$

Proof: For γ we have the representation

$$\gamma : z = (x_1 + iH(x_1))^{1/m}, \ b' < x_1 < b'',$$

for suitable b', b''. Hence the strict monotonicity of $x_1(\xi)$ in $[-1, 1] \cap \overline{U}$ insures that γ is simple. By (5),

$$\left| \frac{dz}{d\xi} \right| = \frac{1}{m} | 1 + iH'(x_1(\xi)) | \, | F_1(\xi) |^{\frac{1-m}{m}} \frac{\partial x_1}{\partial \xi} > 0, \text{ a. e. } \xi \in \overline{U} \cap [-1, 1],$$

which implies the correspondence between null sets. Rectifiability of γ is shown with Lemma 2.1 (i) and (7) is shown with Lemma 2.2 (i).

Corollary 2: With the hypotheses of Theorem 1,

(i) The arc α_λ is rectifiable for almost all λ, $0 \leq \lambda \leq | F_1(b) |$, and

(ii) For almost every λ, $0 \leq \lambda \leq |F_1(b)|$, there is a $C_\lambda < \infty$ such that

$$|F_j'(\zeta)| \leq C_\lambda, \ \zeta \in \alpha_\lambda, j = 1, 2, 3, \text{ and } |F_1'(\zeta)| \geq \frac{1}{C_\lambda}, \ \zeta \in \alpha_\lambda.$$

PROOF: For (i), let $\varphi(z)$ denote the inverse to $F_1(\zeta)^{1/m}$ (cf. [13]).

For each $\varrho \leq R = |F_1(b)|^{1/m}$, let c_ϱ denote the arc of $|z| \leq \varrho$ in $\bar{\Omega}$. Then the length l_ϱ of $\alpha_\lambda = \varphi(c_\varrho)$, $\lambda = \varrho^m$, satisfies

$$l_\varrho^2 = \left(\int_{c_\varrho} |\varphi'(\varrho e^{i\theta})| \varrho \, d\theta\right)^2 \leq 2\pi\varrho \int_{c_\varrho} |\varphi'(\varrho e^{i\theta})|^2 \varrho \, d\theta.$$

Hence

$$\int_0^R \frac{l_\varrho^2}{\varrho} \, d\varrho \leq 2\pi \int_0^R \int_{c_\varrho} |\varphi'(\varrho e^{i\theta})|^2 \varrho \, d\theta \, d\varrho \leq 2\pi A,$$

where A denotes the area of $U \subset G$. Since the integrand l_ϱ^2/ϱ is finite almost everywhere, l_ϱ is finite almost everywhere, $0 \leq \varrho \leq R$.

To prove (ii) we utilize two well known theorems. Let E be the set of ϱ, $0 \leq \varrho \leq R$, such that

a. γ has a tangent at $c_\varrho \cap \gamma = \{F_1(a_\varrho)^{1/m}, F_1(b_\varrho)^{1/m}\}$, and

b. $\lim_{\zeta \to a_\varrho} F_j'(\zeta)$, $\lim_{\zeta \to b_\varrho} F_j'(\zeta)$, $\zeta \in \alpha_\lambda$, $\lambda = \varrho^m$, exist for

$j = 1, 2, 3$, and the limits are not zero when $j = 1$.

Since γ is rectifiable, the set of points ϱ not satisfying *a.* has measure zero. By a Theorem of Lindelöf ([11], p. 357), φ preserves angles at each point where γ has a tangent. It follows that the set $\alpha_\lambda \cap \{|\zeta - a_\varrho| < \delta\}$, for a suitable $\delta > 0$ depending on a_ϱ, is contained in a sector with vertex at a_ϱ for almost every ϱ, $0 \leq \varrho \leq R$. The same is true for almost every b_ϱ, $0 \leq \varrho \leq R$.

According to a well known Theorem ([11], p. 314), a function h of Hardy class satisfies $\lim_{\zeta \to \xi} h(\zeta) = h(\xi)$ ($\neq 0, \neq \infty$) uniformly in any sector with vertex at ξ for almost every ξ. Hence the complement of E has measure zero.

For any $\varrho \in E$ with α_λ, $\lambda = \varrho^m$, we have that for some $C_\lambda > 0$

$$| F_j' (\zeta) | \leq C_\lambda < \infty, \ \zeta \in \alpha_\lambda, \ j = 1, 2, 3, \text{ and}$$

$$\frac{1}{C_\lambda} \leq | F_1' (\zeta) |, \ \zeta \in \alpha_\lambda, \text{ since } F_1' (\zeta) \text{ is known not to vanish in } U.$$

§ 3. **Estimations.**

Limitations for the conformal representation determined in § 2 are provided in this paragraph. In what follows, let S be a minimal surface with a rectifiable boundary which contains the arc Γ described by (4). For convenience, we suppose that the modulus of continuity $\omega(t)$ satisfies $\omega(B) \leq 2^{-5}$. With m the positive integer and U the subdomain of G determined in Theorem 1, we set $z = x + iy = F_1(\zeta)^{1/m}$, $\Omega = \sqrt[m]{F_1(U)}$, γ the image of $[-1, 1] \cap \overline{U}$, and $f_j(z) = F_j(\zeta)$, $j = 1, 2, 3$. Since m is positive, Ω does not intersect the negative imaginary axis. We assume that Ω is bounded by γ and an arc of $|z| = R > 0$, with R^m satisfying the conclusions of Corollary 2. Note that $f_1(z) = z^m$.

LEMMA 3.1 : Set $c_\Gamma = 4(2 + 4\omega(B)^2)^{1/2}$, with $\omega(B) \leq 2^{-5}$.

(i) Let $\gamma : z = x(\sigma) + iy(\sigma)$, where σ denotes arc length on γ. Then

$$\left| \frac{dx}{d\sigma} - 1 \right| \leq \frac{1}{2} c_\Gamma \, \omega(|x_1|) \text{ and } \left| \frac{dy}{d\sigma} \right| \leq \frac{1}{2} c_\Gamma \, \omega(|x_1|), \text{ a. e. on } \gamma.$$

(ii) There is a Lipschitz function $h(x)$ so that $\gamma : z = x + ih(x)$, $a < x < b$, and

$$| h'(x) | \leq c_\Gamma \, \omega(|x_1|) < \tan \frac{\pi}{8} < \frac{1}{2}, \text{ a. e. } a < x < b.$$

(iii) For $t \in \gamma$ and fixed $z \in \gamma$, determine a branch of the logarithm so that $t - z = re^{i\theta}$, with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ for a fixed t satisfying $\mathrm{Re}(t - z) > 0$. Then $\cos 2\theta \geq \frac{1}{2}$ and $\left| \frac{d\sigma}{dr} \right| \leq c_1 < 2$ for $t \in \gamma$, where c_1 depends only on Γ.

(iv) For $z, z' \in \overline{\Omega}$ there is a path C from z to z' in $\overline{\Omega}$ such that

$$\left| \int_C t^s \, dt \right| \leq 5^{s-1} c_1 \max(|z|^s, |z'|^s) \, |z - z'|, \ s \geq 0,$$

and there is a path L from 0 to z in $\bar{\Omega}$ such that

$$(s+1)\left|\int\limits_L t^s\,dt\right| \leq c_1\,|z|^{s+1},\ s \geq 0.$$

This Lemma is proven in an elementary manner. To begin, use the parameterization

$$\gamma : z = (x_1 + iH(x_1))^{1/m},\ b' < x_1 < b'',\ \text{for suitable } b',\, b''.$$

LEMMA 3.2: Let $\varphi(z)$ be analytic in Ω and satisfy

$$\varphi(z) = \frac{1}{2\pi i}\int\limits_{\partial\Omega}\frac{1}{(t-z)}\,\varphi(t)\,dt,\ z \in \Omega, \text{ and}$$

$$|\varphi(t)| \leq A \text{ a.e. on } \partial\Omega.$$

Then there is a subdomain $\Omega' = \Omega \cap \{|z| < R'\}$ and a $c' > 0$ such that

$$|\varphi(z)| \leq c'A \text{ for } z \in \Omega'.$$

PROOF. Let $z = x + iy \in \gamma$ and choose $\varrho > 0$ so small that $z + i\varrho \in \Omega$. Then $z - i\varrho \notin \Omega$; hence,

$$\begin{aligned}\varphi(z+i\varrho) &= \frac{1}{2\pi i}\int\limits_{\partial\Omega}\left(\frac{1}{t-(z+i\varrho)} - \frac{1}{t-(z-i\varrho)}\right)\varphi(t)\,dt \\ &= \frac{1}{\pi}\int\limits_{\partial\Omega}\frac{\varrho}{(t-z)^2+\varrho^2}\,\varphi(t)\,dt.\end{aligned}$$

Choose a branch of $\log(t-z)$ so that $t - z = re^{i\theta}$, $t \in \gamma$, as in

LEMMA 3.1 (iii). Then

$$\frac{1}{2}r^2 + \varrho^2 \leq |(t-z)^2 + \varrho^2|,\ t \in \gamma.$$

Hence

$$\left|\frac{1}{\pi}\int\limits_\gamma\frac{\varrho}{(t-z)^2+\varrho^2}\,\varphi(t)\,dt\right| \leq \frac{A}{\pi}\,c_1\int\limits_{-\infty}^{\infty}\frac{\varrho}{\frac{1}{2}r^2+\varrho^2}\,dr \leq A\,c_1.$$

It is easy to see that there is an R', $0 < R' \leq R$, so that

$$\left| \frac{1}{\pi} \int\limits_{\partial\Omega - \gamma} \frac{\varrho}{(t-z)^2 + \varrho^2} \varphi(t)\, dt \right| \leq c_2 A, \; c_2$$

independent of φ, when $|z + i\varrho| \leq R$. Set $c' = \max(c_1, c_2)$.

LEMMA 3.3. Let $\varphi(z)$ be analytic in Ω and satisfy

$$\varphi(z) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{1}{t-z} \varphi(t)\, dt, \; z \in \Omega, \text{ and}$$

$$|\varphi(t)| \leq A\, |t|^k \text{ a. e. on } \partial\Omega, \; k > 0 \text{ an integer.}$$

Then there is a subdomain Ω' such that

$$|\varphi(z)| \leq c' A\, |z|^k \text{ for } z \in \Omega'.$$

PROOF: In view of the previous lemma, it suffices to show that

$$\frac{\varphi(z)}{z^k} = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{1}{t-z} \frac{\varphi(t)}{t^k}\, dt, \; z \in \Omega.$$

The domain Ω' and constant c' will then be those of the previous lemma. Expanding in partial fractions and integrating,

$$\frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{\varphi(t)}{t^k}\, dt = \frac{\varphi(z)}{z^k} - \frac{1}{2\pi i} \sum_1^k z^{j-k-1} \int\limits_{\partial\Omega} t^{-j} \varphi(t)\, dt, \; z \in \Omega.$$

Note that all the integrals in the expression above are convergent. We show that those on the right hand side vanish. For $t = |t|\, e^{i\theta} \in \gamma$, $|\sin\theta| \leq \frac{1}{2}$ by Lemma 3.1 (ii). Hence for $\delta > 0$,

$$|t + i\delta|^2 = |t|^2 - 2\sin\theta\, |t|\, \delta + \delta^2 \geq \frac{1}{2} |t|^2.$$

For $t \in \partial\Omega - \gamma$ and δ sufficiently small, we also have $|t + i\delta|^2 \geq \frac{1}{2} |t|^2$. Hence for $t \in \gamma$ and δ sufficiently small,

$$|t + i\delta|^{-j} \leq 2^{j/2} |t|^{-j}.$$

Therefore, the functions

$$(t + i\delta)^{-j} \varphi(t) \longrightarrow t^{-j} \varphi(t),$$

pointwise a. e. as $\delta \longrightarrow 0$, and

$$|(t + i\delta)^{-j} \varphi(t)| \leq 2^{-j/2} A, \quad \text{a. e.,} \quad 1 \leq j \leq k.$$

Hence, since $-i\delta \notin \Omega$,

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{\varphi(t)}{t^j} dt = \lim_{\delta \to 0} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{1}{(t + i\delta)^j} \varphi(t)\, dt = 0.$$

LEMMA 3.4. For $z \in \gamma$, $|f_j'(z)| \leq m c_\Gamma |z|^{m-1}$, a. e., $j = 2, 3$.

PROOF. With s as arc length of S on Γ and σ as arc length on γ,

$$\left(\frac{ds}{d\sigma}\right)^2 = (1 + \psi_2'(x_1)^2 + \psi_3'(x_1)^2)\left(\frac{\partial x_1}{\partial\sigma}\right)^2, \quad \text{a. e.} \quad z \in \gamma.$$

Since the isothermal relations hold a. e. on γ (cf. (3) and Corollary 1),

$$\left(\frac{ds}{d\sigma}\right)^2 = \sum_1^3 \left(\frac{\partial x_j}{\partial x}\right)^2 \quad \text{a. e. on } \gamma$$

$$= \sum_1^3 \left(\frac{\partial x_j}{\partial y}\right)^2 \quad \text{a. e. on } \gamma.$$

Hence, $|f_j'(z)|^2 = \left(\frac{\partial x_j}{\partial x}\right)^2 + \left(\frac{\partial x_j}{\partial y}\right)^2, \quad$ a. e. on γ

$$\leq 2\left(\frac{ds}{d\sigma}\right)^2, \quad \text{a. e. on } \gamma$$

$$\leq m^2 c_\Gamma^2 |z|^{2(m-1)}, \quad \text{a. e. on } \gamma.$$

LEMMA 3.5. There is a subdomain $\Omega' = \{z \in \Omega;\ |z| < R'\}$ such that $|f_j'(z)| \leq M|z|^{m-1}$, $z \in \Omega'$, $j = 2, 3$, where $M > 0$ is a constant.

PROOF. We first show that $f_j'(z)$ is (essentially) bounded on $\partial\Omega$. For $z \in \bar{\Omega}$, $|z| = R$,

$$f_j'(z) = m \frac{F_j'(\zeta)}{F_1'(\zeta)} F_1(\zeta)^{(m-1)/m}, \ \zeta \in \alpha_\lambda, \ \lambda = R^m, \tag{8}$$

Since we have assumed that α_λ satisfies the conclusions of Corollary 2, $|f_j'(z)| \leq m C_\lambda^2 R^{m-1} < \infty$ for $|z| = R$, $z \in \overline{\Omega}$, $j = 2, 3$. In view of Lemma 3.4, $f_j'(z)$ is essentially bounded on $\partial \Omega$.

The $f_j(z)$, $j = 2, 3$, are bounded analytic functions in Ω, continuous in $\overline{\Omega}$. Hence

$$f_j'(z) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{1}{(t-z)^2} f_j(t)\, dt, \quad z \in \Omega, \quad j = 2, 3.$$

Since $\int_{\partial \Omega} |f_j'(t)|\, |dt| < \infty$, $j = 2, 3$, by (8), the above expression may be integrated by parts to yield that

$$f_j'(z) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{1}{(t-z)} f_j'(t)\, dt, \quad z \in \Omega, \quad j = 2, 3.$$

The Lemma follows from Lemma 3.3.

§ 4. Boundary Singular Points.

Let S be a minimal surface with a rectifiable boundary ∂S. In this section, we show that when the arc $\Gamma \subset \partial S$ (cf. (4)) has a continuous tangent, certain singular points, defined below, form a discrete subset of Γ. The assumption that Γ be smooth is essential, as simple examples illustrate. Suppose, then, that ∂S contains the arc

$$\Gamma: x_2 = \psi_2(x_1), \quad x_3 = \psi_3(x_1), \quad -B < x_1 < B,$$

where the $\psi_j(x_1) \in C^1([-B, B])$ and $\psi_j(0) = \psi_j'(0) = 0$, $j = 2, 3$. Let $\beta(t)$ denote the modulus of continuity of the ψ_j' in $[-B, B]$.

To Γ we associate a right handed moving trihedral $e_k(P) = (a_{k1}, a_{k2}, a_{k3})$, $k = 1, 2, 3$, where $e_1(P)$ is the unit tangent vector to Γ at P and the functions $a_{kj} = a_{kj}(P)$ are continuous. Note that $\lim_{P \to 0} a_{kj}(P) = \delta_{kj}$, the Kronecker delta. For any fixed P with the distance $\overline{OP}$ sufficiently small, there is a subarc $\Gamma_P \subset \Gamma$ which contains the origin and C^1 functions $\varphi_j(y_1)$, $j = 2, 3$, so that

$$\Gamma_P: y_1 e_1(P) + \varphi_2(y_1)\, e_2(P) + \varphi_3(y_1)\, e_3(P) + P, \quad -l < y_1 < l.$$

The modulus of continuity of $\varphi_j'(y_1)$ is $\omega(t) = K\beta(Kt)$ for a constant $K \geq 1$, independent of P for $\overline{OP}$ sufficiently small. Moreover, for $\overline{OP}$ sufficiently small, the arcs Γ_P, $P \in \Gamma$, contain a common subarc Γ_0 with $0 \in \Gamma_0$.

Using the orthogonal matrix $A_P = (a_{kj}(P))$ we determine a conformal representation $\mathfrak{y}(\zeta) = (y_1(\zeta), y_2(\zeta), y_3(\zeta))$ by

$$\mathfrak{y}(\zeta) = A_P(\mathfrak{x}(\zeta) - P).$$

Suppose that $P = \mathfrak{x}(\xi_P)$. Let $G_j(\zeta) = y_j(\zeta) + iy_j^*(\zeta)$, $j = 1, 2, 3$, where $y_j^*(\zeta)$ is the harmonic conjugate to $y_j(\zeta)$ and $G_j(\xi_P) = 0$. Then

$$G_j(\zeta) = \sum_{k=1}^{3} a_{jk}(P)(F_k(\zeta) - F_k(\xi_P)), \quad j = 1, 2, 3. \tag{9}$$

The hypotheses of Theorem 1 are fulfilled for the conformal representation $\mathfrak{y}(\zeta)$ at $\zeta = \xi_P$. Hence, there is an open subset U_P of G, bounded by a segment $[a, b]$ of the $\operatorname{Im}\zeta = 0$ axis containing ξ_P and a Jordan arc in G joining b to a, and an odd integer $m_P > 0$ such that $w = G_1(\zeta)^{1/m_P}$ is a conformal map of U_P onto a domain Ω_P in the $w = u + iv$ plane. We refer to the parameter w defined in the domain Ω_P obtained in this way as the local parameter for S at P. We call P a « singular point of S on Γ », or briefly, « a singular point », if $m_P > 1$.

In the case where the tangent to Γ satisfies a Hölder condition at a singular point, the notion of singular point coincides with that of branch point for a minimal surface ([8] p. 234). This, and related questions not germane to the present study, will be developed elsewhere.

In our notation, we shall systematically suppress the dependence of the $a_{ij}(P)$ and other functions on P. We assume that $\sup \omega(t) < \frac{1}{32}$.

THEOREM 2. Let S be a minimal surface with rectifiable boundary ∂S. Suppose that ∂S contains the C^1 arc

$$\Gamma:\ x_2 = \varphi_2(x_1),\ x_3 = \varphi_3(x_1),\ -B < x_1 < B.$$

Then the set of singular points of S on Γ is discrete.

As an immediate

COROLLARY 3. Let S be a minimal surface with a C^1 boundary ∂S. If S has N interior branch points and M singular points on ∂S, then $M + N < \infty$.

PROOF OF THEOREM. We assume $\psi_j(0) = \psi_j'(0) = 0$, $j = 2, 3$. Let z be the local parameter at $0 \in \Gamma$ and $m = m_0$ be the integer determined by Theorem 1. Let $\Omega = \Omega_0$ be the parameter domain of z, $\gamma = \gamma_0$ the arc of $\partial\Omega$ whose image is a subarc of γ, and $f_j(z)$, $j = 1, 2, 3$, the analytic functions whose real parts form a conformal representation of S at 0. We shall show that for non-zero $z \in \gamma \cap \{|z| < R\}$, R sufficiently small, $\mathfrak{x}(z) = P \in \Gamma$ is not a singular point.

Let w be the local parameter at $P = \mathfrak{x}(z_P) \in \Gamma$. With $m_P = n$, there is a subset $U = \Omega \cap \{|z - z_P| < \varrho\}$ which is mapped conformally onto a subset $V = V_P \subset \Omega_P$ by

$$w = w(z) = G_1((F_1^{1/m})^{-1}(z))^{1/n}.$$

The mapping $w = w(z)$ is a topological map of $\overline{U}$ onto $\overline{V}$. According to (9),

$$(10)\qquad \begin{aligned} w(z)^n &= a_{11}(z^m - z_P^m) + a_{12}(f_2(z) - f_2(z_P)) + a_{13}(f_3(z) - f_3(z_P)) \\ g_j(w) &= a_{j1}(z^m - z_P^m) + a_{j2}(f_2(z) - f_2(z_P)) + a_{j3}(f_3(z) - f_3(z_P)) \end{aligned} \qquad w \in \overline{V},$$

for $j = 2, 3$; $a_{ij} = a_{ij}(P)$.

First we establish a limitation on $g_j'(w)w^{1-n}$ and $g_j(w)w^{-n}$ which does not depend on $P \in \Gamma$ for $\overline{OP}$ sufficiently small. By Lemma 5.3, there is an $M > 0$ so that $|f_j'(z)z^{1-m}| \leq M$ for $z \in \Omega' = \Omega \cap \{|z| < R'\}$. Since the matrix A_P is orthogonal and $\lim_{P \to 0} a_{ij}(P) = \delta_{ij}$, we may choose $p > 0$ so small that $\overline{PO} < p$ implies

$$P = \mathfrak{x}(z_P) \quad \text{with} \quad z_P \in \gamma \cap \overline{\Omega}'$$

$$(11)\qquad a_{11} \geq \frac{3}{4}$$

$$\sqrt{2}\sqrt{1 - a_{11}^2}\, c_1 \left(4\frac{M}{m} + 2\right) < \frac{1}{4},$$

where $c_1 \geq 1$ is the constant from Lemma 3.1 which depends only on Γ. Therefore, for $\overline{PO} < p$ and $z \in \overline{\Omega}'$,

$$\left| a_{11} + \frac{a_{12}}{m} f_2'(z) z^{1-m} + \frac{a_{13}}{m} f_3'(z) z^{1-m} \right| \geq$$

$$a_{11} - \left(\left| a_{12} \frac{f_2'(z)}{mz^{m-1}} \right| + \left| a_{13} \frac{f_3'(z)}{mz^{m-1}} \right| \right) \geq$$

$$a_{11} - \sqrt{1 - a_{11}^2} \sqrt{2} \frac{M}{m} > \frac{1}{2} .$$

Therefore, by differentiating $g_j(w)$ and employing (10),

$$\text{(12)} \quad \frac{g_j'(w)}{nw^{n-1}} = \frac{ma_{j1} z^{m-1} + a_{j2} f_2'(z) + a_{j3} f_3'(z)}{ma_{11} z^{m-1} + a_{12} f_2'(z) + a_{13} f_3'(z)} \text{ for } w = w(z) \in \overline{V}, \qquad j = 2, 3.$$

$$\text{(13)} \quad \left| \frac{g_j'(w)}{nw^{n-1}} \right| \leq 2 \left(|a_{j1}| + |a_{j2}| \frac{M}{m} + |a_{j3}| \frac{M}{m} \right) \leq \left(2 + 4 \frac{M}{m} \right), \ w \in \overline{V}.$$

The inequality above and (12) are valid except for a set of measure zero in ∂V. By Lemma 3.1 (iv),

$$| g_j(w) | \leq 2c_1 \left(1 + 2 \frac{M}{m} \right) | w |^n , \ w \in \overline{V}, \qquad j = 2, 3.$$

The remainder of the proof is a simple examination of the equation, implied by (10),

$$z^m - z_P^m = a_{11} w^n + a_{21} g_2(w) + a_{31} g_3(w), \qquad w = w(z) \in \overline{V}.$$

For $| z - z_P |$ small and $z \neq z_P$, $z^m - z_P^m \neq 0$. Hence a single valued branch of $\log (z^m - z_P^m)$ may be defined for $0 < | w | < r$, $w \in \overline{V}$, with r sufficiently small. Given $\varepsilon > 0$, choose points $w' = w(z')$, $w'' = w(z'')$ in γ_P with $0 < | w' |, | w'' | < r$ so that

$$| \arg w' | < \frac{\pi}{4n} \varepsilon, \ | \arg w'' - \pi | < \frac{\pi}{4n} \varepsilon \text{ and}$$

$$(m - 1) | \arg z' - \arg z_P | < \varepsilon\pi/4, \ (m - 1) | \arg z'' - \arg z_P | < \varepsilon\pi/4.$$

Let C be a path from w' to w'' in $V \cap \{0 < | w | < r\}$. Then the variation of the argument of $z^m - z_P^m$ on C,

$$\Delta_C \arg (z^m - z_P^m) = \Delta_C \arg (a_{11} w^n + a_{21} g_2(w) + a_{31} g_3(w))$$

$$= \Delta_C \arg w^n + \Delta_C \arg (a_{11} + a_{21} g_2(w) w^{-n} + a_{31} g_3(w) w^{-n}).$$

When $\overline{OP} < p$, $w \in \bar{V}$,

$$\operatorname{Re}(a_{11} + a_{21}\, g_2(w)\, w^{-n} + a_{31}\, g_3(w)\, w^{-n}) \geq$$

$$a_{11} - \sqrt{2}\sqrt{1 - a_{11}^2}\, c_1 \left(2 + 4\,\frac{M}{m}\right) > \frac{1}{2},$$

and $|\operatorname{Im}(a_{11} + a_{21}\, g_2(w)\, w^{-n} + a_{31}\, g_3(w) w^{-n})| < \frac{1}{4}$. Hence

$$|\Delta_O \arg (a_{11} + a_{21}\, g_2(w)\, w^{-n} + a_{31}\, g_3(w) w^{-n})| < 2 \operatorname{arc\,tan} \frac{1}{2} < \frac{\pi}{2}.$$

Therefore, $\Delta_O \arg (z^m - z_P^m) > \left(n - \frac{1}{2} - \frac{\varepsilon}{2}\right)\pi$.

Since $|h'(x)| \leq \frac{1}{4} < \tan \frac{\pi}{8}$, by Lemma 3.1, the complement of Ω contains a sector with vertex at z_P and central angle $q \geq \frac{3}{4}\pi$.

Therefore

$$\Delta_O \arg (z^m - z_P^m) = \Delta_O \arg (z - z_P) + \Delta_O \arg \sum_0^{m-1} \left(\frac{z}{z_P}\right)^k$$

$$< \frac{5}{4}\pi + (m-1)|\arg z' - \arg z_P| + (m-1)|\arg z'' - \arg z_P|$$

$$< \frac{5}{4}\pi + \frac{\varepsilon}{2}\pi.$$

Hence $n < \frac{7}{4} + \varepsilon < 2$. Therefore $n = 1$. Q. E. D.

COROLLARY 4. Let S be a minimal surface whose boundary contains the C^1 arc

$$\Gamma: x_2 = \psi_2(x_1),\ x_3 = \psi_3(x_1),\ -B < x_1 < B,\ \psi_j(0) = \psi_j'(0) = 0,\ j = 2, 3.$$

(i) There is a $p > 0$ such that if $0 < \overline{OP} < p$ for $P \in \Gamma$, then P is not a singular point of S on Γ.

(ii) Let $f_j(z)$ and $g_j(w)$ be the conformal representations for S at 0 and P, $\overline{OP} < p$, respectively. Suppose that $w = w(z)$ is a conformal map of $U \subset \Omega_0$ onto $V \subset \Omega_P$. Then for $w \in V$ and almost all $w \in \partial V$,

$$| g_j'(w) | \leq 2 \left(1 + 2 \frac{M}{m}\right), j = 2, 3 \,;\, M = \sup_{\overline{U}} | f_k'(z) z^{1-m} |,\ k = 2, 3.$$

PROOF. This follows from (11) and (13).

§ 5. Hölder Continuity of the First Derivatives.

The Hölder continuity is shown initially in the domain of the local parameter with a diminished exponent. By applying Kellogg's Theorem ([11], p. 361), the diminished exponent is replaced by the original one. Using Kellogg's Theorem once more, we prove Hölder continuity of the original conformal representation. The main result is Theorem 4.

Throughout, we consider a minimal surface S whose boundary ∂S contains the arc

$$\Gamma: x_2 = \psi_2(x_1),\ x_3 = \psi_3(x_1),\ -B < x_1 < B,$$

where $\psi_j(x_1) \in C^{1,\alpha}([-B, B]),\ 0 < \alpha < 1$; and $\psi_j(0) = \psi_j'(0) = 0, j = 2, 3$. We assume that the Hölder constant of the functions $\varphi_j'(y_1), j = 2, 3$, the derivatives of the functions defining the arc Γ_P (cf. § 4), is the same as the Hölder constant for the $\psi_j'(x_1)$, $j = 2, 3$.

As a shorthand, we refer to the analytic functions $f_j(z)$ (or $g_j(w)$), $j = 1, 2, 3$, whose real parts form the conformal representation for S at $0 \in \Gamma$ (or $P \in \Gamma$) as « the conformal representation of S at 0 (or P) ».

Let z denote the local parameter at $0 \in \Gamma$, Ω the domain of z, γ the preimage of the arc of Γ in $\partial\Omega$, and $f_j(z), j = 1, 2, 3$, the conformal representation of S near 0. Let $m > 0$ be the integer determined by Theorem 1.

LEMMA 5.1. Let σ denote the arc length of γ. Then

$$\frac{dz}{d\sigma} = \left\{\frac{m + \bar{z}^{1-m} \sum_2^3 \psi_j'(x_1) \overline{f_j'(z)}}{m + z^{1-m} \sum_2^3 \psi_j'(x_1) f_j'(z)}\right\}^{1/2} \left(\frac{|z|}{z}\right)^{m-1}, \quad \text{a. e. for } z \in \gamma.$$

PROOF. For $z \in \gamma$, there is an absolutely continuous $H(x_1)$ such that $f_1(z) = x_1(z) + iH(x_1(z))$ by Lemma 2.1. Hence $\gamma : z = (x_1 + iH(x_1))^{1/m}$,

$b' < x_1 < b''$, for suitable b', b''. Therefore

$$\frac{dz}{dx_1} = \frac{1}{m}(1 + iH'(x_1))\, z^{1-m}, \qquad \text{a. e. } z \in \gamma.$$

By (3) and the Corollary to Theorem 1, the arc length s of S on Γ satisfies $\left(\frac{ds}{d\sigma}\right)^2 \neq 0$ a. e. on γ. It follows that $\frac{\partial x_1}{\partial \sigma} \neq 0$ a. e. on γ. From the isothermal relations, we obtain that

$$\frac{\partial x_1}{\partial \sigma} = \frac{f_1'(z) + \sum_2^3 \psi_j'(x_1) f_j'(z)}{1 + \psi_2'(x_1)^2 + \psi_3'(x_1)^2} \frac{dz}{d\sigma}, \quad \text{a. e. on } \gamma.$$

For this formulation, refer to [5]. With $f_1'(z) = mz^{m-1}$,

$$\frac{1}{1 + iH'(x_1)} = \frac{1 + \frac{1}{m} z^{1-m} \Sigma\, \psi_j'(x_1) f_j'(z)}{1 + \Sigma\, \psi_j'(x_1)^2}, \quad \text{a. e. on } \gamma.$$

Hence

$$\frac{dz}{d\sigma} = \left(\frac{dz}{dx_1}\right) \left| \frac{dz}{dx_1} \right|^{-1} = \left\{ \frac{dz}{dx_1} \left(\overline{\frac{dz}{dx_1}}\right)^{-1} \right\}^{1/2}, \quad \text{a. e. on } \gamma$$

By Lemma 3.1, $\left| \frac{dz}{d\sigma} - 1 \right| < \frac{1}{2}$ a. e. on γ; so there is a branch of the square root such that

$$\frac{dz}{d\sigma} = \left\{ \frac{m + \bar{z}^{1-m} \sum_2^3 \psi_j'(x_1) \overline{f_j'(z)}}{m + z^{1-m} \sum_2^3 \psi_j'(x_1) f_j'(z)} \right\}^{1/2} \left(\frac{|z|}{z}\right)^{m-1}, \text{ a. e. on } \gamma.$$

Lemma 5.2. There are $\varepsilon > 0$ and $\lambda_0 > 0$ such that when $\overline{OP} < \varepsilon$ and $\lambda < \lambda_0$,

(i) $w(z) = a_{11}(z^m - z_P{}^m) + a_{12}(f_2(z) - f_2(z_P)) + a_{13}(f_3(z) - f_3(z_P))$ is a conformal map of

$$U_\lambda = \{ z : -\lambda < \arg z < \lambda, |z| < r \} \cap \Omega$$

onto a subset V_P of Ω_P, where r is independent of P.

(ii) For $z, z' \in U_\lambda$, $|w(z) - w(z')| \leq \left(m + \frac{1}{4}\right) r^{m-1} |z - z'|$.

(iii) $f_j'(z)\, z^{1-m}, j = 2, 3,$ are bounded in $\bar{\Omega} \cap \{ |z| \leq r \}$.

PROOF. We remark that for $r > 0$ and $\lambda < \dfrac{\pi}{4m}$,

$$| z^m - z'^m | \geq \max\ (| z |^{m-1}, | z' |^{m-1}) \, | z - z' |, z, z' \in U_\lambda .$$

For $z \in \Omega$,

$$w'(z) = m a_{11} z^{m-1} + a_{12} f_2'(z) + a_{13} f_3'(z) =$$

$$= m\, (a_{11} + B(z))\, z^{m-1} .$$

Choose $\varepsilon, r > 0$ so small that

$$m c_1\, 5^{m-1} \sup_{U_\lambda} | B(z) | < \frac{1}{4} \text{ and } a_{11} > \frac{3}{4} .$$

Hence (iii) is satisfied. Now for $z, z' \in U_\lambda$,

$$w(z) - w(z') = \int_{z'}^{z} m\, (a_{11} + B(t))\, t^{m-1}\, dt$$

$$| w(z) - w(z') | \geq \Big| \int_{z'}^{z} m a_{11}\, t^{m-1}\, dt \Big| - \int_C m\, | B(t) | \, | t |^{m-1}\, | dt |,$$

where C is the path from z' to z determined in Lemma 3.1 (iv).

$$\int_C m\, | B(t) | \, | t |^{m-1} | dt | \leq m c_1\, 5^{m-1} \max (| z |^{m-1}, | z' |^{m-1}) \, | z - z' |$$

$$\leq \frac{1}{4} \max (| z |^{m-1}, | z' |^{m-1}) \, | z - z' |.$$

Hence,

$$| w(z) - w(z') | \geq a_{11} \, | z^m - z'^m | - \frac{1}{4} \max\ (| z |^{m-1}, | z' |^{m-1}) \, | z - z' |$$

$$> \frac{1}{2} (| z |^{m-1}, | z' |^{m-1}) \, | z - z' |.$$

Therefore $w(z)$ is univalent in U_λ. In addition,

$$|w(z) - w(z')| \leq \left(ma_{11} + \frac{1}{4}\right) \max\left(|z|^{m-1}, |z'|^{m-1}\right)|z - z'|$$

$$\leq \left(m + \frac{1}{4}\right) r^{m-1} |z - z'|.$$

An analogous conclusion holds for a set

$$U'_\lambda = \{z : \pi - \lambda < \arg z < \pi + \lambda,\ |z| < r'\} \cap \Omega.$$

Suppose that $r = \min(r, r')$ and let $\Omega' = \Omega \cap \{|z| < r\}$.

The estimate below is a well known fact in potential theory. We present a proof because conclusion occurs in an unusual form. For efficiency, we introduce the notations

$$D(\varrho, q) = \{\zeta : 0 < \arg \zeta < q, |\zeta| < \varrho\}, \text{ where } \frac{\pi}{2} < q \leq \pi \text{ and } 0 < \varrho \leq 1$$

$$\Delta(\xi, \varrho, \delta) = \{\zeta : \delta < \arg(\zeta - \xi) < \pi - \delta, |\zeta| < \varrho\} \subset D(\varrho, q), \text{ where } \pi - q < \delta < \frac{\pi}{2} \text{ and } 0 \leq \varrho, \xi \leq 1.$$

LEMMA 5.3. Let $\varphi(\zeta) = u(\zeta) + iv(\zeta)$ be bounded and analytic in $D(\varrho_1, q)$. Suppose that for a fixed ξ_0, $0 \leq \xi_0 \leq \varrho_2 < \varrho_1$, and $0 < \beta < 1$

$$|u(t)| \leq C|t - \xi_0|^\beta, \text{ a. e. } t \in \partial D(\varrho_1, q).$$

Then

$$|\varphi(\zeta) - \varphi(\zeta')| \leq cC|\zeta - \zeta'|^\beta, \zeta, \zeta' \in \bar{\Delta}(\xi_0, \varrho, \delta); \varrho \leq \varrho_2.$$

The constant c depends on $\beta, \delta, q, \varrho_1 - \varrho_2$.

PROOF. With τ chosen so that $0 < \tau < \pi(1 - \beta)$, $\sin \beta(\theta + \tau) \geq \frac{1}{M} > 0$ for $0 \leq \theta \leq \pi$. Therefore the harmonic function $w(\zeta) = |\zeta - \xi_0|^\beta \sin \beta(\theta + \tau)$, $\theta = \arg \zeta$, is positive on $\partial D(\varrho_1, q)$ except at $\zeta = \xi_0$. By hypothesis,

$$-MC\,w(t) \leq u(t) \leq MC\,w(t), \text{ a. e. } t \in \partial D(\varrho_1, q).$$

Integrating this inequality with respect to the Poisson kernal of $D(\varrho_1, q)$, we find that

$$- MC\,w(\zeta) \leq u(\zeta) \leq MC\,w(\zeta),\ \zeta \in D(\varrho_1, q).$$

Hence,

$$|u(\zeta)| \leq MC\,|\zeta - \xi_0|^\beta, \quad \zeta \in D(\varrho_1, q).$$

When $\zeta \in \Delta(\xi_0, \varrho_2, \delta)$, $D(\varrho_1, q)$ contains the disc

$$|\zeta - t| \leq r = |\zeta - \xi_0| \left| \cos \frac{\delta + q}{2} \right|.$$

Since by Green's Theorem,

$$|u_\xi(\zeta)| \leq \frac{2}{r} \sup |u(\zeta + re^{i\theta})|$$

we have that

$$|u_\xi(\zeta)| \leq \frac{2\left(1 + \left|\cos \dfrac{\delta + q}{2}\right|\right)^\beta}{\left|\cos \dfrac{\delta + q}{2}\right|} MC\,|\zeta - \xi_0|^{\beta - 1},\ \zeta \in \Delta(\xi_0, \varrho_2, \delta).$$

The same estimate holds for $u_\eta(\zeta)$ and therefore $|\varphi'(\zeta)| \leq c_3\,C\,|\zeta - \xi_0|^{\beta-1}$, $\zeta \in \Delta(\xi_0\,\varrho_2, \delta)$. Although there is a general method for concluding that $\varphi \in C^{0,\beta}(\overline{\Delta}(\xi_0, \varrho_2, \delta))$, (cf. [6], pp. 51-53), since ζ is restricted to a sector a direct integration is possible. For $\zeta, \zeta' \in \overline{\Delta}(\xi_0, \varrho_2, \delta)$, let $L: t = \zeta + s(\zeta' - \zeta)$, $0 \leq s \leq 1$, be the line segment between them. Since $t \in \overline{\Delta}$, $\delta \leq \arg(t - \xi_0) = \theta \leq \pi - \delta$. We define a branch of $(t - \xi_0)^\beta$, Hölder continuous in $\overline{\Delta}$, by $(t - \xi_0)^\beta = |t - \xi_0|^\beta\, e^{i\beta\theta}$ for a fixed t. Consequently,

$$\begin{aligned}
\frac{1}{\beta}\, C_\beta\, |\zeta - \zeta'|^\beta &\geq \frac{1}{\beta} |(\zeta' - \xi_0)^\beta - (\zeta - \xi_0)^\beta| \\
&= \left| \int_L (t - \xi_0)^{\beta - 1}\, dt \right| \\
&\geq \left| Im\ e^{-ia} \int_L (t - \xi_0)^{\beta-1}\, dt \right|,\ a = \arg(\zeta' - \zeta)
\end{aligned}$$

$$= \int_L | t - \xi_0 |^{\beta-1} \sin (1 - \beta) \theta \, | dt |$$

$$\geq \sin (1 - \beta) \delta \int_L | t - \xi_0 |^{\beta-1} | dt |.$$

Here, C_β is the Hölder constant of $(t - \xi_0)^\beta$. The lemma follows.

LEMMA 5.4. Let $z = F(\zeta)$ map $D = \{\operatorname{Im} \zeta > 0, |\zeta| < 1\}$ onto $\Omega' = \{ |z| < r\} \cap \Omega$ with $F(0) = 0$ and $\gamma \cap \bar{\Omega}'$ the image of $[-1, 1]$. Then F and F^{-1} are Hölder continuous.

PROOF. Here, r is the radius from Lemma 5.2. This lemma is a consequence of Warschawski [13] and [12], Lemma 1, once a certain chord-arc length ratio for $\partial\Omega'$ is established. We must show that there is a constant $K > 0$ satisfyng

(14) $$\Delta\sigma \leq K | z - z' |, z, z' \in \partial\Omega',$$

where $\Delta\sigma$ denotes the length of the shorter arc of $\partial\Omega'$ between z and z'. Since $\gamma : z = x + ih(x)$, with $|h'(x)| < \tan \frac{\pi}{8}$, for $z, z' \in \gamma$,

$$l \leq \sec \frac{\pi}{8} | z - z' |, \quad l = \text{length of the arc } \widehat{zz'} \text{ of } \gamma.$$

For two points z, z' on the arc $|z| = r$, the condition (14) is obvious. So, let $z \in \gamma \cap \bar{\Omega}', |z'| = r$, and a be the end point of $\gamma \cap \bar{\Omega}'$ closest to z'. Then the length L of $\widehat{zaz'}$ in $\partial\Omega'$ satisfies

$$L \leq \sec \frac{\pi}{8} | z - a | + 2 \left(r \sin \frac{\pi}{8} \right)^{-1} | z' - a |.$$

Also,

$$| z - a |, | z' - a | \leq \left(\sin \frac{3}{8} \pi \right)^{-1} | z - z' |.$$

Hence

$$\Delta\sigma \leq L \leq K | z - z' |.$$

LEMMA 5.5. Let $D_\varrho = \{\zeta \in D, |\zeta| < \varrho\}$ and $\Omega_\varrho = F(D_\varrho)$. Let $\nu = \nu(\varrho)$ and $\nu' = \nu'(\varrho)$ denote the Hölder exponents of $F | \bar{D}_\varrho$, the restriction of

F to $\overline{D}_\varrho$, and $F^{-1} \mid \overline{\Omega}_\varrho$. Then for each $\varrho_1 \leq 1$ there is a $\varrho \leq \varrho_1$ such that

$$f_j'(z)\, z^{1-m} \in C^{0,\varkappa}(\overline{\Omega}_\varrho),\ j = 2, 3, \text{ and } \gamma \cap \overline{\Omega}_\varrho \text{ is a } C^{1,\varkappa} \text{ arc}$$

$$\text{with } \varkappa = \alpha\, \nu(\varrho_1)\, \nu'(\varrho_1).$$

PROOF. We note first that the existence of a tangent to γ at $z = 0$ implies that $F(\zeta)$ is conformal at $\zeta = 0$ by a theorem of Lindelöf. Consequently, the image of

$$U_\lambda = \{z : -\lambda < \arg z < \lambda, \quad |z| < r, z \in \Omega\}$$

under $\zeta = F^{-1}(z)$ contains a sector

$$W = \{\zeta \in D : 0 < \arg \zeta < \mu\} \text{ for some } \mu > 0.$$

An analogous statement holds for the set $U_\lambda' = \{z : \pi - \lambda < \arg z < \pi + \lambda, |z| < r, z \in \Omega\}$.

We derive an estimate for the local parameter representations of S. With σ as arc length on γ,

$$x_j(x, y) = \psi_j(x_1(x, y)), \quad z = x + iy \in \gamma, \qquad j = 2, 3,$$

$$\left(\frac{\partial x_j}{\partial x} + \frac{\partial x_j}{\partial y} h'(x)\right) \frac{dx}{d\sigma} = \psi_j'(x_1)\left(\frac{\partial x_1}{\partial x} + \frac{\partial x_1}{\partial y} h'(x)\right) \frac{dx}{d\sigma}, \quad \text{a. e.} \quad \text{on} \quad \gamma.$$

Since $\dfrac{dx}{d\sigma} \neq 0$ a. e, on γ,

$$\frac{\partial x_j}{\partial x} + \frac{\partial x_j}{\partial y} h'(x) = \psi_j'(x_1)\left(\frac{\partial x_1}{\partial x} + \frac{\partial x_1}{\partial y} h'(x)\right), \quad \text{a. e.} \quad \text{on} \quad \gamma,$$

$$\left|\frac{\partial x_j}{\partial x}\right| \leq \frac{3}{2} |\psi_j'(x_1)|\, |f_1'(z)| + |h'(x)|\, |f_j'(z)|, \quad \text{a. e} \quad \text{on} \quad \gamma$$

$$\leq \frac{3}{2} m\, |z|^{m-1} \omega(|x_1|) + m c_\Gamma^2 |z|^{m-1} \omega(|x_1|), \quad \text{a. e.} \quad \text{on} \quad \gamma$$

by Lemmas 3.1 and 3.4. Therefore, with A as the Hölder constant of the tangent to Γ,

$$\left|\frac{\partial x_j}{\partial x}\right| \leq \left(\frac{3}{2} + c_\Gamma^2\right) m\, A\, |z|^{m-1} |z|^{m\alpha} \quad \text{a. e.} \quad \text{on} \quad \gamma.$$

Hence

$$\operatorname{Re} z^{1-m} f_j'(z) = \frac{\partial x_j}{\partial x}(z) \,|\, z\,|^{1-m} \cos(m-1)\,\theta + \frac{\partial x_j}{\partial y}(z) \,|\, z\,|^{1-m} \sin(m-1)\theta,$$

(15) $$|\operatorname{Re} z^{1-m} f_j'(z)| \leq \left(\frac{3}{2} + 2c_\Gamma^2\right) mA \,|\, z\,|^{m\alpha}, \quad \text{a.e.} \quad \text{on} \quad \gamma, \quad j = 2, 3.$$

This estimate also holds if $\alpha = 1$.

Let w denote the local parameter at $P = \mathfrak{r}(z_P)$, $|z_P| < r$, and $g_j(w)$, $j = 1, 2, 3$, be the conformal representation of S at P. Since A, that is $\omega(t)$, does not depend on P, the conclusion of the computation above may be applied to $g_j(w)$. Consequently,

(15′) $$|\operatorname{Re} g_j'(w)| \leq \left(\frac{3}{2} + 2c_\Gamma^2\right) A \,|\, w\,|^{\alpha}, \quad \text{a.e} \quad \text{on} \quad \gamma_P\,.$$

At this point, the analysis is transformed to D. Set $Y_j(\zeta) = z^{1-m} f_j'(z)$, $z = F(\zeta)$, $z \in \Omega'$, $j = 2, 3$, and

$$h_j(\zeta) = h_j(\zeta, P) = \frac{a_{j1} + \frac{1}{m} a_{j2}\, Y_2(\zeta) + \frac{1}{m} a_{j3}\, Y_3(\zeta)}{a_{11} + \frac{1}{m} a_{12}\, Y_2(\zeta) + \frac{1}{m} a_{13}\, Y_3(\zeta)},$$

$$a_{jk} = a_{jk}(P), \qquad j = 2, 3.$$

By (15),

$$|\operatorname{Re} Y_j(\xi)| \leq C_1 \,|\, \xi\,|^{\beta}, \quad \text{a.e} \quad -\varrho_1 \leq \xi \leq \varrho_1\,, \text{ with } \beta = \alpha\nu \text{ and } C_1$$

a positive constant, $j = 2, 3$. This estimate is easily extended to the semicircle $|\zeta| = \varrho_1$, $\operatorname{Im} \zeta > 0$ since $Y_j(\zeta)$ is bounded. Namely, for a $C_2 \geq C_1$,

$$|\operatorname{Re} Y_j(t)| \leq C_2 \,|\, t\,|^{\beta}, \quad \text{a.e.} \quad t \in \partial D\varrho_1\,, \qquad j = 2, 3.$$

So by Lemma 5.3, $Y_j(\zeta) \in C^{0,\beta}(\overline{\Delta}(0, \varrho_2, \pi - q))$, $j = 2, 3$, for some ϱ_2, $0 < \varrho_2 < \varrho_1$, and any q, $\frac{\pi}{2} < q < \pi$. We choose q and δ satisfying $\pi - q < \delta < \mu$ and ϱ, $\varrho < \varrho_2 < \varrho_1$. Let $\xi_P = F^{-1}(z_P)$ where $\mathfrak{r}(z_P) = P$. In view of (12), $h_j(\xi) = g_j'(w)$ a.e. for $0 \leq \xi \leq 1$. Using (15′) and Lemma 5.2 (ii), we obtain that for $0 \leq \xi_P \leq \varrho$,

$$|\operatorname{Re} h_j(\xi)| \leq C_3 \,|\, \xi - \xi_P\,|^{\beta}, \quad \text{a.e.} \quad 0 \leq \xi \leq \varrho_2\,, \qquad j = 2, 3,$$

with C_3 independent of P. Since $Y_j(\zeta) \in C^{0,\beta}(\bar{\Delta}(0, \varrho_2, q))$,

$$|\operatorname{Re} h_j(t)| \leq C_4 |t - \xi_P|^\beta, \quad t = |t| e^{iq}, \quad 0 \leq |t| \leq \varrho_2,$$

with C_4 independent of P. Consequently, for some $C_5 > 0$, independent of P,

$$|\operatorname{Re} h_j(t)| \leq C_5 |t - \xi_P|^\beta, \quad \text{a. e. } t \in \partial D(\varrho_2, q).$$

Applying Lemma 5.3 again, $h_j(\zeta) \in C^{0,\beta}(\bar{\Delta}(\xi_P, \varrho, \delta))$, with Hölder constant independent of P.

In the sector W, the valid relations

$$\frac{1}{m} Y_j(\zeta) = \frac{a_{1j} + a_{2j} h_2(\zeta, P) + a_{3j} h_3(\zeta, P)}{a_{11} + a_{21} h_2(\zeta, P) + a_{31} h_3(\zeta, P)}, \qquad a_{kj} = a_{kj}(P), \qquad j = 2, 3,$$

imply that $Y_j(\zeta) \in C^{0,\beta}(\bar{\Delta}(\xi_P, \varrho, \delta) \cap \bar{W})$, $j = 2, 3$, for each ξ_P, $0 \leq \xi_P \leq \varrho$, but with Hölder constant independent of P. Since $\pi - q < \delta < \mu$, it is already known that $Y_j(\zeta) \in C^{0,\beta}(\bar{\Delta}(0, \varrho, \mu))$, $j = 2, 3$, and hence $Y_j(\zeta) \in C^{0,\beta}(\bar{\Delta}(\xi_P, \varrho, \delta))$, with Hölder constant independent of P. It follows in an elementary way that $Y_j(\zeta) \in C^{0,\beta}(\bar{D}(\varrho, q))$, $j = 2, 3$. Using the same argument, $Y_j(\zeta) \in C^{0,\beta}(\bar{D}'(\varrho, q))$, $D'(\varrho, q) = \{\zeta \in D : q < \arg \zeta < \pi, \ |\zeta| < \varrho\}$, for q, $0 < q < \frac{\pi}{2}$. Hence $Y_j(\zeta) \in C^{0,\beta}(\bar{D}_\varrho)$, $j = 2, 3$.

Therefore, $f_j'(z) z^{1-m} \in C^{0,\varkappa}(\bar{\Omega}_\varrho)$, $\Omega_\varrho = F(D_\varrho)$, $\varkappa = \beta\nu' = \alpha\nu\nu'$. By Lemma 5.1, $\gamma \cap \bar{\Omega}_\varrho$ is a $C^{1,\varkappa}$ curve.

In the proofs of the next two theorems, we shall employ a well-known, and easily demonstrated, local variant of Kellogg's Theorem: Let G be a simply connected domain whose boundary contains an open $C^{1,\varkappa}$ arc L and let F map D_ϱ conformally onto G so that $(-\varrho, \varrho)$ corresponds to L. Then for each ξ, $-\varrho < \xi < \varrho$, there is a subdomain $V = \{\zeta \in \bar{D}_\varrho ; |\zeta - \xi| \leq \varepsilon\}$ such that $F \in C^{1,\varkappa}(V)$ and $F^{-1} \in C^{1,\varkappa}(F(V))$. Here, as usual, $D_\varrho = \{|\zeta| < \varrho, \operatorname{Im} \zeta > 0\}$.

THEOREM 3: Let S be a minimal surface whose boundary contains the arc

$$\Gamma : x_2 = \psi_2(x_1), \qquad x_3 = \psi_3(x_1), \qquad -B < x_1 < B,$$

where $\psi_j(x_1) \in C^{1,\alpha}([-B, B])$, $0 < \alpha < 1$; $\psi_j(0) = \psi_j'(0) = 0$, $j = 2, 3$. Let z be the local parameter of S at $0 \in \Gamma$, Ω the domain of z, and $f_j(z)$, $j = 1, 2, 3$, the conformal representation for S at 0. Let m be the integer determined by Theorem 1 and γ the arc of $\partial\Omega$ whose image is in Γ.

Then there is a subdomain $\Omega'' \subset \Omega$ such that $f_j'(z) z^{1-m} \in C^{0,\alpha}(\bar{\Omega}'')$, $j = 1, 2, 3$, and $\bar{\Omega}'' \cap \gamma$ is a $C^{1,\alpha}$ arc.

Proof. Let $z = F(\zeta)$ map D onto $\Omega' = \{z \in \Omega, |z| < r\}$, as in Lemma 5.5. According to this lemma, there is a $\varrho_1 < 1$ such that $\gamma \cap \bar{\Omega}_{\varrho_1}$ is a $C^{1,\varkappa}$ arc. Hence, by the local invariant of Kellogg's Theorem, there is a $\varrho_2 \leq \varrho_1$ such that $F \in C^{1,\varkappa}(\bar{D}_{\varrho_2})$ and $F^{-1} \in C^{1,\varkappa}(\bar{\Omega}_{\varrho_2})$. In particular, F and F^{-1} are Lipschitz in these domains, so that $\nu(\varrho_2) = \nu'(\varrho_2) = 1$. Applying Lemma 5.5 once more, there exists a $\varrho < \varrho_2$ such that $f_j'(z) z^{1-m} \in C^{0,\alpha}(\bar{\Omega}_\varrho)$. By Lemma 5.1, $\gamma \cap \bar{\Omega}_\varrho$ is a $C^{1,\alpha}$ arc. The theorem is proved with $\Omega'' = \Omega_\varrho$.

Theorem 4. Let S a minimal surface whose boundary contains arc

$$\Gamma: x_2 = \psi_2(x_1),\ x_3 = \psi_3(x_1),\ -B < x_1 < B,$$

where

$$\psi_j(x_1) \in C^{1,\alpha}([-B, B]),\ 0 < \alpha < 1;\ \psi_j(0) = \psi_j'(0) = 0, \qquad j = 2, 3.$$

Let $F_j(\zeta)$ be a conformal representation of S for $\operatorname{Im} \zeta > 0$ with $F_j(0) = 0$, $j = 1, 2, 3$.

Then there is a subdomain $N = \{|\zeta| \leq R, \operatorname{Im} \zeta \geq 0\}$ such that $F_j(\zeta) \in C^{1,\alpha}(N)$, $j = 1, 2, 3$.

Proof. We are assuming, of course, that the image of an interval $a \leq \xi \leq b$ containing $\zeta = 0$ is monotonely mapped onto a subset of Γ containing the origin by the conformal representation $\mathfrak{r}(\zeta) = \operatorname{Re}(F_1, F_2, F_3)$.

By Theorem 3, $\gamma \cap \bar{\Omega}''$ is a $C^{1,\alpha}$ arc. Hence by the local variant of Kellogg's Theorem, there is a neighborhood $N = \{|\zeta| \leq R, \operatorname{Im} \zeta \geq 0\}$ such that the conformal map $z = F_1(\zeta)^{1/m}$ from N to $F_1(N)^{1/m}$ is of class $C^{1,\alpha}$ in N. Here m is the integer determined by Theorem 1. So

$$\frac{dz}{d\zeta} = \frac{1}{m} F_1(\zeta)^{\frac{1}{m}-1} F_1'(\zeta) \in C^{0,\alpha}(N).$$

It follows that $F_1'(\zeta) \in C^{0,\alpha}(N)$ since $F_1(\zeta)^{1/m}$ is Lipschitz. Now, for $j = 2, 3$,

$$F_j'(\zeta) = f_j'(z) \frac{dz}{d\zeta} = \frac{1}{m} f_j'(z) z^{1-m} F_1'(\zeta) \in C^{0,\alpha}(N) \text{ by}$$

Theorem 2. Q. E. D.

We say that the piecewise $C^{1,\alpha}$ curve Γ has a cusp at O if Γ is the union of two curves

$$\Gamma' : x_2 = \psi_2(x_1),\ x_3 = \psi_3(x_1), \qquad 0 \leq x_1 < B,$$

and

$$\Gamma'' : x_2 = \varphi_3(x_1),\ x_3 = \varphi_3(x_1), \qquad 0 \leq x_1 < B,$$

where

$$\psi_j,\ \varphi_j \in C^{1,\alpha}([0, B]),\ 0 < \alpha < 1\ ;\ \psi_j(0) = \psi_j'(0) = \varphi_j(0) = \varphi_j'(0) = 0,\ j = 2, 3.$$

THEOREM 4′ : Let S be a minimal surface whose boundary contains the piecewise $C^{1,\alpha}$ arc $\Gamma = \Gamma' \cup \Gamma''$ which has a cusp at 0. Let $F_j(\zeta)$, be a conformal representation of S for $\operatorname{Im} \zeta > 0$ with $F_j(0) = 0$, $j = 1, 2, 3$, such that Γ' and Γ'' are the topological images of $(-1, 0]$ and $[0, 1)$ respectively. Suppose that the integer m « determined by Theorem 1 » is positive.

Then there is a subdomain $N = \{|\zeta| \leq R,\ \operatorname{Im} \zeta \geq 0\}$ such that $F_j(\zeta) \in C^{1,\alpha}(N)$, $j = 1, 2, 3$.

For such a minimal surface, there is a theorem analogous to Theorem 1, except that the integer m is even or zero. This latter case occurs, for example, in the conformal mapping of $\operatorname{Im} \zeta > 0$ into a domain bounded by a curve with a cusp having exterior angle 2π. For such a conformal mapping, the conclusion of Theorem 4′ also fails.

§ 6. Higher Derivatives.

Higher derivatives are obtained using the representations (9). A brief description of this procedure is given here. The proof is valid also for conformal maps of plane domains. No attempt to reprove Lewy's result [5] is made here.

THEOREM 5. Let S be a minimal surface whose boundary contains the arc

$$\Gamma : x_2 = \psi_2(x_1),\ x_3 = \psi_3(x_1),\ -B < x_1 < B,$$

where

$$\psi_j(x_1) \in C^{n,\alpha}([-B, B]),\ 0 < \alpha < 1 \leq n\ ;\ \psi_j(0) = \psi_j'(0) = 0\ ;$$

$j = 2, 3$. Let $F_j(\zeta)$ be a conformal representation of S for $\operatorname{Im} \zeta > 0$, $F_j(0) = 0$, $j = 1, 2, 3$. Then there is a subset $N = \{|\zeta| \leq R,\ \operatorname{Im} \zeta \geq 0\}$ such that $F_j(\zeta) \in C^{n,\alpha}(N)$, $j = 1, 2, 3$.

We proceed by Lemmas.

LEMMA 6.1. Let $\varphi(\zeta) = u(\zeta) + iv(\zeta)$ be analytic in D_{ϱ_0} and continuous in $\overline{D_{\varrho_0}}$, $\varrho_0 \leq 1$. Suppose that for a fixed ξ_0, $|\xi_0| \leq \varrho_1 < \varrho_0$, and α, $0 < \alpha < 1$,

$$|u(t)| \leq C\,|t - \xi_0|^{1+\alpha} \text{ for } t \in \partial D_{\varrho_0}.$$

Then for any sector

$$\Delta : \delta < \arg(\zeta - \xi_0) < \pi - \delta, \; |\zeta| \leq \varrho_1,$$

$$|\varphi'(\zeta)| \leq cC, \; \zeta \in \overline{\Delta},$$

where c depends on $\delta\,(> 0)$, α, ϱ_0, ϱ_1.

PROOF: Since $u(\zeta)$ is bounded and harmonic in D_{ϱ_0}, we may write

$$u(\zeta) = \frac{1}{\pi}\int_{-\varrho_0}^{\varrho_0} \frac{\eta}{|t - \zeta|^2}\, u(t)\, dt + \operatorname{Re} h(\zeta), \text{ where}$$

$h(\zeta)$ is analytic for $|\zeta| < \varrho_0$. Hence

$$\varphi'(\zeta) = -\frac{i}{\pi}\int_{-\varrho_0}^{\varrho_0} \frac{(\zeta - t)^2}{|\zeta - t|^4}\, u(t)\, dt + h'(\zeta), \; \zeta \in D_{\varrho_0}.$$

Since for $\zeta \in \Delta$, $\dfrac{|t - \xi_0|}{|t - \zeta|} \leq (\sin\delta)^{-1}$, we see that

$$\begin{aligned} |\varphi'(\zeta)| &\leq \frac{C}{\pi}\left(\frac{1}{\sin\delta}\right)^{1+\alpha} \int_{-\varrho_0}^{\varrho_0} |t - \xi_0|^{\alpha - 1}\, dt + \sup_{|\zeta| \leq \varrho_1} |h'(\zeta)| \\ &\leq c\,C \end{aligned}$$

REMARK 1. For a function $w \in C^{s,\beta}(I)$, I an interval, let

$$|w|_{s,\beta} = \max_{0 \leq k \leq s} \sup_{I} |D^k w| + \sup_{t, t' \in I} \frac{|D^k w(t) - D^k w(t')|}{|t - t'|^\beta}, \; D = \frac{d}{dt}.$$

Let $g(t) \in C^{n,\tau}([-\varrho, \varrho])$, $f(x) \in C^{n+1,\alpha}([-B, B])$, $0 < \alpha \leq \tau \leq 1 \leq n$, with $g(t_0) = f(0) = f'(0) = 0$ for a t_0, $|t_0| < \varrho$, and $|g(t)| < B$. Then there are a, b, c whose moduli depend on $|g|_{n,\tau}$, $|f|_{n+1,\alpha}$ such that for $u(t) = f(g(t))$,

$$|D^n u(t) + a + b(t - t_0)| \leq c\,|t - t_0|^{1+\alpha}, \; |t| \leq \varrho.$$

If $\tau = 1$, then $|D^{n+1} u(t) + b| \leq (1 + \alpha)\, c\, |t - t_0|^\alpha$, a. e. $|t| \leq \varrho$.

REMARK 2. Let $g(t) \in C^{n,\tau}([-\varrho,\varrho])$, $h(t) \in C^{n-1,\tau}([-\varrho,\varrho])$, $v(x) \in C^{n,\alpha}([-B,B])$, $0 < \alpha \leq \tau \leq 1 \leq n$, with $v(0) = g(t_0) = 0$ for a t_0, $|t_0| < \varrho$. Then there are a, b, c whose moduli depend on $|g|_{n,\tau}$, $|h|_{n-1,\tau}$, $|v|_{n,\alpha}$ such that for $u(t) = v(g(t))\, h(t)$

$$|D^{n-1} u(t) + a + b(t - t_0)| \leq c\,|t - t_0|^{1+\alpha}, \; |t| \leq \varrho.$$

If $\tau = 1$, then $|D^n u(t) + b| \leq (1 + \alpha)\, c\, |t - t_0|^\alpha$, a e. $|t| \leq \varrho$.

LEMMA 6.2: Suppose that $\Gamma \in C^{n+1,\alpha}$ and $F_j(\zeta) \in C^{n,\tau}(\bar{D}_{\varrho_0})$ for each τ, $0 < \tau < 1 \leq n$. Then $F_j^{(n+1)}(\zeta)$ are bounded analytic functions in $\bar{D}_\varrho$ for each $\varrho < \varrho_0$, $j = 1, 2, 3$.

PROOF: We shall show that $G_j^{(n+1)}(\zeta)$ has bounded modulus, independent of P, in each sector $\Delta(\xi_P, \varrho, \delta)$ for a suitable δ. Given ϱ, choose ϱ_1, $\varrho < \varrho_1 < \varrho_0$ and $\delta > 0$. For each $\xi_P \in [-\varrho, \varrho]$, there are functions $\varphi_j(y_1) \in C^{n+1,\alpha}([-l, l])$, $\varphi_j(0) = \varphi_j'(0) = 0$, $j = 2, 3$, so that a subarc $\Gamma_0' \subset \Gamma$ is given by

$$\Gamma_0 : \; y_2 = \varphi_2(y_1), \; y_3 = \varphi_3(y_1), \; -l < y_1 < l.$$

Hence by the first remark, there are a_j, b_j, c_0 with moduli bounded independently of ξ_P such that

$$|D^n y_j(t) + a_j + b_j(t - \xi_P)| \leq c_0\, |t - \xi_P|^{1+\alpha}, \quad -\varrho_0 \leq t \leq \varrho_0, \quad j = 2, 3.$$

Hence there is a constant $C \geq c_0$ and independent of P such that

$$|D^n y_j(t) + a_j + b_j(t - \xi_P)| \leq C\, |t - \xi_P|^{1+\alpha}, \; t \in \partial D_{\varrho_0}, \qquad j = 2, 3.$$

By Lemma 6.1, $|G_j^{(n+1)}(\zeta) + b_j| \leq cC$ in $\Delta(\xi_P, \varrho, \delta)$, $j = 2, 3$.

According to (3),

$$\frac{\partial y_1^*}{\partial t}(t) = -\sum_{j=2}^{3} \varphi_j'(y_1(t))\, \frac{\partial y_j^*}{\partial t}(t), \qquad -\varrho_0 \leq t \leq \varrho_0.$$

The second remark applies to this case. Again by Lemma 6.1, we find that $|G_1^{(n+1)}(\zeta)| \leq cC'$ in $\Delta(\xi_P, \varrho, \delta)$, $j = 2, 3$. By (9), $F_j^{(n+1)}(\zeta)$ are bounded in modulus in each $\Delta(\xi_P, \varrho, \delta)$, $j = 1, 2, 3$, with bound independent of P for $|\xi_P| \leq \varrho$.

PROOF OF THEOREM 5: For a proof by induction, we assume that $\Gamma \in C^{n+1,\alpha}$ and $F_j \in C^{n,\tau}(\bar{D}_{\varrho_n})$ for each τ, $0 < \tau < 1 \leq n$. Choose $\varrho_{n+1} < \varrho_n$, let $\varrho = \frac{1}{2}(\varrho_n + \varrho_{n+1})$, and fix δ, $0 < \delta < \frac{\pi}{2}$. For each ξ_P, $|\xi_P| \leq \varrho_{n+1}$, there are b_j such that

$$|D^{n+1} y_j(t) + b_j| \leq (1 + \alpha)\, C\, |t - \xi_P|^{\alpha}, \text{ a. e. } t \in \partial D_{\varrho_n}, \qquad j = 2, 3,$$

in view of the preceding Lemma and Remark 1. Hence by Lemma 5.3 $G_j^{(n+1)}(\zeta) \in C^{0,\alpha}(\bar{\Delta}(\xi_P, \varrho, \delta))$ with Hölder constant independent of P for $|\xi_P| \leq \varrho_{n+1}$, $j = 2, 3$.

By using the second remark, we see that $G_1^{(n+1)}(\zeta) \in C^{0,\alpha}(\bar{\Delta}(\xi_P, \varrho, \delta))$. Hence $F_j^{(n+1)} \in C^{0,\alpha}(\bar{\Delta}(\xi_P, \varrho, \delta))$ for each ξ_P, $|\xi_P| \leq \varrho_{n+1}$. By an elementary, argument it follows that $F_j^{(n+1)} \in C^{0,\alpha}(\bar{D}_{\varrho_{n+1}})$. (*)

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