ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

R. O., JR. WELLS

Concerning the envelope of holomorphy of a compact differentiable submanifold of a complex manifold

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 23, nº 2 (1969), p. 347-361

<http://www.numdam.org/item?id=ASNSP_1969_3_23_2_347_0>

CONCERNING THE ENVELOPE OF HOLOMORPHY OF A COMPACT DIFFERENTIABLE SUBMANIFOLD OF A COMPLEX MANIFOLD

R. O. WELLS, JR. (*)

Introduction.

A classical theorem due to Hartogs says that any function holomorphic in a neighborhood of the unit sphere S^{2n-1} in C^n (for $n > 1$) can be continued analytically to the interior of the sphere. This result can be phrased by saying that the envelope of holomorphy of S^{2n-1} contains the interior of S^{2n-1}, an open set in C^n. More generally, it can be shown that any compact hypersurface S in C^n has an envelope of holomorphy $E(S)$ which contains an open set in C^n. Intuitively, $E(S)$ is the largest set (not necessarily contained in C^n due to multiple-valued continuation) to which all functions holomorphic in a neighborhood of S can be continued analytically. A precise definition is given in Section 1.

Our main result in this paper is that any compact smooth submanifold M in a Stein manifold X of real codimension *two* has the property that its envelope of holomorphy $E(M)$ contains a smooth submanifold of X of real codimension *one* (Theorem 4.1). This is a precise analogue of the Hartogs theorem for hypersurfaces S as stated above, where $E(S)$ jumps one dimension to an open set. Moreover, there are examples to show that, in general, $E(M)$ can jump at most one dimension (see Remark 2.7).

Recent work initiated by H. Lewy in [6] and [7] and stimulated by Bishop's fundamental contribution [1] has led to a general local theory concerning envelopes of holomorphy of differentiable submanifolds of C^n (see [3], [10], [11], and [12]). One basic result states that a local submanifold of

Pervenuto alla Redazione il 25 Luglio 1968.

(*) This research was supported by Army Grant No. DA-ARO-D-31-124-G-866 at Brandeis University.

generic holomorphic tangent dimension with a nonvanishing Levi form has a local envelope of holomorphy containing a one higher dimensional submanifold. In this paper we use global data on compact submanifolds M and deduce somewhere on M the local data necessary to use the local theory. The parts of $E(M)$ we construct are therefore local in nature and do not indicate the global extent.

In Section 1 we introduce the concept of envelope of holomorphy $E(K)$ of a compact subset K of a Stein manifold X and define K to be holomorphically convex when $K = E(K)$ (following [5]). We show that no compact oriented smooth submanifold M in a Stein manifold X is holomorphically convex if $\dim_R M > \dim_C X$. This is a simple cohomology argument using a result from [5], and is analogous to a similar theorem for polynomial convexity in [2]. In Section 2 we show that for « almost all » embeddings of a compact smooth manifold M in a complex manifold X, where $\dim_R M > \dim_C X$, $E(M)$ contains a smooth submanifold $N \subset X$ where $\dim_R N = \dim_R M + 1$ (here « almost all » is in the category sense). In Section 4 we show this is true for *all* embeddings of a compact smooth manifold M in a Stein manifold X, where $\dim_C X > 2$ and M has real codimension two in X (Theorem 4.1). The techniques used here also work for codimension one, but not for codimension higher than two. The proof of Theorem 4.1 depends on a result of independent interest (Theorem 3.1) concerning the Šilov boundary $\check{S}(A(M))$ of the holomorphic function algebra $A(M)$ for a smooth submanifold M in a Stein manifold X: namely, $\check{S}(A(M))$ must contain a nonempty open subset of the submanifold M. This is similar to results of Bremermann and Rossi concerning peak points on certain types of smooth hypersufaces in C^n. Theorem 3.1 is reduced to a theorem in C^n by the proper embedding theorems for Stein manifolds, and the proof there involves finding extremal points of a certain kind, leading to holomorphic peak points.

1. Holomorphic Extension and Holomorphic Convexity.

Let X be a complex manifold [1] and let $\mathcal{O} = \mathcal{O}_X$ denote the sheaf of germs of holomorphic functions on X. Suppose U is an open subset of X, then we let $\mathcal{O}(U)$ denote the Frechet algebra of holomorphic functions on U. If K is any compact subset of X, we let

$$\mathcal{O}(K) = \varinjlim_{U \supset K} \mathcal{O}(U)$$

---

(1) We shall assume all differentiable manifolds under consideration have a countable basis for their topology.

be the inductive limit with the natural (locally convex) inductive limit topology. Thus $\mathcal{O}(K)$ is a topological algebra over C, and we denote by $E(K)$ the spectrum [2] of $\mathcal{O}(K)$. Let $g: K \longrightarrow E(K)$ be the usual evaluation map,

$$g(x) f = f(x), \quad \text{for } f \in \mathcal{O}(K).$$

We shall say that K is *holomorphically convex* if and only if g is a bijective map onto K. Suppose X is now a Stein manifold [3], then it is shown in [5] that there is a natural map $\pi: E(K) \longrightarrow X$, so that the following is a commutative diagram

$$\begin{array}{ccc} & E(K) & \\ {}^{g}\nearrow & & \searrow^{\pi} \\ K & \xrightarrow{i} & X \end{array}$$

where i is the inclusion map. Moreover $E(K)$ is a compact Hausdorff space and g is a homeomorphism onto its image in $E(K)$. One can define (by inductive limits) a « holomorphic » sheaf $\mathcal{O}_{E(K)}$ over $E(K)$ so that the restriction map

$$g^*: \Gamma(E(K), \mathcal{O}_{E(K)}) \longrightarrow \Gamma(K, \mathcal{O}) = \mathcal{O}(K)$$

is an algebraic isomorphism (see [5]). This turns out to be a proper generalization of the now classical theory of envelopes of holomorphy for domains spread over C^n (see [4], Chapter I). Thus we call $E(K)$ the *envelope of holomorphy* of K. For a given $K \subset X$ we would like to know the structure of $E(K)$, which is related to the question of holomorphic convexity of K, i. e., when $K = E(K)$ (under the identification of K with its image $g(K)$ in $E(K)$).

If X is not Stein, one can still define the envelope of holomorphy $E(K)$ to be the spectrum of $\mathcal{O}(K)$, but much less is known about it. However, we can introduce the concept of extension or extendibility of a compact set K in X. Suppose K is compact in a complex manifold X, and suppose there is some compact set $K' \underset{\neq}{\supset} K$, and such that K' is connected to K. We shall say that K is *extendible* to K' if the natural restriction map

$$\mathcal{O}(K') \longrightarrow \mathcal{O}(K)$$

is onto. It follows that $\mathcal{O}(K') \cong \mathcal{O}(K)$ and hence K' is naturally injected into $E(K)$. We say K is *extendible* if there is some $K' \underset{\neq}{\supset} K$ to which K is extendible (or to which K extends). If Q is any set in X, then we say that K

---

(2) The nonzero continuous homomorphisms of $\mathcal{O}(K)$ into C.

is extendible to Q if Ki s exendible to some set $K' \supset Q$, where $K' \underset{\neq}{\supset} K$, and K' is connected to K. From these definitions it follows immediately that:

(1.1) If K is extendible, then K is not holomorphically convex.

(1.2) If K is holomorphically convex, then K is not extendible.

However it is unknown whether extendibility of K follows from the assumption that K is not holomorphically convex. In fact, it is unknown whether $\pi(E(K)) = K$ implies $K = E(K)$ (4).

In this paper we study primarily the extendibility (and consequently the partial structure of the envelope of holomorphy) of compact C^∞ submanifolds of a complex manifold. By a C^m *k-manifold* embedded in a complex manifold X we shall mean an m-times continuously differentiable submanifold (not necessarily closed) of real dimension k of the underlying differentiable manifold in X, where $0 < k < 2 \dim_C X$, and $1 \leq m \leq \infty$. A compact C^m k manifold M means compact without boundary. Our first theorem concerns the holomorphic convexity of compact submanifolds of X.

(1.3) THEOREM: Let M be a compact oriented C^1 k-manifold embedded in a Stein manifold X, and suppose that $k > \dim_C X$, then M is not holomorphically convex.

The proof is a simple application of the following theorem proved in [5].

(1.4) THEOREM: Suppose K is a compact holomorphically convex subset of a Stein manifold X, then

$$H^q(K, C) = 0, \; q > \dim_C X.$$

PROOF OF THEOREM 1.2: Suppose M were holomorphically convex, then by Theorem 1.4 we have $H^k(M, C) = 0$, which contradicts the fact that for orientable compact (connected) k-manifolds, $H^k(M, C) \cong C$.

q. e. d.

(1.5) REMARK:

1. Under the hypotheses of Theorem 1.3 (letting $X = C^n$), it was known that M could not be polynomially convex if $k \geq n$ (see [2]).

---

(3) See e. g., [4] for basic concepts of several complex variables.

(4) This is however true in the classical case where D is a domain in C^n, and $E(D)$ is the spectrum of $\mathcal{O}(D)$.

2. If $T = \{z \in C^n : |z_i| = 1\}$, then T is a compact oriented n-manifold embedded in C^n, and T is holomorphically convex. So the dimensions in (1.3) are the best possible.

This theorem, along with various examples, lead us to make the following

(1.6) CONJECTURE: Suppose M is a compact C^∞ k-manifold embedded in a complex manifold X, where $k > n = \dim_C X > 1$, then:

a) M is extendible.

b) M is extendible to a C^m submanifold of X of real dimension $(k + 1)$, for any positive m.

c) M is extendible to a C^∞ submanifold of X of real dimension $(k + 1)$.

(1.7) REMARK:

1. Examples show that, in gneral, M can be extended to only one higher dimensions (see Remark 2.7).

2. If M is *generic* at each point (see Definition 2.1 below), then *a*) was proven in [12]. Also *b*) for $k = n + 1$ was proven in [12] and a proof for general k is given in [3]. But these techniques do not give *c*).

3. For $k = 2n - 1$, *a*), *b*), *c*) reduce to the classical Hartogs' theorems for compact hypersurface which bound a domain in $C^n, n > 1$.

As stated in the introduction, we prove *b*) for « almost all » embeddings of a given k-manifold M in X, for $k > n = \dim_C X > 2$, (Section 2), and for all embeddings of codimension 2 of a k-manifold in a Stein manifold X, for $\dim_C X > 2$, (Section 4).

2. Embeddings of compact manifolds in a complex manifold.

Let M be a C^∞ k-manifold and let X be a complex manifold of complex dimension n. Let $E(M, X)$ denote the set of all embeddings of M in X (which may be empty for a given M and X). Letting $C^\infty(M, X)$ denote the topological space of C^∞ maps from M to X with the usual C^∞ topology [5], we let $E(M, X)$ have the induced topology. If M is compact, then $E(M, X)$ is an open set in $C^\infty(M, X)$. If Y is a differentiable manifold, then we will denote the real tangent bundle of Y by $T(Y)$ with fibre $T_y(Y)$, for $y \in Y$.

Suppose $f \in E(M, X) \neq \emptyset$, for a given M and X, then let M_f denote the submanifold of X given by $f(M)$. Suppose $x \in M$, then, since f is an

---

[5] $C^\infty(M, X)$ is metrizable and complete with respect to the usual C^∞ metric of uniform convergence on compact subsets of all derivatives.

embedding, the R-linear map, (the real Jacobian)

$$df_p : T_p(M) \longrightarrow T_{f(p)}(X)$$

has maximal rank. Since $T_{f(p)}(X)$ has the structure of an n-dimensional vector space over C, it's clear that df_p extends naturally (by linearity) to a C-linear map (the complex Jacobian)

$$\widetilde{df}_p : T_p(M) \otimes_R C \longrightarrow T_{f(p)}(X).$$

(2.1) DEFINITION: We say that $x = f(p)$ is a *generic point* of M_f if $\widetilde{df}_p$ has maximal rank as a C-linear map.

This definition agrees with that given in [3] and [12], and we shall, of course, say that $x \in M_f$ is *non-generic* if x is not generic. For real hypersurfaces in X, *all* points are generic, however for higher codimension there are topological obstructions to this being the case in general (see [13]).

Locally we can express M_f in the following manner. For $x \in M_f$, there is a neighborhood U of x in X and a C^∞ map $g : U \longrightarrow R^{2n-k}$, so that

$$M_f \cap U = \{x \in U : g(x) = 0\},$$

and moreover the real Jacobian

$$dg_x : T_x(X) \longrightarrow T_{g(x)}(R^{2n-k})$$

has maximal rank at each point $x \in M_f \cap U$. Also we have the complex Jacobian

$$\partial g_x : T_x(X) \longrightarrow T_{g(x)}(R^{2n-k}) \otimes_R C,$$

where $d = \partial + \bar{\partial}$, as usual, and $\partial g = (\partial g_1, \dots, \partial g_{2n-k})$. Then let

$$H_x(M_f) = \operatorname{Ker} \partial g_x$$

which is a subspace of $T_x(M_f) = \operatorname{Ker} dg_x$. We call $H_x(M_f)$ the *holomorphic tangent space* to M_f at x (cf. [3], [12]), and note that $H_x(M_f)$ is the maximal complex subspace of $T_x(X)$ contained in $T_x(M_f)$. We have the relationship, setting $x = f(p)$,

$$\operatorname{rank}_C \widetilde{df}_p = (k - n) + \operatorname{rank}_C \partial g_x,$$

which is easy to check. If $\operatorname{rank}_C \widetilde{df}_p$ is constant for p near $p_0 \in M$, then we say that M_f is a *CR-submanifold* of X near $x_0 = f(p_0)$ (cf. [3]). In this case we can define the *Levi form* at any x near x_0,

$$L_x(M_f): H_x(M_f) \longrightarrow T_x(M_f) \otimes C / H_x(M_f) \otimes C$$

defined by $L_x(M_f)(t) = \pi_p\{[Y, \bar{Y}]_p\}$, where Y is a local section of $H(M_f)$ such that $Y_p = t$, $[Y, \bar{Y}]_p$ is the Lie bracket evaluated at p, and $\pi_p: T_x(X) \otimes C \longrightarrow T_x(X) \otimes C / H_x(M_f) \otimes C$ is the natural projection (see [3], [11], [12]). Suppose M_f is generic at x, then it follows easily from the definition of the Levi form that $L_x(M_f) \neq 0$ if and only if the vector

$$\partial\bar{\partial} g(t, \bar{t}) = \begin{bmatrix} \partial\bar{\partial} g_1(t, \bar{t}) \\ \vdots \\ \partial\bar{\partial} g_{2n-k}(t, \bar{t}) \end{bmatrix} \tag{2.2}$$

is not zero for some $t \in H_x(M_f)$, and this is all we shall need in this paper. We now state the following result which relates these various concepts. In the following theorems M and X are as described above, with $\dim_R M = k$ and $\dim_C X = n$.

(2.3) THEOREM: Suppose $k > n$ and $f \in E(M, X) \neq \emptyset$. If $x \in M_f$ is a generic point and $L_x(M_f) \neq 0$, then any compact neighborhood K of x in M_f is extendible to a C^m-submanifold $N^{(m)}$ of X where $\dim_R N^{(m)} = k + 1$, and m may be any positive integer.

This theorem is proved in [12] for $k = n + 1$, and in [13] for arbitrary k.

(2.4) REMARK: In this theorem, we do not need to assume M is C^∞, but it is necessary (for the proof) to have M_f be a $C^{l(n)}$ submanifold where $l(n)$ depends linearly on $n = \dim_C X$, due to the fact that Sobolev's lemma is used in the proof. Therefore we restrict ourselves to C^∞ submanifolds in the hypotheses. On the other hand, the present proof does not allow us to conclude that M_f extends to some C^∞ submanifold. We now have the following proposition concerning M and X.

(2.5) PROPOSITION: If $k > n$, and if M is compact then there is an open dense set $U \subset E(M, X)$ such that if $f \in U$, then there is a point $x \in M_f$ such that:

a) x is generic

b) $L_x(M_f) \neq 0$.

Assuming this proposition we have immediately, in conjunction with Theorem 2.3, (2.6) Theorem: Suppose M is a compact C^∞ k-manifold, and $k > n = \dim_C X$, where X is a complex manifold such that $E(M, X) \neq \emptyset$. Then there is an open dense set $U \subset E(M, X)$ such that if $f \in U$, then M_f is extendible to a C^m-submanifold $N^{(m)}$ of X, where $\dim_R N^{(m)} = k + 1$, and m may be any positive integer.

(2.7) Remark: Under the hypotheses of the above theorem, $N^{(m)}$ cannot be more than $(k+1)$-dimensional in general, as we shall show by an example, although there is very likely an open dense set of embeddings $U \subset E(M, X)$ so that $E(M_f)$ contains an open set in X. As an example, let S^{k-1} be the standard $(k-1)$-sphere in R^k, and let B^k be the closed unit ball in R^k. Then we have

$$S^{k-1} \subset B^k \subset R^k \subset R^k \times R^{2n-k} = R^{2n},$$

where we assume that $n < k - 1 < 2n - 1$. We shall put a complex structure on R^{2n} so that $E(S^{k-1}) = B^k$, with respect to this complex structure. If $k = 2l$ is even, then we merely give R^k a complex structure, C^l, and we have

$$S^{2l-1} \subset B^{2l} \subset C^l \times C^{n-l} = C^n.$$

If $k = 2l + 1$ is odd, then we have $S^{2l} \subset R^{2l+1}$, which we consider as $C^l \times R$, and similarly $R^{2n-k} = R^{2(n-l-1)+1}$, which we can consider as $R \times C^{n-l-1}$, thus we obtain

$$S^{2l} \subset B^{2l} \subset C^l \times R \times R \times C^{n-l-1},$$

and we give $R \times R$ a complex structure also obtaining an embedding $S^{2l} \subset C^n$. In each embedding it is easy to see by classical methods of several complex variables that $E(S^k) = B^{k+1}$.

Proof of Proposition 2.5: First, it's clear that the set of $f \in E(M, X)$ which satisfy conditions $a)$ and $b)$ in Proposition 2.5 is an open set. So we must only prove that for any given $f_0 \in E(M, X)$, we can perturb f_0 so that $a)$ and $b)$ hold. One can apply the general Thom transversality theory to such questions, but since we only need to work locally we can give a direct elementary proof.

We shall first perturb f_0 to rind a generic point (hence the name *generic*!). Let x_0 be any point in M. Take local coordinates $t = (t_1, \dots, t_k)$ at x_0, so that $x_0 = (0, \dots, 0)$ in R^k, and let $z = (z_1, \dots, z_n)$ be local coordinates at $f(x_0) \in X$, so that $f(x_0) = (0, \dots, 0) \in C^n$. Then $\widetilde{df}_{x_0}$ can be represented as a linear map

$$A_0 : C^k \longrightarrow C^n,$$

where C^k is the complexification of R^k. If A_0 has maximal rank, then there is nothing to prove, since x_0 would then be generic. Suppose A_0 is not of maximal rank, then there is a sequence of linear maps A_ν of maximal rank which converge to A_0 in the usual topology of the set of all linear maps from C^k to C^n, since the set of linear maps of lower rank is nowhere dense Then, letting φ be a C_0^∞ function in R^k with $\varphi \equiv 1$ near $t = 0$, we set

$$f^\nu(t) = f_0(t) + \varphi(t)(A_\nu - A_0)\cdot t.$$

Then at $t = 0$, we have, in these coordinates,

$$\widetilde{df}^{(\nu)}_{x_0} = A_0 + (A_\nu - A_0) = A_\nu.$$

Also, $f^{(\nu)}$ is a map which is defined on all of M since φ had compact support. Moreover, $f^{(\nu)}$ converges to f_0 in the C^∞ topology on M, and $\widetilde{df}^{(\nu)}_{x_0}$ has maximal rank at x_0. For ν large we have $f^{(\nu)} \in E(M, X)$, and thus the set of $f \in E(M, X)$ with at least one generic point are dense.

Suppose now that $f \in E(M, X)$ and x_0 is a generic point of M_f. Let U be a neighborhood of x_0 and g a C^∞ map defined in U,

$$g : U \longrightarrow R^{2n-k}$$

so that

$$M_f \cap U = \{x \in U : g(x) = 0\}$$

and

$$dg_x : T_x(M_f) \longrightarrow T_{g(x)}(R^{2n-k})$$

has maximal rank at each $x \in M_f \cap U$. Suppose U is a coordinate patch, which we may consider as a domain in C^n, with coordinates $z = (z_1, \dots, z_n)$ where $x_0 = (0, \dots, 0)$. Then let $\varphi \in C_0^\infty(U)$, with $\varphi = 1$ near $z = 0$. For small constant ε we set

$$g_\varepsilon^1(z) = g^1(z) + \varepsilon\varphi(z)\,|z|^2$$

$$g_\varepsilon^j(z) = g^j(z), \qquad j = 2, \dots, 2n-k,$$

where $g = (g^1, \dots, g^{2n-k})$, and then

$$g_\varepsilon : U \longrightarrow R^{2n-k}.$$

Since φ has compact support in U, it's clear that for small ε, $g_\varepsilon^{-1}(0)$ is a closed C^∞ submanifold of U which coincides with $M_f \cap U$ near the boundary of U. Thus we can define a new perturbed submanifold $\widetilde{M}$ with the property that $\widetilde{M} = M_f$ outside U, and $\widetilde{M} = g_\varepsilon^{-1}(0)$ in U. For small ε, this new submanifold $\widetilde{M}$ is diffeomorphic to M, and we let $\widetilde{M} = M_{f_\varepsilon}$, for some $f \in E(M, X)$ As $\varepsilon \to 0$, it's clear that $f_\varepsilon \to f$ in $E(M, X)$.

We must now show that $L_x(M_{f_\varepsilon}) \neq 0$, for some x near x_0. Either i) $L_x(M_f) \neq 0$ for some generic x near x_0 or ii) $L_x(M_f) \equiv 0$ in a neighborhood of x_0. In the first case we have *b*) is satisfied for $\varepsilon = 0$. Suppose we have ii), then this is equivalent to saying, by (2.2), that $\partial\bar{\partial} g_x(t, \bar{t})$ vanishes for all $t \in H_x(M_f)$, for all x near x_0 in M_f. We now compute $\partial\bar{\partial} g_\varepsilon^1$ at $z = 0$, noting that $z = 0 \in M_{f_\varepsilon}$, for all ε; therefore x_0 in X is still a point of our perturbed submanifold M_{f_ε}. We note that

$$\partial g_\varepsilon(0) = \partial g(0),$$

since

$$\partial(|z|^2)|_{z=0} = 0,$$

and

$$\partial g_\varepsilon^1(0) = \partial g^1(0) + \varepsilon\partial(|z|^2)|_{z=0} = \partial g^1(0).$$

Thus $H_{x_0}(M_f) = H_{x_0}(M_{f_\varepsilon})$. We obtain, therefore, for $0 \neq t \in H_{x_0}(M_{f_\varepsilon})$, at $z = 0$,

$$\partial\bar{\partial} g_\varepsilon(t, \bar{t}) = \partial\bar{\partial} g(t, \bar{t}) + \varepsilon \sum_i dz_i \wedge d\bar{z}_i(t, \bar{t}),$$

and the first term vanishes by ii) above, and the second term is positive for $\varepsilon > 0$; hence

$$\partial\bar{\partial} g_\varepsilon(t, \bar{t}) \neq 0.$$

for small $\varepsilon \neq 0$. It's clear that x_0 is still a generic point of M_{f_ε}, and thus *a*) and *b*) are satisfied for M_{f_ε} at x_0. q. e. d.

3. Holomorphic Peak Points of Smooth Submanifolds.

Let K be a compact subset of a complex manifold X. We shall call a point $x \in K$ a *holomorphic peak point* if there exists a function $f \in \mathcal{O}(K)$ such that, for any $y \in K - \{x\}$, we have

$$|f(y)| < |f(x)|.$$

Let $h(K)$ denote the set of holomorphic peak points on K. Let $C(K)$ be the Banach algebra of continuous complex-valued functions on K, with the maximum norm, and let $A(K)$ be the closure of the image of $\mathcal{O}(K)$ in $C(K)$.

Let $\check{S}(A(K))$ be the *Šilov boundary* (6) of $A(K)$, then it's clear that

$$h(K) \subset \check{S}(A(K)),$$

and moreover, $h(K)$ is sometimes dense in $\check{S}(A(K))$. (see [8]). For some sets K it is well known that $\check{S}(A(K))$ is a « lower dimensional set » in K. For instance, if K is the topological boundary $\partial\Delta$ of the unit polydisc Δ in C^n, $n > 1$, then $\check{S}(A(K))$ is the distinguished boundary of Δ, which has real dimension n. Our next result shows that this is due to the fact that $\partial\Delta$ is not a smooth submanifold of C^n.

(3.1) THEOREM: Let M be a compact C^2 submanifold of a Stein manifold X. Then $h(M)$ contains a non-empty open subset of M.

(3.2) REMARK: It's perhaps true that $h(M)$ is open in M, but our proof does not give us this.

PROOF (7): By the Remmert-Bishop proper embedding theorem for Stein manifolds it's clear that we may assume that M is a compact C^2 submanifold of C^n, for some $n \geq 1$. (see [4], Chapter VII).

Since M is a compact set in C^n, the function $r(z) = |z|$ assumes a maximum R at some point $z_0 \in M$. Let

$$B = \{z : |z| \leq R\}$$

$$S = \{z : |z| = R\},$$

and suppose that by translation z_0 becomes the origin in C^n. Identifying $T_0(C^n)$ with C^n, we have the subspaces

$$T_0(M) \subset T_0(S) \subset C^n.$$

Let Q denote the orthogonal complement to $T_0(S)$ in C^n, let $W = Q \oplus T_0(M)$, and let π be the orthogonal projection from C^n onto W. Let $\pi(M) = K$, which is a compact subset of W, and let $S' = S \cap W$ be the sphere of radius R (centered at $-z_0$ in the new coordinates) and we have $K \subset \pi(B) = B'$.

It follows readily that for some neighborhood U of $0 \in M$, $\pi \mid U \cap M$ is a diffeomorphism onto its image, although in general π will be a many to

---

(6) See [4], Chapter I.

(7) The author would like to thank T. Sherman for a helpful suggestion concerning this proof.

one map. Of course we have $\pi(0) = 0 \in K$, and set $N = \pi(U \cap M)$, which is a hypersurface in W defined near 0. Note that

$$T_0(N) = T_0(M) = T_0(S') \subset W.$$

Consider now in some coordinate system on W centered at 0, with real coordinates $(x_0, \dots, x_k)$, $(k = \dim_R M)$, where $\frac{\partial}{\partial x_0}$ is normal to $T_0(N)$ at 0, and $x = (x_1, \dots, x_k)$ are coordinates in $T_0(N)$. In these coordinates N and S' can be represented near 0 as the graphs of C^2 functions

$$S' = \{(x_0, x) : x_0 = h(x)\}$$

$$N = \{(x_0, x) : x_0 = g(x)\}.$$

And locally we have near $x = 0$,

$$g(x) \geq h(x) \tag{3.3}$$

since $K \subset B'$. Since the real Hessian of h, $(h_{x_i x_j})(0)$, $i, j = 1, \dots, k$, is positive definite at 0, it follows easily from (3.3) that the real Hessian of g is positive definite at 0, and hence in some neighborhood of 0. Let $K' = K - N$, then we have

$$T_0(N) \cap K = \{0\}$$

$$T_0(N) \cap K' = \emptyset.$$

There is a neighborhood V of $T_0(N)$ in the Grassmannian manifold of affine k-dimensional subspaces of W such that any affine subspace P in V will not intersect K', since K' is compact. For $p \in N$ sufficiently close to 0, we have that the affine subspace $T_p(N)$ centered at p lies in V. Also for p sufficiently close to 0, the affine subspace $T_p(M)$ intersects N at only the point p, since the real Hessian of g is positive definite near 0. Thus for some neighborhood N' of 0 in N, we have for each point $p \in N'$ an (affine) real hyperplane $W_p (= T_p(N)) \subset W$ which intersects K at only the point p. Moreover $K - \{p\}$ lies on one side of the hyperplane W_p. It follows that

$$H_p = \pi^{-1}(W_p)$$

is a (real) hyperplane in C^n which intersects M at only the single point $p' = \pi^{-1}(p) \cap M$, and $M - \{p'\}$ lies on one side of H_p. Therefore we have that $N'' = \pi^{-1}(N') \cap M$ is a neighborhood of 0 in M with the property that for each point $q \in N''$ there is a holomorphic linear function $l_q(z)$ such that

$$M - \{q\} \subset \{z : \operatorname{Re} l_q(z) < 0\},$$

and $l(q) = 0$. Then define

$$f_q(z) = \exp(l_q(z)) \tag{3.4}$$

and we have for $z \in M - \{q\}$, $q \in N''$,

$$|f_q(z)| < |f_q(q)| = 1,$$

and q is a holomorphic peak point. q. e. d..

(3.5) COROLLARY: Under the hypotheses of Theorem 3.1, $\check{S}(A(M))$ contains a non-empty open set.

(3.6) COROLLARY: Let M be a C^2 compact submanifold of C^n, and let $P(M)$ be the algebra of uniform limits of polynomials on M. Then $\check{S}(P(M))$ contains a non-empty open set.

PROOF: The function $f_q(z)$ given by (3.4) which peaked at q is an entire function. q. e. d..

4. Compact Submanifolds of Real Codimension Two.

We shall use the results of the previous sections to prove the following theorem.

(4.1) THEOREM: Suppose X is a Stein manifold of complex dimension $n > 2$, and suppose M is a compact C^∞ submanifold of X of real codimension two in X. Then M is extendible to a C^m-submanifold $N^{(m)}$ of X of real codimension one, where m is any positive integer; i. e., $E(M)$ contains a smooth submanifold of X of one higher dimension than M.

(4.2) REMARK: In view of Remark 1.5.2, it's clear that for $n = 2$, the result above cannot hold as stated. However such a result is presumably true for higher real codimension of M, in view of Theorem 2.5, where we have a similar result for « almost all » submanifolds, a much weaker theorem. We shall see however that the proof we give breaks down completely for higher real codimension as one of the key lemmas is false in general.

We shall prove the theorem by first reducing it to the following proposition in view of Theorem 2.3. Let M and X be as in Theorem 4.1 for the remainder of this section.

(4.3) PROPOSITION: There exists a point $x \in M$ such that:

a) x is a generic point of M.

b) $L_x(M) \neq 0$.

We shall prove this proposition by a sequence of short lemmas. Let S be the set of nongeneric points of M, and let G be the set of generic points of M.

(4.4) LEMMA: The interior of S in M is a complex submanifold of X of dimension $n-1$.

PROOF: At a point $x \in S$, we have that $\dim_C H_x(M) = n-1$, and thus $H_x(M) = T_x(M)$. It follows from a classical result due to Levi-Civita that int S in M is a complex submanifold of X of complex dimension $n-1$, where the basic fact used is that the real tangent space $T_x(M)$ at each point $x \in \operatorname{int} S$ is a C-linear subspace of $T_x(X)$, (see e. g. [9]). q. e. d..

(4.5) LEMMA: G is non-empty.

PROOF: This follows trivially from Lemma 4.4, since otherwise, $M = S$ would be a compact complex submanifold of X of complex dimension $n-1$, which is impossible. q. e. d..

(4.6) REMARK: There is an example of a compact 5-manifold M^5 in C^4 such that $G(M^5)$ is empty. Namely, let S^5 be the standard 5-sphere in C^3, and consider the embedding

$$S^5 \subset C^3 \subset C^3 \times C = C^4.$$

Then at a generic point of a 5-manifold M in C^4, we must have

$$\dim_C H_x(M) = 1,$$

but

$$\dim_C H_x(S^5) = 2,$$

so all points of S^5 are non-generic.

(4.7) LEMMA: There is a point $x \in G$ such that $L_x(M) \neq 0$.

PROOF: Suppose not, then we use the fact that if $L_x(M) \equiv 0$ for $x \in G$, then G is locally a 2-parameter family of complex submanifolds of X of complex dimension $n-2 > 0$. (see [3], [9], [12]). Therefore G and int S, which is a complex submanifold by Lemma 4.4, cannot contain any holomorphic peak points of M, since this would violate the maximum principle. But this contradicts Theorem 3.1. which asserts that the set of holomorphic peak points on M has a non-empty interior. q. e. d..

BIBLIOGRAPHY

[1] E. Bishop, *Differentiable manifolds in complex Euclidean space*, Duke Math. J., **32** (1965), 1-22.

[2] A. Browder, *Cohomology of maximal ideal spaces*, Bull. Amer. Math. Soc. **67** (1961), 515-516.

[3] S. Greenfield, *Cauchy-Riemann equations in several variables*, Annali d. Scu. Norm. Sup. di Pisa, **22** (1968), 275-314.

[4] R. C. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, N. J., 1962.

[5] R. Harvey and R. O. Wells, Jr., *Compact holomorphically convex subsets of a Stein manifold*, Trans. Amer. Math. Soc. **136** (1969), 509-516.

[6] H. Lewy, *On the local character of the sulution of an atypical linear differential equation, in three variables and a related theorem for regular functions of two complex variables,* Ann. of Math., **64** (1956), 514-522.

[7] H. Lewy, *On hulls of holomorphy*, Comm. Pure Appl. Math. **13** (1960), 587-591.

[8] H. Rossi, *Holomorphically convex sets in several complex variables*, Ann. of Math., **74** (1961), 470-493.

[9] F. Sommer, *Komplex-analytische Blätterung reeler Mannigfaltigkeiten im* C^n, Math. Ann. **136** (1958), 111-133.

[10] B. Weinstock, *On Holomorphic Extension from Real Submanifolds of Complex Euclidean Space,* thesis, M. I. T., 1966.

[11] R. O. Wells, Jr., *On the local holomorphic hull of a real submanifold in several complex variables*, Comm. Pure Appl. Math. **19** (1966), 145-165.

[12] R. O. Wells, Jr., *Holomorphic hulls and holomorphic convexity of differentiable submanifolds,* Trans. Amer. Math. Soc. **132** (1968), 245-262.

[13] R. O. Wells, Jr., *Compact real submanifolds of a complex manifold with a nondegenerate holomorphic tangent bundle*, Math. Ann. **179** (1969), 123-129.

Brandeis University and
Rice University