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ON TOPOLOGIZING MEASURE SPACES VIA DIFFERENTIATION BASES

A. M. BRUCKNER [1] and MELVIN ROSENFELD

1. Introduction.

Let $(X, \mathcal{M}, \mu)$ be a totally σ-finite complete measure space. A number of authors have developed theories of differentiation of measures on such spaces by imposing a so called differentiation basis $(\mathcal{J}, \Longrightarrow)$ on the space. Here $\mathcal{J}$ denotes a family of sets of positive measure and $\Longrightarrow$ denotes an abstract notion of contraction of nets (i. e. Moore-Smith generalized sequences) of sets in $\mathcal{J}$ to points of X. The first to develop such a theory for abstract spaces was de Possel [15]. Some subsequent developments of the subject can be found in Morse [12], Denjoy [1], Pauc [14], Hayes and Pauc [6] and Trjitzinsky [19], [20]. Applications of the abstract theory to the theory of Martingales can be found in a number of articles including Krickeberg and Pauc [10]. The purpose of this article is to develop another application of the theory. Specifically, we shall use a differentiation basis to define, in a natural way, a family $\mathcal{C}$ of « continuous » functions. The family $\mathcal{C}$ determines a topology $\mathcal{T}$ with the property that $\mathcal{C}$ is exactly the class of function which are $\mathcal{T}$-continuous. We study some of the aspects of this topology in section 4, following short preliminary and motivational sections. Then in section 5 we consider several examples to illustrate the theory. We devote section 6 to a study of the case in which the measure space $(X, \mathcal{M}, \mu)$ is separable. In this case there is always a differentiation basis of a particularly simple kind-a so called net structure. We show in this case that $\mathcal{T}$ can be pseudo metrized by a pseudo metric ϱ which is in certain ways compatible with the measure μ. In particular, the class $\mathcal{C}$ is sufficiently large to approximate the class of all measurable functions in the

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Lusin sense. It follows from our results that if one considers differentiation with respect to net structures in abstract measure spaces (see for example Munroe [13] or Gurevič and Shilov [4]) one may assume the space to be pseudo metrized in such a way that the theory of differentiation with respect to nets in metric measure spaces (see, for example, Saks [17]) applies. We end with section 7 in which we state a number of open problems.

2. Preliminaries.

Throughout this article $(X, \mathcal{M}, \mu)$ will denote a totally σ-finite measure space and $(\mathcal{I}, \Longrightarrow)$ will denote a differentiation basis for the space. This means that $\mathcal{I}$ is a family of sets of finite positive measure and $\Longrightarrow$ is a notion of contraction of nets (generalized sequences) of sets in $\mathcal{I}$ to points of X, such that the following two conditions are satisfied:

(i) if $x \in X$, there exists a net $\{I_\alpha\}$ of elements of $\mathcal{I}$ contracting to x; in symbols, $I_\alpha \Longrightarrow x$.

(ii) Any subnet of a net contracting to a point x also contracts to x.

Let σ be a real valued function defined (at least) on the sets of $\mathcal{I}$. We define the upper and lower derivatives of σ with respect to μ at a point $x \in X$ by

$$\bar{D}\sigma(x) = \sup \left\{ \limsup \frac{\sigma(I_\alpha)}{\mu(I_\alpha)} \right\}$$

and

$$\underline{D}\sigma(x) = \inf \left\{ \liminf \frac{\sigma(I_\alpha)}{\mu(I_\alpha)} \right\}$$

where the limits superior and inferior are taken over a net $\{I_\alpha\}$ contracting to x and the supremum and infimum are taken over the family of all such nets.

If $\bar{D}\sigma(x)$ and $\underline{D}\sigma(x)$ are finite and equal, we denote this number by $D\sigma(x)$ and we say σ is differentiable with respect to μ at x and we call $D\sigma(x)$ the derivative of σ with respect to μ at x.

Many authors have studied the problem of determining conditions on a differentiation basis which will guarantee that the Fundamental Theorem of Calculus is valid. Although the settings vary somewhat from one author to another, an irreducible requirement for a satisfactory Fundamental Theorem is that $(\mathcal{I}, \Longrightarrow)$ posses a certain weak Vitali property. (See, for example, Hayes and Pauc [6] for an exhaustive study of this question). This property can be stated in several forms. We state below, the form most convenient for our purposes. In the sequel, some of our results will involve this Vitali property; others will not.

DEFINITION 2.1. A subset $\mathcal{V}$ of $\mathcal{J}$ is called a Vitali covering of a set $A \subset X$ provided for each $x \in A$ there exists a net of elements of $\mathcal{V}$, say $\{J_\alpha(x)\}$, such that $J_\alpha(x) \Longrightarrow x$.

Now let μ^* denote the outer measure generated by μ.

DEFINITION 2.2. The differentiation basis $(\mathcal{J}, \Longrightarrow)$ possesses the weak Vitali property provided the following condition is met: If A is any subset of X, $\varepsilon > 0$, and $\mathcal{V}$ is a Vitali covering of A, then there exist a countable coltection $\{J_k\}$ of sets in $\mathcal{V}$ such that $\mu^*(A) = \mu^*(A \cap \cup J_k)$ and $\Sigma \mu(J_k) < < \mu^*(A) + \varepsilon$.

If, in the preceding definition, the sets $\{J_k\}$ can be taken to be disjoint, we say that $(\mathcal{J}, \Longrightarrow)$ possesses the strong Vitali property.

3. A motivating example.

In section 4 we shall develop a notion of continuity in measure spaces which we wish to motivate in the present section.

Let $(X, \mathcal{M}, \mu)$ be the linear Lebesgue measure space and let $\mathcal{J}$ denote the family of open intervals. Let $I_k \Longrightarrow x$ mean that $x \in I_k$ for all k and $\lim_{k \to \infty} \delta(I_k) = 0$, where $\delta(I_k)$ denotes the diameter of I_k. (Note that contraction involves ordinary sequences in this example). Suppose f is summable on sets of finite measure and σ is defined for all $E \in \mathcal{M}$ by $\sigma(E) = \int_E f \, d\mu$.

Then for almost every real number x, $D\sigma(x) = \lim_{I_k \Longrightarrow x} \dfrac{\sigma(I_k)}{\mu(I_k)} = f(x)$. This limit where it exists, is just the ordinary derivative of the function F defined by $F(x) = \int_0^x f(t)\, dt$. If the limit exists for all x, then f is just the ordinary derivative of F. In this case, f need not be continuous, of course. It is easy to prove, however, that f is continuous if and only if for every x, and every sequence $\{E_k\}$ of sets of positive measure such that $x \in E_k$ for all k and $\lim_{k \to \infty} \delta(E_k) = 0$, we have $\lim_{k \to \infty} \dfrac{\sigma(E_k)}{\mu(E_k)} = f(x)$. One can use this criterion of continuity as a definition of continuity at least in the special case at hand. It is this type of notion which we wish to take as our basic notion of continuity. We shall see that this notion meets the minimum requirement of any satisfactory notion-namely, that in a large number of cases of important measure spaces already furnished with a topology $\mathcal{J}$, a differen tiation basis can be constructed which gives rise to a famliy of continuous functions which generated $\mathcal{T}$.

We mention in passing, that in certain settings, the class of approximately continuous functions admits of a similar characterization: a bounded function f is approximately continuous if and only if $\lim\limits_{E_k \Longrightarrow x} \dfrac{\sigma(E_k)}{\mu(E_k)} = f(x)$ for every sequence of sets of positive measure converging to x *regularly*. We shall not pursue these ideas further in the present article, but we refer the interested reader to Rosenthal [16].

4. The class $\mathcal{C}$ of continuous functions and its topology $\mathcal{T}$.

Following the ideas suggested by the preceding section, we now define the class $\mathcal{C}$.

DEFINITION 4.1. A net $\{E_\alpha\}$ of sets of positive measure is said to contract to a point x provided for each α, there exists a set $I_\alpha \in \mathcal{T}$ such that $E_\alpha \subset I_\alpha$ and $I_\alpha \Longrightarrow x$. We write $E_\alpha \Longrightarrow x$.

Now let f be any function which is summable on every set of finite measure and let σ be its integral.

DEFINITION 4.2. The function f is said to be in class $\mathcal{C}$ provided $\lim\limits_{E_\alpha \Longrightarrow x} \dfrac{\sigma(E_\alpha)}{\mu(E_\alpha)} = f(x)$ for every $x \in X$ and every net $\{E_\alpha\}$ contracting to x.

The functions in $\mathcal{C}$ admit of a very simple characterization.

THEOREM 4.3. A function f is in $\mathcal{C}$ if and only if for every $x_0 \in X$ if $a < f(x_0) < b$, and $I_\alpha \Longrightarrow x_0$ then there exists a β such that if $\alpha > \beta$ then $a \leq f(x) \leq b$ for almost every $x \in I_\alpha$.

PROOF: Suppose $f \in \mathcal{C}$, $a < f(x_0) < b$ and $I_\alpha(x_0) \Longrightarrow x_0$ yet frequently $\mu(A_\alpha) > 0$ where $A_\alpha = \{x \in I_\alpha(x_0) : f(x) > b\}$. The subnet $\{B_\gamma\}$ of the sets A_α with positive measure contracts to x_0. Hence $\lim\limits_{B_\gamma \Longrightarrow x_0} \int_{B_\gamma} f\, d\mu/\mu(B_\gamma) \geq b > f(x_0)$, contradicting the fact that $f \in \mathcal{C}$. Thus eventually $\mu(A_\alpha) = 0$. Similary, eventually $\mu(x \in I_\alpha(x_0) : f(x) < a\}) = 0$, and necessity follows.

Conversely if f satisfies the conditions of the theorem, $A_\alpha \Longrightarrow x_0$ and $a < f(x_0) < b$ then eventually $a \leq \int_{A_\alpha} f\, d\mu \leq b$ and hence

$$\lim_{A_\alpha \Longrightarrow x_0} \int_{A_\alpha} f\, d\mu/\mu(A_\alpha) = f(x_0)$$

which completes our proof.

We shall find it convenient to use the notation $A \dot{\subset} B$ to mean $\mu(A \setminus B) = 0$. Thus, our criterion in Theorem 4.3 can be written in the form $I_\alpha \dot{\subset} f^{-1}([a, b])$.

REMARK: The « almost everywhere » inclusion in Theorem 4.3 can be replaced by strict inclusion provided $(\mathcal{J}, \Longrightarrow)$ satisfies the following condition: if $I_\alpha \Longrightarrow x$, $y \in I_\beta$ for some β, and $J_\alpha \Longrightarrow y$ then for γ sufficiently large $\mu(J_\gamma \cap I_\beta) > 0$. To see this suppose $f \in \mathcal{C}$ and $a < f(x_0) < b$. By Theorem 4.3 for α sufficiently large $I_\alpha(x_0) \dot{\subset} f^{-1}[a, b]$. If for some $\varepsilon > 0$ there exists a $y \in I_\alpha(x_0)$ such that $f(y) > b + \varepsilon$ and if $I_\beta(y) \Longrightarrow y$, then eventually $I_\beta(y) \dot{\subset} f^{-1}[b + \varepsilon, \infty)$. But by hypothesis $I_\beta(y) \cap I_\alpha(x_0)$ has positive measure and hence cannot be a.e. contained in two disjoint sets contradicting the existence of y. So for sufficiently large α, $f(y) \leq b$ for all $y \in I_\alpha(x_0)$. Similarly for sufficiently large α, $a \leq f(y)$ for all $y \in I_\alpha(x_0)$. Hence $I_\alpha(x_0) \subset f^{-1}[a, b]$.

DEFINITION 4.4. Let $\mathcal{T}$ be the smallest topology with respect to which all functions in $\mathcal{C}$ are continuous.

THEOREM 4.5. The class of functions continuous with respect to the topology $\mathcal{T}$ is exactly the class $\mathcal{C}$.

PROOF: Clearly each $f \in \mathcal{C}$ is continuous in the $\mathcal{T}$ topology Conversely, if g is continuous in the $\mathcal{T}$ topology and $a < g(x_0) < b$, we must show that if $I_\alpha(x_0) \Longrightarrow x_0$ then $I_\alpha(x_0) \dot{\subset} g^{-1}[a, b]$ for α sufficiently large. But $g^{-1}(a, b)$ is an open set containing x_0 and therefore from the definition of $\mathcal{T}$, $x_0 \in f_1^{-1}(a_1, b_1) \cap \dots \cap f_n^{-1}(a_n, b_n) \subset g^{-1}(a, b)$ for some functions $f_1, \dots, f_n$ in $\mathcal{C}$ and real numbers $a_1, \dots, a_n$; $b_1, \dots, b_n$. But then there exists an $\varepsilon > 0$ such that for each $i = 1, \dots, n$, $a_i + \varepsilon < f_i(x_0) < b_i - \varepsilon$ and therefore for α sufficiently large $I_\alpha(x_0) \dot{\subset} f_1^{-1}[a_1 + \varepsilon, b_1 - \varepsilon] \cap \dots \cap f_n^{-1}[a_n + \varepsilon, b_n - \varepsilon] \subset$ $\subset f_1^{-1}(a_1, b_1) \cap \dots \cap f_n^{-1}(a_n, b_n) \subset g^{-1}[a, b]$ which completes our proof.

We note that the sets of the form $f^{-1}(0, \infty)$ for $f \in \mathcal{C}$ form a basis for $\mathcal{T}$ and that $\mathcal{T}$ is completely regular. These are strictly topological results true for any family of functions generating a topology.

REMARK: There are other ways of topologizing X. For example, the family $\mathcal{J}$ generates a topology for X. Our approach has the virtue, however, that changing the elements of $\mathcal{J}$ on sets of measure zero does not change the topology. We assume, of course, that a net of « changed » sets I'_α contracts to a point x if and only if the net of original sets I_α contracts to x. Changing elements of $\mathcal{J}$ on sets of measure zero can alter the topology generated by the sets in $\mathcal{J}$. For example, in euclidean 2-dimensional space, the family of open squares generate the euclidean topology while the family of closed squares does not. It is easy to verify that using our approach, either of these families gives rise to the euclidean topology. Under certain conditions, the topology generated by $\mathcal{J}$ is the same as the topology determined by $\mathcal{C}$. In particular, this is always the case when $(\mathcal{J}, \Longrightarrow)$ donotes a net structure (see section 6, below).

THEOREM 4.6. If $(\mathcal{J}, \Longrightarrow)$ possesses the weak Vitali property, then every open set is mesurable.

PROOF: A necessary (although not sufficient) condition for a set W to be $\mathcal{J}$-open is that for every $p \in W$ and $I_\alpha \Longrightarrow p$, eventually $I_\alpha \dot\subset W$. Let W be an open set and let $\mathcal{V}$ be the subset of $\mathcal{J}$ consisting of those $I \in \mathcal{J}$ essentially contained in W (i.e., $I \dot\subset W$). Since the differentiation basis satisfies the weak Vitali property, there is a countable collection $\{J_n\} \subset \mathcal{V}$ such that $\mu^*(W \setminus \cup J_n) = 0$. Since $\mu^*(J_n \setminus W) = 0$ for every n, W differs from a measurable set by a set of measure zero and therefore is measurable, which completes our proof.

Even under the hypothesis of Theorem 4.6, it is possible for $\mathcal{J}$ to be trivial. The reason for this is firstly that the abstract notion of contraction is very general and secondly that the weak Vitali property is independent of the meaning of contraction on any null set. Thus even if one has a very natural notion of contraction, one can change this notion on a null set in such a way that one does not destroy the Vitali property but does destroy any desirable property of $\mathcal{J}$. See Example 5.5.

It is natural question to determine those topological spaces $(X, \mathcal{J})$ that arise as in Definition 4.4. They are not completely arbitrary. Clearly $\mathcal{J}$ must be completely regular. Moreover, since $(X, \mathcal{M}, \mu)$ is σ-finite and every open set contains a set of positive measure, $\mathcal{J}$ has property (s) (for Souslin). That is, any disjoint collection of sets in $\mathcal{J}$ must be at most denumerable. This property is called the countable chain condition in Kelley [8]. If $\mathcal{J}$ is a metric topology, then $\mathcal{J}$ has property (s) iff $\mathcal{J}$ is separadle iff every isolated set (i.e. a set whose relative topology is discrete) in X is denumerable iff $\mathcal{J}$ is second countable. Clearly a separable metric space has the following property: any non-denumerable collection of open sets has a non-denumerable subcollection with a non-empty intersection. After Marczewski [11] we say $\mathcal{J}$ has property (k) if any non-denumerable subcollection of $\mathcal{J}$ has a non-denumerable subcollection $\mathcal{S}$ such that $S_1 \cap S_2 \neq \emptyset$ for any two sets S_1, S_2 in $\mathcal{S}$.

Any collection of sets (not necessarily a topology) satisfying condition (k) also satisfies condition (s). But the converse is not true: Sierpinski [18] gave an example of a collection $\mathcal{K}$ of sets such that any uncountable subcollection of $\mathcal{K}$ contains two sets which are not disjoint and two sets that are disjoint. Hence $\mathcal{K}$ satisfies (s) but not (k). Knaster [9] shows that if $\mathcal{K}$ is the class of intervals of a continuous ordered set, then the problem (s) implies (k) is equivalent to Souslin's problem [9]. we show now that $\mathcal{J}$ has property (k), a consequence of the fact that since $(X, \mathcal{M}, \mu)$ is σ-finite, the collection of sets with positive measure has property (k).

DEFINITION 4.7. A collection $\mathcal{K}$ of sets will be said to have property (D_n) if any n distinct sets from $\mathcal{K}$ must include two which are not disjoint.

LEMMA 4.8. Let $n \geq 2$ and let $\mathcal{K}$ be a non-denumerable collection of subsets of X such that $\mathcal{K}$ satisfies property (D_{n+1}). Then there is a non-denumerable subcollection $\mathcal{K}_1$ of $\mathcal{K}$ satisfying property (D_n).

PROOF: By Zorn's lemma there exists a subcollection $\mathcal{A}$ of $\mathcal{K}$ maximal by inclusion among those subcollections of $\mathcal{K}$ satisfying property (D_n). If $\mathcal{A}$ is non-denumerable, then let $\mathcal{K}_1 = \mathcal{A}$. Thus suppose $\mathcal{A}$ is denumerable. By the maximality of $\mathcal{A}$, for each of the sets $E \in \mathcal{K} \setminus \mathcal{A}$ there is a collection of $n-1$ distinct sets of $\mathcal{A}$, say $A_{i_1} \ldots , A_{i_{n-1}}$ such that $\{E, A_{i_1}, \ldots , A_{i_{n-1}}\}$ is pairwise disjoint. Since $\mathcal{K} \setminus \mathcal{A}$ is non denumerable and the set of collections of $n-1$ sets from $\mathcal{A}$ is denumerable, there is at least one collection of $n-1$ sets from $\mathcal{A}$, say $A_{i_1}, \ldots , A_{i_{n-1}}$ such that $\mathcal{B} = \{E \in \mathcal{K} \setminus \mathcal{A} : \{E, A_{i_1}, \ldots, A_{i_{n-1}}\}$ is pairwise disjoint$\}$ is non-denumerable. But $\mathcal{B}$ satisfies property (D_2) and therefore certainly property (D_n). For if B_1 and B_2 are distinct sets in $\mathcal{B}$,) $\{B_1, B_2, A_{i_1}, \ldots , A_{i_{n-1}}\}$ cannot be disjoint since $\mathcal{B}$ satisfies property $(D_{n+1}$ and the only two sets that could possibly meet are B_1 and B_2. Thus we let $\mathcal{B} = \mathcal{K}_1$ which completes our proof.

The following corollary answers affirmatively a question posed by Marczewski [11].

COROLLARY 4.9 If $(X, \mathcal{M}, \mu)$ is σ-finite, then $\{E \in \mathcal{M} : \mu(E) > 0\}$ satisfies property (k).

PROOF: It is elementary to show that we may assume $\mu(X) > \infty$. If $\mathcal{S}$ is any non-denumerable collection of sets in $\mathcal{M}$ with positive measure, then there exists an $\varepsilon > 0$ such that $\mathcal{K} = \{E \in \mathcal{S} : \mu(E) \geq \varepsilon\}$ is non-denumerable. $\mathcal{K}$ satisfies property (D_{n+1}) if $n\varepsilon \geq \mu(X)$ and therefore by Lemma 4.8 $\mathcal{K}$ contains a non-denumerable subcollection satisfying property (D_2). Hence no two sets in $\mathcal{K}$ are disjoint which completes our proof.

5 Examples.

We shall now give several examples of topological spaces $(X, \mathcal{T})$ arising as in Definition 4.4 with the view of showing which topological properties are not consequences of our definition. Since we are most interested in those topologies which arise from differentiation bases satisfying the weak Vitali condition, most of our examples will be of this type. First a simple observation.

THEOREM 5.1. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. If $\mathcal{T}_0$ is a completely regular topology on X such that each nonempty open set is measurable and has positive measure, and if $(\mathcal{I}, \Longrightarrow)$ is a differentiation basis such that each $I \in \mathcal{I}$ is $\mathcal{T}_0$ open and $I_\alpha \Longrightarrow p$ means the set $\{I_\alpha\}$ is directed by inclusion and forms a $\mathcal{T}_0$ neighborhood basis for p, then $\mathcal{T}_0 = \mathcal{T}$, the topology of Definition 4.4.

PROOF: Since both $\mathcal{T}$ and $\mathcal{T}_0$ are completely regular we need only show the two topologies have the same class of continuous functions. Since the differentiation basis satisfies the conditions of the remark following Theorem 4.3, a function f is $\mathcal{T}$-continuous if and only if for $f(x) \in (a, b)$, $x \in X$, there is a $\mathcal{T}_0$ open set containing x which f maps into (a, b), and this of course is true if and only if f is $\mathcal{T}_0$ continuous which completes our proof.

EXAMPLES 5.2. Let S be a set of cardinality greater than c, the cardinal number of the real numbers. Let $X = [0,1]^S$ with the product topology. We define μ on rectangles by $\mu(\{x : x(s_i) \in E_i \text{ for all } i = 1, \ldots, n\}) = \lambda(E_1)\lambda(E_2)\ldots\lambda(E_n)$ where $E_1, \ldots, E_n$ are Lebesgue measurable subsets of $[0, 1]$ and λ is Lebesgue measure on $[0,1]$. We then can extend μ by Caratheodory's method. Theorem 5.1 applies if we let the differentiation basis consist of the open sets. Thus $(X, \mathcal{T})$ is not separable nor first countable. However, as far as we know at present, $(X, \mathcal{T})$ cannot be defined via a differentiation basis that satisfies the weak Vitali condition.

It is interesting to note that in the classical torus space, $[0, 1]^S$ with S the set of positive integers (see [2, 17]), one can construct a differentiation basis for which the Vitali theorem holds. This basis does not give rise to the product topology, however. The differentiation basis is not formed by taking the family of all « intervals » as elements of the differentiation basis. (Here an interval is any rectangle as defined above with each E_i being an interval in $[0, 1]$). In fact, Jessen [7] has shown that this family of intervals does not satisfy the weak Vitali property even for the case where S is the set of positive integers.

EXAMPLE 5.3. Let X be the set of pairs of real numbers, and let $\mathcal{T}_0$ be the topology generated by all sets of the form $\{(x, y) : a_1 \leq x < a_2$ and $b_1 \leq y < b_2\}$. With $(X, \mathcal{M}, \mu)$ being the Lebesgue measure space and the differentiation basis consisting of the above defined half open rectangles, the conditions of 5.1 are met and $(X, \mathcal{T}) = (X, \mathcal{T}_0)$. This differentiation basis does satisfy the Vitali condition. The space $(X, \mathcal{T})$ is separable, has an uncountable isolated set, is first countable and is not normal. For the analogous space in one dimension the continuous functions are easy to describe: they are those functions which in the usual sense are continuous from the right.

EXAMPLE 5.4. Let $X = (-1, 1) \times (-1, 1)$ and $\mathcal{M}_0$ be the Borel subsets of X and let $\mu(A) = \lambda_2(A) + \lambda_1(A)$ for $A \in \mathcal{M}_0$ where $\lambda_2(A)$ is the two dimensional Lebesgue measure of A and $\lambda_1(A)$ is the one dimensional Lebesgue measure of $\{x \in (-1, 1) : (x, 0) \in A\}$. Let $\mathcal{M}$ be the μ-completion of $\mathcal{M}_0$. We define as a differentiation basis $(\mathcal{T}, \Longrightarrow) : I_\alpha \Longrightarrow p = (x_0, y_0)$ means $\{I_\alpha\}$ is a subnet of the sequence $I_k = \left\{(x, y) : (x - x_0)^2 + (y - y_0)^2 < \frac{1}{k}\right\}$ if $(x_0, y_0) \neq (0, 0)$ and $I_k = \left\{(x, y) : x^2 + y^2 < \frac{1}{k} \text{ or } y = 0\right\}$ if $(x_0, y_0) = (0, 0)$. The continuous functions for this differentiation basis are just those functions continuous in the ordinary sense and which are constant on the x-axis. The topology $(X, \mathcal{T})$ they induce is separable, not Hausdorff and not first countable.

EXAMPLE 5.5. Let $X = (-1, 1)$ with Lebesgue measure μ and differentiation basis consisting of open intervals where $I_\alpha \Longrightarrow x$ for $x \neq 0$ means $x \in I_\alpha$ for all α and diam (I_α) converges to 0 and $I_\alpha \Longrightarrow 0$ means $I_\alpha = X$ for all α. The continuous functions for this differentiation basis are constant and hence $(X, \mathcal{T})$ is trivial.

EXAMPLE 5.6. Let $(X, \mathcal{T}_0)$ be defined as follows: let X be the real line. A Lebesgue measurable set is $\mathcal{T}_0$ open if each point of the set is a point of density of the set. Let μ be Lebesgue measure. We let $(\mathcal{T}, \Longrightarrow)$ consist of those $\mathcal{T}_0$-open sets A such that $\mu(A)/\text{diam}\, A \geq \frac{1}{2}$ with the 5.1 sense of contraction to a point. Then $(X, \mathcal{T}_0)$ is completely regular and the $\mathcal{T}_0$-continuous functions are just the approximately continuous functions (see

Goffmann, Neugebauer, and Nishiura [3]. The differentiation basis posseses the weak Vitali property since the family of intervals possesses this property and $\mathcal{J}$ is regular with respect to the intervals. Thus the conditions of Theorem 5.1 are satisfied so that $(X, \mathcal{J}) = (X, \mathcal{J}_0)$. This topological space is not separable or first countable. In this example no differentiation basis can be chosen so that it yields the $\mathcal{J}_0$ topology and is also « sequential » in that $I_k \Longrightarrow p$ for each $p \in X$.

6. Separable measure spaces.

We turn now to the case in which $(X, \mathcal{M}, \mu)$ is a separable and σ-finite measure space, and we make that assumption throughout this section. We shall construct a very simple sort of differentiation basis called a net structure for which the strong Vitali property holds. A net structure can be described as follows.

Let $\mathcal{N} = \{\mathcal{N}_1, \mathcal{N}_2, ...\}$ where each $\mathcal{N}_i$ is a partition of X into sets of finite positive measure and $\mathcal{N}_{i+1}$ refines $\mathcal{N}_i$. We call $\mathcal{N}$ a net structure. We define a differentiation basis $(\mathcal{J}, \Longrightarrow)$ from $\mathcal{N}$ as follows: $I_a \Longrightarrow x$ means $\{I_a\}$ is a subnet of the sequence $\{I_k\}$ where I_k is the element of $\mathcal{N}_k$ containing x. $\mathcal{J}$ has the following property: For any subcollection $\mathcal{V}$ of $\mathcal{J}$ there is a subcollection $\mathcal{O}$ of $\mathcal{V}$ such that the elements of $\mathcal{O}$ are mutually disjoint and $\bigcup_{I \in \mathcal{O}} I = \bigcup_{I \in \mathcal{V}} I$. This is a consequence of the fact that for $I_1, I_2 \in \mathcal{J}$, $I_1 \cap I_2$ is either $\emptyset$, I_1 or I_2. For we merely let $\mathcal{O} = \{I \in \mathcal{V}: I \not\subset J$ for any $J \in \mathcal{V}$ other than $I\}$. We now construct a net for $(X, \mathcal{M}, \mu)$, first with the assumption that $\mu(X) < \infty$.

Since $(X, \mathcal{M}, \mu)$ is separable, there exist a countable collection $\mathcal{A} = \{A_n\}$ of sets in $\mathcal{M}$ such that for every $\varepsilon > 0$ and $E \in \mathcal{M}$, there exists an n such that $\mu(E \triangle A_n) < \varepsilon$. We define $\mathcal{N}_n$ to be the partition of X obtained by the relation $x \equiv y$ if and only if for every $k = 1, ..., n$ $x \in A_k$ if and only if $y \in A_k$. $\mathcal{N} = \{\mathcal{N}_1, \mathcal{N}_2, ...\}$ only fails to be a net in that some of the elements in $\bigcup \mathcal{N}_i$ might have zero measure. But by an obvious argument we can redefine each A_i on a set with measure zero so that every element in $\bigcup \mathcal{N}_i$ has positive measure. Now if $\mu(X) = \infty$, since X is σ-finite, we may write X as the disjoint union of sets X_n with finite positive measure and each X_n having a net $\mathcal{N}^n = \{\mathcal{N}_1^n, \mathcal{N}_2^n, ...\}$. We define $\mathcal{N} = \{\mathcal{N}_1, \mathcal{N}_2, ...\}$ where $\mathcal{N}_i = \bigcup_{n=1}^{\infty} \mathcal{N}_i^n$.

We now show that $(\mathcal{J}, \Longrightarrow)$ satisfies the strong Vitali condition: Let B be any subset of X, not necessarily measurable. Let $\overline{B}$ be a measurable cover of B. Because of the σ-finite condition, it suffices to suppose $\overline{B}$ has finite measure. Let $\mathcal{V}$ be a Vitali covering of B. Let $\{\varepsilon_n\}$ be a sequenee of positive numbers. We choose inductively $A_{n_1}, A_{n_2}, \ldots$ so that $\mu(\bar{B} \triangle A_{n_1}) < \varepsilon_1$, $\mu\left(\left[\bar{B} \setminus \bigcup_{j=1}^{k} A_{n_j}\right] \triangle A_{n_{j+1}}\right) < \varepsilon_{j+1}$. Then for $C = \bigcup A_{n_j}$, $\mu(\bar{B} \setminus C) \leq \inf \varepsilon_j$, $u(C) \leq \Sigma \mu(A_{n_j}) \leq \mu(\bar{B}) + \varepsilon_1 + (\varepsilon_1 + \varepsilon_2) + (\varepsilon_2 + \varepsilon_3) + (\varepsilon_3 + \varepsilon_4) + \ldots$. Hence given $\varepsilon > 0$, there exists a set C, a union of sets from $\{A_n\}$ and therefore a union of elements of $\mathcal{J}$, such that $\mu(\bar{B} \setminus C) = 0$ and $\mu(C) \leq \mu(\bar{B}) + \varepsilon$. Let Q denote those sets of $\mathcal{V}$ contained in C and let $C' = \bigcup_{I \in Q} I$. Since $\mathcal{V}$ is a Vitali covering of B, $B \setminus C' = B \setminus C$. Thus $\mu^*(B \setminus C') = 0$ and $\mu(C') \leq \mu^*(B) + \varepsilon$. Since C' can also be written as a countable disjoint union of elements of $\mathcal{V}$, $(\mathcal{J}, \Longrightarrow)$ satisfies the Vitali condition.

Since the strong Vitali property holds for $\mathcal{J}$, it is true (see de Possel [15]) that the Fundamental Theorem of Calculus holds for every summable function f; that is, if f is summable on X and $\sigma(E) = \int_E f\, d\mu$ for every $E \in \mathcal{M}$, then $D\sigma(x) = f(x)$ a.e..

Now let $\mathcal{J}$ be the topology generated by $\mathcal{C}$. Each $I \in \mathcal{J}$ is open. In fact, $\mathcal{J}$ forms a base for $\mathcal{T}$. To see this, first, since the conditions of the remark following Theorem 4.3 are satisfied, if W is an open set and $p \in W$, then there exists $I \in \mathcal{J}$ such that $p \in I \subset W$. Conversely if $I \in \mathcal{J}$ and χ is the characteristic function for I, then χ is continuous. For if $I_\alpha \Longrightarrow x$ then either eventually $\chi \equiv 1$ on I_α or $\chi \equiv 0$ on I_α depending on whether x is in I or not.

We now show that the topological space $(X, \mathcal{J})$ is pseudo-metrizable. For each n we define $\varrho_n(x, y)$ to be 0 if x and y belong to the same set in $\mathcal{N}_n$ and to be 1 if not. We define $\varrho(x, y) = \sum_{n=1}^{\infty} \frac{\varrho_n(x, y)}{2^n}$. $(X, \mathcal{J})$ is topologized by this pseudo metric: for let $y \in X$ and let I_n be the element of $\mathcal{N}_n$ containing y. Then

$$\left\{x : \varrho(x, y) < \frac{1}{2^n}\right\} \subset I_n \subset \left\{x : \varrho(x, y) \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n}\right\}.$$

It is easy to check that ϱ is a metric if and only if each decreasing sequence of sets of $\mathcal{J}$ contains at most one point, and that (X, ϱ) is complete if and only if each decreasing sequence of sets of $\mathcal{J}$ contains at least one point.

We note that a decreasing sequence of sets in $\mathcal{J}$ might intersect in more than one point. Unless this intersection is an atom, however, it must have measure zero. (This is a consequence of the separability of the measure space and the way in which the net structure was defined). In any case, the derivative of an integral σ is constant on any such intersection.

We turn now to certain questions concerning the « compatibility » of μ with $\mathcal{J}$. Although μ need not be a regular measure (see the example following Theorem 6.2 below), it does satisfy a certain regularity condition.

THEOREM 6.1. If $E \in \mathcal{M}$ and $\varepsilon > 0$ there exists an open set G such that $E \dot{\subset} G$ and $\mu(G \triangle E) < \varepsilon$.

PROOF: We first recall that if I and J are in $\mathcal{J}$ then either $I \cap J = \emptyset$ or $I \subset J$ or $J \subset I$. It follows that the characteristic function of a set I in $\mathcal{J}$ is continuous, and therefore I is both open and closed. Now, if A is any set in the countable basis for $(X, \mathcal{M}, \mu)$, our construction of the net structure shows that A is a finite union of sets in $\mathcal{J}$. Therefore A is open and closed. Because of σ-finiteness, for $\varepsilon_1 > 0$ and $E \in \mathcal{M}$ there exists a set A which is a countable union of sets in $\mathcal{A}$ and hence open such that $\mu(A \triangle E) < \varepsilon_1$. It follows from a straightforward argument that there exists a set B which is a countable union of sets in $\mathcal{A}$ (and therefore open) such that $E \dot{\subset} B$ and $\mu(B \triangle E) < \varepsilon$ which completes our proof.

We now show that the class of continuous functions is sufficiently large to approximate the class of measurable functions in the Lusin sense. In this connection we should say that Lusin's theorem takes several forms, the strongest of which is that for $\varepsilon > 0$ and for f an arbitrary measurable function there is a continuous function g such that $f = g$ on an closed set whose complement has measure less than ε. This result does not hold in our setting (see the example following Theorem 6.2), but it does hold if one replaces the word « closed » by the word « measurable ».

THEOREM 6.2. Let $\varepsilon > 0$ and let f be a mesurable function on the separable σ-finite measure space $(X, \mathcal{M}, \mu)$. Let $\mathcal{J}$ be the net structure defined above. Then there exists a continuous function g and a measurable set E such that $\mu(X \setminus E) < \varepsilon$ and $f = g$ on E.

PROOF: First suppose $(X, \mathcal{M}, \mu)$ is a finite measure space. Let $f = \sum_{k=1}^{n} \alpha_k \chi_{E_k}$ where $E_k \in \mathcal{M}$ for all $k = 1, \dots, n$ and $E_j \cap E_k = \emptyset$ for $i \neq j$. Let $\delta > 0$. We choose $F_1, F_2, \dots, F_n$ from $\mathcal{A}$ such that $\mu(F_k \triangle E_k) \leq \delta$ for $k = 1, \dots, n$. Then each F_k is closed and open. We define g to be continuous such that $g \equiv \alpha_1$ on F_1, $g \equiv \alpha_2$ on $F_2 \setminus F_1, \dots, g = \alpha_n$ on $F_n \setminus (F_1 \cup \dots \cup F_{n-1})$. Now

$$\{x \in X : g(x) \neq f(x)\} \subset (E_1 \triangle F_1) \cup (E_2 \triangle F_2 \cup F_2 \cap F_1)$$
$$\dots \cup (E_n \triangle F_n \cup F_n \cap (F_1 \cup \dots \cup F_{n-1})).$$

Since $\{E_k\}$ is a disjoint collection and $\mu(E_k \triangle F_k) \leq \delta$ for all $k = 1, \ldots, n$, therefore

$$\mu(\{x \in X : g(x) \neq f(x)\} \leq (\delta) + (\delta + \delta) + \ldots + (\delta + \delta) = (2n - 1)\delta.$$

Since δ is arbitrary, we see that for any $\varepsilon > 0$ there exists a continuous function g such that $\mu(\{x : f(x) \neq g(x)\}) \leq \varepsilon$.

Let h be a measurable function. By Egoroff's theorem, for all n there exists a step function f_n such that $\mu\left(\left\{x \in X : |f_n(x) - h(x)| \geq \frac{1}{n}\right\}\right) \leq \frac{1}{n}$. But the above paragraph shows that there exists a continuous function g_n such that $\mu\left(\left\{x \in X : |g_n(x) - h(x)| \geq \frac{1}{n}\right\}\right) \leq \frac{2}{n}$. Hence the sequence $\{g_n\}$ converges to h in measure. Hence a subsequence $\{g_{n_k}\}$ converges to h almost uniformly. For each k let $\bar{g}_k = g_{n_k}$ and let $\varepsilon > 0$ be given. There exists a set E such that $\mu(X \setminus E) < \varepsilon$ and $\bar{g}_n \mid E$ converges uniformly to $h \mid E$. By passing to a subsequence of $\{\bar{g}_n\}$ if necessary, we may suppose $\sup\limits_{x \in E} |\bar{g}_n(x) - h(x)| \leq \frac{1}{2} n$. Let

$$K = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \left\{x : |\bar{g}_n(x) - \bar{g}_{n+m}(x)| \leq \frac{2}{2^n}\right\}.$$

K is a closed set containing E and $\bar{g}_n$ converges uniformly on K to a function $g \mid K$ which is the restriction of a continuous function g. Then $g \mid E = h \mid E$ and $\mu(X \setminus E) < \varepsilon$.

We now consider the general case in which $(X, \mathcal{M}, \mu)$ is σ-finite. Since X is a disjoint denumerable union of open sets of finite measure, any function which is continuous on each of these sets is continuous on X. The result follows immediatly.

We next show that the somewhat stronger form of Lusin's theorem is not valid, namely we cannot insist that the set E be closed. More precisely: there is a finite separable measure space $(X, \mathcal{M}, \bar{\mu})$ and a net structure on this space and a measurable function f so that any closed subset K of X such that $f|K$ is continuous must satisfy $\bar{\mu}(K) = 0$. To see this let $X = [0, 1]$, μ be Lebesgue measure, $\mathcal{S}$ denote the Lebesgue measurable sets and let M be a subset of X such that $\mu_*(M) = 0$ and $\mu^*(M) = 1$. Let $\mathcal{M}$ be the σ-ring generated by $\mathcal{S} \cup \{M\}$, namely $\{(E \cap M) \triangle (F \setminus M) : E, F \in \mathcal{S}\}$ and define $\bar{\mu}$ on $\mathcal{M}$ by $\bar{\mu}((E \cap M) \triangle (F \setminus M)) = \mu(F)$. The verification that $(X, \mathcal{M}, \bar{\mu})$ is a measure space extending $(X, \mathcal{S}, \mu)$ is sketched in Halmos [5 : p. 71]. $(X, \mathcal{S}, \mu)$ is separable and $\mathcal{S}$ is dense in $\mathcal{M}$, since $\bar{\mu}(F \triangle [(E \cap M) \triangle (F \setminus M)] = \bar{\mu}((F \setminus M) \triangle (F \setminus M)) = 0$

If we choose as a countable dense set for $\mathcal{M}$, a sequence $\{A_n\}$ from $\mathcal{S}$, then the open sets in the topology generated by the net structure determined by $\{A_n\}$ are elements of $\mathcal{S}$. Any continuous function must then be $\mathcal{S}$ measurable. We let $f = \chi_M$. Suppose K is a closed subset of X and $f \mid K$ is continuous, say the restriction of the continuous function g to K. Then $M \cap K = \{x \in X : g(x) = 1\} \cap K$ and is measurable. Thus both $K \cap M$ and $K \setminus M$ are measurable. Since $\mu_*(M) = 0$ we have $\mu(K \cap M) = 0$ and since $\mu^*(M) = 1$ we have $\mu(K \setminus M) = 0$ and hence $\mu(K) = 0$.

REMARK : It follows immediately from Theorem 6.2 that every measurable function is equivalent to a function in Baire classe 2. We need only note that if f is measurable, there exists a sequence $\{g_n\}$ of continuous functions which converge a.e. to f. The function $g(x) = \limsup_{n \to \infty} g_n(x)$ in then in Baire class 2 and $g = f$ a.e

Several authors have written developments of the theory of differentiation of integrals with respect to net structures in abstract measure spaces; see, for example, the texts Munroe [13] and Gurevič and Shilov [4]. The results we established above indicate that for certain purposes one may assume the measure space to be pseud metrized. In particular, the net structure we constructed satisfies all the conditions required of a net structure in metric measure spaces of finite measure as developed in Saks [17]. (The fact that we have a pseudo metric space rather than a metric is inessential to Saks' development). Thus the results that Saks obtains are valid in our setting.

7. Open problems.

The results of the previous sections suggest a number of problems. We list a few of these.

1. For a fixed measure space $(X, \mathcal{M}, \mu)$, under what circumstances will two differentiation bases give rise to the same topology?

We saw in section 6 that a separable measure space always admits of a differentiation basis for which the strong Vitali property holds and such that the resulting topology $\mathcal{T}$ is pseudo-metrizable. These results suggest the following question.

2. When is $\mathcal{T}$ metrizable?

We say in section 4 that $\mathcal{T}$ must always satisfy certain conditions related to, but weaker than separability. In case $(X, \mathcal{M}, \mu)$ is separable, the topology $\mathcal{T}$ derived from the net structure is clearly separable. This is true for any metric topology coming from a differentiation basis. On the other hand, Example 5.2 shows $\mathcal{T}$ need not be separable in general.

3. Under what circumstances must $\mathcal{T}$ be separable?

4. Let $(X, \mathcal{M}, \mu)$ be a measure space furnished with a topology $\mathcal{T}^*$. Under wath circumstances does $\mathcal{T}^*$ come from a differentiation basis with a Vitali property? (Theorem 5.1 shows that if $\mathcal{T}^*$ is completely regular and every nonempty open set has positive measure then $\mathcal{T}^*$ comes from a differentiation basis, but such a basis need not possess a Vitali property).

5. If $(X, \mathcal{M}, \mu)$ is separable, then as we saw in section 6 the net structure gives rise to a pseudo-metrizable topology. It is clear that if the cardinality of X is larger than c, this pseudo metric cannot be a metric. If the cardinality of X is c, must it always be possible to construct that net in such a way as to make ϱ a legitimate metric?

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