ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

GERD GRUBB

A characterization of the non local boundary value problems associated with an elliptic operator

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 22, nº 3 (1968), p. 425-513
<http://www.numdam.org/item?id=ASNSP_1968_3_22_3_425_0>

A CHARACTERIZATION OF THE NON-LOCAL BOUNDARY VALUE PROBLEMS ASSOCIATED WITH AN ELLIPTIC OPERATOR

GERD GRUBB

Introduction.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ, and let A be an elliptic differential operator of order $2m$, with smooth coefficients defined in $\bar{\Omega}$. (The boundedness of Ω is assumed for convenience, but is not essential for the main results of the paper). Together with A are given m boundary differential operators B_j, $j = 0, \dots, m-1$, such that the system $\{B_j\}$ is normal and covers A (terminology as in Schechter [28]). We assume that the boundary value problem $Au = f$ in Ω, $B_j u = 0$ in Γ, $j = 0, \dots, m-1$, is uniquely solvable for given f (f and u in suitable function spaces).

With A one can associate certain operators in $L^2(\Omega)$. The maximal operator A_1 is: A defined on the set of $u \in L^2(\Omega)$ for which Au (defined weakly) $\in L^2(\Omega)$, and the minimal operator A_0 is defined as the closure of: A defined on the C^∞ functions with compact support in Ω. The linear operators $\widetilde{A}$ between A_0 and A_1 will be called realizations of A. An example of a realization is the closure of: A defined on smooth functions u which satisfy the boundary condition $B_j u = 0$ in Γ, $j = 0, \dots, m-1$. This realization is determined by a *boundary condition;* more generally one views the operators in the family of realizations as representing abstract bondary conditions. This is justified by the fact that the domains $D(\widetilde{A})$ are determined by the behavior of the functions $u \in D(\widetilde{A})$ near the boundary Γ, since $D(A_1) \subset H^{2m}_{\mathrm{loc}}(\Omega)$, and $D(A_0) = H^{2m}_0(\Omega)$. (The Sobolev-spaces $H^s(\Omega)$, $H^s_0(\Omega)$, $H^s(\Gamma)$ (s real) are defined in detail in I § 1).

Pervenuto alla Redazione il 18 Dicembre 1967 ed in forma definitiva il 12 Febbraio 1968.

The realizations given by boundary conditions in terms of boundary differential operators (like B_j) have been investigated thoroughly in recent years. Numerous references should be given here; we will just mention [1], [3], [18], [23], [24], [28], which are of use in the present paper.

More recently, certain non-local boundary value problems (called non-local because they contain boundary operators that are not necessarily differential operators) have been discussed; mostly in the form of generalized versions of

$$\frac{\partial u}{\partial n} = Ku \quad \text{on} \quad \Gamma \tag{1}$$

(the typical boundary condition for $m = 1$), where K is a possibly non-local operator in Γ, see [2], [4], [5], [6], [12], [29]. (The paper [29] also treats a different type of boundary condition).

Thus, by the introduction of more and more general boundary conditions, more and more general realizations have been considered. The present paper treats a converse problem: to find, to an arbitrary realization, a specific boundary condition, expressed in terms of boundary operators from $D(A_1)$ to $\mathcal{D}'(\Gamma)$ and operators between spaces over Γ, that it represents. It is shown how this is possible for all closed realizations; some non-closed realizations are included in a natural way. The result is given in the form of a 1-1 correspondence between the closed realizations $\widetilde{A}$ and the closed operators L between certain spaces related to the spaces $H^s(\Gamma)$ (III § 2).

The correspondence between $\widetilde{A}$ and L becomes more interesting by the fact that it preserves properties of the operators such as dimension of null-space, closedness of range and codimension of range; and the adjoint $\widetilde{A}^*$ corresponds analogously to the adjoint L^* (III § 2). Moreover, the correspondence preserves regularity (III § 3) (i. e., the property of having $D(\widetilde{A})$ (graphtopology) continuously imbedded in $H^s(\Omega)$, $s > 0$, corresponds to a similar property of L), and, if A is formally selfadjoint, it preserves spectral and numerical properties (III § 4).

The theory requires introduction of a certain non-local boundary operator from $D(A_1)$ into $(L^2(\Gamma))^m$, for which a Green's formula holds for *all* $u \in D(A_1)$, *all* $v \in D(A_1')$ (III § 1) (A' is the formal adjoint of A).

Since the main part of earlier work on non-local boundary conditions was concerned with the condition (1), we have included some considerations concerning this particular type of problem. In III § 2 we give a necessary and sufficient condition that a closed realization represents this type. In III § 6 are given some further results on this kind of boundary value problem, deduced from our theory; they overlap with [29].

The immediate background for our theory is the theory for the non-homogeneous boundary value problem

$$(2) \qquad \begin{cases} Au = f \\ B_j u = g_j, \quad j = 0, \dots, m-1, \end{cases}$$

as developed by Lions and Magenes in [24], [24′]. Those of their results that we use, and some underlying results, are briefly presented in Chapter I.

Chapter II contains the Hilbert space theory (abstract theory) which is the basis for our approach. It has points in common with Vishik [32], but differs in a way that makes it possible to treat all closed realizations, not only those with closed range. Notation, and some preliminary results for this chapter, are explained in the Appendix.

Finally, Chapter III combines I and II to give the main results.

The present paper is a revised edition of the author's doctoral dissertation [15]. The main change is that we have omitted considerations concerning sesquilinear forms, in particular a systematic discussion of the realizations associated with sesquilinear forms in $L^2(\Omega)$ (not in general continuous on $H^m(\Omega)$) for the case $A = A'$, strongly elliptic, given in [15]. Other changes: we use a more general notion of ellipticity here, and base our considerations on the boundary value problem (2) instead of the Dirichlet problem from the start; also some results have been sharpened.

The author would like to express her warmest gratitude to her advisor, Professor Ralph S. Phillips, for his advice and encouragement throughout the work.

Chapter I. - PRELIMINARIES

Our assumptions are basically the same as those in Lions-Magenes [24] V, except that we do not work with L^p-spaces with $p \neq 2$.

§ 1. Spaces.

Let Ω be a bounded domain in $\mathbb{R}^n$ (with generic point $x = (x_1, \dots, x_n)$); the boundary Γ is assumed to be an $n - 1$ dimensional C^∞ manifold. Let us note at this point that the assumptions of smoothness can be weakened considerably; we will not make any efforts in this direction. The condition that Ω be bounded is not essential; cases where Ω is unbounded (and where the coefficients of the differential operators satisfy the various conditions in some uniform sense) could also be treated by our methods. Only in certain applications (Theorem III 2.5 and part of III § 6) will the boundedness be of importance.

We denote by $\mathcal{D}(\bar{\Omega})$ the space of functions which are infinitely differentiable in $\bar{\Omega}$, by $\mathcal{D}(\Omega)$ the space of functions belonging to $\mathcal{D}(\bar{\Omega})$ which have compact support in Ω, and by $\mathcal{D}'(\Omega)$ Schwartz' space of distributions in Ω.

Let p be the multi-index $\{p_1, \dots, p_n\}$, then D^p denotes the differential operator

$$D^p = \frac{\partial^{p_1}}{\partial x_1^{p_1}} \cdots \frac{\partial^{p_n}}{\partial x_n^{p_n}};$$

it is of order $|p| = p_1 + \dots + p_n$.

We shall need several types of Sobolev spaces; the definitions given here follow Hörmander [18], with a slightly different notation.

Let s be any real number. Then $H^s(\mathbb{R}^n)$ is defined as the space of $u \in \mathcal{S}'$ ($\mathcal{S}'$ is the space of temperate distributions in $\mathbb{R}^n$) for which

$$(1 + |\xi|^2)^{s/2} \hat{u}(\xi) \in L^2(\mathbb{R}^n);$$

here $\hat{u}(\xi)$ denotes the Fourier transform of $u(x)$. $H^s(\mathbb{R}^n)$ is a Hilbert space with the norm

$$|u|_s = |(1 + |\xi|^2)^{s/2} \hat{u}(\xi)|_{L^2(\mathbb{R}^n)}.$$

Now define

(1.1) $H^s(\Omega)$ = the space of $u \in \mathcal{D}'(\Omega)$, for which there exist $U \in H^s(\mathbb{R}^n)$ with $U = u$ in Ω; $H^s(\Omega)$ is a Hilbert space with the norm $|u|_s = \inf |U|_s$, the infimum taken over all such U;

(1.2) $H_0^s(\Omega)$ = the space of $u \in H^s(\mathbb{R}^n)$ which have support in $\bar{\Omega}$; $H_0^s(\Omega)$ is a closed subspace of $H^s(\mathbb{R}^n)$.

Here $\mathcal{D}(\bar{\Omega})$ is dense in $H^s(\Omega)$ and $\mathcal{D}(\Omega)$ is dense in $H_0^s(\Omega)$. Moreover, $H^{-s}(\Omega)$ and $H_0^s(\Omega)$ are strong anti-duals with respect to an extension of

$$\int_\Omega u\bar{v}\,dx, \quad u \in \mathcal{D}(\bar{\Omega}), \quad v \in \mathcal{D}(\Omega).$$

When s is a positive integer, $H^s(\Omega)$ is the space of $u \in L^2(\Omega)$ for which $D^p u$ (defined in the distribution sense) is in $L^2(\Omega)$ for all $|p| \leq s$, and the norm is equivalent with

$$\left(\sum_{|p| \leq s} |D^p u|_0^2\right)^{\frac{1}{2}}.$$

When s and s' are real numbers with $s \leq s'$, $H^{s'}(\Omega) \subset H^s(\Omega)$ algebraically and topologically (i. e., the imbedding is continuous); furthermore, $H^{s'}(\Omega)$ is dense in $H^s(\Omega)$.

$H^s(\Gamma)$ is defined by local coordinates, using the definition of $H^s(\mathbb{R}^{n-1})$ (see e. g. Hörmander [18]). $\mathcal{D}(\Gamma)$ is dense in $H^s(\Gamma)$, for all s. It follows from the boundedness of Γ that every distribution $\in \mathcal{D}'(\Gamma)$ belongs to $H^s(\Gamma)$ for some s.

$H^s(\Gamma)$ and $H^{-s}(\Gamma)$ are strong anti-duals with respect to an extension of

$$\int_\Gamma \varphi\bar{\psi}\,d\sigma, \quad \varphi, \quad \psi \in \mathcal{D}(\Gamma);$$

the duality will be denoted by $\langle \varphi, \psi \rangle_{s\ -s}$.

We shall often have to consider product spaces

$$\prod_{j=0}^{m-1} H^{s_j}(\Gamma), \quad s_j \in \mathbb{R}, \quad j = 0, \dots, m-1.$$

The summation will nearly always be over $j = 0, \dots, m-1$, where m is a fixed number, in that case we just write

$$\Pi H^{s_j}(\Gamma).$$

If the norm in $H^{s_j}(\Gamma)$ is $|\cdot|_{s_j}$, $j = 0, \dots, m-1$, we denote the norm

$$\left(\sum_{j=0}^{m-1} |\varphi_j|_{s_j}^2\right)^{\frac{1}{2}} \text{ of } \varphi = \{\varphi_0, \dots, \varphi_{m-1}\} \in \Pi H^{s_j}(\Gamma) \text{ by } |\varphi|_{\{s_j\}}.$$

Also, the duality between $\Pi H^{s_j}(\Gamma)$ and $\Pi H^{-s_j}(\Gamma)$ will be denoted by

$$\langle \underset{\{s_j\}}{\varphi}, \underset{\{-s_j\}}{\psi} \rangle, \text{ which stands for } \sum_{j=0}^{m-1} \langle \underset{s_j}{\varphi_j}, \underset{-s_j}{\psi_j} \rangle.$$

Many of the results in the following are based on the interpolation theory for the spaces $H^s(\Omega)$, $H^s(\Gamma)$ and related spaces. We shall only use it directly a few times, each time in the form of the following theorem. (s_j and u_j, $j = 0, \dots, m-1$, denote real numbers; m a positive integer. $(\mathcal{D}'(\Gamma))^m$ denotes the product space of $\mathcal{D}'(\Gamma)$ with itself m times).

THEOREM 1.1. *Let r_0 and r_1 be real numbers ($r_0 \leq r_1$). Let K be an operator in $(\mathcal{D}'(\Gamma))^m$ which maps $\Pi H^{r+s_j}(\Gamma)$ continuously into ΠH^{r+u_j} for $r = r_0$ and $r = r_1$. Then K maps $\Pi H^{r+s_j}(\Gamma)$ continuously into $\Pi H^{r+u_j}(\Gamma)$ for all $r_0 \leq r \leq r_1$.*

I § 2. Boundary differential operators.

Denote by γ_0 the mapping of $u \in \mathcal{D}(\bar{\Omega})$ into its boundary value $\gamma_0 u = u|_\Gamma \in \mathcal{D}(\Gamma)$, and denote by γ_j ($j = 1, 2, \dots$) the mapping of $u \in \mathcal{D}(\bar{\Omega})$ into its j'th interior normal derivative $\gamma_j u = \dfrac{\partial^j u}{\partial n^j} \in \mathcal{D}(\Gamma)$. A fundamental « trace theorem » is the following:

THEOREM 2.1. *Let s be a real number $> \frac{1}{2}$ and let r be the largest integer with $r < s - \frac{1}{2}$. The mapping $\gamma = \{\gamma_0, \dots, \gamma_r\}$, defined on $\mathcal{D}(\bar{\Omega})$, extends by continuity to a continuous mapping, which we will also denote by $\gamma = \{\gamma_0, \dots, \gamma_r\}$, of $H^s(\Omega)$ onto $\prod_{j=0}^{j=r} H^{s-j-\frac{1}{2}}(\Gamma)$. The kernel of γ is exactly $H_0^s(\Omega)$.*

(Various parts of the theorem have been proved by many autors — for references see Lions-Magenes [24] III or [24′]; we here present a very strong version due to Uspenskii [31], the last statement due to Lions-Magenes [24′] Theorem 1.11.5).

REMARK 2.1. In this way, γ_j is defined on all spaces $H^s(\Omega)$ with $s > j + \frac{1}{2}$. Since $\mathcal{D}(\bar{\Omega})$ is a dense subset of $H^s(\Omega)$ for all s, and $H^{s'}(\Omega) \subset H^s(\Omega)$ algebraically, topologically and densely, whenever $s' \geq s$, the definition of γ_j on $H^s(\Omega)$ is an extension of γ_j defined on $H^{s'}(\Omega)$, $s' \geq s > j + \frac{1}{2}$.

By a boundary differential operator B_j of order m_j with coefficients in $\mathcal{D}(\Gamma)$ (j and m_j are nonnegative integers) we shall mean an operator of the form

$$B_j u = \sum_{|p| \leq m_j} b_{jp} \gamma_0 D^p u, \tag{2.1}$$

where the b_{jp} are functions in $\mathcal{D}(\Gamma)$ (not all zero for $|p| = m_j$). B_j maps $\mathcal{D}(\bar{\Omega})$ into $\mathcal{D}(\Gamma)$. It can also be written as

$$B_j u = b_j \gamma_{m_j} u + \sum_{k=0}^{k=m_j-1} T_{jk} \gamma_k u, \tag{2.2}$$

where $b_j \in \mathcal{D}(\Gamma)$ and the T_{jk} are (« tangential ») differential operators in Γ of orders $\leq m_j - k$, with coefficients in $\mathcal{D}(\Gamma)$.

Let m be a positive integer, and let $\{B_j\}_{j=0}^{j=m-1}$ be a system of m boundary differential operators. We say that the system is *normal* if the orders m_j are distinct, and the functions b_j as in (2.2) have inverses (which then belong to $\mathcal{D}(\Gamma)$). We say that the system is a *Dirichlet system of order m* if, furthermore, the orders m_j fill out the set $0, \dots, m-1$. (For details see Aronszajn-Milgram [3] and Schechter [28]).

One has easily from Theorem 2.1 :

COROLLARY 2.1. *Let m be a positive integer and let B_j be a boundary operator of order $m_j \leq 2m - 1$. The mapping $u \mapsto B_j u$, defined on $\mathcal{D}(\bar{\Omega})$, extends by continuity to a continuous mapping, also denoted by B_j, of $H^{2m}(\Omega)$ into $H^{2m-m_j-\frac{1}{2}}(\Gamma)$.*

The situation will usually be that we have two normal systems $\{B_j\}_{j=0}^{m-1}$ and $\{C_j\}_{j=0}^{m-1}$ (denoting the order of C_j by μ_j), such that $\{B_0, \dots, B_{m-1}, C_0, \dots, C_{m-1}\}$ is a Dirichlet system of order $2m$. One can then derive from Theorem 2.1:

PROPOSITION 2.1. *Let m be a positive integer and let $\{B_j\}_{j=0}^{m-1}$ (orders m_j) and $\{C_j\}_{j=0}^{m-1}$ (orders μ_j) be normal systems such that $\{B_0, \dots, B_{m-1}, C_0, \dots, C_{m-1}\}$*

is a Dirichlet system of order 2m. Then $\{B_0, \dots, B_{m-1}, C_0, \dots, C_{m-1}\}$ *maps* $H^{2m}(\Omega)$ *onto* $\prod_{j=0}^{m-1} H^{2m-m-\frac{1}{2}}(\Gamma) \times \prod_{j=0}^{m-1} H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$.

In the proof one transforms the problem of constructing u from boundary data $B_j u = \varphi_j \in H^{2m-m_j-\frac{1}{2}}(\Gamma)$, $C_j u = \psi_j \in H^{2m-\mu_j-\frac{1}{2}}(\Gamma), j=0, \dots, m-1$, into the construction of u from boundary data $\gamma_j u = \varrho_j \in H^{2m-j-\frac{1}{2}}(\Gamma)$, $j = 0, \dots, 2m-1$; which is possible according to a lemma by Aronszajn-Milgram [3], since the orders are distinct and the coefficients b_j (as in (2.2)) are invertible. (For details, see e. g. Schechter [28]). One then applies Theorem 2.1 with $s = 2m$.

I § 3. Assumptions; and some results on elliptic differential operators.

There is given a differential operator A of order $2m$ (m is a positive integer), with coefficients $a_{pq}(x) \in \mathcal{D}(\overline{\Omega})$:

$$A = \sum_{|p|, |q| \leq m} (-1)^{|p|} D^p a_{pq}(x) D^q ; \tag{3.1}$$

which is *properly elliptic* in $\overline{\Omega}$ (as defined in Schechter [28], or [24] V p. 8). The formal adjoint A' is defined by

$$A' = \sum_{|p|, |q| \leq m} (-1)^{|p|} D^p \overline{a_{qp}(x)} D^q . \tag{3.2}$$

it is then also properly elliptic in Ω.

There are given four systems each consisting of m boundary differential operators with coefficients in $\mathcal{D}(\Gamma)$: $B = \{B_0, \dots, B_{m-1}\}$, $C = \{C_0, \dots, C_{m-1}\}$, $B' = \{B'_0, \dots, B'_{m-1}\}$, $C' = \{C'_0, \dots, C'_{m-1}\}$, where B_j is of order m_j, C_j is of order μ_j, B'_j is of order $2m - \mu_j - 1$, and C'_j is of order $2m - m_j - 1$, for $j = 0, \dots, m-1$. It is assumed that all four systems are *normal*, that $\{B_0, \dots, B_{m-1}, C_0, \dots, C_{m-1}\}$ and $\{B'_0, \dots, B'_{m-1}, C'_0, \dots, C'_{m-1}\}$ are *Dirichlet systems*, and that they together with A and A' satisfy *Green's formula*

$$\int_\Omega (Au\,\overline{v} - u\,\overline{A'v})\,dx = \sum_{j=0}^{m-1} \int_\Gamma (C_j u\,\overline{B'_j v} - B_j u\,\overline{C'_j v})\,d\sigma \tag{3.3}$$

for all $u, v \in H^{2m}(\Omega)$.

We will say that the system $\{A, B\}$ satisfies the hypothesis $(\mathcal{C})$ if

$(\mathcal{C})$ The system of boundary operators $B = \{B_j\}$ *covers* A (Schechter [28], or [24] V p. 11).

REMARK 3.1. The following properties are mentioned in Lions-Magenes [24] V; we also refer to Schechter [28] and Aronszajn-Milgram [3]:

If $\{A, B\}$ satisfies $(\mathcal{C})$ then also $\{A', B'\}$ satisfies $(\mathcal{C})$. For the special case where $B = \gamma = \{\gamma_0, \dots, \gamma_{m-1}\}$. one has: $\{A, \gamma\}$ satisfies $(\mathcal{C})$ for any properly elliptic A.

We will now introduce several operators *in* $L^2(\Omega)$, associated with the formal differential operators A and A'.

The minimal operator A_0 [resp. A_0'] is defined as: the closure as an operator in $L^2(\Omega)$, of A [resp. A'] defined on $\mathcal{D}(\Omega)$. The maximal operator A_1 is defined by: $D(A_1) = \{u \in L^2(\Omega) \mid Au \in L^2(\Omega)$ (Au defined in the distribution sense)$\}$; $Au = A_1 u$ for $u \in D(A_1)$; A_1' is defined analogously. Then $A_1 = (A_0')^*$ and $A_1' = A_0^*$. (For further explanation, see e.g. Hörmander [17]). Note that A_0, A_1, A_0' and A_1' are closed operators.

Because of the ellipticity of A, one can prove that the functions in $D(A_1)$ satisfy: $u \in H^{2m}(\Omega_1)$ for every open subset $\Omega_1 \subset \Omega$ with $\overline{\Omega}_1 \subset \Omega$ (interior regularity, Friedrichs [14]); and that $D(A_0) = H_0^{2m}(\Omega)$ (also in the sense that the graphnorm and the $H^{2m}(\Omega)$-norm are equivalent on $D(A_0)$).

The linear operators $\widetilde{A}$ with $A_0 \subset \widetilde{A} \subset A_1$ will be called the *realizations of* A. Similarly, the operators $\widetilde{A}'$ with $A_0' \subset \widetilde{A}' \subset A_1'$ are the realizations of A'. Clearly, the adjoint of a realization of A is a realization of A', and vice versa.

Let $\widetilde{A}$ be a realization of A, and let $u \in D(A_1)$. For any open subset Ω_1 of Ω with $\overline{\Omega}_1 \subset \Omega$ one can, because of the interior regularity, find $u \in H_0^{2m}(\Omega) = D(A_0)$ such that $u = u_1$ in Ω_1; therefore solely the behavior of *u near the boundary* Γ determines whether it belongs to $D(\widetilde{A})$. This justifies the statement that each realization corresponds to an « abstract boundary condition ». Our aim is to give a concrete formulation of this idea; in fact to show how every (closed) realization corresponds to a boundary condition, expressed in terms of an operator between certain spaces over Γ, of the type described in § 1. To do this, we shall need the basic results in the theory for local nonhomogeneous boundary value problems developed by Lions and Magenes [24], [24']. Their theory builds, among other things, on the regularity results by Schechter [28], Agmon-Douglis-Nirenberg [1], and uses interpolation theory.

We begin with describing the fundamental regularity results.

Define the operators A_β and A_β' by

$$D(A_\beta) = \{u \in H^{2m}(\Omega) \mid Bu = 0\}, \quad A_\beta \subset A_1 \tag{3.4}$$

and

$$D(A_\beta') = \{u \in H^{2m}(\Omega) \mid B' u = 0\}, \quad A_\beta' \subset A_1'. \tag{3.5}$$

Since $Bu = 0$ and $B'u = 0$ for $u \in H_0^{2m}(\Omega) = D(A_0) = D(A_0')$, A_β is a realization of A, and A_β' is a realization of A'. It follows from (3.3) that A_β and A_β' are contained in each others adjoints.

B is so far defined for $u \in H^{2m}(\Omega)$; for general $u \in D(A_1)$ we now define what we mean by $Bu = 0$ by (weak definition):

(3.6) For $u \in D(A_1)$, $Bu = 0$ if and only if $u \in D((A_\beta')^*)$.

Then one has (Schechter [28], Agmon-Douglis-Nirenberg [1]):

THEOREM 3.1. *If $\{A, B\}$ satisfies $(\mathcal{C})$, then $u \in D(A_1)$ with $Bu = 0$ weakly imply $u \in H^{2m}(\Omega)$ with $Bu = 0$ in the ordinary sense (as in Corollary 2.1).*

This shows that $D((A_\beta')^*) \subset D(A_\beta)$, which together with the already known: $A_\beta \subset (A_\beta')^*$ gives

$$A_\beta = (A_\beta')^*.$$

In the course of the proof of Theorem 3.1 one finds that the graphtopology on $D(A_\beta)$ is equivalent with the $H^{2m}(\Omega)$-topology; we can also note here that it is a consequence of (3.4) and the statement of Theorem 3.1, by the closed graph theorem (using that $D(A_\beta)$ equals algebraically a closed subspace of $H^{2m}(\Omega)$, and the graphtopology is weaker than the $H^{2m}(\Omega)$-topology). Recall Remark 2.1, that $\{A, B\}$ satisfying $(\mathcal{C})$ implies $\{A', B'\}$ satisfying $(\mathcal{C})$. Then similar considerations apply to A_β'. Altogether:

COROLLARY 3.1. *If $\{A, B\}$ satisfies $(\mathcal{C})$, then A_β and A_β' are adjoints, and the graphtopology on $D(A_\beta)$ $[D(A_\beta')]$ is equivalent with the $H^{2m}(\Omega)$-topology.*

We will say that the system $\{A, B\}$ satisfies the hypothesis $(\mathcal{U})$ if

$(\mathcal{U})$ the problem

$$\left\{\begin{aligned} Au &= f \\ Bu &= 0 \quad \text{(weakly)} \end{aligned}\right. \tag{3.7}$$

has a *unique* solution $u \in D(A_1)$ for all $f \in L^2(\Omega)$.

Note that when $\{A, B\}$ satisfies $(\mathcal{C})$, (3.7) is the boundary value problem represented by A_β, and then $\{A, B\}$ satisfies $(\mathcal{U})$ if and only if $0 \in \varrho(A_\beta)$.

We will not try to list sufficient conditions for the validity of $(\mathcal{U})$ here. A discussion can be found in Lions-Magenes [24] V, where further references to the literature are given.

(3.7) contains the homogeneous boundary condition $Bu = 0$; to treat the corresponding nonhomogeneous boundary value problem one needs an extension of the definition of B to all $u \in D(A_1)$.

DEFINITION 3.1. *Let s be a real number. We define*

$$D_A^s(\Omega) = \{u \in H^s(\Omega) \mid Au \in H^0(\Omega)\}$$

as a Hilbert space with the norm

$$|u|_{D_A^s(\Omega)} = (|u|_s^2 + |Au|_0^2)^{\frac{1}{2}}.$$

We define the space

$$Z_A^s(\Omega) = \{u \in H^s(\Omega) \mid Au = 0\},$$

it is a closed subspace of $D_A^s(\Omega)$ as well as of $H^s(\Omega)$.

Note that $D_A^s(\Omega) \subset H^s(\Omega)$ algebraically and topologically for all s. When $s \geq 2m$ one has in fact $D_A^s(\Omega) = H^s(\Omega)$, since $u \in H^s(\Omega)$ implies $Au \in H^{s-2m}(\Omega)$; the identity also holds topologically, by the closed graph theorem. Note that $D_A^0(\Omega)$ is identical $D(A_1)$.

THEOREM 3.2. *The boundary operator $B = \{B_0, \dots, B_{m-1}\}$ defined on $H^{2m}(\Omega)$ can be extended to an operator, also denoted by B, which maps $D_A^s(\Omega)$ continuously into $\prod_{j=0}^{m-1} H^{s-m_j-\frac{1}{2}}(\Gamma)$ for all real s.*

For all $s < 2m$, $H^{2m}(\Omega)$ is dense in $D_A^s(\Omega)$, so that here B is an extension by continuity of B defined on $H^{2m}(\Omega)$.

Similarly, one obtains by extension by continuity operators C, B' and C', mapping $D_A^s(\Omega)$ and $D_{A'}^s(\Omega)$ continuously into $\Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$, $\Pi H^{s-2m+\mu_j+\frac{1}{2}}(\Gamma)$ and $\Pi H^{s-2m+m_j+\frac{1}{2}}(\Gamma)$, respectively, for all real s.

We refer to Lions-Magenes [24] V, VI and [24′] for details. The extension is defined such that it is consistent with the weak definition of $Bu = 0$ for $u \in D(A_1)$ given in (3.6). (In fact the extension is defined by a clever duality argument, using an analogue of Green's formula (3.3)). Note that B maps $D(A_1) = D_A^0(\Omega)$ continuously into $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$.

REMARK 3.2. For $s \geq 2m$, the above statement can actually be deduced from Corollary 2.1, by interpolation between integer cases.

For $s \leq 2m$, the theorem gives new information. Here the present statement is an improvement from the results in [24] V, VI, in that it is not prescribed that $s - \frac{1}{2}$ be different from an integer, as was the case in [24]. The improved version follows from some stronger results in Lions and Magenes book [24′], of which the authors have kindly shown me the manuscript. In the same way, Theorem 3.3 below is valid for all s.

With the new definition of B, C, B' and C', Green's formula can be extended as follows:

COROLLARY 3.3. *Let* $s \in [0, 2m]$. *For* $u \in D_A^s(\Omega)$, $v \in D_{A'}^{2m-s}(\Omega)$, *one has*

$$(3.8)\qquad (Au, v) - (u, A'v) = \langle Cu, B'v \rangle_{\{s-\mu_j-\frac{1}{2}\}\,\{-s+\mu_j+\frac{1}{2}\}} - \langle Bu, C'v \rangle_{\{s-m_j-\frac{1}{2}\}\,\{-s+m_j+\frac{1}{2}\}}.$$

PROOF. The formula follows from (3.3) by extension by continuity, using Theorem 3.2.

REMARK 3.3. The above extension of Green's formula is in a sense the best possible. By this we mean that even though B, C, B' and C' are *defined* on $D_A^0(\Omega)$ resp $D_{A'}^0(\Omega)$, the formula does not make sense for all pairs $[u, v] \in D_A^s(\Omega) \times D_{A'}^{s'}(\Omega)$ unless $s + s' \geq 2m$. Proof for the case where B *and* C cover A (thus B' *and* C' cover A'): Bu and $C'v$ must belong to dual spaces in order that (3.8) makes sense. By Proposition III 5.2 (which uses the boundedness of Ω) mentioned in III § 5, $Bu \in \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ if and only if $u \in D_A^s(\Omega)$ ($s \in [0, 2m]$, u is assumed to be in $D_A^0(\Omega)$), and $C'v \in \Pi H^{-s+m_j+\frac{1}{2}}(\Gamma)$ if and only if $v \in D_{A'}^{2m-s}(\Omega)$.

Finally, we present the fundamental result of Lions and Magenes [24] V, VI, [24′] which gives a precise description of the non-homogeneous boundary value problem, when the given differential operators are sufficiently nice:

THEOREM 3.3. *Assume that both* $\{A, B\}$ *and* $\{A', B'\}$ *satisfy* $(\mathcal{C})$ *and* $(\mathcal{U})$.

(i) *Let* $s \leq 2m$. *The mapping* $\{A, B\}$ *is an isomorphism of* $D_A^s(\Omega)$ *onto* $H^0(\Omega) \times \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$, *and* $\{A', B'\}$ *is an isomorphism of* $D_{A'}^s(\Omega)$ *onto* $H^0(\Omega) \times \Pi H^{s-2m+\mu_j+\frac{1}{2}}(\Gamma)$.

(ii) *Let* $s \geq 2m$. *The mapping* $\{A, B\}$ *is an isomorphism of* $H^s(\Omega)$ *onto* $H^{s-2m} \times \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$, *and the mapping* $\{A', B'\}$ *is an isomorphism of* $H^s(\Omega)$ *onto* $H^{s-2m}(\Omega) \times \Pi H^{s-2m+\mu_j+\frac{1}{2}}(\Gamma)$.

Here, the really important contribution is statement (i); statement (ii) follows by interpolation from the results in [1] and is used in the process of proving (i).

CHAPTER II. - ABSTRACT THEORY

§ 1. Basic results.

Definitions, and certain special results that we shall need, are given in the Appendix.

Let H be a Hilbert space with norm denoted by $|u|$, inner product (u, v). We assume that the following operators are given in H: A pair of closed, densely defined, unbounded operators A_0, A_0' satisfying $A_0 \subset (A_0')^*$, $A_0' \subset A_0^*$, and a closed operator A_β which has a bounded, everywhere defined inverse, and which satisfies:

$$A_0 \subset A_\beta \subset (A_0')^* \text{ (1)}.$$

Denote $(A_0')^* = A_1$ and $A_0^* = A_1'$, then $A_0 \subset A_\beta \subset A_1$, $A_0' \subset A_\beta^* \subset A_1'$; A_β^* has a bounded everywhere defined inverse.

The set of linear operators $\widetilde{A}$ in H satisfying $A_0 \subset \widetilde{A} \subset A_1$ will be denoted by $\mathcal{M}$; the set of linear operators $\widetilde{A}'$ with $A_0' \subset \widetilde{A}' \subset A_1'$ will be denoted by $\mathcal{M}'$. Clearly, $\widetilde{A} \in \mathcal{M}$ implies $\widetilde{A}^* \in \mathcal{M}'$, and $\widetilde{A}' \in \mathcal{M}'$ implies $(\widetilde{A}')^* \in \mathcal{M}$.

In this § we will give a characterization of the closed operators $\widetilde{A} \in \mathcal{M}$ in terms of operators T from $Z(A_1)$ to $Z(A_1')$; in the form of a $1-1$ correspondence. Some properties of the correspondence will be deduced.

We will use the following simplification of notation, whenever convenient: When $u \in D(\widetilde{A})$ for some $\widetilde{A} \in \mathcal{M}$ we write Au, instead of $\widetilde{A}u$, and when $v \in D(\widetilde{A})$ for some $\widetilde{A}' \in \mathcal{M}'$ we write $A'v$ instead of $\widetilde{A}'v$. The graphnorms in any of the spaces $D(\widetilde{A})$, $\widetilde{A} \in \mathcal{M}$, will be denoted by

$$|u|_A = (|u|^2 + |Au|^2)^{\frac{1}{2}},$$

similarly the graphnorm in $D(\widetilde{A}')$, $\widetilde{A}' \in \mathcal{M}'$, will be denoted by

$$|v|_{A'} = (|v|^2 + |A'v|^2)^{\frac{1}{2}}.$$

---

(1) Note that then A_0 also has a bounded inverse. Conversely, if A_0 and A_0' have bounded inverses, and $A_0 \subset (A_0')^*$, then there exists A_β with $0 \in \varrho(A_\beta)$, $A_0 \subset A_\beta \subset (A_0')^*$. (Description in Vishik [32]).

LEMMA 1.1. *$D(A_1) = D(A_\beta) \dot{+} Z(A_1)$ and $D(A_1') = D(A_\beta^*) \dot{+} Z(A_1')$, direct topological sums in the graphtopologies.*

PROOF: Since $D(A_\beta) \subset D(A_1)$ and $Z(A_1) \subset D(A_1)$, $D(A_\beta) + Z(A_1) \subset \subset D(A_1)$. Now let $u \in D(A_1)$. If u can be decomposed into $u = u_\beta + u_\zeta$, where $u_\beta \in D(A_\beta)$ and $u_\zeta \in Z(A_1)$, then $Au = Au_\beta$ so $u_\beta = A_\beta^{-1} Au$. This shows that the decomposition is unique, and it also indicates how the general element u of $D(A_1)$ can be decomposed: If $u \in D(A_1)$, let $u_\beta = A_\beta^{-1} Au$, then $u_\zeta = u - u_\beta$ satisfies $Au_\zeta = Au - Au_\beta = 0$.

Thus $D(A_1) = D(A_\beta) \dot{+} Z(A_1)$, direct sum. To show that it is a direct topological sum we have to show that the mapping $[u_\beta, u_\zeta] \frown\!\searrow u = u_\beta + u_\zeta$ is an isomorphism of $D(A_\beta) \times Z(A_1)$ onto $D(A_1)$ (graphtopologies). The mapping is clearly continuous:

$$| u_\beta + u_\zeta |_A \leq | u_\beta |_A + | u_\zeta |_A = | u_\beta |_A + | u_\zeta |$$

for all $u_\beta \in D(A_\beta)$, $u_\zeta \in Z(A_1)$. Since A_β and A_1 are closed, the spaces $D(A_\beta) \times Z(A_1)$ and $D(A_1)$ are Hilbert spaces; it then follows by the closed graph theorem that the mapping is an isomorphism. Thus

$$D(A_1) = D(A_\beta) \dot{+} Z(A_1), \quad \text{direct topological sum.}$$

An analogous proof shows that

$$D(A_1') = D(A_\beta^*) \dot{+} Z(A_1'), \quad \text{direct topological sum.}$$

Since we will use these decompositions again and again, we make the following definitions:

The projection of $D(A_1)$ onto $D(A_\beta)$ defined by Lemma 1.1 is called pr_β, the projection onto $Z(A_1)$ is called pr_ζ. For $u \in D(A_1)$ we also write $pr_\beta u = u_\beta$; $pr_\zeta u = u_\zeta$. The projections of $D(A_1')$ onto $D(A_\beta^*)$ and $Z(A_1')$ defined by Lemma 1.1 are denoted $pr_{\beta'}$ resp. $pr_{\zeta'}$, and we write $pr_{\beta'} v = v_{\beta'}$, $pr_{\zeta'} v = v_{\zeta'}$, whenever convenient.

We will still reserve the terminology U_X for the *orthogonal* projection of a subspace or element U into a closed subspace X of H.

LEMMA 1.2. *For $u \in D(A_1)$, $v \in D(A_1')$ one has*

$$(Au, v) - (u, A'v) = (Au, v_{\zeta'}) - (u_{\zeta'}, A'v).$$

PROOF:

$$\begin{aligned}(Au, v) - (u, A'v) &= (Au, v_{\beta'} + v_{\zeta'}) - (u_\beta + u_\zeta, A'v) \\ &= (Au_\beta, v_{\beta'}) - (u_\beta, A'v_{\beta'}) + (Au, v_{\zeta'}) - (u_\zeta, A'v)\end{aligned}$$

since $Au_\zeta = 0$, $A'v_{\zeta'} = 0$;

$$= (Au, v_{\zeta'}) - (u_\zeta , A'v),$$

since u_β and $v_{\beta'}$ belong to domains of adjoint operators.

PROPOSITION 1.1. *Let $\widetilde{A} \in \mathcal{M}$, $\widetilde{A}^* \in \mathcal{M}'$ be a pair of adjoint operators. Define $V = pr_\zeta\, D(\widetilde{A})$ and $W = pr_{\zeta'}\, D(\widetilde{A}^*)$ (so $V \subset Z(A_1)$, $W \subset Z(A_1')$, closed subspaces). Then*

(i) *the equations*

$$\left\{\begin{aligned} &D(T) = pr_\zeta\, D(\widetilde{A}) \\ &Tu_\zeta = (Au)_W\,, \quad all\ u \in D(\widetilde{A}) \end{aligned}\right. \tag{1.1}$$

define an operator T: $V \longrightarrow W$, and the equations

$$\left\{\begin{aligned} &D(T_1) = pr_{\zeta'}\, D(\widetilde{A}^*) \\ &T_1 v_{\zeta'} = (A'v)_V\,, \quad all\ v \in D(\widetilde{A}^*) \end{aligned}\right. \tag{1.2}$$

define an operator T_1: $W \longrightarrow V$;

(ii) *the operators T and T_1 are adjoints.*

PROOF: Let $u \in D(\widetilde{A})$, $v \in D(\widetilde{A}^*)$. It then follows from Lemma 1.2 that

$$0 = (Au, v) - (u, A'v) = (Au, v_{\zeta'}) - (u_\zeta , A'v),$$

and therefore, since $u_\zeta \in V$, $v_{\zeta'} \in W$:

$$((Au)_W\,, v_{\zeta'}) = (u_\zeta\,, (A'v)_V\,), \quad \text{all } u \in D(\widetilde{A}), \quad v \in D(\widetilde{A}^*). \tag{1.3}$$

Now if $u_\zeta = 0$ then, by (1.3), $((Au)_W\,, v_{\zeta'}) = 0$ for all $v \in D(\widetilde{A}^*)$, from which it follows, since $W = pr_{\zeta'} D(\widetilde{A}^*)$, that $(Au)_W = 0$. Thus $(Au)_W$ is a function of u_ζ for all $u \in D(\widetilde{A})$. We can then define the operator T by

$$Tu_\zeta = (Au)_W$$

for all $u \in D(A)$. Clearly $D(T) = pr_\zeta\, D(\widetilde{A}) \subset V$, and $R(T) \subset W$.

Similarly, it follows from (1.3) that $(A'v)_V$ is a function of $v_{\zeta'}$ for all $v \in D(\widetilde{A}^*)$. This defines the operator T_1 satisfying (1.2) and the proof of (i) is completed.

We will now prove that T and T_1 are adjoints. By symmetry it suffices to prove, e. g., that T is the adjoint of T_1.

Since $T_1: W \to V$ is densely defined in W, the adjoint $T_1^*: V \to W$ exists. It follows from (1.3) that

$$(Tv, w)_W = (v, T_1 w)_V, \quad \text{all } v \in D(T), \quad w \in D(T_1).$$

Thus $T \subset T_1^*$.

Now let $z \in D(T_1^*)$. Define $x = z + A_\beta^{-1} T_1^* z$; then clearly $x \in D(A_1)$ with $x_\beta = A_\beta^{-1} T_1^* z$, $x_\zeta = z$. One has for all $v \in D(\widetilde{A}^*)$:

$$\begin{aligned}(Ax, v) - (x, A'v) &= (Ax, v_{\zeta'}) - (x_\zeta, A'v) \quad \text{by Lemma 1.2} \\ &= (T_1^* z, v_{\zeta'}) - (z, A'v) \\ &= (T_1^* z, v_{\zeta'}) - (z, (A'v)_V) \quad \text{since } z \in V \\ &= (T_1^* z, v_{\zeta'}) - (z, T_1 v_{\zeta'}) \quad \text{by the def. of } T_1, \\ &= 0.\end{aligned}$$

Thus $x \in D(\widetilde{A}^{**}) = D(\widetilde{A})$. Consequently, by the definition of T, $x_\zeta = z \in D(T)$ and $Tx_\zeta = (Ax)_W = (T_1^* z)_W = T_1^* z$, i.e., $Tz = T_1^* z$. It follows that $T_1^* \subset T$.

We have then proved that $T_1^* = T$, which completes the proof of the proposition.

Proposition 1.1. shows how every adjoint pair $\widetilde{A} \in \mathcal{M}$, $\widetilde{A}^* \in \mathcal{M}'$, gives rise to an adjoint pair $T: V \to W$, $T^*: W \to V$, with $V \subset Z(A_1)$ and $W \subset Z(A_1')$, closed subspaces. The next proposition shows that any adjoint pair $T: V \to W$, $T^*: W \to V$, where V and W are arbitrary closed subspaces of $Z(A_1)$ resp. $Z(A_1')$, is reached in this way.

PROPOSITION 1.2. *Let V be a closed subspace of $Z(A_1)$, W a closed subspace of $Z(A_1')$ and $T: V \to W$, $T^*: W \to V$ a pair of adjoint operators (not necessarily bounded). Then the operators $\widetilde{A}$ and $\widetilde{A}'$ in H determined by*

$$D(\widetilde{A}) = \{u \in D(A_1) \mid u_\zeta \in D(T),\ (Au)_W = Tu_\zeta\}, \quad \widetilde{A} \in \mathcal{M} \tag{1.4}$$

and

$$D(\widetilde{A}') = \{v \in D(A_1') \mid v_{\zeta'} \in D(T^*),\ (A'v)_V = T^* v_{\zeta'}\}, \quad \widetilde{A}' \in \mathcal{M}' \tag{1.5}$$

are adjoints, and the operators derived from $\widetilde{A}$, $\widetilde{A}^*$ by Proposition 1.1 are exactly $T: V \to W$ and $T^*: W \to V$.

PROOF: For $u \in D(\widetilde{A})$, $v \in D(\widetilde{A}')$ we have by Lemma 1.2:

$$\begin{aligned}(Au, v) - u, A'v) &= (Au, v_{\zeta'}) - (u_\zeta, A'v)\\ &= ((Au)_W, v_{\zeta'}) - (u_\zeta, (A'v)_V), \quad \text{since } u_\zeta \in V,\ v_{\zeta'} \in W,\\ &= (Tu_\zeta, v_{\zeta'}) - (u_\zeta, T^* v_{\zeta'}) \quad \text{by the def. of } \widetilde{A} \text{ and } \widetilde{A}',\\ &= 0.\end{aligned}$$

Thus $\widetilde{A}$ and $\widetilde{A}'$ are contained in each others adjoints.

We now have to prove that $\widetilde{A}^* \subset \widetilde{A}'$ and that $(\widetilde{A}')^* \subset \widetilde{A}$; by symmetry it suffices to prove, e. g., that $\widetilde{A}^* \subset \widetilde{A}'$.

Let $v \in D(\widetilde{A}^*)$. Since A_0 is closed with a bounded inverse, and has A_1' as adjoint,

(1.6) $$H = R(A_0) \oplus Z(A_1') \quad \text{(orthogonal sum)};$$

thus $R(A_0) \perp W$. Therefore any element u of the form $u = z + A_\beta^{-1} Tz + w$, where $z \in D(T)$ and $w \in D(A_0)$, is in $D(\widetilde{A})$, since $u_\zeta = z \in D(T)$ and $(Au)_W = (Tz + A_0 w)_W = Tz$. We have for all such u:

$$\begin{aligned}0 &= (Au, v) - (u, A'v)\\ &= (Tz + Aw, v_{\zeta'}) - (z, A'v)\\ &= (Tz, v_{\zeta'}) - (z, (A'v)_V).\end{aligned}$$

Thus $(Tz, v_{\zeta'}) = (z, (A'v)_V)$ for all $z \in D(T)$. This shows that $v_{\zeta'} \in D(T^*)$ with $T^* v_{\zeta'} = (A'v)_V$, whence, by definition, $v \in D(\widetilde{A}')$. Thus $\widetilde{A}^* \subset \widetilde{A}'$.

The last statement of the proposition is obvious.

Finally, every pair T, T^* stems from only one pair $\widetilde{A}$, $\widetilde{A}^*$:

LEMMA 1.3. *Let $\widetilde{A} \in \mathcal{M}$ and $\widetilde{A}^* \in \mathcal{M}'$ be a pair of adjoint operators, and let $T: V \to W$ and $T^*: W \to V$ be derived from $\widetilde{A}$ and $\widetilde{A}^*$ as in Proposition* 1.1. *Then*

$$D(\widetilde{A}) = \{u \in D(A_1) \mid u_\zeta \in D(T),\ (Au)_W = Tu_\zeta\}$$

$$D(\widetilde{A}^*) = \{v \in D(A_1') \mid v_{\zeta'} \in D(T^*),\ (A'v)_V = T^* v_{\zeta'}\}.$$

PROOF: Let $\widetilde{A}_1$ and $\widetilde{A}_1^*$ be the operators defined by (1.4) resp. (1.5) in Proposition 1.2. It follows from the definitions of T and T^* in Proposition 1.1, that

$$D(\widetilde{A}) \subset D(\widetilde{A}_1) \tag{1.7}$$

and

$$D(\widetilde{A}^*) \subset D(\widetilde{A}_1^*). \tag{1.8}$$

Since $\widetilde{A}$ and $\widetilde{A}_1 \in \mathcal{M}$, (1.7) shows that $\widetilde{A} \subset \widetilde{A}_1$. Then $\widetilde{A}^* \supset \widetilde{A}_1^*$. This, together with (1.8) implies $\widetilde{A}^* = \widetilde{A}_1^*$. Then also $\widetilde{A} = \widetilde{A}_1$.

The Propositions 1.1 and 1.2, and Lemma 1.3 together imply the theorem:

THEOREM 1.1. *There is a $1-1$ correspondence between all pairs o- adjoint operators $\widetilde{A}$, $\widetilde{A}^*$ with $\widetilde{A} \in \mathcal{M}$, $\widetilde{A}^* \in \mathcal{M}'$, and all pairs of adjoint opef rators T, T^* with $T: V \to W$, $T^*: W \to V$, where V denotes a closed subspace of $Z(A_1)$ and W denotes a closed subspace of $Z(A_1')$; the correspondence being given by:*

$$D(\widetilde{A}) = \{u \in D(A_1) \mid u_\zeta \in D(T),\ (Au)_W = Tu_\zeta\}$$

$$D(\widetilde{A}^*) = \{v \in D(A_1') \mid v_{\zeta'} \in D(T^*),\ (A'v)_V = T^* v_{\zeta'}\}.$$

In this correspondence, $D(T) = pr_\zeta D(\widetilde{A})$, $D(T^*) = pr_{\zeta'} D(\widetilde{A}^*)$, $V = \overline{pr_\zeta D(\widetilde{A})}$ and $W = \overline{pr_{\zeta'} D(\widetilde{A}^*)}$.

The formulation of Theorem 1.1 is completely symmetric $\widetilde{A}$ and $\widetilde{A}^*$, and in T and T^*. Since $\widetilde{A}^*$ is actually determined by $\widetilde{A}$, and $T^*: W \to V$ is determined by $T: V \to W$, an immediate consequence of Theorem 1.1 is:

COROLLARY 1.1. *There is a $1-1$ correspondence between all closed operators $\widetilde{A} \in \mathcal{M}$ and all operators $T: V \to W$ satisfying*

(i) *V is a closed subspace of $Z(A_1)$, W is a closed subspace of $Z(A_1')$;*

(ii) *T is densely defined in V and closed;*

the correspondence being given by

$$D(\widetilde{A}) = \{u \in D(A_1) \mid u_\zeta \in D(T),\ (Au)_W = Tu_\zeta\}. \tag{1.9}$$

In this correspondence, $D(T) = pr_\zeta D(\widetilde{A})$ (so $V = \overline{pr_\zeta D(\widetilde{A})}$).

Furthermore, if $\widetilde{A}$ *corresponds to* $T: V \rightarrow W$ *in the above sense, then* $\widetilde{A}^*$ *corresponds to* $T^*: W \rightarrow V$ *by*

$$(1.10) \qquad D(\widetilde{A}^*) = \{v \in D(A_1') \mid v_{\zeta'} \in D(T^*),\ (A'v)_V = T^* v_{\zeta'}\}$$

and $D(T^*) = pr_{\zeta'} D(\widetilde{A}^*)$ (so $W = \overline{pr_{\zeta'} D(\widetilde{A}^*)}$).

Remark 1.1. If T is given as a closed operator from $Z(A_1)$ to $Z(A_1')$ we can choose W as any closed subspace of $Z(A_1')$ for which $R(T) \subset W$, and define $\widetilde{A}$ by (1.9). Then $\widetilde{A}$ corresponds to $T: V \rightarrow W$, where $V = \overline{D(T)}$. Of course, different choices of W give different operators $\widetilde{A}$, whereas V is necessarily equal to $\overline{D(T)}$. Therefore V need not be explicitly mentioned in this connection; however, V is important when we consider $\widetilde{A}^*$, since it is the range space for T^*, and enters in (1.10).

We will give another description of the connection between $\widetilde{A}$ and T, which also sheds light on the topological structures. To do this we need the following lemma:

Lemma 1.4. *Let* W *be a closed subspace of* $Z(A_1')$, *and let* T *be any operator with* $D(T) \subset Z(A_1)$ *and* $R(T) \subset W$. *Then the following two sets* D_1 *and* D_2 *are identical:*

$$D_1 = \{u \in D(A_1) \mid u_\zeta \in D(T),\ (Au)_W = Tu_\zeta\}$$

$$(1.11) \quad D_2 = \{u = z + A_\beta^{-1}(Tz + f) + v \mid z \in D(T), f \in Z(A_1') \ominus W,\ v \in D(A_0)\}.$$

Moreover, the elements $z \in D(T)$, $f \in Z(A_1') \ominus W$ *and* $v \in D(A_0)$ *are uniquely determined by* u *in* (1.11).

Proof: In the proof we use the earlier mentioned fact that $H = R(A_0) \oplus Z(A_1')$.

Let $u \in D_2$, i. e., $u = z + A_\beta^{-1}(Tz + f) + v$, with $z \in D(T)$, $f \in Z(A_1') \ominus W$, $v \in D(A_0)$. Then clearly $u \in D(A_1)$. Since $z \in Z(A_1)$ and $A_\beta^{-1}(Tz + f) + v \in D(A_\beta)$, $u_\zeta = z$. Also, $Au = Tz + f + Av$, where $Tz \in W$, $f \in Z(A_1') \ominus W$ and $v \in R(A_0)$; then since

$$(1.12) \qquad H = W \oplus (Z(A_1') \ominus W) \oplus R(A_0),$$

we find that $(Au)_W = Tz$. It follows that $u \in D_1$.

Conversely, let $u \in D_1$. Decompose

$$(1.13) \qquad Au = (Au)_W + (Au)_{Z(A_1') \ominus W} + (Au)_{R(A_0)}.$$

Set $(Au)_{Z(A_1') \ominus W} = f$, and set $A_\beta^{-1}[(Au)_{R(A_0)}] = v$. Then $v \in D(A_0)$, since A_0 is $1-1$.

Now $u = u_\zeta + u_\beta$, where $u_\beta = A_\beta^{-1} Au$. By assumption, $u_\zeta \in D(T)$ and $(Au)_W = Tu_\zeta$. Then

$$\begin{aligned} u &= u_\zeta + A_\beta^{-1} Au \\ &= u_\zeta + A_\beta^{-1}(Tu_\zeta + f + Av) \qquad \text{by (1.13)} \\ &= u_\zeta + A_\beta^{-1}(Tu_\zeta + f) + v \end{aligned}$$

where $u_\zeta \in D(T)$, $f \in Z(A_1') \ominus W$ and $v \in D(A_0)$. Thus u belongs to D_2.

The uniqueness is easily shown (by use of (1.12)).

We can now prove:

Theorem 1.2. *Let $\widetilde{A}$ correspond to $T: V \to W$ as in Corollary 1.1. Then $u \in D(\widetilde{A})$ if and only if*

$$u = z + A_\beta^{-1}(Tz + f) + v \textit{ for some } z \in D(T), f \in Z(A_1') \ominus W, v \in D(A_0)$$

Here z, f and v are uniquely determined by u, and the mapping

$$[z, f, v] \curvearrowright u = z + A_\beta^{-1}(Tz + f) + v$$

is an isomorphism of $D(T) \times (Z(A_1') \ominus W) \times D(A_0)$ onto $D(\widetilde{A})$, when the spaces are provided with the graphtopologies.

Proof: The first part of the theorem follows immediately from Corollary 1.1 together with Lemma 1.4. Lemma 1.4 also gives that the mapping $[z, f, v] \curvearrowright u = z + A_\beta^{-1}(Tz + f) + v$ of $D(T) \times (Z(A_1') \ominus W) \times D(A_0)$ onto $D(\widetilde{A})$ is $1-1$; it only remains to prove the continuity of the mapping and its inverse.

Let the sequence $[z^n, f^n, v^n]$ converge to $[z, f, v]$ in $D(T) \times (Z(A_1') \ominus W) \times \times D(A_0)$. This means that $z^n \to z$, $Tz^n \to Tz$, $f^n \to f$, $v^n \to v$ and $Av^n \to Av$, in H. Then

$$u^n = z^n + A_\beta^{-1}(Tz^n + f^n) + v^n \to z + A_\beta^{-1}(Tz + f) + v = u,$$

since A_β^{-1} is continuous, and clearly

$$Au^n = Tz^n + f^n + Av^n \longrightarrow Tz + f + Av = Au.$$

Thus $u^n \longrightarrow u$ in $D(\widetilde{A})$. Since the spaces are metric, this proves that the mapping $[z, f, v] \frown\!\searrow u$ is continuous. By the closed graph theorem (recall $\widetilde{A}$ and T are closed, so $D(\widetilde{A})$ and $D(T) \times (Z(A_1') \ominus W) \times D(A_0)$ are Hilbert spaces) the inverse mapping is also continuous, and the proof is completed.

COROLLARY 1.2. *Let $\widetilde{A}$ correspond to T as in Corollary 1.1. and let $\{u^n\}$ be a sequence in $D(\widetilde{A})$. Then $u^n \longrightarrow u$ in $D(\widetilde{A})$ if and only if $u_\beta^n \longrightarrow u_\beta$ in $D(A_\beta)$ and $u_\zeta^n \longrightarrow u_\zeta$ in $D(T)$ (graphtopologies).*

PROOF : By Theorem 1.2, $u^n = z^n + A_\beta^{-1}(Tz^n + f^n) + v^n$, where $[v^n, f^n, v^n]$ is a uniquely determined element of $D(T) \times (Z(A_1') \ominus W) \times D(A_0)$; then $u_\zeta^n = z^n$ and $u_\beta^n = A_\beta^{-1}(Tz^n + f^n) + v^n$. The corollary follows by a straightforward application of Theorem 1.2.

REMARK 1.2. Note however, that $D(\widetilde{A})$ is in general *not* the direct sum of $D(T)$ and a subspace of $D(A_\beta)$. When $u \in D(\widetilde{A})$, the component u_β usually depends (partially) on the component u_ζ since it is of the form $u_\beta = A_\beta^{-1}(Tu_\zeta + f) + v$ (for some $f \in Z(A_1') \ominus W$, $v \in D(A_0)$).

More properties of the correspondence between $\widetilde{A}$ and T are given in the following.

THEOREM 1.3. *Let $\widetilde{A}$ correspond to $T: V \longrightarrow W$ as in Corollary 1.1. Then*

(i) *$Z(\widetilde{A}) = Z(T)$, so $\widetilde{A}$ is $1-1$ if and only if T is $1-1$*

(ii) *$H \diagdown R(\widetilde{A}) = W \diagdown R(T)$. Thus in particular,*

a) $\widetilde{A}$ is onto if and only if T is onto;

b) $R(\widetilde{A})$ is closed if and only if $R(T)$ is closed;

c) $R(\widetilde{A})$ and $R(T)$ have the same codimension.

PROOF :

(i) Let $u \in Z(\widetilde{A})$. Then $u = u_\zeta$ and $Tu_\zeta = (Au)_W = 0$, so $u \in Z(T)$. Let $u \in Z(T)$. Then $u \in Z(A_1) \cap D(\widetilde{A}) = Z(\widetilde{A})$.

(ii) By Theorem 1.2, the general element of $R(\widetilde{A})$ is

$$g = Tz + f + Av$$

where $z \in D(T)$, $f \in Z(A_1') \ominus W$ and $v \in D(A_0)$ independently. Here f runs through $Z(A_1') \ominus W$ and Av runs through $R(A_0) = H \ominus Z(A_1')$, so $f + Av$ runs through $H \ominus W$. Therefore the elements of H that g does not reach are exactly the elements of W that Tz does not reach, i. e.,

$$H \diagdown R(\widetilde{A}) = W \diagdown R(T).$$

COROLLARY 1.3. *$\widetilde{A}$ and T have the same index. $\widetilde{A}$ is a semi-Fredholm operator if and only if T is one, and $\widetilde{A}$ is a Fredholm operator if and only if T is one.*

(The index of an operator S is defined as the dimension of $Z(S)$ minus the codimension of $R(S)$, if both are finite. S is called a semi-Fredholm operator if S is closed, $R(S)$ is closed, and the dimension of $Z(S)$ is finite; it is called a Fredholm operator if furthermore the codimension of $R(S)$ is finite).

Theorem 1.3 states that $\widetilde{A}$ is $1-1$ if and only if $T: V \longrightarrow W$ is $1-1$. For this case one has:

THEOREM 1.4. *Let $\widetilde{A}$ correspond to $T: V \longrightarrow W$ as in Corollary 1.1. Assume that $\widetilde{A}$ is $1-1$ (or, equivalently, T is $1-1$). Define $T^{(-1)}$ as the linear extension of*

$$T^{(-1)} f = \begin{cases} T^{-1} f & \text{when } f \in R(T) \\ 0 & \text{when } f \in H \ominus W \end{cases}$$

Then

$$\widetilde{A}^{-1} = A_\beta^{-1} + T^{(-1)} \text{ (defined on } R(\widetilde{A})).$$

PROOF: Let $f \in R(\widetilde{A})$. Let $u = \widetilde{A}^{-1} f$, $v = A_\beta^{-1} f$. Then $u = v + z$ where $z \in Z(A_1)$.

By the definition of T, $z \in D(T)$ and

$$Tz = (Au)_W = f_W.$$

Therefore $f_W \in R(T)$ and

$$T^{(-1)} f = T^{-1} f_W = z.$$

Inserting this in $u = v + z$ we find

$$\widetilde{A}^{-1} f = A_\beta^{-1} f + T^{(-1)} f$$

which proves the theorem.

With this theorem one can bring perturbation theorems into use. As an example we mention the obvious

COROLLARY 1.4. *Let A_β^{-1} be a compact operator. Then $\widetilde{A}^{-1}$ exists and is compact if and only if T^{-1} exists and is compact.*

Other examples are given in Theorem 2.2 below.

II § 2. The symmetric case.

Let us in this § assume once and for all: $\widetilde{A}$ corresponds to $T: V \to W$ as in Corollary 1.1.

In the case where $A_1 = A_0^*$ (or equivalently $A_0' = A_0$, or $A_1' = A_1$), A_β can be chosen to be selfadjoint, which makes the set-up particularly simple. Then the two decompositions mentioned in Lemma 1.1 *coincide*, and the operators $T: V \to W$ are operators *in* $Z(A_1)$. It may now happen that $V \subset W$ or $V = W$. In the first case the numerical range $\nu(T)$ of T (see Appendix) can be defined; in the second also the spectrum $\sigma(T)$. We will show how properties of these are reflected in properties of $\nu(\widetilde{A})$ and $\sigma(\widetilde{A})$.

Throughout this § we assume, in addition to the assumptions of § 1:

$$A_1 = A_0^*. \tag{2.1}$$

Then $\mathcal{M} = \mathcal{M}' =$ the set of operators between A_0 and A_1. In particular, $\widetilde{A} \in \mathcal{M}$ is selfadjoint if and only if $\widetilde{A}$ is a selfadjoint extension of A_0.

Let A_β be a selfadjoint operator $\in \mathcal{M}$ such that $0 \in \varrho(A_\beta)$ (that such an operator exists when A_0 is symmetric with bounded inverse was proved first by Calkin [9], see Riesz-Nagy [26] p. 336). One then obtains immediately from Theorem 1.1 that $\widetilde{A} \in \mathcal{M}$ is selfadjoint if and only if $V = W$ and $T: V \to V$ is selfadjoint. In a more common terminology:

THEOREM 2.1. *Let A_0 be a closed, symmetric, densely defined operator with bounded inverse, and let $A_1 = A_0^*$. Let A_β be a selfadjoint extension of A_0 with $0 \in \varrho(A_\beta)$; denote the decomposition $D(A_1) = D(A_\beta) \dotplus Z(A_1)$ by $u = u_\beta + u_\zeta$.*

(i) *Let V be any closed subspace of $Z(A_1)$ and let T be any selfadjoint operator in V. Then the operator $\widetilde{A} \subset A_1$ defined by*

$$D(\widetilde{A}) = \{u \in D(A_1) \mid u_\zeta \in D(T),\ (Au)_V = Tu_\zeta\} \tag{2.2}$$

is a selfadjoint extension of A_0.

(ii) *Conversely, any selfadjoint extension* $\widetilde{A}$ *of* A_0 *defines a selfadjoint operator* T *in* V (= *the closure of* $\{u_\zeta \mid u \in D(\widetilde{A})\}$) *by* (2.2).

$\widetilde{A}$ *and* T *correspond uniquely to each other.*

REMARK 2.1 This theorem gives a complete characterization of all selfadjoint extensions of A_0, a problem earlier discussed by Calkin [9], Krein [22], Birman [7], Vishik [32] (in some cases with partial, and more complicated results). The above result has the advantage of being easily translatable into a theorem on boundary conditions (Theorem III. 4.1).

We give two examples of perturbation theorems for selfadjoint operators applied to Theorem 1.4 :

THEOREM 2.2. *Assume* (2.1) *and* A_β *selfadjoint. Let* $\widetilde{A} \in \mathcal{M}$ *be selfadjoint with* $0 \in \varrho(\widetilde{A})$, *and let* $\widetilde{A}$ *correspond to* $T: V \to V$ *as in Theorem* 2.1.

(i) *If* A_β^{-1} *is compact, then* $\widetilde{A}$ *and* T *have the same essential spectrum.*

(ii) *If* T^{-1} *is of trace class, then* $\widetilde{A}^{-1}$ *and* A_β^{-1} *have the same absolutely continuous spectrum.*

PROOF : Recall (Theorem 1.3) that $0 \in \varrho(\widetilde{A})$ if and only if $0 \in \varrho(T)$. Let $T^{(-1)}$ be defined as the linear extension of

$$T^{(-1)} f = \begin{cases} T^{-1} f & \text{for } f \in V \\ 0 & \text{for } f \in H \ominus V. \end{cases}$$

Then it follows from Theorem 1.4 that

$$\widetilde{A}^{-1} = A_\beta^{-1} + T^{(-1)}.$$

(i) The proof is an application of the theorem of Weyl (Riesz-Nagy [26] p. 362): If S_1 is bounded selfadjoint and S_2 is compact selfadjoint then S_1 and $S_1 + S_2$ have the same essential spectrum.

Let $S_1 = T^{(-1)}$, $S_2 = A_\beta^{-1}$. Then if A_β^{-1} is compact, $\widetilde{A}^{-1}$ and $T^{(-1)}$ have the same essential spectrum. Since $0 \in \varrho(\widetilde{A})$ and $0 \in \varrho(T)$ it follows that $\widetilde{A}$ and T have the same essential spectrum.

(ii) The proof is an application of the theorem of Rosenblum-Kato [27], [20]: If S_1 is selfadjoint and S_2 is selfadjoint and of trace class (i. e., has finite absolute trace), then S_1 and $S_1 + S_2$ have the same absolutely continuous spectrum.

Let $S_1 = A_\beta^{-1}$, $S_2 = T^{(-1)}$. Clearly $T^{(-1)}$ is of trace class if and only if T^{-1} is of trace class.

REMARK 2.2. A version of Theorem 1.4, with selfadjoint *positive* A_β and $\widetilde{A}$, was proved by Birman [7]; for such operators Theorem 2.2 is a consequence of his result. He gave in [8] an interesting application of Theorem 2.2 (ii).

For the characterization of more general $\widetilde{A} \in \mathcal{M}$, the following lemma will be very useful:

LEMMA 2.1. *Assume* (2.1) *and* A_β *selfadjoint.*
If $V \subset W$ *one has for* $u, v \in D(\widetilde{A})$:

$$(Au, v) = (Au_\beta, v_\beta) + (Tu_\zeta, v_\zeta). \tag{2.3}$$

In particular

$$\operatorname{Re}(Au, u) = (Au_\beta, u_\beta) + \operatorname{Re}(Tu_\zeta, u_\zeta) \tag{2.4}$$

$$\operatorname{Im}(Au, u) = \operatorname{Im}(Tu_\zeta, u_\zeta). \tag{2.5}$$

PROOF: For $u, v \in D(\widetilde{A})$

$$(Au, v) = (Au_\beta, v_\beta) + (Au, v_\zeta).$$

If $V \subset W$, $v_\zeta \in W$ so $(Au, v_\zeta) = ((Au)_W, v_\zeta) = (Tu_\zeta, v_\zeta)$.
Then

$$(Au, v) = (Au_\beta, v_\beta) + (Tu_\zeta, v_\zeta).$$

(2.4) and (2.5) follow from the fact that (Au_β, u_β) is real.
For the symmetric $\widetilde{A} \in \mathcal{M}$ we then get

THEOREM 2.3. *Assume* (2.1) *and* A_β *selfadjoint. The following statements are equivalent*:

(i) $\widetilde{A}$ *is symmetric*
(ii) $V \subset W$ *and* T *is symmetric as an operator in* W.

PROOF: If $V \subset W$ and T is symmetric, then by Lemma 2.1:
$\operatorname{Im}(Au, u) = \operatorname{Im}(Tu_\zeta, u_\zeta) = 0$ for all $u \in D(\widetilde{A})$, thus $\widetilde{A}$ is symmetric.

Conversely, if $\widetilde{A}$ is symmetric then $\widetilde{A} \subset \widetilde{A}^*$ so $V = \overline{pr_\zeta D(\widetilde{A})} \subset \overline{pr_\zeta D(\widetilde{A}^*)} = W$; and $\operatorname{Im}(Tu_\zeta, u_\zeta) = 0$ for all $u_\zeta \in D(T)$ by (2.5). This implies that T is symmetric as an operator in W.

COROLLARY 2.3. *Assume* (2.1) *and* A_β *selfadjoint.* A *is maximal symmetric if and only if* $V = W$ *and* T *is maximal symmetric.*

PROOF: Let $V = W$ and let T be maximal symmetric. If $\widetilde{A}_e$ is a symmetric extension of $\widetilde{A}$, $\widetilde{A}_e^* \subset \widetilde{A}^*$; then the corresponding operator $T_e: V_e \rightarrow W_e$ satisfies $V \subset V_e \subset W_e \subset W = V$, so that T_e is a symmetric extension of T. By the assumption on T, $T_e = T$, which shows that $\widetilde{A}$ is maximal symmetric.

Conversely let $\widetilde{A}$ be maximal symmetric. Then $V \subset W$ and T is symmetric in W. If $V \neq W$ we can extend T trivially to a symmetric operator $T_e: W \longrightarrow W$ by letting $T_e z = 0$ for $z \in W \ominus V$. T_e corresponds to a proper symmetric extension $\widetilde{A}_e$ of $\widetilde{A}$, in contradiction to $\widetilde{A}$ being maximal symmetric. Thus $V = W$, and the maximality of $\widetilde{A}$ implies the maximality of T.

Lemma 2.1 indicates that there is a close connection between the numerical ranges (and then also spectra) of $\widetilde{A}$ and T (2). Theorems 2.1 and 2.3 show that the spectra resp. numerical ranges of $\widetilde{A}$ and T are simultaneously contained in the real axis. Further results can be obtained if we assume positivity of A_0:

$$m(A_0) = \inf \{(Au, u) \mid u \in D(A_0), |u| = 1\} > 0. \tag{2.6}$$

Then there exist positive selfadjoint extensions A_β, by Friedrichs' lemma below. Friedrichs' lemma singles out one particular selfadjoint extension A_γ with the same lower bound as A_0:

LEMMA 2.2 (*Friedrichs* [13]) *Assume* (2.1) *and* (2.6). *There exists one and only one selfadjoint operator* $A_\gamma \in \mathcal{M}$ *which satisfies:*

For every $u \in D(A_\gamma)$ *there exists a sequence* $\{u^n\} \subset D(A_0)$ *such that* $u^n \longrightarrow u$ *and* $(A(u^n - u), u^n - u) \longrightarrow 0$.

This operator A_γ *satisfies* $m(A_\gamma) = m(A_0)$.

A proof of Lemma 2.2 is given in Riesz-Nagy [26] p. 325-331.

---

(2) The definition and the relevant properties of the numerical range and spectrum of an operator are given in the Appendix (where the results are denumerated by A.1 A.2 etc.).

Outline of results.

Let us first note that one always has

$$\nu(\widetilde{A}) \supset \nu(A_0), \tag{2.7}$$

simply because $\widetilde{A}$ is an extension of A_0. This limits the arbitrariness of $\nu(\widetilde{A})$; in particular since $\nu(\widetilde{A})$ is convex (Appendix, Lemma A. 1). Under the assumption of (2.6), (2.7) implies

$$\nu(\widetilde{A}) \supset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > m(A_0), \operatorname{Im}\lambda = 0\}. \tag{2.8}$$

1^0 Properties carrying over from $\widetilde{A}$ to T. Assume (2.1) and (2.6). For the case where $A_\beta = A_\gamma$ we prove that $\nu(T) \subset \bar{\nu}(\widetilde{A})$. If furthermore $\sigma(\widetilde{A}) \subset \subset \bar{\nu}(\widetilde{A})$ (this holds if merely one point in each component of the complement of $\bar{\nu}(\widetilde{A})$ belongs to $\varrho(\widetilde{A})$), then $\sigma(T) \subset \bar{\nu}(\widetilde{A})$. For the case $A_\beta \neq A_\gamma$ we have no results in this direction.

2^0 Properties carrying over from T to $\widetilde{A}$. Assume (2.1) and (2.6) and let A_β be selfadjoint with $m(A_\beta) > 0$. Let $\nu(T)$ be contained in a closed halfplane π_T with the property : the closed halfline $\{\operatorname{Re}\lambda \leq -m(A_\beta), \operatorname{Im}\lambda = 0\}$ is either exterior or part of the boundary of π_T. Then $\nu(\widetilde{A})$ is contained in a certain closed halfplane $\pi_{\widetilde{A}}$ parallel to π_T (such that $\pi_{\widetilde{A}}$ contains the halfline $\{\operatorname{Re}\lambda \geq m(A_0), \operatorname{Im}\lambda = 0\}$, by (2.8)). Here, if the boundary of π_T intersects the positive real axis, then the same holds for the boundary of $\pi_{\widetilde{A}}$; if the boundary of π_T contains 0, then so does the boundary of $\pi_{\widetilde{A}}$.

If both $\nu(T)$ and $\sigma(T)$ are contained in π_T, then also $\sigma(\widetilde{A})$ is contained in $\pi_{\widetilde{A}}$.

The above results can be combined to obtain results about angles and more complicated convex figures.

1^0 *Properties carrying over from $\widetilde{A}$ to T.*

We here make the basic assumption :

$$\text{(2.1) and (2.6) hold} ;\ \underline{A_\beta = A_\gamma}. \tag{2.9}$$

PROPOSITION 2.1. *Assume* (2.9). *If there exist* $\lambda \in \mathbb{C}$, $c > 0$, *such that*

$$|(Au, u) - \lambda |u|^2| \geq c |u|^2 \text{ for all } u \in D(\widetilde{A}), \tag{2.10}$$

then

$$\text{(i)} \quad V \subset W$$

$$\text{(ii)} \quad |(Tz, z) - \lambda |z|^2 \geq c|z|^2 \quad \textit{for all} \quad z \in D(T). \tag{2.11}$$

PROOF: By Theorem 1.2, the general element of $D(\widetilde{A})$ is

$$u = z + A_\gamma^{-1}(Tz + f) + v, \quad z \in D(T), \ f \in Z(A_1) \ominus W, \ v \in D(A_0).$$

Here $u_\zeta = z$, $u_\gamma = A_\gamma^{-1}(Tz + f) + v$. Therefore for $u \in D(\widetilde{A})$

$$\begin{aligned}(Au, u) &= (Au, u_\gamma + u_\zeta) \\ &= (Au_\gamma, u_\gamma) + (Tz + f + Av, z).\end{aligned}$$

Since $v \in D(A_0)$ and $R(A_0) \perp Z(A_1)$, this can be written as:

$$(Au, u) = (Au_\gamma, u_\gamma) + (Tz, z) + (f, z).$$

Now assume that for some $\lambda \in \mathbb{C}$, $c > 0$, (2.10) holds, i.e.,

$$\left| \frac{(Au, u)}{|u|^2} - \lambda \right| \geq c, \qquad \text{all } u \in D(\widetilde{A}) \setminus \{0\}.$$

Then

$$\left| \frac{(Au_\gamma, u_\gamma) + (Tz, z) + (f, z)}{|u_\gamma + z|^2} - \lambda \right| \geq c$$

for all $z \in D(T)$, $f \in Z(A_1) \ominus W$, $v \in D(A_0)$ ($[z, f, v] \neq (0, 0, 0]$). By Lemma 2.2 we can, for fixed f, z, choose v such that (Au_γ, u_γ) is arbitrarily small (with $|u_\gamma|^2 \leq m(A_\gamma)^{-1}(Au_\gamma, u_\gamma)$).

This implies that in fact

$$\left| \frac{(Tz, z) + (f, z)}{|z|^2} - \lambda \right| \geq c \tag{2.12}$$

for all $z \in D(T) \setminus \{0\}$, all $f \in Z(A_1) \ominus W$.

Now, if (f, z) were $\neq 0$ for some $z \in D(T)$, $f \in Z(A_1) \ominus W$, then the expression

$$\left| \frac{(Tz, z) + (kf, z)}{|z|^2} - \lambda \right|$$

would equal 0 for some $k \in \mathbb{C}$. This contradicts (2.12); therefore $(f, z) = 0$ for all $z \in D(T)$, all $f \in Z(A_1) \ominus W$.

Recall $\overline{D(T)} = V$; we have then proved that $V \perp Z(A_1) \ominus W$ or, $V \subset W$. Then

$$\left| \frac{(Tz, z)}{|z|^2} - \lambda \right| \geq c \text{ for all } z \in D(T) \setminus \{0\},$$

i.e.,

$$|(Tz, z) - \lambda |z|^2| \geq c\, |z^2| \text{ for all } z \in D(T).$$

THEOREM 2.4. *Assume* (2.9). *If* $\nu(\widetilde{A})$ *is not all of* $\mathbb{C}$, *then* $V \subset W$, *and*

$$\nu(T) \subset \overline{\nu}(\widetilde{A}). \tag{2.13}$$

PROOF: Let $\nu(\widetilde{A}) \neq \mathbb{C}$, then also $\overline{\nu}(\widetilde{A}) \neq \mathbb{C}$, since $\nu(\widetilde{A})$ is convex (it is in fact contained in a halfplane). Now $\lambda \in \mathbb{C} \setminus \overline{\nu}(\widetilde{A})$ if and only if λ satisfies (2.10) with positive c (c can be chosen as dist $(\lambda, \nu(\widetilde{A}))$). Then Proposition 2.1 shows that $V \subset W$, and that $\lambda \in \mathbb{C} \setminus \overline{\nu}(\widetilde{A})$ implies (2.11), i.e., $\lambda \in \mathbb{C} \setminus \overline{\nu}(T)$.

COROLLARY 2.4. *Assume* (2.9). *If for some* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $m(e^{i\theta} \widetilde{A}) > -\infty$, *then* $V \subset W$ *and* $m(e^{i\theta} T) \geq m(e^{i\theta} \widetilde{A})$. $\Big($*For all* $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$; $m(e^{i\theta}\widetilde{A}) = -\infty$ by (2.8)$\Big)$.

COROLLARY 2.5. *Assume* (2.9), *and let* $V \subset W$. *If* $m(e^{i\theta} T) = -\infty$ *for some* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, *then also* $m(e^{i\theta} \widetilde{A}) = -\infty$. *If* $m(e^{i\theta} T) = -\infty$ *for all* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ *then* $\nu(\widetilde{A}) = \mathbb{C}$.

If $\varkappa$ is a subset of $\mathbb{C}$, we denote the conjugate set $\{\bar{\lambda} \mid \lambda \in \varkappa\}$ by $\varkappa'$.

THEOREM 2.5. *Assume* (2.9). *Let* $\nu(\widetilde{A}) \neq \mathbb{C}$, *and let* $\varkappa$ *be a component of* $\mathbb{C} \setminus \overline{\nu}(\widetilde{A})$, *for which* $\varkappa \cap \varrho(\widetilde{A}) \neq \Phi$. *Then* $V = W$ *and*

$$\varkappa \subset \varrho(T), \qquad \varkappa' \subset \varrho(T^*).$$

PROOF: By proposition A.3, $\varkappa' \subset \mathbb{C} \setminus \overline{\nu}(\widetilde{A}^*)$. Applying Theorem 2.4 to $\widetilde{A}$ and $\widetilde{A}^*$ we obtain

$$V \subset W, \qquad \overline{\nu}(T) \subset \overline{\nu}(\widetilde{A}),$$

$$W \subset V, \qquad \overline{\nu}(T^*) \subset \overline{\nu}(\widetilde{A}^*).$$

Thus $V = W$. Moreover, since $\varkappa \subset \mathbb{C} \setminus \bar{\nu}(\tilde{A})$ and $\varkappa' \subset \mathbb{C} \setminus \bar{\nu}(\tilde{A}^*)$ it follows that $\varkappa \subset \mathbb{C} \setminus \bar{\nu}(T)$ and $\varkappa' \subset \mathbb{C} \setminus \bar{\nu}(T^*)$. Applying Lemma A.3 to the operators T and T^*, we then obtain

$$\varkappa \subset \varrho(T), \qquad \varkappa' \subset \varrho(T^*).$$

COROLLARY 2.6. *Assume* (2.9). *If for some* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $e^{i\theta} \tilde{A}$ *is maximal lower bounded, then so is* $e^{i\theta} T$, *and* $m(e^{i\theta} T) \geq m(e^{i\theta} \tilde{A})$.

For applications of this corollary, note that a closed operators S is maximal nonnegative [maximal positive] if and only if it is nonnegative [positive] and maximal lower bounded (Corollary A.4).

From Theorem 2.5 we deduce the following result, which for the adjoint is particularly informative in the case where $\bar{\nu}(\tilde{A})$ is not a halfplane:

THEOREM 2.6. *Assume* (2.9). *Let* $\bar{\nu}(\tilde{A}) \neq \mathbb{C}$, *and let* $\sigma(\tilde{A}) \subset \bar{\nu}(\tilde{A})$ (*this holds if merely one point in each component of* $\mathbb{C} \setminus \bar{\nu}(\tilde{A})$ *is in* $\varrho(\tilde{A})$). *Then*

$$\sigma(T) \subset \bar{\nu}(\tilde{A}) \text{ and } \sigma(T^*) \subset \bar{\nu}(\tilde{A})'.$$

Here $\mathbb{C} \setminus \bar{\nu}(\tilde{A})'$ *is a component of* $\mathbb{C} \setminus \bar{\nu}(\tilde{A}^*)$, *and if* $\bar{\nu}(\tilde{A})$ *is not equal to a halfplane, then in fact*

$$\bar{\nu}(\tilde{A})' = \bar{\nu}(\tilde{A}^*).$$

PROOF: By application of Theorem 2.5 to each component of $\mathbb{C} \setminus \bar{\nu}(\tilde{A})$, we get $\mathbb{C} \setminus \bar{\nu}(\tilde{A}) \subset \varrho(T)$ and $\mathbb{C} \setminus \bar{\nu}(\tilde{A})' \subset \varrho(T^*)$; hence

$$\sigma(T) \subset \bar{\nu}(\tilde{A}), \qquad \sigma(T^*) \subset \bar{\nu}(\tilde{A})'.$$

The last statement follows from Proposition A.3 and Corollary A.3.

2^0 *Properties carrying over from* T *to* $\tilde{A}$.

In this section we assume:

(2.14) (2.1) and (2.6) hold; A_β is selfadjoint with $m(A_\beta) > 0$.

THEOREM 2.7. *Assume* (2.14). *Let* $V \subset W$, *and let* $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. *If* $m(e^{i\theta} T) > -\cos\theta \; m(A_\beta)$, *then*

$$m(e^{i\theta} \tilde{A}) \geq \frac{\cos\theta \; m(A_\beta) \; m(e^{i\theta} T)}{\cos\theta \; m(A_\beta) + m(e^{i\theta} T)}. \tag{2.15}$$

PROOF: Let $u \in D(A) \setminus \{0\}$. Then, by Lemma 2.1

$$\frac{\mathrm{Re}\,(e^{i\theta}\,\tilde{A}u,\,u)}{|\,u\,|^2} = \frac{\mathrm{Re}\,(e^{i\theta}\,Au_\beta\,,\,u_\beta) + \mathrm{Re}\,(e^{i\theta}\,Tu_\zeta\,,\,u_\zeta)}{|\,u_\beta + u_\zeta\,|^2}$$

$$\frac{\mathrm{Re}\,(e^{i\theta}\,\tilde{A}u,\,u)}{|\,u\,|^2} \geq \frac{\cos\theta\ m(A_\beta)\,|\,u_\beta\,|^2 + m(e^{i\theta}\,T)\,|\,u_\zeta\,|^2}{|\,u_\beta + u_\zeta\,|^2}. \tag{2.16}$$

It follows immediately that

$$\frac{\mathrm{Re}\,(e^{i\theta}\,\tilde{A}u,\,u)}{|\,u\,|^2} \geq \cos\theta\ m\,(A_\beta) \text{ if } u_\zeta = 0. \tag{2.17}$$

Let us now assume $u_\zeta \neq 0$. In the case $m\,(e^{i\theta}\,T) \geq 0$, we obtain from (2.16)

$$\frac{\mathrm{Re}\,(e^{i\theta}\,\tilde{A}u,u)}{|\,u\,|^2} \geq \frac{\cos\theta\ m\,(A_\beta)\,|\,u_\beta\,|^2 + m\,(e^{i\theta}\,T)\,|\,u_\zeta\,|^2}{(|\,u_\beta\,| + |\,u_\zeta\,|)^2}$$

$$= \frac{\cos\theta\ m\,(A_\beta)\,t^2 + m\,(e^{i\theta}\,T)}{(t+1)^2},$$

where $t = |\,u_\beta\,|\,|\,u_\zeta\,|^{-1}$. Since the function $f(t) = (at^2 + b)(t+1)^{-2}$, $a > 0$, $b \geq 0$, defined for $t \geq 0$, obtains its minimum at $t = ba^{-1}$ with $f(ba^{-1}) = ab\,(a+b)^{-1}$, we find that

$$\frac{\mathrm{Re}\,(e^{i\theta}\,\tilde{A}u,\,u)}{|\,u\,|^2} \geq \frac{\cos\theta\ m(A_\beta)\ \ m(e^{i\theta}\,T)}{\cos\theta\ m(A_\beta) + m(e^{i\theta}\,T)}, \text{ when } m\,(e^{i\theta}\,T) \geq 0. \tag{2.18}$$

In the case $0 > m\,(e^{i\theta}\,T) > -\cos\theta\ m(A_\beta)$ we proceed a little differently. Here,

$$\frac{\mathrm{Re}\,(e^{i\theta}\,\tilde{A}u,u)}{|\,u\,|^2} \geq 0 \text{ if } \cos\theta\ m(A_\beta)\,|\,u_\beta\,|^2 + m\,(e^{i\theta}\,T)\,|\,u_\zeta\,|^2 \geq 0. \tag{2.19}$$

Otherwise, $t = |\,u_\beta\,|\,|\,u_\zeta\,|^{-1} < (-\,m\,(e^{i\theta}\,T))^{\frac{1}{2}}\,(\cos\theta\ m(A_\beta))^{-\frac{1}{2}} < 1$, and

$$\frac{\mathrm{Re}\,(e^{i\theta}\,\tilde{A}u,\,u)}{|\,u\,|^2} \geq \frac{\cos\theta\ m(A_\beta)\,t^2 + m\,(e^{i\theta}\,T)}{(t-1)^2},$$

since the denumerator is negative. The function $g\,(t) = (at^2 + b)\,(t-1)^{-2}$, $a > 0$, $0 > b > -a$, defined for $0 \leq t < (-\,b)^{\frac{1}{2}}\ a^{-\frac{1}{2}}$, obtains its minimum at

$t = -ba^{-1}$ with $g(-ba^{-1}) = ab(a+b)^{-1}$. Thus we also here get

$$(2.20)\quad \begin{cases} \dfrac{\operatorname{Re}(e^{i\theta}\tilde{A}u, u)}{|u|^2} \geq \dfrac{\cos\theta\, m(A_\beta)\; m(e^{i\theta}T)}{\cos\theta\, m(A_\beta) + m(e^{i\theta}T)}, \text{ when} \\ 0 > m(e^{i\theta}T) > -\cos\theta\, m(A_\beta), \cos\theta\, m(A_\beta)\,|u_\beta|^2 + m(e^{i\theta}T)\,|u_\zeta|^2 < 0. \end{cases}$$

Since $ab(a+b)^{-1} \leq a$ when $b > -a$, and $ab(a+b)^{-1} < 0$ when $0 > b > -a$, a comparison of (2.17), (2.18), (2.19) and (2.20) gives that

$$\frac{\operatorname{Re}(e^{i\theta}\tilde{A}u, u)}{|u|^2} \geq \frac{\cos\theta\, m(A_\beta)\; m(e^{i\theta}T)}{\cos\theta\, m(A_\beta) + m(e^{i\theta}T)}, \quad \text{all } u \in D(\tilde{A}) \setminus \{0\},$$

which proves the theorem.

COROLLARY 2.7. *Assumptions of Theorem 2.7. If $m(e^{i\theta}T)$ is $> 0, \geq 0$ or $> -\cos\theta\, m(A_\beta)$, then $m(e^{i\theta}\tilde{A})$ is $> 0, \geq 0$ or $> -\infty$, respectively.*

REMARK 2.3. Theorem 2.7 extends a result by Birman [7], proved under the assumptions that $A_\beta = A_\gamma$, T is selfadjoint, $\theta = 0$.

REMARK 2.4. For a geometric interpretation of Theorem 2.7, note that $m(e^{i\theta}S) > -\infty$ $\left(\text{some } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$ means that $\nu(S)$ is contained in the closed halfplane π_S whose boundary intersects the real axis at the point $\cos^{-1}\theta\, m(e^{i\theta}S)$ under the angle $\frac{\pi}{2} - \theta$; π_S contains large positive real points. When $m(e^{i\theta}S) > -\cos\theta\, m(A_\beta)$, the point $-m(A_\beta)$ is exterior to π_S.

The result for the case $\theta = \pm\frac{\pi}{2}$ is rather trivial, and it does not use positivity of A_β:

THEOREM 2.8. *Assume* (2.1) *and A_β selfadjoint. Let $V \subset W$ and let $\theta = \frac{\pi}{2}$ or $-\frac{\pi}{2}$. Then $m(e^{i\theta}T) \geq 0$ if and only if $m(e^{i\theta}\tilde{A}) = 0$.*

PROOF: Equation (2.5) in Lemma 2.1 shows that $m(e^{i\theta}\tilde{A}) \geq 0$ if and only if $m(e^{i\theta}T) \geq 0$. Moreover, it follows from (2.7) that $m(e^{i\theta}\tilde{A}) \leq 0$ for all T; this completes the proof.

We will now use the connection between the spectrum and the numerical range.

THEOREM 2.9. *Assume* (2.14). *Let* $V = W$, *and let* $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. *If* $e^{i\theta} T$ *is maximal lower bounded with* $m(e^{i\theta} T) > -\cos\theta\, m(A_\beta)$, *then* $e^{i\theta} \widetilde{A}$ *is maximal lower bounded (the lower bound estimated by* (2.15)).

PROOF: If $e^{i\theta} T$ is maximal lower bounded, then $e^{i\theta} T - m(e^{i\theta} T)$ is maximal nonnegative, by Proposition A.4 in the appendix. Then also the adjoint $e^{-i\theta} T^* - m(e^{i\theta} T)$ is nonnegative, by Corollary A.5. When $m(e^{i\theta} T) > -\cos\theta\, m(A_\beta)$, we can apply Theorem 2.7 to $e^{i\theta} T$ and $e^{-i\theta} T^*$ to obtain that $e^{i\theta} \widetilde{A}$ and $e^{-i\theta} \widetilde{A}^*$ are lower bounded. Then an application of Corollary A.5 gives that $e^{i\theta} \widetilde{A}$ is in fact maximal lower bounded.

Application of a similar technique to Theorem 2.8 gives

THEOREM 2.10. *Assume* (2.1) *and* A_β *selfadjoint. Let* $V = W$ *and let* $\theta = \frac{\pi}{2}$ *or* $-\frac{\pi}{2}$. *Then* $e^{i\theta} T$ *is maximal nonnegative if and only if* $e^{i\theta} \widetilde{A}$ *is maximal nonnegative.*

Finally, we note that Theorem 2.7 - 2.10 can be employed to give results about angles and other convex sets, e. g.,

COROLLARY 2.8. *Let* α *denote a closed angle which is the intersection of two closed halfplanes for which the halfaxis* $\{\mathrm{Re}\,\lambda \le -m(A_\beta),\ \mathrm{Im}\,\lambda = 0\}$ *is exterior or part of the boundary. If* $\nu(T)$ *is contained in* α, *and one point of* $\mathbb{C} \setminus \alpha$ *is in* $\varrho(T)$, *then* $\nu(\widetilde{A})$ *and* $\sigma(\widetilde{A})$ *are contained in an angle* α_1, *obtained from* α *by parallel-translation. Here, if both boundary lines are not parallel to the real axis,* $0 \notin \alpha$ *implies* $0 \notin \alpha_1$; *in general* $0 \notin$ *interior of* α *implies* $0 \notin$ *interior of* α_1.

More general convex sets can be treated by computations on the formula (2.15).

II § 3. A discussion of the non-closed operators.

The characterization of operators $\widetilde{A} \in \mathcal{M}$ in terms of operators T between the nullspaces of A_1 and A_1' given in § 1 is limited to closed $\widetilde{A} \in \mathcal{M}$; in the following we will show how far it can be extended to include non-closed operators.

Let $\widehat{A}$ be a closed operator $\in \mathcal{M}$, and let $\widetilde{A} \in \mathcal{M}$ with $\overline{\widetilde{A}} = \widehat{A}$. $\widehat{A}$ corresponds by Corollary 1.1 to an operator $\widehat{T}: V \to W$, closed densely defined, where $V = \overline{pr_\zeta D(\widehat{A})}$ and $W = \overline{pr_{\zeta'} D(\widehat{A}^*)}$. Note that one also has $V =$

$= \overline{pr_\zeta D(\widetilde{A})}$ and $W = \overline{pr_{\zeta'} D(\widetilde{A}^*)}$; for $\widetilde{A}^* = \widehat{A}^*$ and $\overline{pr_\zeta D(\widetilde{A})} = \overline{pr_\zeta D(\widehat{A})}$ since pr_ζ is continuous from $D(A_1)$ to $Z(A_1)$ in the graphtopology.

Theorem 1.2 states that

$$D(\widehat{A}) = \{u = z + A_\beta^{-1}(\widehat{T}z + f) + v \mid [z, f, v] \in D(\widehat{T}) \times (Z(A_1') \ominus W) \times D(A_0)\};$$

now $D(\widetilde{A})$ is a certain dense subset of this set. The exact possibilities for $D(\widetilde{A})$ are given in the following lemma:

LEMMA 3.1. *Let $\widehat{A}$ and $\widetilde{A} \in \mathcal{M}$ such that $\widehat{A}$ is closed and corresponds to $\widehat{T}$: $V \to W$ by Corollary 1.1, and $\widetilde{A} \subset \widehat{A}$. Then $\overline{\widetilde{A}} = \widehat{A}$ if and only if*

$$D(\widetilde{A}) = \{u = z + A_\beta^{-1}(\widehat{T}z + f) + v \mid [z, f] \in F, v \in D(A_0)\}, \tag{3.1}$$

where F is a dense subspace of $D(\widehat{T}) \times (Z(A_1') \ominus W)$ (graphtopology).

PROOF: Since $A_0 \subset \widetilde{A} \subset \widehat{A}$, $D(\widetilde{A})$ is certainly of the form (3.1) for some subspace $F \subset D(\widehat{T}) \times (Z(A_1') \ominus W)$. Then by the isomorphism in Theorem 1.2, $D(\widetilde{A})$ is dense in $D(\widehat{A})$ if and only if $F \times D(A_0)$ is dense in $D(\widehat{T}) \times (Z(A_1') \ominus W) \times D(A_0)$, i.e., if and only if F is dense in $D(\widehat{T}) \times (Z(A_1') \ominus W)$ (graphtopologies).

The remaining discussion will be divided into two cases, according to whether $Z(A_1') \ominus W$ equals $\{0\}$ or not.

1° $W = Z(A_1')$.

When $W = Z(A_1')$, i.e., $\overline{pr_{\zeta'} D(\widetilde{A}^*)} = Z(A_1')$, then it follows from Lemma 3.1 that the set F mentioned there is of the form $F = D \times \{0\}$, where D is a dense subset of $D(\widehat{T})$ (graphtopology). Denote the restriction of $\widehat{T}$ to D by T (so that $D = D(T)$), then Lemma 3.1 shows that

$$D(\widetilde{A}) = \{u = z + A_\beta^{-1} Tz + v \mid z \in D(T), v \in D(A_0)\} \tag{3.2}$$

or, equivalently, using Lemma 1.4;

$$D(\widetilde{A}) = \{u \in D(A_1) \mid u_\zeta \in D(T), (Au)_{Z(A_1')} = Tu_\zeta\}. \tag{3.3}$$

Note that $\overline{T} = \widehat{T}$.

We have then shown that when $\overline{pr_{\zeta'} D(\widetilde{A}^*)} = Z(A_1')$, then there exists a closable operator $T: Z(A_1) \longrightarrow Z(A_1')$ such that $\widetilde{A}$ is determined by one of the equivalent formulas (3.2) and (3.3).

Conversely, when a closable operator $T: Z(A_1) \longrightarrow Z(A_1')$ is given, the operator $\widetilde{A} \in \mathcal{M}$ defined by (3.2) or (3.3) satisfies $\overline{pr_{\zeta'} D(\widetilde{A}^*)} = Z(A_1')$:

Let the closable operator $T: Z(A_1) \longrightarrow Z(A_1')$ be given, and define $\widetilde{A}$ by (3.2). It now follows by Theorem 1.2 that $\overline{\widetilde{A}}$ is determined by

$$D(\overline{\widetilde{A}}) = \{u = z + A_\beta^{-1} \, \overline{T} z + v \mid z \in D(\overline{T}),\ v \in D(A_0)\}.$$

Moreover, $\overline{\widetilde{A}}$ corresponds to $\overline{T}: V \longrightarrow Z(A_1')$ in the sense of Corollary 1.1, with $V = \overline{D(T)}$ (closure in H). Then, by Theorem 1.1, $(\overline{\widetilde{A}})^*$ *corresponds to* $(\overline{T})^*: Z(A_1') \longrightarrow V$, which shows that $\overline{pr_{\zeta'} D(\widetilde{A}^*)} = \overline{pr_{\zeta'} D(\overline{\widetilde{A}}^*)} = Z(A_1')$.

Altogether we have proved

Theorem 3.1. *An operator $\widetilde{A} \in \mathcal{M}$ (closed or not) satisfies $\overline{pr_{\zeta'} D(\widetilde{A}^*)} = Z(A_1')$ if and only if there exists a* c l o s a b l e *operator $T: Z(A_1) \longrightarrow Z(A_1')$ such that*

$$(3.4) \qquad (D(\widetilde{A}) = \{u \in D(A_1) \mid u_\zeta \in D(T),\ (Au)_{Z(A_1')} = Tu_\zeta\}.$$

$\widetilde{A}$ and T determine each other by (3.4).

When $\widetilde{A}$ corresponds to T in this way, $\overline{\widetilde{A}}$ corresponds to $\overline{T}: V \longrightarrow Z(A_1')$ in the sense of Corollary 1.1, *with $V = \overline{D(T)}$.*

Remark 3.1. The correspondence between the $\widetilde{A} \in \mathcal{M}$ for which $\overline{pr_{\zeta'} D(\widetilde{A}^*)} = Z(A_1')$, and the closable operators $T: Z(A_1) \longrightarrow Z(A_1')$, given in the above theorem, is easily seen to be *inclusion preserving.*

2^0 $W \neq Z(A_1')$.

We saw that when $\overline{pr_{\zeta'} D(\widetilde{A}^*)} = Z(A_1')$, then $\widetilde{A}$ *can* be characterized by an operator $T: Z(A_1) \longrightarrow Z(A_1')$. This is not always the case when $pr_{\zeta'} D(\widetilde{A}^*)$ is not dense in $Z(A_1')$. The set F mentioned in Lemma 3.1 is now a dense subset of $D(\widehat{T}) \times (Z(A_1') \ominus W)$. There are three possibilities: 1) F may be of the form $D(T) \times (Z(A_1') \ominus W)$, where $\overline{T} = \widehat{T}$; 2) F may be of the form $D(T) \times Z_1$, where $\overline{T} = \widehat{T}$ and $\overline{Z}_1 = Z(A_1') \ominus W$ (but $Z_1 \neq Z(A_1') \ominus W$); or 3) F may not even be the product of a subspace of $D(\widehat{T})$ and a subspace of $Z(A_1') \ominus W$.

1) In the first case, it is easily seen that $\tilde{A}$ is characterized by T, W, by the formula

$$D(\tilde{A}) = \{u \in D(A_1) \mid u_\zeta \in D(T),\ (Au)_W = Tu_\zeta\}, \tag{3.5}$$

and conversely, (3.5) determines T when $\tilde{A}$ is known. Here $\bar{\tilde{A}} = \hat{A}$ and $\bar{T} = \hat{T}$; so we get the theorem

THEOREM 3.2. *Let V be a closed subspace of $Z(A_1)$, W a closed subspace of $Z(A_1')$, and $T: V \longrightarrow W$ a closable, densely defined operator. Then $T: V \longrightarrow W$ corresponds to an operator $\tilde{A} \in \mathcal{M}$ by*

$$D(\tilde{A}) = \{u \in D(A_1) \mid u_\zeta \in D(T),\ (Au)_W = Tu_\zeta\};$$

and $\bar{T}: V \longrightarrow W$ corresponds to $\bar{\tilde{A}}$ in the sense of Corollary 1.1.

2) In the second case, $\tilde{A}$ cannot be deseribed solely by T and W; information about which part of $Z(A_1') \ominus W$ that is used is also required. Therefore there is not a $1-1$ correspondence between operators $\tilde{A}$ and T as in the previous theorems.

3) The third case contains the remaining types of operators. We will not discuss these further, except that we will mention an important example:

Let T be a *non-closable* operator with $D(T) \subset Z(A_1)$, $R(T) \subset Z(A_1')$. Define the operator $\tilde{A} \in \mathcal{M}$ by

$$D(\tilde{A}) = \{u \in D(A_1) \mid u_\zeta \in D(T),\ (Au)_{Z(A_1')} = Tu_\zeta\}.$$

It follows from Theorem 3.1 that $pr_{\zeta'} D(\tilde{A}^*)$ is *not* dense in $Z(A_1')$. Thus $\bar{\tilde{A}}$ corresponds to an operator $T_1: \overline{D(T)} \longrightarrow W$, where $W \neq Z(A_1')$. One can show that T_1 is an extension of $pr_W \circ T$, and that F (for $\tilde{A}$) consists of the pairs $[z, Tz - T_1 z]$ where $z \in D(T)$; thus is obviously not a product space of a subspace of $D(T_1)$ and a subspace of $Z(A_1') \ominus W$.

(One has in this case that $\tilde{A}^*$ corresponds to $T^{(*)}: W \longrightarrow \overline{D(T)}$, where $T^{(*)}$ is the adjoint of $T: \overline{D(T)} \longrightarrow Z(A_1')$, and $W = \overline{D(T^{(*)})}$. Then $\bar{\tilde{A}} = \tilde{A}^{**}$ corresponds to $T_1: \overline{D(T)} \longrightarrow W$, where T_1 is the adjoint of $T^{(*)}: W \longrightarrow \overline{D(T)}$. Further details are given in [15]).

CHAPTER III. - GENERAL BOUNDARY VALUE PROBLEMS

§ 1. The operators P and M.

Let A, A', B, B', C, C', denote the differential, resp. boundary differential, operators introduced in Chapter I. Throughout this chapter we will assume :

Basic Assumption : Both of the systems $\{A, B\}$ and $\{A', B'\}$ satisfy the hypotheses $(\mathcal{C})$ and $(\mathcal{U})$.

Then the operators $A_0, A_0', A_1, A_1', A_\beta$ and $A_\beta^* = A_\beta'$, defined in Chapter I, satisfy the hypotheses of Chapter II § 1. With the notations used there, $\mathcal{M}$ is the set of realizations of A, and $\mathcal{M}'$ is the set of realizations of A'. Recall that A_β satisfies

(1.1) $$D(A_\beta) = \{u \in D(A_1) \mid Bu = 0\} = \{u \in H^{2m}(\Omega) \mid Bu = 0\},$$

and A_β' satisfies

(1.2) $$D(A_\beta') = \{u \in D(A_1') \mid B'u = 0\} = \{u \in H^{2m}(\Omega) \mid B'u = 0\};$$

A_β and A_β' are adjoints (Corollary I 3.1).

In the present chapter we will translate the $1 - 1$ correspondence between operators $\widetilde{A} \in \mathcal{M}$ and operators T between the nullspaces of A_1 and A_1', given in Chapter II, into a characterization of the realizations of A in terms of boundary value problems. The fundamental property of our set-up that makes this possible is that the boundary operator B maps $Z(A_1)$ isomorphically onto the space of distributions on the boundary $\Pi H^{-m_j - \frac{1}{2}}(\Gamma)$.

More generally, one has :

PROPOSITION 1.1. $B\,[B']$ maps $Z_A^s(\Omega)\,[Z_{A'}^s(\Omega)]$ isomorphically onto $\Pi H^{s-m_j-\frac{1}{2}}(\Gamma)\,[\Pi H^{s-2m+\mu_j+\frac{1}{2}}(\Gamma)]$, for all real s.

PROOF : The proof is a straightforward application of the results of Lions and Magenes quoted in Chapter I.

1^0 $s \leq 2m$.

By Theorem I 3.3 (i) the mapping $\{A, B\}$ is an isomorphism of $D_A^s(\Omega)$ (Definition I 3.1) onto $H^0(\Omega) \times \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$; then the inverse of $\{A, B\}$ maps

$\{0\} \times \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ isomorphically onto a closed subspace of $D_A^s(\Omega)$. This space is exactly $\{u \in D_A^s(\Omega) \mid Au = 0\}$ with the topology of $D_A^s(\Omega)$; by Definition I 3.1 it is equal to the space $Z_A^s(\Omega)$. It follows that B maps $Z_A^s(\Omega)$ isomorphically onto $\Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$.

2^0 $s \geq 2m$.

By Theorem I 3.3 (ii), $\{A, B\}$ maps $H^s(\Omega)$ isomorphically onto $H^{s-2m}(\Omega) \times \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$; thus $\{A, B\}^{-1}$ maps $\{0\} \times \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ isomorphically onto the space $\{u \in H^s(\Omega) \mid Au = 0\}$, provided with the topology of $H^s(\Omega)$. This space is exactly $Z_A^s(\Omega)$ (Definition I 3.1). It follows that B maps $Z_A^s(\Omega)$ isomorphically onto $\Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$.

The analogous arguments can be applied to B'.

Lemma 1.1 *$C\,[C']$ maps $Z_A^s(\Omega)\,[Z_{A'}^s(\Omega)]$ continuously into $\Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$ $[\Pi H^{s-2m+m_j+\frac{1}{2}}(\Gamma)]$, for all real s.*

Proof: Follows immediately from Theorem I 3.2.

The results in Chapter II were derived from a rewriting of $(Au, v) - (u, A'v)$:

$$(1.3) \qquad (Au, v) - (u, A'v) = (Au, v_{\zeta'}) - (u_\zeta, A'v),$$

where $u \in D(A_1)$ and $v \in D(A_1')$ are decomposed according to Lemma II 1.1. We would like to have a formula analogous to (1.3), but with boundary terms appearing on the right. As noted in Remark I 3.3, Greens formula

$$(1.4) \qquad (Au, v) - (u, A'v) = \langle Cu, B'v \rangle - \langle Bu, C'v \rangle$$

cannot in general be extended to hold for all pairs $[u, v] \in D(A_1) \times D(A_1')$. However, it is possible to get around this difficulty by introducing certain non-local boundary operators M and M', related to B, C, B' and C', to take the place of C and C' in (1.4).

M and M' will be introduced in connection with certain operators P and P', acting in the boundary.

Definition 1.1. *Let $s \in \mathbb{R}$.*

Let $\varphi \in \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$. Then $P\varphi \in \Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$ is defined by: $P\varphi = Cu$, where u is the solution in $Z_A^s(\Omega)$ of $Bu = \varphi$.

Similarly, let $\psi \in \Pi H^{s-2m+\mu_j+\frac{1}{2}}(\Gamma)$. *Then* $P'\psi \in \Pi H^{s-2m+m_j+\frac{1}{2}}(\Gamma)$ *is defined by*: $P'\psi = C'v$, *where* v *is the solution in* $Z^s_{A'}(\Omega)$ *of* $B'v = \psi$.

For each real s, this definition makes sense because of Proposition 1.1. and Lemma 1.1. Moreover, it easily seen that if $s' > s$, then the definition for s is an extension of the definition for s' (cf. Remark I 2.1). In this way, P is defined as an operator in $(\mathcal{D}'(\Gamma))^m$.

P is an $m \times m$ matrix of operators P_{kl}; $k, l = 1, \dots, m$, in $\mathcal{D}'(\Gamma)$, which are in general non-local (i.e., the support of $P_{kl}\varphi$ need not be contained in the support of φ). If Ω is the halfspace $\{(x_1, \dots, x_n) \mid x_n > 0\}$ and A, B and C have constant coefficients, the P_{kl} are singular integral operators. (This case of unbounded Ω can be included in the above procedures.) This makes it plausible that P in general is a pseudo-differential operator in Γ, in the sense of Kohn-Nirenberg [21] and Hörmander [19] (the theories extended to the manifold Γ). We will not go further into this here.

P has the order—property:

THEOREM 1.1. *For all real* s, P *maps* $\Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ *continuosly into* $\Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$ *and* P' *maps* $\Pi H^{s-2m+\mu_j+\frac{1}{2}}(\Gamma)$ *continuously into* $\Pi H^{s-2m+m_j+\frac{1}{2}}(\Gamma)$.

PROOF: The statement is an immediate consequence of Proposition 1.1 and Lemma 1.1.

More properties of P and P' are given in § 5. In the rest of this § we actually only use the definition and continuity of P and P' for $s = 0$, i.e., $P: \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \to \Pi H^{-\mu_j-\frac{1}{2}}(\Gamma)$; $P': \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma) \to \Pi H^{-2m+m_j+\frac{1}{2}}(\Gamma)$.

We will now introdnce the boundary operator M, defined on $D(A_1) = D^0_A(\Omega)$. This can be done in several equivalent ways; we begin with one that uses P explicitly.

DEFINITION 1.2. *For any* $u \in D(A_1)$, Mu *is defined as*

$$Mu = Cu - PBu.$$

Similarly, for $v \in D(A'_1)$, $M'v$ *is defined as*

$$M'v = C'v - P'B'v.$$

Since the mappings $C: D(A_1) \to \Pi H^{-\mu_j-\frac{1}{2}}(\Gamma)$, $B: D(A_1) \to \Pi H^{-m_j-\frac{1}{2}}(\Gamma)$ and $P: \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \to \Pi H^{-\mu_j-\frac{1}{2}}(\Gamma)$ are everywhere defined and continuous, M clearly maps $D(A_1)$ continuosly into $\Pi H^{-\mu_j-\frac{1}{2}}(\Gamma)$.

However, a closer look reveals that the range of M is contained in the space of functions $\Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$; and that in fact M maps $D(A_1)$ *continuosly onto* $\Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$. This is mentioned in the following theorem, which is a compilation of the relevant properties of M.

THEOREM 1.2.

(i) *The following three definitions of M are equivalent:*

a) *For $u \in D(A_1)$, $Mu = Cu - PBu$.*

b) *For $u \in D(A_1)$, $Mu = Cu_\beta$ (according to the decomposition in Lemma II 1.1).*

c) *Let $u \in D(A_1)$, then Mu is the unique element of $\Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ for which*

$$(Au, v) = \langle \underset{\{2m-\mu_j-\frac{1}{2}\}}{Mu} , \underset{\{-2m+\mu_j+\frac{1}{2}\}}{B'v} \rangle \quad \text{for all } v \in Z(A_1').$$

(ii) *M maps $D(A_1)$ continuously onto $\Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$. On $D(A_\beta)$, M coincides with C, and maps $D(A_\beta)$ continuously onto $\Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$. (Graph-topologies on $D(A_1)$ and $D(A_\beta)$.)*

(iii) *The kernel of M: $D(A_1) \to \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ is $D(A_0) + Z(A_1)$.*

(iv) *$M' = C' - P'B'$ can be defined similarly, and has properties analogous to those of M; in particular it maps $D(A_1')$ continuously onto $\Pi H^{m_j+\frac{1}{2}}(\Gamma)$. The following « Green's formula » holds for all pairs $u \in D(A_1)$, $v \in D(A_1')$:*

$$(1.5) \qquad (Au, v) - (u, A'v) = \langle \underset{\{2m-\mu_j-\frac{1}{2}\}}{Mu} , \underset{\{-2m+\mu_j+\frac{1}{2}\}}{B'v} \rangle - \langle \underset{\{-m_j-\frac{1}{2}\}}{Bu} , \underset{\{m_j+\frac{1}{2}\}}{M'v} \rangle .$$

(v) *Define the realizations A_M and $A'_{M'}$ by:*

$$(1.6) \qquad A_M \in \mathcal{M} \; ; \; D(A_M) = \{u \in D(A_1) \mid Mu = 0\}$$

$$(1.7) \qquad A'_{M'} \in \mathcal{M}'; \; D(A'_{M'}) = \{v \in D(A_1') \mid M'v = 0\}.$$

Then A_M and $A'_{M'}$ are adjoints.

PROOF :

(i) Let $u \in D(A_1)$ and let $u = u_\beta + u_\zeta$, where $u_\beta \in D(A_\beta)$, $u_\zeta \in Z(A_1)$ as in Lemma II 1.1.

Since $Bu_\beta = 0$ (see (1.1)), $Cu - PBu = Cu_\beta + Cu_\zeta - PBu_\zeta$, which equals Cu_β by the definition of P. Thus a) and b) are equivalent definitions of M.

Now, if $v \in Z(A_1')$

$$(Au, v) = (Au_\beta, v) = (Au_\beta, v) - (u_\beta, A'v)$$

$$= \langle \underset{\{2m-\mu_j-\frac{1}{2}\}}{Cu_\beta}, \underset{\{-2m+\mu_j+\frac{1}{2}\}}{B'v} \rangle - \langle \underset{\{2m-m_j-\frac{1}{2}\}}{Bu_\beta}, \underset{\{-2m+m_j+\frac{1}{2}\}}{C'v} \rangle,$$

since $u_\beta \in H^{2m}(\Omega)$ (see (1.1)), so that we can apply Green's formula I (3.8) with $s = 2m$;

$$= \langle \underset{\{2m-\mu_j-\frac{1}{2}\}}{Cu_\beta}, \underset{\{-2m+\mu_j+\frac{1}{2}\}}{B'v} \rangle, \text{ since } Bu_\beta = 0.$$

This shows that if $\psi = Cu_\beta$ then

$$(Au, v) = \langle \underset{\{2m-\mu_j-\frac{1}{2}\}}{\psi}, \underset{\{-2m+\mu_j+\frac{1}{2}\}}{B'v} \rangle, \quad \text{all } v \in Z(A_1'). \tag{1.8}$$

However, since B' maps $Z(A_1')$ isomorphically onto $\Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$, equation (1 8) determines a unique element $\psi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ for each $u \in D(A_1)$. This implies that b) and c) are equivalent definitions of M.

(ii) We will use definition b), recalling that $D(A_\beta)$ is a closed subspace of $H^{2m}(\Omega)$ (see (1.1) or Corollary I 3.1). Since the mapping pr_β is continuous from $D(A_1)$ onto $D(A_\beta)$, and C is continuous from $H^{2m}(\Omega)$ into $\Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$, $M = C \circ pr_\beta$ maps $D(A_1)$ continuously into $\Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$. Also by b), $Mu = Cu$ when $u \in D(A_\beta)$ and therefore M maps $D(A_\beta)$ continuously into $\Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$. That this last mapping is surjective follows from Proposition I 2.1 : for given $\varphi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ there exists $u \in H^{2m}(\Omega)$ with $Bu = 0$, $Cu\,(= Mu) = \varphi$.

(iii) When $u \in D(A_0)$, $Bu = Cu = 0$, whence $Mu = 0$ by a). When $u \in Z(A_1)$, $Mu = 0$ by c).

Conversely, let $u \in D(A_1)$ with $Mu = 0$. Then, by c), $(Au, v) = 0$ for all $v \in Z(A_1')$, i.e., $Au \in Z(A_1')^{\perp}$. Since $Z(A_1')^{\perp} = R(A_0)$ (as noted in II (1.6)), $Au \in R(A_0)$. Then $u = u_0 + u_1$ where $u_0 \in D(A_0)$, $u_1 \in Z(A_1)$.

(iv) M' has analogous properties to those of M.

Let $u \in D(A_1)$, $v \in D(A_1')$. Then

$$(Au, v) - (u, A'v) = (Au, v_{\zeta'}) - (u_{\zeta}, A'v) \qquad \text{(Lemma II 1.2)}$$

$$= \langle \underset{\{2m-\mu_j-\frac{1}{2}\}}{Mu}, \underset{\{-2m+\mu_j+\frac{1}{2}\}}{B'v} \rangle - \langle \underset{\{-m_j-\frac{1}{2}\}}{Bu}, \underset{\{m_j+\frac{1}{2}\}}{M'v} \rangle,$$

since $B'v_{\beta'} = 0$, $Bu_{\beta} = 0$.

(v) Let A_M and $A'_{M'}$ be defined by (1.6) and (1.7). For $u \in D(A_M)$, $v \in D(A'_{M'})$ one has by (iv):

$$(Au, v) - (u, A'v) = 0$$

so A_M and $A'_{M'}$ are contained in each others adjoints. Recall that $A_M \in \mathcal{M}$ implies $(A_M)^* \in \mathcal{M}'$, and $A'_{M'} \in \mathcal{M}'$ implies $(A'_{M'})^* \in \mathcal{M}$. Let $u \in D((A'_{M'})^*)$, then this means that $u \in D(A_1)$ and

$$(Au, v) - (u, A'v) = 0 \text{ for all } v \in D(A'_{M'}).$$

Since $Z(A_1') \subset D(A'_{M'})$ (by (iii)), one has in particular:

$$(Au, v) = 0 \text{ for all } v \in Z(A_1'),$$

whence, by definition c), $Mu = 0$. Thus $u \in D(A_M)$, which completes the proof that $A_M = (A'_{M'})^*$. The proof that $A'_{M'} = (A_M)^*$ is analogous.

REMARK 1.1. Note that the operator A_M defined in Theorem 1.2 (v) is an example of a realization of A determined by a non-local boundary condition; moreover its domain is not in $H^s(\Omega)$ for any $s > 0$, since it contains $Z(A_1)$ (by (iii)).

PROPOSITION 1.2. *The set of equations*

$$\left\{ \begin{aligned} Bu &= \varphi \\ Mu &= \psi \end{aligned} \right. \tag{1.9}$$

has a solution $u \in D(A_1)$ *for any pair* $\varphi \in \Pi H^{-m_j-\frac{1}{2}}(\Gamma)$, $\psi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$. *The solution is unique modulo* $D(A_0)$.

Proof: Let $\varphi \in \Pi H^{-m_j-\frac{1}{2}}(\Gamma)$ and $\psi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ be given. If $u \in D(A_1)$ is a solution of (1.9) then

$$Bu = Bu_\zeta = \varphi$$

$$Mu = Mu_\beta = \psi$$

and conversely, if $u = v + z$ where $v \in D(A_\beta)$, $z \in Z(A_1)$ and

$$Bz = \varphi \tag{1.10}$$

$$Mv = \psi, \tag{1.11}$$

then u is a solution of (1.9).

Since $D(A_0) \subset D(A_\beta)$ and $Z(A_1) \cap D(A_\beta) = \{0\}$, it follows from Theorem 1.1 (ii), (iii), that M maps $D(A_\beta)$ onto $\Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ with kernel $D(A_0)$. Thus (1.11) has a solution $v \in D(A_\beta)$, uniqne modulo $D(A_0)$. By Proposition 1.1, (1.10) has a unique solution $z \in Z(A_1)$. It follows that (1.9) has a solution $u \in D(A_1)$, unique modulo $D(A_0)$.

Corollary 1.2. *The set of equations*

$$\begin{cases} Bu = \varphi \\ Cu = \varrho \end{cases} \tag{1.12}$$

has a solution $u \in D(A_1)$ *for* $\varphi \in \Pi H^{-m_j-\frac{1}{2}}(\Gamma)$, $\varrho \in \Pi H^{-\mu_j-\frac{1}{2}}(\Gamma)$, *if and only if* $\varrho - P\varphi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$.

Proof: If $u \in D(A_1)$ is a solution of (1.12), then $\varrho - P\varphi = Cu - PBu = Mu \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ by Theorem 1.1 (ii).

Conversely, if $\varrho - P\varphi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$, then a solution $u \in D(A_1)$ of

$$\begin{cases} Bu = \varphi \\ Mu = \varrho - P\varphi \end{cases}$$

exists according to Proposition 1.2; this u also satifies (1.12).

III § 2. **Fundamental results.**

Let V be a closed subspace of $Z(A_1)$, and W a closed subspace of $Z(A_1')$. It was proved in Proposition 1.1 that B maps $Z(A_1)$ isomorphically onto $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$; consequently B maps V isomorphically onto a closed subspace X of $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$. Denote the strong antidual of X by X'. Since X is a Hilbert space, the strong antidual of X' is X; moreover, a norm in X together with the duality leads to an identification of X with X'. However, we will avoid to make use of this identification since it requires a specific norm in X, cf. Remark 2.1 below.

Since $X \subset \Pi H^{-m_j-\frac{1}{2}}(\Gamma)$, every element $\varphi \in \Pi H^{m_j+\frac{1}{2}}(\Gamma)$ defines a unique element $\varphi_1 \in X'$ by

$$\langle \underset{\{m_j+\frac{1}{2}\}}{\varphi}\ ,\ \underset{\{-m_j-\frac{1}{2}\}}{\psi} \rangle = \langle \underset{X'}{\varphi_1}\ ,\ \underset{X}{\psi} \rangle \text{ for all } \psi \in X. \tag{2.1}$$

φ_1 is the restriction of φ, considered as a functional on $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$, to the subspace X. To describe the connection between φ and φ_1 we will either use (2.1) or simply say that

$$\varphi = \varphi_1 \text{ on } X. \tag{2 2}$$

From B we can deduce an isomorphism E of V onto X' by the formula

$$\langle \underset{X'}{Ev_1}\ ,\ \underset{X}{Bv_2} \rangle = (v_1\ ,\ v_2)_V\ , \qquad \text{all } v_1\ ,\ v_2 \in V\ .$$

Similarly, let $B'W = Y$ (closed subspace of $\Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$), then B' gives rise to an isomorphism F of W onto Y' by

$$\langle \underset{Y'}{Fw_1}\ ,\ \underset{Y}{B'\,w_2} \rangle = (w_1\ ,\ w_2)_W\ , \qquad \text{all } w_1\ ,\ w_2 \in W.$$

Now if T is a linear operator with domain in V and range in W, then T defines an operator $L : X \longrightarrow Y'$ by

$$D(L) = BD(T), \quad LBv = FTv, \text{ all } v \in D(T).$$

We prefer to write the definition of L in the following equivalent form :

DEFINITION 2.1a. *Let $V \subset Z(A_1)$, $W \subset Z(A_1')$, closed subspaces, and let T be an operator with $D(T) \subset V$, $R(T) \subset W$. Denote $BV = X$ and $B'W = Y$. Then $T : V \longrightarrow W$ gives rise to the operator $L : X \longrightarrow Y'$ defined by*

$$D(L) = BD(T)$$

$$\underset{Y'}{\langle LBv}, \underset{Y}{B'w \rangle} = (Tv, w)_W, \qquad \text{all } v \in D(T),\ w \in W.$$

Similarly :

DEFINITION 2.1b. *Let V, W, X, Y be as in Definition 2.1a, and let T_1 be an operator with $D(T_1) \subset W$, $R(T_1) \subset V$. Then $T_1 : W \longrightarrow V$ gives rise to the operator $L_1 : Y \longrightarrow X'$ defined by*

$$D(L_1) = B'D(T_1)$$

$$\underset{X'}{\langle L_1} B'w, \underset{X}{Bv \rangle} = (T_1 w, v)_V, \quad \text{all } w \in D(T_1), v \in V.$$

REMARK 2.1. Note that the definitions are independent of any particular norms in X and Y. We have aimed at this for the following reason : In the consideration of the space $\Pi H^{m_j + \frac{1}{2}}(\Gamma)$ and its strong dual $\Pi H^{-m_j - \frac{1}{2}}(\Gamma)$, the *duality*, i.e., the sesquilinear form

$$\underset{\{-m_j - \frac{1}{2}\}}{\langle \varphi}, \underset{\{m_j + \frac{1}{2}\}}{\psi \rangle}$$

is given as an extension of the L^2-inner product in $(\mathcal{D}(\Gamma))^m$ (with respect to the surface measure on Γ), whereas there is an arbitrary choice between equivalent *norms in* $\Pi H^{m_j + \frac{1}{2}}(\Gamma)$, for instance corresponding to different systems of local coordinates. Therefore, results that do not depend on the choice of a particular norm in the spaces $H^s(\Gamma)$ ($s \neq 0$), are in general the most useful ones.

LEMMA 2.1. *Definition* 2.1a *establishes a* $1 - 1$ *correspondence between all linear operators $T : V \longrightarrow W$ ($V \subset Z(A_1)$, $W \subset Z(A_1')$, closed subspaces), and all linear operators $L : X \longrightarrow Y'$ ($X \subset \Pi H^{-m_j - \frac{1}{2}}(\Gamma)$, $Y \subset \Pi^{-2m + \mu_j + \frac{1}{2}}(\Gamma)$, closed subspaces).*

Similarly, Definiton 2.1b *establishes a* $1 - 1$ *correspondence between all linear operators $T_1 : W \longrightarrow V$ (V and W as above) and all linear operators $L_1 : Y \longrightarrow X'$ (X and Y as above).*

Proof: See Remark 2.2 below.

Remark 2.2. Consideration of the isomorphisms $B: V \to X$, $E: V \to X'$, $B': W \to Y$ and $F: W \to Y'$, shows Lemma 2.1, and moreover shows the fact that (loosely speaking) the correspondences in Lemma 2.1 preserve all those properties of the operators that can be expressed independently of particular norms in $\Pi H^{-m_j - \frac{1}{2}}(\Gamma)$ and $\Pi H^{-2m+\mu_j + \frac{1}{2}}(\Gamma)$ (cf. Remark 2.1). Since the proof for each property, we shall need to consider, is immediate, we will not state this fact explicitly in a theorem.

If $L: X \to Y'$ is densely defined, we define the adjoint L^* as an operator *from* Y *to* X' by: L^* is the operator with largest domain satisfying

$$\langle L\varphi, \psi \rangle_{Y' \; Y} = \langle \varphi, L^* \psi \rangle_{X \; X'}, \qquad \text{all } \varphi \in D(L),\ \psi \in D(L^*).$$

(Here we use the notation $\langle \xi, \eta \rangle_{X \; X'} = \overline{\langle \eta, \xi \rangle_{X' \; X}}$.)

One has:

Lemma 2.2. *Let $T: V \to W$ be densely defined, so that $T^*: W \to V$ exists. If $T: V \to W$ corresponds to $L: X \to Y'$ by Definition* 2.1a, *then $T^*: W \to V$ corresponds to $L^*: Y \to X'$ by Definition* 2.1b.

The proof is easy and will be omitted. We further note:

Lemma 2.3.

(i) *When $u \in D(A_1)$, $Bu = Bu_\zeta$. In particular, if $\widetilde{A} \in \mathcal{M}$, then $B(pr_\zeta D(\widetilde{A})) = BD(\widetilde{A})$.*

(ii) *Let $T: V \to W$ correspond to $L: X \to Y'$ as in Definition* 2.1a. *If $u \in D(A_1)$, then $u_\zeta \in D(T)$ if and only if $Bu \in D(L)$.*

Proof:

(i) When $u \in D(A_1)$, $u = u_\beta + u_\zeta$ where $u_\beta \in D(A_\beta)$, $u_\zeta \in Z(A_1)$. By the definition of A_β, $Bu_\beta = 0$; thus $Bu = Bu_\zeta$.

(ii) Let $T: V \to W$ correspond to $L: X \to Y'$ as in Definition 2.1a, and let $u \in D(A_1)$. Then $u_\zeta \in D(T)$ if and only if $Bu_\zeta \in D(L)$. The statement follows from the fact that $Bu_\zeta = Bu$.

Lemma 2.4. *Let $T: V \to W$ correspond to $L: X \to Y'$ as in Definition* 2.1a. *Then the folowing two sets D_1 and D_2 are identical:*

$$D_1 = \{u \in D(A_1) \mid u_\zeta \in D(T),\ (Au)_W = Tu_\zeta\}$$

$$D_2 = \{u \in D(A_1) \mid Bu \in D(L),\ Mu = LBu \ \text{on} \ Y\}.$$

PROOF : Let $u \in D(A_1)$ with $u_\zeta \in D(T)$. Since this is equivalent with $u \in D(A_1)$, $Bu \in D(L)$, by Lemma 2.3, it is enough to prove that $(Au)_W = Tu_\zeta$ if and only if $Mu = LBu$ on Y (in the terminology introduced in (2.1)-(2.2)).

Let u satisfy $(Au)_W = Tu_\zeta$ or, equivalently,

$$(Au, w) = (Tu_\zeta, w) \text{ for all } w \in W. \tag{2.3}$$

Since $W \subset Z(A_1')$, $(Au, w) = \langle \underset{\{2m-\mu_j-\frac{1}{2}\}}{Mu}, \underset{\{-2m+\mu_j+\frac{1}{2}\}}{B'w} \rangle$, by the definition of M (Theorem 1.2 (i) c)). Also, by the definition of L, $(Tu_\zeta, w) = \langle \underset{Y'}{LBu_\zeta}, \underset{Y}{B'w} \rangle$, which equals $\langle \underset{Y'}{LBu}, \underset{Y}{B'w} \rangle$ by Lemma 2.3 (i). Therefore, (2.3) is equivalent with

$$\langle \underset{\{2m-\mu_j-\frac{1}{2}\}}{Mu}, \underset{\{-2m+\mu_j+\frac{1}{2}\}}{B'w} \rangle = \langle \underset{Y'}{LBu}, \underset{Y}{B'w} \rangle, \qquad \text{all } w \in W. \tag{2.4}$$

Since B' maps W isomorphically onto Y, (2.4) is equivalent with

$$\langle \underset{\{2m-\mu_j-\frac{1}{2}\}}{Mu}, \underset{\{-2m+\mu_j+\frac{1}{2}\}}{\psi} \rangle = \langle \underset{Y'}{LBu}, \underset{Y}{\psi} \rangle, \qquad \text{all } \psi \in Y, \tag{2.5}$$

i. e.,

$$Mu = LBu \quad \text{on} \quad Y. \tag{2.6}$$

This shows that u satisfies $(Au)_W = Tu_\zeta$ if and only if $Mu = LBu$ on Y.

With Lemma 2.1-2.4 we have the complete machinery to « translate » the results of Chapter II § 1 and 3 into statements involving boundary conditions. The proofs of the theorems below are quite straightforward, so we will only indicate details in a few cases.

Corollary II 1.1 carries into:

THEOREM 2.1. *There is a $1-1$ correspondence between all closed operators $\widetilde{A} \in \mathcal{M}$ and all operators $L: X \to Y'$ satisfying*

(i) *X is a closed subspace of $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$, Y is a closed subspace of $\Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$;*

(ii) *L is densely defined and closed;*

the correspondence being given by

$$D(\widetilde{A}) = \{u \in D(A_1) \mid Bu \in D(L),\ Mu = LBu \text{ on } Y\}.$$

In this correspondence, $D(L) = BD(\widetilde{A})$ (so $X = \overline{BD(\widetilde{A})}$).

Furthermore, if $\widetilde{A}$ *corresponds to* $L: X \to Y'$ *in the above sense, then* $\widetilde{A}^*$ *corresponds to* $L^*: Y \to X'$ *by*

$$D(\widetilde{A}^*) = \{v \in D(A_1') \mid B'v \in D(L^*),\ M'v = L^*B'v \text{ on } X\};$$

here $D(L^*) = B'D(\widetilde{A}^*)$ *(so* $Y = \overline{B'D(\widetilde{A}^*)}$*).*

REMARK 2.3. As in Remark II 1.1 we note that for $\widetilde{A}$ we only need to know L as an operator from $\Pi H^{-m_j - \frac{1}{2}}(\Gamma)$ to Y', since $X = \overline{D(L)}$; however, X is important for the definition of L^*, and the correspondence with $\widetilde{A}^*$.

Noting that by Lemma 2.3 (for $\widetilde{A}^*$ and B') and the fact that B' is an isomorphism of $Z(A_1')$ onto $\Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$, $pr_{\zeta'} D(\widetilde{A}^*)$ is dense in $Z(A_1')$ if and only if $B'D(\widetilde{A}^*)$ is dense in $\Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$, we obtain from Theorem II 3.1 the following theorem:

THEOREM 2.2. *An operator* $\widetilde{A} \in \mathcal{M}$ *(closed or not) satisfies* $\overline{B'D(\widetilde{A}^*)} = \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$ *if and only if there exists a closable operator* $L: \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \to \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ *such that*

$$D(\widetilde{A}) = \{u \in D(A_1) \mid Bu \in D(L),\ Mu = LBu\}. \tag{2.7}$$

$\widetilde{A}$ *and* L *determine each other by* (2.7); *the correspondence is inclusion preserving.*

When $\widetilde{A}$ *corresponds to* L *in this way,* $\overline{\widetilde{A}}$ *corresponds to* $\overline{L}: X \to \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ *in the sense of Theorem* 2.1, *with* $X =$ *the closure of* $D(L)$ *in* $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$.

Theorem II 3.2 is easily carried into

THEOREM 2.3. *Let* X *be a closed subspace of* $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$, Y *a closed subspace of* $\Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$ *and* $L: X \to Y'$ *a closable, densely defined* *operator. Then L corresponds to an operator $\tilde{A}$ by*

$$D(\tilde{A}) = \{u \in D(A_1) \mid Bu \in D(L),\ Mu = LBu \text{ on } Y\},$$

and $\bar{L}: X \to Y'$ corresponds to $\bar{\tilde{A}}$ in the sense of Theorem 2.1.

Theorem 2.2 and 2.3 enable us to include certain non-closed cases in the following description of the correspondence between properties of $\tilde{A}$ and L. Note that the realizations considered in Theorem 2.2 are a subclass of those considered in Theorem 2.3, namely those for which $Y = \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$. We have stated Theorem 2.2 in preparation for the discussion of « pure conditions » later in this §.

THEOREM 2.4. *Let $\tilde{A}$ correspond to $L: X \to Y'$ as in Theorem 2.1 or Theorem 2.3. Then*

(i) *$\tilde{A}$ is $1-1$ if and only if L is $1-1$; in fact*

$$\dim Z(\tilde{A}) = \dim Z(L);$$

(ii) *$\tilde{A}$ maps $D(\tilde{A})$ onto H if and only if L maps $D(L)$ onto Y';*

(iii) *$R(\tilde{A})$ is closed if and only if $R(L)$ is closed;*

(iv)

$$\operatorname{codim} R(\tilde{A}) = \operatorname{codim} R(L).$$

PROOF: Theorem II 1.3 is easily extended to the non-closed cases described in Theorem II 3.2. Then Theorem II 1.3 implies the above statements for the correspondence in Theorem 2.1 and 2.3.

COROLLARY 2.4. *Let $\tilde{A}$ correspond to $L: X \to Y'$ as in Theorem 2.1 or 2.3. Then $\tilde{A}$ and L have the same index; $\tilde{A}$ is a semi-Fredholm operator if and only if L is one, and $\tilde{A}$ is a Fredholm operator if and only if L is one.*

A complete translation of Theorem II 1.4 seems unnecessary here (it requires a specific norm in $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$ and $\Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$, and leads to a somewhat unnatural statement), however, we will mention the consequence of Corollary II 1.4:

Theorem 2.5. (*Uses the boundedness of Ω*). *Let $\widetilde{A}$ correspond to $L: X \longrightarrow Y'$ as in Theorem* 2.1 *or* 2.3. *If $\widetilde{A}^{-1}$ exists* (*or, equivalently, L^{-1} exists*), *then $\widetilde{A}^{-1}$ is compact if and only if L^{-1} is compact.*

Proof: Since Ω is bounded, the imbedding of $H^{2m}(\Omega)$ into $H^0(\Omega)$ is compact, and therefore A_β^{-1} is compact. Thus $\widetilde{A}^{-1}$ and T^{-1} are simultaneously compact, by Corollary II 1.4, so the same holds for $\widetilde{A}^{-1}$ and L^{-1}.

Remark 2.4. As mentioned in I § 1, the fundamental results can be proved also in (sufficiently nice) cases where Ω is unbounded. In that way one would get a characterization of Fredholm operators also in such unbounded eases, even though Theorem 2.5 could not be extended.

Let us describe how some well known (local) boundary conditions fit into our theory. (More detailed descriptions in [15]).

1) The boundary condition $Bu = 0$ determines the operator A_β, which corresponds to $L: X \longrightarrow Y'$ with $X = Y = \{0\}$. For, $X = \overline{D(L)} = \overline{BD(A_\beta)} = \{0\}$ and $Y = \overline{B'D(A_\beta^*)} = \overline{B'D(A'_\beta)} = \{0\}$.

2) The condition $Cu = 0$ is equivalent with $Mu + PBu = 0$ or $Mu = - PBu$. Let $\widetilde{A}$ denote the realization determined by the boundary condition $Cu = 0$, then $\widetilde{A}$ is closed by Theorem I. 3.2; and corresponds in the sense of Theorem 2.1 to the operator $L: \Pi H^{-m_j - \frac{1}{2}}(\Gamma) \longrightarrow \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ defined by

$$D(L) = \{\varphi \in \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \mid P\varphi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)\}$$

$$L\varphi = - P\varphi \quad \text{for} \quad \varphi \in D(L).$$

3) The so-called « mixed conditions »: $Bu = 0$ on Γ_1, $Cu = 0$ on Γ_2, where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Sigma$ (nontrivial disjoint union; Γ_1 and Γ_2 are open subsets of Γ, Σ is an $n-2$ dimensional C^∞ manifold), give rise to closed operators $\widetilde{A}$, which correspond to $L: X \longrightarrow Y'$ with X and Y defined by

$$X = \{\varphi \in \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \mid \operatorname{supp}\varphi \subset \overline{\Gamma_2}\},$$

$$Y = \{\psi \in \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma) \mid \operatorname{supp}\psi \subset \overline{\Gamma_2}\}$$

and L defined by

$$D(L) = \{\varphi \in X \mid P\varphi \in Y'\}$$

$$L\varphi = - P\varphi \text{ on } Y, \quad \text{for} \quad \varphi \in D(L).$$

(Here $P\varphi \in Y'$ is understood in the following way: When $\varphi \in X$, $P\varphi \in$ $\in \Pi H^{-\mu_j-\frac{1}{2}}(\Gamma) \subset (\mathcal{D}'(\Gamma))^m$, so $\langle P\varphi, \psi \rangle$ is defined for all $\psi \in (\mathcal{D}(\Gamma))^m$. Now $Y = \Pi H_0^{-2m+\mu_j+\frac{1}{2}}(\Gamma_2)$ (cf. definition I (1.2), negative s), so that $(\mathcal{D}(\Gamma_2))^m$ is dense in Y. We say that $P\varphi \in Y'$ if $\langle P\varphi, \psi \rangle$ is continuous on $\psi \in (\mathcal{D}(\Gamma_2))^m$ with respect to the topology of Y. — Note that Y' is the space of functions $\Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma_2)$).

In the last example $X \neq \Pi H^{-m_j-\frac{1}{2}}(\Gamma)$ and $Y \neq \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$. When $Y = \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$, the boundary condition states that Mu is a function of Bu (not just coincident with LBu as functionals on a subspace of $\Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$). Since $Cu = Mu + PBu$ (Theorem 1.2 (i)), this also means that Cu is a function of Bu, or,

$$Cu = KBu, \; u \in D(\widetilde{A}), \tag{2.8}$$

for some operator K in $(\mathcal{D}'(\Gamma))^m$.

The earlier studies of non-local boundary value problems [2], [4], [5], [6], [12], [29]) have been concerned mainly with boundary conditions of the type (2.8) ([29] does include other types that (2.8)); the term « non-local » referring to the fact that K is not required to be a differential operator. We will therefore give this problem some special attention. Here we change our point of view slightly, in that we will discuss *all* realizations $\widetilde{A}$ that can be described by a boundary condition of the type (2.8); then we shall have to consider closed as well as non-closed $\widetilde{A}$, and we do not have a correspondence $\widetilde{A} \longleftrightarrow L$ to start with. In the remainder of this § it is described how such realizations fit into our theory. Some further results will be given in § 6.

Since the « mixed problem » described above in 3) is obviously not in this class, we have chosen (for lack of better terminology) to call boundary conditions of the type (2.8) « pure conditions ». More precisely:

DEFINITION 2.2. *An operator $\widetilde{A} \in \mathcal{M}$ will be said to represent a pure condition if and only if $Bu = 0$ implies $Cu = 0$ when $u \in D(\widetilde{A})$.*

Recalling that $Mu = Cu - PBu$ for $u \in D(A_1)$, and M maps $D(A_1)$ into $\Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ (Theorem 1.2) one easily proves:

LEMMA 2.5. *Let $\tilde{A} \in \mathcal{M}$. The following statements are equivalent*:

(i) $\tilde{A}$ *satisfies Definition* 2.2;

(ii) *there exists an operator* $K: \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \to \Pi H^{-\mu_j-\frac{1}{2}}(\Gamma)$ *such that*

$$D(\tilde{A}) = \{u \in D(A_1) \mid Bu \in D(K),\ Cu = KBu\}; \tag{2.9}$$

(iii) *there exists an operator* $L: \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \to \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ *such that*

$$D(\tilde{A}) = \{u \in D(A_1) \mid Bu \in D(L),\ Mu = LBu\}. \tag{2.10}$$

When $\tilde{A} \in \mathcal{M}$ satisfies (2.9) we say that $\tilde{A}$ represents the pure condition $Cu = KBu$; when $\tilde{A} \in \mathcal{M}$ satisfies (2.10) we say that $\tilde{A}$ represents the pure condition $Mu = LBu$.

Let $\tilde{A}$ satisfy Definition 2.2, and let K and L be operators for which $\tilde{A}$ satisfies (2.9) resp. (2.10). There is a fundamental difference between the way in which $\tilde{A}$ is related to K and the way in which it is related to L. Recall Proposition 1.2: the problem

$$\begin{cases} Bu = \varphi \\ Mu = \psi \end{cases}$$

has a solution $u \in D(A_1)$ for *all* pairs $[\varphi, \psi] \in \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \times \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$. Therefore all of $D(L) \times R(L)$ can be reached by $[Bu, Mu]$ when $u \in D(A_1)$; and consequently is not only $\tilde{A}$ determined by L, but also L determined by $\tilde{A}$ in (2.10).

Considering (2.9), we recall Corollary 1.2: the problem

$$\begin{cases} Bu = \varphi \\ Cu = \varrho \end{cases}$$

has a solution $u \in D(A_1)$ for $[\varphi, \varrho] \in \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \times \Pi H^{-\mu_j-\frac{1}{2}}(\Gamma)$ if and only if $\varrho - P\varphi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$, and then u is a solution of

$$\begin{cases} Bu = \varphi \\ Mu = \varrho - P\varphi. \end{cases}$$

This shows that whereas K determines $\widetilde{A}$ by (2.9), only the part of $D(K)$, for which $K\varphi - P\varphi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$, enters in the definition of $\widetilde{A}$; outside of this K may be arbitrarily chosen. Moreover, we see that the connection between K and L is exactly:

$$\begin{cases} D(L) = \{\varphi \in D(K) \mid K\varphi - P\varphi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)\} \\ L\varphi = K\varphi - P\varphi \text{ for } \varphi \in D(L). \end{cases}$$

Altogether, we have found

PROPOSITION 2.1.

(i) *There is a* $1-1$ *correspondence between all* $\widetilde{A}$ *satisfying Definition 2.2 and all operators* $L: \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \to \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$, *the correspondence being given by* (2.10). *In this correspondence,* $D(L) = BD(\widetilde{A})$.

(ii) A *satisfies* (2.9) *for an operator* $K: \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \to \Pi H^{-\mu_j-\frac{1}{2}}(\Gamma)$ *if and only if* $\widetilde{A}$ *satisfies* (2.10) *for the operator* L *defined by*

$$\text{(2.11)} \qquad \begin{cases} D(L) = \{\varphi \in D(K) \mid K\varphi - P\varphi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)\} \\ L\varphi \quad = K\varphi - P\varphi \text{ for } \varphi \in D(L). \end{cases}$$

We can now restrict the attention to the connection between $\widetilde{A}$ and L. The question to consider is how the properties of $\widetilde{A}$ and L are related, or rather: To what extent can our previous theory be applied? It turns out that there are two radically different possibilities, according to whether $L: \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \to \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ is *closable* or not. When L is closable, we are in the case described by Theorem 2.2; one then has that $\overline{\widetilde{A}}$ represents the pure condition $Mu = \overline{L}Bu$, and the whole theory about correspondences between properties of $\widetilde{A}$ and L can be applied (with $\overline{D(L)}$ as the domain space for L, strictly speaking). When L is not closable, an accurate use of Theorem 2.2 will show that $\overline{\widetilde{A}}$ does *not* represent a pure condition even though $\widetilde{A}$ does, so that the operator $L_1 : X_1 \to Y_1'$ to which $\overline{\widetilde{A}}$ corresponds by Theorem 2.1 is *not* an extension of L; a rather pathological situation. A criterion for $\widetilde{A}$ determining which case we are in, is whether

$B' D(\widetilde{A}^*)$ is dense in $\Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$ or not. The precise result is given in the following:

THEOREM 2.6. *Let $\widetilde{A} \in \mathfrak{M}$. The following statements are equivalent:*

(i) *$\widetilde{A}$ represents a pure condition $Mu = LBu$ where $L : \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \to \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ is closable;*

(ii) *$\overline{\widetilde{A}}$ represents a pure condition;*

(iii) *$B' D(\widetilde{A}^*)$ is dense in $\Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$.*

In the affirmative case, $\overline{\widetilde{A}}$ represents the pure condition $Mu = \bar{L}Bu$ where L is the operator appearing in (i).

PROOF: The proof uses Theorem 2.2. In fact, the equivalence of (i) and (iii) is a mere restatement of the first part of Theorem 2.2. It is also seen from Theorem 2.2, that (i) implies (ii), and that $\overline{\widetilde{A}}$ represents the pure condition $Mu = \bar{L}Bu$ if (i) holds. Finally, since $\widetilde{A}^* = (\overline{\widetilde{A}})^*$, Theorem 2.2 applied to $\overline{\widetilde{A}}$ shows that (ii) implies (iii).

As a corollary one gets a description of the pathological case:

COROLLARY 2.6. *When $\widetilde{A}$ is determined by a pure condition $Mu = LBu$ with $L : \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \to \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ not closable then $\overline{\widetilde{A}}$ corresponds to an operator $L_1 : X_1 \to Y_1'$ with $Y_1 \neq \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$ (and $X_1 = \overline{D(L)} \supset D(L_1) \supset D(L)$); L_1 is (clearly) not an extension of L.*

PROOF: None of the statements (i)-(iii) of Theorem 2.6 hold in this case. Let $\overline{\widetilde{A}}$ correspond to $L_1 : X_1 \to Y_1'$ by Theorem 2.1. Then since $\bar{\widetilde{A}}^* = \widetilde{A}^*$,

$$Y_1 = \overline{B' D(\overline{\widetilde{A}}^*)} = \overline{B' D(\widetilde{A}^*)} \neq \Pi H^{-2m+\mu_j+\frac{1}{2}}.$$

Concerning X_1 we first note that $D(L_1) = BD(\overline{\widetilde{A}})$ (Theorem 2.1) and $D(L) = BD(\widetilde{A})$ (Proposition 2.1 (i)); now the closures of $BD(\overline{\widetilde{A}})$ and $BD(\widetilde{A})$ in $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$ are equal since B is continuous from $D(A_1)$ with the *graph-topology* onto $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$. Thus $X_1 = \overline{D(L_1)} = \overline{D(L)}$. Since $BD(\overline{\widetilde{A}}) \supset BD(\widetilde{A})$, $D(L_1) \supset D(L)$.

The next corollaries are easily shown.

COROLLARY 2.7. *Let $\tilde{A}$ be a closed operator $\in \mathcal{M}$. Then $\tilde{A}$ represents a pure condition if and only if $B' D(\tilde{A}^*)$ is dense in $\Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$.*

COROLLARY 2.8. *Let $\tilde{A} \in \mathcal{M}$ represent the pure condition $Mu = LBu$. Then $\tilde{A}$ is closed if and only if L is closed.*

About the adjoint one has:

THEOREM 2.7. *Let $\tilde{A}$ represent the pure condition $Mu = LBu$ (L closable or not). Then $\tilde{A}^*$ represents a pure condition (with obvious notation) if and only if $D(L)$ is dense in $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$, i.e., if and only if $BD(\tilde{A})$ is dense in $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$.*

In the affirmative case, $\tilde{A}^*$ represents the pure condition $M'u = L^*B'u$.

PROOF: Applying the $\mathcal{M}'$-analogue of Corollary 2.7 to $\tilde{A}^*$ (which is closed) we see that $\tilde{A}^*$ represents a pure condition if and only if $BD(\tilde{A}^{**}) = BD(\bar{\tilde{A}})$ is dense in $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$. When L is closable, $BD(\bar{\tilde{A}}) = D(\bar{L})$ by Theorem 2.6, so $\overline{BD(\bar{\tilde{A}})} = \overline{D(\bar{L})}$; when L is not closable, $\overline{BD(\bar{\tilde{A}})} = D(L_1)$ where $\overline{D(L_1)} = \overline{D(L)}$ by Corollary 2.6, so that also here $\overline{BD(\bar{\tilde{A}})} = \overline{D(L)}$. This proves the first part of the theorem.

The second part follows, when L is closable, from $\bar{L}^* = L^*$; in the nonclosable case it can be deduced from the last statement in II § 3.

III § 3. Regularity.

This § is concerned with the question of regularity, i.e., the smoothness properties of the solutions of the various boundary value problems. We here consider inclusions $D(\tilde{A}) \subset H^s(\Omega)$ and $D(L) \subset \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ and show how these correspond to each other when $0 \leq s \leq 2m$. For $s > 2m$ one never has $D(\tilde{A}) \subset H^s(\Omega)$, since $D(\tilde{A}) \supset D(A_0) = H_0^{2m}(\Omega)$; here we just see what $D(L) \subset \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ implies for $\tilde{A}$. Other types of regularity results (e.g., concerning the property $Au \in H^s(\Omega) \Longrightarrow u \in H^{s+t}(\Omega)$, varying s and t) will be discussed in a subsequent paper.

We distinguish between two types of inclusions, exemplified as follows:

Let F be an operator from $H^0(\Omega)$ to a Hilbert space K, and let $s > 0$. Then $D(F) \subset H^s(\Omega)$ (algebraically) means that $D(F)$ is a subset of $H^s(\Omega)$; and $D(F) \subset H^s(\Omega)$ (algebraically and topologically) means that furthermore the inclusion mapping is continous, when $D(F)$ is provided with the graph-topology; i. e.,

$$|u|_s \leq c(|u|_0 + |Fu|_K), \; c > 0, \text{ all } u \in D(F).$$

For an operator $L: X \longrightarrow Y'$ (X a closed subspace of $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$) the graph topology on $D(L)$ is of course determined by a norm $|\varphi|_L = (|\varphi|^2_{\{-m_j-\frac{1}{2}\}} + |L\varphi|^2_{Y'})^{\frac{1}{2}}$; here one is interested in inclusions $D(L) \subset \subset \Pi H^{s-m_j-\frac{1}{2}}$, $s > 0$.

THEOREM 3.1. *Let $\widetilde{A}$ correspond to L as in Theorem* 2.1-3 *or Proposition* 2.1 (i).

(i) Let $0 \leq s \leq 2m$. *Then $D(\widetilde{A}) \subset H^s(\Omega)$ (alg.) if and only if $D(L) \subset \subset \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ (alg.).*

(ii) *Let $s \geq 2m$. Then $D(L) \subset \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ (alg.) implies*

$$\{u \in D(\widetilde{A}) \mid Au \in H^{s-2m}(\Omega)\} \subset H^s(\Omega) \, (alg.).$$

PROOF: We observe first by inspection of Theorem 2.1-3 and Proposition 2.1 (i), that in all cases one has

$$D(L) = BD(\widetilde{A}). \tag{3.1}$$

Now let $u \in D(A_1)$, and consider the decomposition $u = u_\beta + u_\zeta$ with $u_\beta \in D(A_\beta)$, $u_\zeta \in Z(A_1)$.

For $0 \leq s \leq 2m$, we note that $D(A_\beta) \subset H^{2m}(\Omega)$, thus $u \in H^s(\Omega)$ if and only if $u_\zeta \in H^s(\Omega) \cap Z(A_1) = Z_A^s$; i. e., if and only if $Bu_\zeta \in \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ (Proposition 1.1). Since $Bu_\zeta = Bu$, it follows that $u \in H^s(\Omega)$ if and only if $Bu \in \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$. Then (i) follows by use of (3.1).

For $s \geq 2m$ we have: $Au \in H^{s-2m}(\Omega)$ implies $u_\beta = A_\beta^{-1} Au \in H^s(\Omega)$ (Theorem I 3.3, $Bu_\beta = 0$). Then, for those u which have $Au \in H^{s-2m}(\Omega)$, $u \in H^s(\Omega)$ if and only if $u_\zeta \in Z_A^s(\Omega)$, i. e., $Bu \in \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$. This leads to (ii).

Lemma 3.1. *Let F be a closable operator from $H^0(\Omega)$ into a Hilbert space K, and let $s > 0$. Then $D(F) \subset H^s(\Omega)$ (alg. and top.) if and only if $D(\bar{F}) \subset H^s(\Omega)$ (alg.).*

Proof. That $D(F) \subset H^s(\Omega)$ (alg. and top.) implies $D(\bar{F}) \subset H^s(\Omega)$ is obvious; on the other hand, if $D(\bar{F}) \subset H^s(\Omega)$ (alg.), the inclusion mapping $D(\bar{F}) \to H^0(\Omega)$ is continuous *with values in* $H^s(\Omega)$, thus continous into $H^s(\Omega)$, by the closed graph theorem.

Theorem 3.2. *Let $\tilde{A}$ correspond to L as in Theorem 2.1-3 (so L is assumed closable).*

(i) *Let $0 \leq s \leq 2m$. Then $D(\tilde{A}) \subset H^s(\Omega)$ (alg. and top.) if and only if $D(L) \subset \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ (alg. and top.).*

(ii). *Let $s \geq 2m$. Then $D(L) \subset \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ (alg. and top.) implies $\{u \in D(\tilde{A}) \mid Au \in H^{s-2m}(\Omega)\} \subset H^s(\Omega)$ (alg. and top.), where $\{u \in D(\tilde{A}) \mid Au \in H^{s-2m}(\Omega)\}$ is provided with the norm $(|u|_0^2 + |Au|_{s-2m}^2)^{\frac{1}{2}}$.*

Proof: Since L is closable, $\bar{\tilde{A}}$ corresponds to $\bar{L}$.

If follows from Lemma 3.1 that $D(\tilde{A}) \subset H^s(\Omega)$ (alg. and top.) if and only if $D(\bar{\tilde{A}}) \subset H^s(\Omega)$(alg.); and that for $s \geq 2m$, $\{u \in D(\tilde{A}) \mid Au \in H^{s-2m}(\Omega)\} \subset H^s(\Omega)$ (alg. and top.) if and only if $\{u \in D(\bar{\tilde{A}}) \mid Au \in H^{s-2m}(\Omega)\} \subset H^s(\Omega)$ (alg.), using that the closure of $\{u \in D(\tilde{A}) \mid Au \in H^{s-2m}(\Omega)\}$ in the norm $(|u|_0^2 + |Au|_{s-2m}^2)^{\frac{1}{2}}$ equals $\{u \in D(\bar{\tilde{A}}) \mid Au \in H^{s-2m}(\Omega)\}$. The technique of the proof of Lemma 3.1 also gives that $D(L) \subset \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ (alg. and top.) if and only if $D(\bar{L}) \subset \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ (alg.).

The theorem now follows by application of Theorem 3.1 to $\bar{\tilde{A}}$ and $\bar{L}$.

Remark 3.1 Note that in the above theorems there are no other explicit assumptions on Y than the defining one: that Y be a closed subspace of $\Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$.

III § 4. The formally selfadjoint case.

When A is formally selfadjoint (i.e., $A = A'$), then $A_0 = A_0'$ and $A_1 = A_1'$. Moreover, the boundary operators B, B', C and C' can be chosen such that $B = B'$ and $C = C'$; then the operator A_β is selfadjoint. (Systems $\{B, C\}$ of this kind are called self-conjugate in Ercolano-Schechter [10]. The index sets $\{m_j\}$, $\{\mu_j\}$ will satisfy $\mu_j = 2m - m_j - 1$, $j = 0, \dots, m - 1$. We refer to [10] for a detailed description.)

In the present § we will add this assumption to the basic assumption stated in § 1:

$$A = A',\ B = B',\ C = C'. \tag{4.1}$$

Then the theory of II § 2 can be applied. Note that we now have $BZ(A_1) = B' Z(A_1') = \Pi H^{-m_j - \frac{1}{2}}(\Gamma)$.

Theorem 4.1 *Assume* (4.1).

Let X be any closed subspace of $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$, and let $L: X \to X'$ be selfadjoint. Then the operator $\widetilde{A} \in \mathcal{M}$ defined by

$$D(\widetilde{A}) = \{u \in D(A_1) \mid Bu \in D(L),\ Mu = LBu \ \text{on}\ X\} \tag{4.2}$$

is selfadjoint.

Conversely, any selfadjoint operator $\widetilde{A} \in \mathcal{M}$ defines a selfadjoint operator $L: X \to X'$ by (4.2); *here $X = \overline{BD(\widetilde{A})}$.*

$\widetilde{A}$ and L correspond uniquely to each other; $D(L) = BD(\widetilde{A})$.

This theorem follows easily from Theorem II 2.1.

With our usual terminology, les us assume in the rest of this § that

$$\widetilde{A} \text{ corresponds to } L: X \to Y' \text{ as in Theorem 2.1.} \tag{4.3}$$

One gets straightforwardly from Lemma II 2.1 and Definition (III) 2.1:

Lemma 4.1. *Assume* (4.1) *and* (4.3).

Let $X \subset Y$. Then for $u, v \in D(\widetilde{A})$

$$(Au, v) = (Au_\beta, v_\beta) + \langle LBu, Bv \rangle_{Y' \ \ Y}.$$

In particular

$$\operatorname{Re}(Au, u) = (Au_\beta, u_\beta) + \operatorname{Re} \langle LBu, Bu \rangle_{Y' \ \ Y},$$

$$\operatorname{Im}(Au, u) = \operatorname{Im} \langle LBu, Bu \rangle_{Y' \ \ Y}.$$

Theorem II 2.3 and Corollary II 2.3 imply:

THEOREM 4.2. *Assume* (4.1) *and* (4.3).

$\widetilde{A}$ *is symmetric if and only if* $X \subset Y$ *and* L *is symmetric (i. e.* $\langle L\varphi, \psi \rangle_{Y' \; Y} = \langle \varphi, L\psi \rangle_{Y \; Y'}$ *for all* $\varphi, \psi \in D(L)$).

Moreover, $\widetilde{A}$ *is maximal symmetric if and only if* $X = Y$ *and* L *is maximal symmetric.*

For the remaining theorems we assume that A_0 is positive, i. e.

$$(4.4) \qquad m(A_0) = \inf \{(Au, u) \mid u \in D(A_0), \ |u| = 1\} > 0.$$

Then there exist positive selfadjoint realizations, in particular the operator A_γ determined by Friedrichs' lemma, Lemma II 2.2. The justification for calling Friedrichs' extension A_γ is that it is exactly the realization $A_{B=\gamma}$, say, which corresponds to $B = B' = \gamma$ (« Dirichlet conditions »). We will indicate a proof:

Denote $\{\gamma_0, \dots, \gamma_{m-1}\} = \gamma$, where $\gamma_j = \dfrac{\partial^j}{\partial n^j}$ (as in Theorem I 2.1). Using that $a_{pq}(x) \in \mathcal{D}(\overline{\Omega})$ one obtains by integration by parts

$$(4.5) \qquad (Au, v) = a(u, v) + \int_\Gamma Nu \cdot \overline{\gamma v} \, d\sigma, \text{ all } u, v \in \mathcal{D}(\overline{\Omega}),$$

where $a(u, v)$ is defined by

$$(4.6) \qquad a(u, v) = \sum_{|p|,\, |q| \leq m} \int_\Omega a_{pq}(x) D^q u \, \overline{D^p v} \, dx ;$$

and $N = \{N_0, \dots, N_{m-1}\}$ is a normal system of boundary operators with C^∞ coefficients and orders $2m - j - 1$, $j = 0, \dots, m-1$. (4.6) actually makes sense for all $u, v \in H^m(\Omega)$; then $a(u, v)$ is a *continuons* symmetric sesquilinear form on $H^m(\Omega)$.

By extension by continuity (using Corollary I 2.1), (4.5) extends to $H^{2m}(\Omega)$:

$$(4.7) \qquad (Au, v) = a(u, v) + \int_\Gamma Nu \cdot \overline{\gamma v} \, d\sigma, \text{ all } u, v \in H^{2m}(\Omega);$$

here γ and N map $H^{2m}(\Omega)$ continuously into $\Pi H^{2m-j-\frac{1}{2}}(\Gamma)$ resp. $\Pi H^{j+\frac{1}{2}}(\Gamma)$.

Define the operator $A_{B=\gamma}$ by

$$A_{B=\gamma} \subset A_1$$

$$D(A_{B=\gamma}) = \{u \in H^{2m}(\Omega) \mid \gamma u = 0\};$$

by the regularity theorem mentioned in Chapter I (see Remark I 3.1 and Corollary I 3.1) it is *selfadjoint*. Moreover, $D(A_{B=\gamma}) \subset H_0^m(\Omega)$, for $D(A_{B=\gamma}) \subset H^{2m}(\Omega) \subset H^m(\Omega)$ and γ defined on $H^m(\Omega)$ extends γ defined on $H^{2m}(\Omega)$ (Remark I 2.1); we then use that $\{u \in H^m(\Omega) \mid \gamma u = 0\} = H_0^m(\Omega)$ (Theorem I 2.1).

Now let $u \in D(A_{B=\gamma})$. Then since $\mathcal{D}(\Omega)$ is dense in $H_0^m(\Omega)$, there exists a sequence $\{u^n\} \subset \mathcal{D}(\Omega) \subset D(A_0)$ so that $u^n \longrightarrow u$ in $H_0^m(\Omega)$. In particular $u^n \longrightarrow u$ in $L^2(\Omega)$. One also has, using (4.7)

$$|(A(u^n - u),\ u^n - u)| = |a(u^n - u,\ u^n - u)| \leq c\,|u^n - u|_m^2,$$

since $a(u, v)$ is continuous on $H^m(\Omega)$. Thus $(A(u^n - u),\ u^n - u) \longrightarrow 0$ for $n \longrightarrow \infty$.

This shows that $A_{B=\gamma}$ satisfies the conditions of Lemma II 2.2, thus

$$A_{B=\gamma} = A_\gamma .$$

We will use the notation A_γ. Note that Lemma II 2.2 implies that $m(A_\gamma) = m(A_0)$.

The results of II § 2 are concerned with numerical ranges and spectra. There is one complication in carrying this over to the correspondence between $\widetilde{A}$ and L, that we would like to point out:

Usually, the *duality* between $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$ and $\Pi H^{m_j+\frac{1}{2}}(\Gamma)$ is given, whereas the *norm* in $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$ is not specified (cf. Remark 2.1). Therefore (if $L: \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \longrightarrow \Pi H^{m_j+\frac{1}{2}}(\Gamma)$) the value of $\langle L\varphi, \varphi\rangle$ is well known, whereas $|\varphi|_{\left|-m_j-\frac{1}{2}\right|}$ is not, so the numerical range of L is not independently defined. Also the identification of $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$ with its dual is not fixed, so that L does not have a well defined resolvent or spectrum.

There are several ways of handling this.

1⁰ We can fix the norm in $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$. This can be done in many ways; particularly suitable for the given set up is the norm that one obtains by demanding that $B: Z(A_1) \longrightarrow \Pi H^{m_j-\frac{1}{2}}(\Gamma)$ be an *isometry*. We can then define the spectrum (and various parts of it) and the numerical range, using the identification between $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$ and $\Pi H^{m_j+\frac{1}{2}}(\Gamma)$ defined by the duality together with this particular norm. All the statements of II § 2 carry over word for word. The results from this approach seem somewhat artificial, not very easily applicable.

2^0 If we notice that the particular norm in $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$ mentioned in 1^0 can actually be described in terms of the norm in $L^2(\Omega)$, we obtain that the corresponding numerical range and lower bound can be defined in terms of the norm in $L^2(\Omega)$ and the duality between Y and Y'; let us therefore define (for $L: X \longrightarrow Y'$ with $X \subset Y$, closed subspaces of $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$):

$$v_Z(L) = \{\langle LBz, Bz \rangle_{Y' \; Y} \mid z \in Z(A_1) \quad \text{with } Bz \in D(L), \underline{|z| = 1}\}, \tag{4.8}$$

$$m_Z(L) = \inf\{\operatorname{Re} \langle LBz, Bz \rangle_{Y' \; Y} \mid z \in Z(A_1) \quad \text{with } Bz \in D(L), |z| = 1\}. \tag{4.9}$$

Now, those spectral properties that are connected with the numerical range and the concept of lower boundedness can still be treated if we work only with $v_Z(L)$ and $m_Z(L)$.

3^0 Finally, it can be shown that positivity, nonnegativity and lower boundedness, as well as the corresponding maximal concepts, can be defined qualitatively, without reference to norm or numerical range. This will lead to qualitative descriptions of the correspondence between spectral (and numerical) properties of $\tilde{A}$ and L. The properties that depend on specific estimates are lost by this approach.

In the following we will develop the ideas mentioned in 2^0 and 3^0, thereby applying the major part of II § 2. However, one type of application has been omitted; the application of perturbation theorems as in Theorem II 2.2 These obviously give rise to statements in the style of 1^0 above; however, one could also get a qualitative statement from Theorem II 2.2 (ii), with suitable definitions. Since this type of idea requires a thorough treatment we have omitted it here.

LEMMA 4.2. *Let* $X \subset Y \subset \Pi H^{-m_j-\frac{1}{2}}(\Gamma)$ *(closed subspaces) and let* $L: X \longrightarrow Y'$. *Let* $|\cdot|_{X,1}$ *and* $|\cdot|_{X,2}$ *be two equivalent hermitian norms in* X. *Then*

$$\inf\{\operatorname{Re} \langle L\varphi, \varphi \rangle \mid \varphi \in D(L), |\varphi|_{X,1} = 1\}$$

and

$$\inf\{\operatorname{Re} \langle L\varphi, \varphi \rangle \mid \varphi \in D(L), |\varphi|_{X,2} = 1\}$$

belong to the same of the four sets

(i) $\{t > 0\}$, (ii) $\{t \geq 0\}$, (iii) $\{t > -\infty\}$, (iv) $\{t = -\infty\}$.

Proof: Follows easily from the fact that there exist positive constants c', c'' so that

$$c' \,|\, \varphi \,|_{X,1} \leq |\, \varphi \,|_{X,2} \leq c'' \,|\, \varphi \,|_{X,1} \quad \text{for all } \varphi \in X.$$

Definition 4.1. *Let $X \subset Y \subset \Pi H^{-m_j - \frac{1}{2}}(\Gamma)$ (closed subspaces) and let $L: X \longrightarrow Y'$. L will be said to be positive, nonnegative, lower bounded or unbounded below, according to whether*

$$\inf \{\operatorname{Re} \langle L\varphi, \varphi \rangle \mid \varphi \in D(L), \ |\, \varphi \,|_X = 1\}$$

is > 0, ≥ 0, $> -\infty$ or $= -\infty$ for all the equivalent norms in X.

The definition makes sense because of Lemma 4.2.

Lemma 4.3. *Let $L: X \longrightarrow Y'$ $(X \subset Y)$ correspond to $T: V \longrightarrow W$ as in Definition 2.1a. Then L is positive, nonnegative, lower bounded or unbounded below, if and only if T is positive, nonnegative, lower bounded or unbounded below, respectively. (One has $m_Z(L) = m(T)$.)*

Proof: Let X be provided with the norm defined by $|\, Bz \,|_X = |z|$; then the statement follows immediately from $\langle LBz, Bz \rangle = (Tz, z)$.

Definition 4.2. *Assumptions of Definition 4.1, with $X = Y$. $L: X \longrightarrow Y'$ will be said to be maximal positive/maximal nonnegative/maximal lower bounded, if it is positive/nonnegative/lower bounded, and has no proper positive/nonnegative/lower bounded extension (respectively).*

Lemma 4.4. *Let $L: X \rightarrow X'$ correspond to $T: V \longrightarrow V$ as in Definition 2.1a. Then L is maximal positive/maximal nonnegative/maximal lower bounded, if and only if T is maximal positive/maximal nonnegative/maximal lower bounded (respectively).*

Proof. The lemma is an easy consequence of Lemma 4.3, Definition 4.2 and Definition A.2 (Appendix).

For the case where L is closed, densely defined, Definition 4.2 can now be replaced by a more useful description.

Proposition. 4.1. *Let $L: X \longrightarrow X'$ be closed. Then L is maximal positive [maximal nonnegative] if and only if L is maximal lower bounded and positive [nonnegative].*

PROOF : By Corollary A.4 the statement holds with $L: X \to X'$ replaced by $T: V \to V$. Then Lemma 4.3 and 4.4 imply that it holds for L.

We also mention the consequence of Corollary A.5 :

PROPOSITION 4.2. *Let $L: X \to X'$ be closed. Then L is maximal positive/maximal nonnegative/maximal lower bounded, if and only if L is densely defined and L and L^* are both positive/nonnegative/lower bounded (respectively).*

The application of these ideas to the results of II § 2 is now quite straightforward, so we will list the theorems without proofs. Recall II (2.7): $\nu(\widetilde{A}) \supset \nu(A_0)$.

From $\widetilde{A}$ to L:

THEOREM 4.3. *Assume* (4.1), (4.3) *and* (4.4), *and assume that* $A_\beta = A_\gamma$.

(i) *If* $\nu(\widetilde{A}) \neq \mathbb{C}$, *then* $X \subset Y$ *and* $\nu_Z(L) \subset \overline{\nu}(\widetilde{A})$.

(ii) *If for some* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $m(e^{i\theta}\widetilde{A}) > -\infty$, *then* $X \subset Y$ *and* $e^{i\theta} L$ *is lower bounded with* $m_Z(e^{i\theta} L) \geq m(e^{i\theta}\widetilde{A})$.

(iii) *If for some* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $e^{i\theta}\widetilde{A}$ *is maximal lower bounded, then* $X = Y$ *and* $e^{i\theta} L$ *is maximal lower bounded with* $m_Z(e^{i\theta} L) \geq m(e^{i\theta}\widetilde{A})$.

Qualitative statements :

THEOREM 4.4. *Assumptions of Theorem* 4.3.

(i) *Let* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. *If one of the following properties holds for* $e^{i\theta}\widetilde{A}$, *then it holds for* $e^{i\theta} L$: *positivity, nonnegativity, lower boundedness, and the corresponding maximal concepts.*

(ii) *Let* $X \subset Y$. *If for some* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $e^{i\theta} L$ *is unbounded below, then so is* $e^{i\theta}\widetilde{A}$. *If* $e^{i\theta} L$ *is unbounded below for all* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, *then* $\nu(\widetilde{A}) = \mathbb{C}$.

From L to $\widetilde{A}$:

THEOREM 4.5. *Assume* (4.1), (4.3) *and* (4.4), *and assume that* A_β *satisfies* $m(A_\beta) > 0$.

(i) *Let* $X \subset Y$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. *If* $e^{i\theta} L$ *is lower bounded with* $m_Z(e^{i\theta} L) > -\cos\theta \; m(A_\beta)$, *then* $e^{i\theta}\widetilde{A}$ *is lower bounded with*

$$m(e^{i\theta}\widetilde{A}) \geq \frac{\cos\theta \; m(A_\beta)\, m_Z(e^{i\theta} L)}{\cos\theta \; m(A_\beta) + m_Z(e^{i\theta} L)}. \tag{4.10}$$

(ii) *Let* $X = Y$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. *If* $e^{i\theta} L$ *is maximal lower bounded with* $m_Z(e^{i\theta} L) > -\cos\theta\ m(A_\beta)$ *then* $e^{i\theta} \tilde{A}$ *is maximal lower bounded, the lower bound estimated by* (4.10).

Qualitative statements:

THEOREM 4.6. *Assumptions of Theorem* 4.5. *Let* $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. *If* $X \subset Y$ *and* $e^{i\theta} L$ *is nonnegative* [*positive*], *then* $e^{i\theta} \tilde{A}$ *is nonnegative* [*positive*]; *if furthermore* $X = Y$ *and* $e^{i\theta} L$ *is maximal nonnegative* [*positive*], *then* $e^{i\theta} \tilde{A}$ *is maximal nonnegative* [*positive*].

THEOREM 4.7. *Assume* (4.1) *and* (4.3).

(i) *Let* $V \subset W$ *and let* $\theta = \frac{\pi}{2}$ *or* $-\frac{\pi}{2}$. *Then* $e^{i\theta} L$ *is nonnegative if and only if* $e^{i\theta} \tilde{A}$ *is nonnegative* (*and then* $m(e^{i\theta} \tilde{A}) = 0$).

(ii) *Let* $V = W$ *and let* $\theta = \frac{\pi}{2}$ *or* $-\frac{\pi}{2}$. *Then* $e^{i\theta} L$ *is maximal nonnegative if and only if* $e^{i\theta} \tilde{A}$ *is maximal nonnegative.*

As mentioned in II § 2 one can by combination obtain theorems about angles and other convex sets. Example:

COROLLARY 4.5. *Assumptions of Theorem* 4.5. *Let* $X = Y$. *If* $-\frac{\pi}{2} \leq \theta_1 < \theta_2 \leq \frac{\pi}{2}$, *and* $e^{i\theta_1} L$ *and* $e^{i\theta_2} L$ *are maximal nonnegative, then* $\sigma(\tilde{A})$ *and* $\nu(\tilde{A})$ *are contained in the angle*

$$\left\{\lambda = re^{i\theta} \mid r \geq 0,\ -\frac{\pi}{2} - \theta_1 \leq \theta \leq \frac{\pi}{2} - \theta_2\right\}.$$

The formula (4.10) can be employed to give results about more general convex sets, using $\nu_Z(L)$.

III § 5. Additional properties of P.

The operator P was defined in § 2 as: the inverse of $B: Z^s_A(\Omega) \to \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ followed by $C: Z^s_A(\Omega) \to \Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$. It was shown how P is defined consistently in this way for all real s, and that P maps

$\Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ continuously into $\Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$ for all real s. P' was defined analogously from A', B' and C'; P' maps $\Pi H^{s-2m+\mu_j+\frac{1}{2}}(\Gamma)$ continuously into $\Pi H^{s-2m+m_j+\frac{1}{2}}(\Gamma)$ for all real s.

In this § we will derive some additional properties of P and P'.

THEOREM 5.1. *For each real s, define P_s as the restriction of P with domain $\Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$, and define P_s' as the restriction of P' with domain $\Pi H^{s-2m+\mu_j+\frac{1}{2}}(\Gamma)$.*

Then for all real s, $P_s\colon \Pi H^{s-m_j-\frac{1}{2}}(\Gamma) \to \Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$ and $P_{2m-s}'\colon \Pi H^{-s+\mu_j+\frac{1}{2}}(\Gamma) \to \Pi H^{-s+m_j+\frac{1}{2}}(\Gamma)$ are adjoint operators.

PROOF: Let $\varphi, \psi \in (\mathcal{D}(\Gamma))^m$. It follows from Proposition 1.1 that the equations $Bu = \varphi$, $B'v = \psi$ have unique solutions $u \in Z_A^{2m}(\Omega)$ and $v \in Z_{A'}^{2m}(\Omega)$. Then

$$\int_\Gamma (P\varphi \cdot \overline{\psi} - \varphi \cdot \overline{P'\psi})\, d\sigma = \int_\Gamma (Cu \cdot \overline{B'v} - Bu \cdot \overline{C'v})\, d\sigma$$

$$= \int_\Omega (Au\overline{v} - u\overline{A'v})\, dx, \qquad \text{by I (3.3)},$$

$$= 0, \quad \text{since } Au = A'v = 0.$$

Since $\mathcal{D}(\Gamma)$ is dense in $H^t(\Gamma)$, all real t, the statement now follows by extension by continuity (recall that the duality between $H^t(\Gamma)$ and $H^{-t}(\Gamma)$ is an extension of the L^2-inner product between functions in $\mathcal{D}(\Gamma)$, as described in Chapter I).

Next we will show how, under certain additional assumptions on our given differential operators A, A', B, B', C, C', the operators P and P' have an « ellipticity »-property (or regularity-property). Here we can either assume that $\{A, C\}$ and $\{A', C'\}$ have the same nice properties ($(\mathcal{C})$ and $(\mathcal{U})$ of Chapter I) as $\{A, B\}$ and $\{A', B'\}$, so that the roles of B and C (or of B' and C') can be interchanged, and P and P' are simply invertible. Or we can make use of the boundedness of Ω and Γ, which has not played an essential part up to this point (the fundamental theory does not use compactness arguments), and use certain results of Lions-Magenes [24] VI, requiring only that C covers A (i. e , $\{A, C\}$ satisfies $(\mathcal{C})$).

The first result is easily described:

THEOREM 5.2. *Assume in addition to the basic assumption that* $\{A, C\}$ *and* $\{A', C'\}$ *satisfy the hypotheses* $(\mathcal{C})$ *and* $(\mathcal{U})$. *Then* P *maps* $\Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ *isomorphically onto* $\Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$ *for all real* s, *and* P' *maps* $\Pi H^{s-2m+\mu_j+\frac{1}{2}}(\Gamma)$ *isomorphically onto* $\Pi H^{s-2m+m_j+\frac{1}{2}}(\Gamma)$ *for all real* s.

PROOF: By interchanging B and C and applying Definition 1.1 and Theorem 1.1 we obtain an operator which is the inverse of P and maps $\Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$ continuously into $\Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ for all real s. The analogous argument applies to P'.

The theory from Lions-Magenes [24] VI that is required for the second result will be stated for the system $\{A, C\}$ (rather than $\{A, B\}$) right away.

Define:

$$Z(A, C) = \{u \in H^{2m}(\Omega) \mid Au = 0,\ Cu = 0\}$$

$$Z(A', C') = \{v \in H^{2m}(\Omega) \mid A'v = 0,\ C'v = 0\}.$$

One has (Schechter [28], Agmon-Douglis-Nirenberg [1]):

PROPOSITION 5.1. (*Uses boundedness of* Ω.) *If* $\{A, C\}$ *satisfies the hypothesis* $(\mathcal{C})$, *then* $Z(A, C)$ *and* $Z(A', C')$ *are finite dimensional and contained in* $\mathcal{D}(\overline{\Omega})$.

Then $Z(A, C)$ is a closed subspace of any of the spaces $D^s_A(\Omega)$, $H^s(\Omega)$, s real, so that the quotient spaces $D^s_A(\Omega)/Z(A, C)$ and $H^s(\Omega)/Z(A, C)$ can be defined as Hilbert spaces with the quotient topology. A and C are defined on $D^s_A(\Omega)/Z(A, C)$ and $H^s(\Omega)/Z(A, C)$ in the obvious way.

For $s \geq 0$, t real, let $\{H^s(\Omega) \times \Pi H^{t-\mu_j-\frac{1}{2}}(\Gamma);\ Z(A', C'),\ B'Z(A', C')\}$ denote the space of distributions $\{f, \varphi\} \in H^s(\Omega) \times \Pi H^{t-\mu_j-\frac{1}{2}}(\Gamma)$ which satisfy

$$(f, g)_{L^2(\Omega)} + \langle \underset{\{t-\mu_j-\frac{1}{2}\}}{\varphi}, \underset{\{-t+\mu_j+\frac{1}{2}\}}{B'g} \rangle = 0 \text{ for all } g \in Z(A', C');$$

it is a closed subspace of $H^s(\Omega) \times \Pi H^{t-\mu_j-\frac{1}{2}}(\Gamma)$.

Lions and Magenes prove (3) ([24] VI, Theorem 4.1, Theorem 8.1 and Remark 8.3):

(3) The condition « $s - \frac{1}{2}$ not integer » has been omitted because of results in [24'], see our Remark I.3.2.

THEOREM 5.3. (*Uses boundedness of Ω*). *Assume that* $\{A, C\}$ *satisfies* $(\mathcal{C})$.

(i) *For all* $s \leq 2m$, $\{A, C\}$ *maps* $D^s_A(\Omega)/Z(A, C)$ *isomorphically onto the space* $\{H^0(\Omega) \times \Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)\,;\; Z(A', C'), B'Z(A', C')\}$.

(ii) *For all* $s \geq 2m$, $\{A, C\}$ *maps* $H^s(\Omega)/Z(A, C)$ *isomorphically onto the space* $\{H^{s-2m}(\Omega) \times \Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)\,;\; Z(A', C'), B'Z(A', C')\}$.

A particular consequence of this result is

PROPOSITION 5.2. (*Uses boundedness of Ω*). *Assume that* $\{A, C\}$ *satisfies* $(\mathcal{C})$.

(i) *Let* $s \leq 2m$. *If for some* $r < s$, $u \in D^r_A(\Omega)$ *with* $Cu \in \Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$, *then* $u \in D^s_A(\Omega)$.

(ii) *Let* $s \geq 2m$. *If for some* $r < s$, $u \in H^r(\Omega)$ *with* $Au \in H^{s-2m}(\Omega)$ *and* $Cu \in \Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$, *then* $u \in H^s(\Omega)$.

PROOF :

(i) It follows from Theorem 5.3 (i) that $\{Au, Cu\} \in \{H^0(\Omega) \times \Pi H^{r-\mu_j-\frac{1}{2}}(\Gamma)\,;\; Z(A', C'), B'Z(A', C')\}$. The assumption on Cu implies that in fact $\{Au, Cu\} \in \{H^0(\Omega) \times \Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)\,;\; Z(A', C'), B'Z(A', C')\}$, using that $B'Z(A', C') \subset (\mathcal{D}(\Gamma))^m$. Another application of Theorem 5.3 (i) then gives that $u = u_0 + z$ where $u_0 \in D^s_A(\Omega)$ and $z \in Z(A, C) \subset \mathcal{D}(\bar{\Omega})$, i.e., $u \in D^s_A(\Omega)$.

(ii) Since in particular $u \in D^r_A(\Omega)$, it follows from Theorem 5.3 (i) that $\{Au, Cu\} \in \{H^0(\Omega) \times \Pi H^{r-\mu_j-\frac{1}{2}}(\Gamma)\,;\; Z(A', C'), B'Z(A', C')\}$. The assumptions on Au and Cu imply that in fact $\{Au, Cu\} \in \{H^{s-2m}(\Omega) \times \Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)\,;\; Z(A', C'), B'Z(A', C')\}$, using that $Z(A', C') \subset \mathcal{D}(\bar{\Omega})$ and $B'Z(A', C') \subset \mathcal{D}(\Gamma)$. An application of Theorem 5.3 (ii) now shows that $u = u_0 + z$ where $u_0 \in H^s(\Omega)$ and $z \in Z(A, C) \subset \mathcal{D}(\bar{\Omega})$, i.e., $u \in H^s(\Omega)$.

We can now obtain the following statement about P:

THEOREM 5.4. (*Uses boundedness of Ω*). *In addition to the basic assumption, assume that* $\{A, C\}$ *satisfies* $(\mathcal{C})$. *Then* P *has the property:*

For each real s, $\varphi \in (\mathcal{D}'(\Gamma))^m$ *with* $P\varphi \in \Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$ *imply* $\varphi \in \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$.

PROOF: Let $\varphi \in (\mathcal{D}'(\Gamma))^m$, then $\varphi \in \Pi H^{r-m_j-\frac{1}{2}}(\Gamma)$ for some $r \in \mathbf{R}$ since Γ is compact. Assume that $P\varphi \in \Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$ for some $s \in \mathbf{R}$. If $r \geq s$ we are through; therefore let $r < s$. By Proposition 1.1, $\varphi = Bu$ for some $u \in Z^r_A(\Omega)$. u has the properties: $u \in H^r(\Omega)$, $Au = 0 \in H^t(\Omega)$, all $t \in \mathbf{R}$, and $Cu = P\varphi \in \Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$. Then Proposition 5.2 shows that $u \in H^s(\Omega)$, i.e., $u \in Z^s_A(\Omega)$. Another application of Proposition 1.1 gives that $\varphi = Bu \in \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$.

III § 6. Some applications.

As mentioned in (III) § 2, previous investigations of non-local boundary conditions have almost solely dealt with the boundary condition $Cu = KBu$, K a given operator in $(\mathcal{D}'(\Gamma))^m$. We called this type of condition a « pure condition » in § 2; it is shown there how K determines uniquely the operator L to which $\widetilde{A}$ corresponds, whereas it is only a certain part of K that is fixed by L or $\widetilde{A}$. The method to treat the problem of how properties of $\widetilde{A}$ depend on properties of K within our framework will be to derive the properties of L from K, these correspond to similar properties of $\widetilde{A}$ by our theory.

Let K be given as an operator in $(\mathcal{D}'(\Gamma))^m$. Let $\widetilde{A}$ be the realization of A determined by

$$D(\widetilde{A}) = \{u \in D(A_1) \mid Bu \in D(K),\ Cu = KBu\}; \tag{6.1}$$

i.e., $\widetilde{A}$ represents the pure condition $Cu = KBu$; then $\widetilde{A}$ also represents the pure condition $Mu = LBu$ where $L : \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \to \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ is defined by:

$$\begin{cases} D(L) = \{\varphi \in D(K) \cap \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \mid K\varphi - P\varphi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)\} \\ L\varphi = K\varphi - P\varphi \quad \text{for} \quad \varphi \in D(L) \end{cases} \tag{6.2}$$

(Proposition 2.1).

We always have $D(L) = BD(\widetilde{A})$ and can apply Theorem 3.1; however, the case where we get most information is the case where L is closable; then the whole preceding theory applies to the correspondence between $\widetilde{A}$ and L.

Because of (6.2) one could say that it is *the difference* $K - P$ that determines the character of A.

Below are given a few results; they should be considered more as an illustration of our theory than a final goal, since we have not made an effort to use all possible aspects of it. The results overlaps with Schechter [29].

Sketch of results.

Assume for simplicity that $m = 1$, $B = B' = \gamma$. Then the operators P and P' map $H^{s-\frac{1}{2}}(\Gamma)$ continuously into $H^{s-\frac{3}{2}}(\Gamma)$ for all real s (Theorem 1.1), i. e., are of order 1. Under additional assumptions (as in Theorem 5.2 or 5.4) they have an « ellipticity » — or regularity — property: $P\varphi \in H^{s-\frac{3}{2}}(\Gamma)$ $[P'\varphi \in H^{s-\frac{3}{2}}(\Gamma)]$ implies $\varphi \in H^{s-\frac{1}{2}}(\Gamma)$.

We assume that K is given as an operator in $\mathcal{D}'(\Gamma)$, and maps $H^s(\Gamma)$ continuously into $H^{s-r}(\Gamma)$ for some fixed real r, and s belonging to a suitable finite interval (in particular this is satisfied if K has order r, i. e., maps $H^s(\Gamma)$ continuously into $H^{s-r}(\Gamma)$ for all real s). Then L is closed. Furthermore:

(i) If $r < 1$ and P has the regularity-property, then $D(L) \subset H^{\frac{3}{2}}(\Gamma)$ (alg. and top.) so that (by Theorem 3.2) $D(\widetilde{A}) \subset H^2(\Omega)$ (alg. and top.). Moreover, $\widetilde{A}^*$ is determined by the boundary condition $C'u = K'B'u$, where K' is a naturally defined adjoint of K; here $D(L^*) \subset H^{3/2}(\Gamma)$ (alg. and top.) so that $D(\widetilde{A}^*) \subset H^2(\Omega)$ (alg. and top.). Since $D(L)$ and $D(L^*)$ are continuously imbedded in $H^{3/2}(\Gamma)$, L and L^* are Fredholm operators (using that Γ is bounded) then $\widetilde{A}$ and $\widetilde{A}^*$ are Fredholm operators.

(ii) If $r > 1$ and K has a regularity-property, then we get the same conclusions as in (i) (for the adjoint we have to assume that K' also has the regularity property).

(iii) If $r = 1$ and P is an isomorphism of $H^{s-\frac{1}{2}}(\Gamma)$ onto $H^{s-\frac{3}{2}}(\Gamma)$ for all s, and $K\colon H^{s-\frac{1}{2}}(\Gamma) \to H^{s-\frac{3}{2}}(\Gamma)$ has a sufficiently small norm for each s, then the conclusions of (i) hold; and in fact $0 \in \varrho(\widetilde{A})$. Using the boundedness of Ω we get that the inverse $\widetilde{A}^{-1}$ is a compact operator.

We proceed to deduce the exact results. Let us mention the following once and for all: We will often use the assumption that an operator K in $(\mathcal{D}'(\Gamma))^m$ maps $\Pi H^{r+s_j}(\Gamma)$ continuously into $\Pi H^{r+u_j}(\Gamma)$, for all r belonging to a closed interval $[r_0, r_1]$. By interpolation (see Theorem I. 1.1), it is actually enough that the property holds for the end-points $r = r_0$ and $r = r_1$.

LEMMA 6.1. *Let K be an operator in $(\mathcal{D}'(\Gamma))^m$ such that $\Pi H^{-m_j-\frac{1}{2}}(\Gamma) \subset D(K)$ and K is continuous from $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$ into $\Pi H^{t-\mu_j-\frac{1}{2}}(\Gamma)$ for some real t. Then $L : \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \to \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$, defined by (6.2), is closed.*

PROOF: Let $\{\varphi^n\}$ be a sequence in $D(L)$ such that

$$\varphi^n \to \varphi \quad \text{in} \quad \Pi H^{-m_j-\frac{1}{2}}(\Gamma), \tag{6.3}$$

$$L\varphi^n \to \psi \quad \text{in} \quad \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma). \tag{6.4}$$

(6.3) implies that $K\varphi^n \to K\varphi$ in $\Pi H^{t-\mu_j-\frac{1}{2}}(\Gamma)$ and $P\varphi^n \to P\varphi$ in $\Pi H^{-\mu_j-\frac{1}{2}}(\Gamma)$ (Theorem 1.1), thus altogether

$$L\varphi^n = K\varphi^n - P\varphi^n \to K\varphi - P\varphi \quad \text{in} \quad \Pi H^{t_1-\mu_j-\frac{1}{2}}(\Gamma),$$

where $t_1 = \min\{t, 0\}$. From (6.4) we obtain that $L\varphi^n \to \psi$ in $\Pi H^{t_1-\mu_j-\frac{1}{2}}(\Gamma)$, since $t_1 < 2m$; thus $K\varphi - P\varphi = \psi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$. This shows that $\varphi \in D(L)$ with $L\varphi = \psi$.

LEMMA 6.2. *Let t, r_0 and r_1 be real numbers ($r_0 \leq r_1$), and let K be a mapping in $(\mathcal{D}'(\Gamma))^m$ which maps $\Pi H^{r-t-m_j-\frac{1}{2}}(\Gamma)$ continuously into $\Pi H^{r-\mu_j-\frac{1}{2}}(\Gamma)$ for all $r \in [r_0, r_1]$. Denote the restriction of K with domain $\Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ by K_s ($s \in [r_0 - t, r_1 - t]$).*

There exists a mapping K' in $(\mathcal{D}'(\Gamma))^m$ such that K' maps $\Pi H^{s-2m+\mu_j+\frac{1}{2}}$ continuously into $\Pi H^{s+t-2m+m_j+\frac{1}{2}}(\Gamma)$ for all $s \in [2m - r_1, 2m - r_0]$, and *such that, if K_s' denotes the restriction of K' with domain* $\Pi H^{s-2m+\mu_j+\frac{1}{2}}(\Gamma)$, *then* $K_{r-t}\colon \Pi H^{r-t-m_j-\frac{1}{2}}(\Gamma) \to \Pi H^{r-\mu_j-\frac{1}{2}}(\Gamma)$ *and* $K'_{2m-r}\colon \Pi H^{-r+\mu_j+\frac{1}{2}}(\Gamma) \to$ $\to \Pi H^{-r+t+m_j+\frac{1}{2}}(\Gamma)$ *are adjoint operators for all* $r \in [r_0, r_1]$.

We will say that K' is the adjoint of K, or, if it is necessary to be more precise, that K' is a $\{t, r_0, r_1\}$*-adjoint of K.*

PROOF: For each $r \in [r_0, r_1]$, K_{r-t} is a continuous mapping of $\Pi H^{r-t-m_j-\frac{1}{2}}(\Gamma)$ into $\Pi H^{r-\mu_j-\frac{1}{2}}(\Gamma)$. It therefore has an adjoint $(K_{r-t})^*$ sending $\Pi H^{-r+\mu_j+\frac{1}{2}}(\Gamma)$ continuously into $\Pi H^{-r+t+m_j+\frac{1}{2}}(\Gamma)$.

When $s' \geq s$, $H^{s'}(\Gamma) \subset H^s(\Gamma)$ alg. and top., and densely, and the duality between $H^s(\Gamma)$ and $(H^s(\Gamma))' = H^{-s}(\Gamma)$ is an extension of the duality between $H^{s'}(\Gamma)$ and $H^{-s'}(\Gamma)$. Therefore, for $r_0 \leq r \leq r' \leq r_1$, $K_r \supset K_{r'}$ implies $(K_r)^* \subset (K_{r'})^*$. Let $K' = (K_{r_1-t})^*$, then $(K_{r-t})^* \subset K'$ for all $r \in [r_0, r_1]$, and $(K_{r-t})^*$ is in fact the restriction of K' with domain $\Pi H^{-r+\mu_j+\frac{1}{2}}(\Gamma)$. With the notation: K_s' is the restriction of K' with domain $\Pi H^{s-2m+\mu_j+\frac{1}{2}}(\Gamma)$, this means that $(K_{r-t})^* = K'_{2m-r}$. Note that when $r \in [r_0, r_1]$, $s = 2m - r \in [2m - r_1, 2m - r_0]$.

LEMMA 6.3. *Assume that the hypotheses of either Theorem 5.2 or Theorem 5.4 hold, so that P has the property:*

$$(6.5)\qquad \varphi \in (\mathcal{D}'(\Gamma))^m \quad \textit{with } P\varphi \in \Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma), \quad \textit{some } s \in \mathbb{R},$$

$$\textit{imply } \varphi \in \Pi H^{s-m_j-\frac{1}{2}}(\Gamma).$$

Let K be an operator in $(\mathcal{D}'(\Gamma))^m$ for which there exists $t > 0$ such that K maps $\Pi H^{s-t-m_j-\frac{1}{2}}(\Gamma)$ *continuously into* $\Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$ *for all* $s \in [0, 2m]$. *Then L, defined by* (6.2), *is closed and satisfies*

$$D(L) = \Pi H^{2m-m_j-\frac{1}{2}}(\Gamma) \textit{ alg. and top. (Notation as in § 3).}$$

PROOF: Since $t > 0$, K maps $\Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ continuously into $\Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$ for all $s \in [0, 2m]$. For $s = 0$ this implies that L is closed

(Lemma 6.1). For $s = 2m$ we see that both K and P map $\Pi H^{2m-m_j-\frac{1}{2}}(\Gamma)$ continuously into $\Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$, therefore L satisfies

$$\Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma) \subset D(L) \text{ alg. and top.} \tag{6.6}$$

(see (6.2)).

Now let $\varphi \in D(L)$. If for some $0 \leq r < 2m$

$$P\varphi \in \Pi H^{r-\mu_j-\frac{1}{2}}(\Gamma), \quad P\varphi \notin \Pi H^{r+\varepsilon-\mu_j-\frac{1}{2}}(\Gamma), \quad \text{all } \varepsilon > 0\,; \tag{6.7}$$

then the same is true for $K\varphi$ since $K\varphi - P\varphi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$. Then $\varphi \notin \Pi H^{r+\varepsilon-t-m_j-\frac{1}{2}}(\Gamma)$ for all $\varepsilon > 0$, by the assumption on K. It now follows from (6.5) that $P\varphi \notin \Pi H^{r+\varepsilon-t-\mu_j-\frac{1}{2}}(\Gamma)$ for all $\varepsilon > 0$. For $\varepsilon = t$ this means that $P\varphi \notin \Pi H^{r-\mu_j-\frac{1}{2}}(\Gamma)$, which contradicts (6.7).

Therefore there is no $0 \leq r < 2m$ for which (6.7) holds. Thus either $P\varphi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ or (6.7) holds for some $r < 0$. The latter case is excluded since $\varphi \in \Pi H^{-m_j-\frac{1}{2}}(\Gamma)$ so that $P\varphi \in \Pi H^{-\mu_j-\frac{1}{2}}(\Gamma)$. Using (6.5) again we see that $P\varphi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ implies $\varphi \in \Pi H^{2m-m_j-\frac{1}{2}}(\Gamma)$.

We have then proved that

$$D(L) \subset \Pi H^{2m-m_j-\frac{1}{2}}(\Gamma) \text{ alg.} \tag{6.8}$$

By the closed graph theorem, (6.6) and (6.8) together imply

$$D(L) = \Pi H^{2m-m_j-\frac{1}{2}}(\Gamma) \text{ alg. and top.}$$

LEMMA 6.4. *Assumptions of Lemma 6.3. Let K' be the adjoint of K, defined according to Lemma 6.2; it maps $\Pi H^{s-2m+\mu_j+\frac{1}{2}}(\Gamma)$ continuously into $\Pi H^{s+t-2m+m_j+\frac{1}{2}}(\Gamma)$ for all $s \in [0, 2m]$. Let $L : \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \to$ $\to \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ be defined by (6.2). Then the adjoint $L^*: \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma) \to$*

$\to \Pi H^{m_j+\frac{1}{2}}(\Gamma)$ *is exactly the operator defined by*

$$(6.9)\qquad \begin{cases} D(L^*) = \{\psi \in \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma) \mid K'\psi - P'\psi \in \Pi H^{m_j+\frac{1}{2}}(\Gamma)\} \\ L^*\psi = K'\psi - P'\psi \quad \textit{for} \quad \psi \in D(L^*). \end{cases}$$

PROOF: With the notation of Lemma 6.2, $K_{r-t}\colon \Pi H^{r-t-m_j-\frac{1}{2}}(\Gamma) \to \Pi H^{r-\mu_j-\frac{1}{2}}(\Gamma)$ and $K'_{2m-r}\colon \Pi H^{-r+\mu_j+\frac{1}{2}}(\Gamma) \to \Pi H^{-r+t+m_j+\frac{1}{2}}(\Gamma)$ are adjoints, for all $r \in [0, 2m]$.

By Lemma 6.3, $L\colon \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \to \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ satisfies

$$D(L) = \Pi H^{2m-m_j-\frac{1}{2}}(\Gamma) \text{ alg. and top.}$$

L^* is defined on the set of $\psi \in \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$ for which there exist $\psi^* \in \Pi H^{m_j+\frac{1}{2}}(\Gamma)$ such that

$$(6.10)\qquad \langle L\varphi, \psi \rangle_{\{2m-\mu_j-\frac{1}{2}\}\{-2m+\mu_j+\frac{1}{2}\}} = \langle \varphi, \psi^* \rangle_{\{-m_j-\frac{1}{2}\}\{m_j+\frac{1}{2}\}}, \quad \text{all } \varphi \in D(L);$$

then $\psi^* = L^*\psi$.

We also have for $\psi \in \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$, $\varphi \in D(L) = \Pi H^{2m-m_j-\frac{1}{2}}(\Gamma)$:

$$\begin{aligned} \langle L\varphi, \psi \rangle_{\{2m-\mu_j-\frac{1}{2}\}\{-2m+\mu_j+\frac{1}{2}\}} &= \langle K\varphi, \psi \rangle_{\{2m-\mu_j-\frac{1}{2}\}\{-2m+\mu_j+\frac{1}{2}\}} - \langle P\varphi, \psi \rangle_{\{2m-\mu_j-\frac{1}{2}\}\{-2m+\mu_j+\frac{1}{2}\}} \\ &= \langle \varphi, K'\psi \rangle_{\{2m-t-m_j-\frac{1}{2}\}\{-2m+t+m_j+\frac{1}{2}\}} - \langle \varphi, P'\psi \rangle_{\{2m-m_j-\frac{1}{2}\}\{-2m+m_j+\frac{1}{2}\}}, \end{aligned}$$

using that K_{2m-t} and K'_0 are adjoints, and that P_{2m} and P'_0 are adjoints (Theorem 5.1). Since $t > 0$ and in fact $\varphi \in \Pi H^{2m-m_j-\frac{1}{2}}(\Gamma)$, the expression can be transformed into

$$\begin{aligned} &= \langle \varphi, K'\psi \rangle_{\{2m-m_j-\frac{1}{2}\}\{-2m+m_j+\frac{1}{2}\}} - \langle \varphi, P'\psi \rangle_{\{2m-m_j-\frac{1}{2}\}\{-2m+m_j+\frac{1}{2}\}} \\ &= \langle \varphi, K'\psi - P'\psi \rangle_{\{2m-m_j-\frac{1}{2}\}\{-2m+m_j+\frac{1}{2}\}}. \end{aligned}$$

Altogether we have found

$$(6.11)\qquad \langle \underset{\{2m-\mu_j-\frac{1}{2}\}}{L\varphi}\ ,\ \underset{\{-2m+\mu_j+\frac{1}{2}\}}{\psi}\rangle = \langle \underset{\{2m-m_j-\frac{1}{2}\}}{\varphi}\ ,\ \underset{\{-2m+m_j+\frac{1}{2}\}}{K'\psi - P'\psi}\rangle ,$$

$$\text{all } \varphi \in D(L), \quad \text{all } \psi \in \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma).$$

By comparison of (6.10) and (6.11) we see that an element $\psi \in \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$ is in $D(L^*)$ if and only if $K'\psi - P'\psi \in \Pi H^{m_j+\frac{1}{2}}(\Gamma)$, and then $L^*\psi = K'\psi - P'\psi$. This proves that L^* is defined by (6.9).

We can now apply the regularity theorem in § 3 to the set-up defined in the preceding lemmas. In order to get the regularity of $\widetilde{A}^*$ we have to assume continuity of K for $s \in [0, 2m+t]$; for the first statements of the theorem $s \in [0, 2m]$ is enough.

THEOREM 6.1. *In addition to the basic assumption, assume that the operators A, A', C and C' satisfy the hypotheses of either Theorem 5.2 or Theorem 5.4.*

Let there be given an operator K in $(\mathcal{D}'(\Gamma))^m$ for which there exists $t > 0$ such that K maps $\Pi H^{s-t-m_j-\frac{1}{2}}(\Gamma)$ continuously into $\Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$ for all $s \in [0, 2m+t]$. Let K' be the adjoint according to Lemma 6.2.

Then the realization $\widetilde{A}$ of A which represents the boundary condition $Cu = KBu$ (i. e., is defined by (6.1)) is closed and satisfies

$$D(\widetilde{A}) \subset H^{2m}(\Omega) \quad \textit{alg. and top.};$$

its adjoint $\widetilde{A}^*$ is exactly the realization of A' which represents the boundary condition $C'u = K'B'u$, and it also satisfies

$$D(\widetilde{A}^*) \subset H^{2m}(\Omega) \quad \textit{alg. and top.}$$

PROOF: By Lemma 6.3, L defined by (6.2) is closed and satisfies

$$(6.12)\qquad D(L) = \Pi H^{2m-m_j-\frac{1}{2}}(\Gamma) \quad \text{alg. and top.}$$

Since $\widetilde{A}$ corresponds to $L: \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \to \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ in the sense of Theorem 2.1, $\widetilde{A}$ is closed and it follows from Theorem 3.2 that

$$(6.13)\qquad D(\widetilde{A}) \subset H^{2m}(\Omega) \quad \text{alg. and top.}$$

By Lemma 6.4, $L^*: \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma) \to \Pi H^{m_j+\frac{1}{2}}(\Gamma)$ is defined by K' by (6.9). Then since $\widetilde{A}^*$ corresponds to L^* (Theorem 2.1), $\widetilde{A}^*$ represents the pure condition $C'u = K'B'u$ (Proposition 2.1). By Lemma 6.2, the assumption on K implies that K' maps $\Pi H^{s-2m+\mu_j+\frac{1}{2}}(\Gamma)$ continuously into $\Pi H^{s+t-2m+m_j+\frac{1}{2}}(\Gamma)$ for all $s \in [2m-(2m+t), 2m-0] = [-t, 2m]$. Then Lemma 6.3 can be applied to L^* and K', to show that

$$D(L^*) = \Pi H^{\mu_j+\frac{1}{2}}(\Gamma) \quad \text{alg. and top.}$$

Now Theorem 3.2, applied to $\widetilde{A}^*$, shows that

$$D(\widetilde{A}^*) \subset H^{2m}(\Omega) \quad \text{alg. and top.}$$

COROLLARY 6.1. *Assumptions of Theorem 6.1. The boundednes of Γ implies that $\widetilde{A}$ and $\widetilde{A}^*$ are Fredholm operators.*

PROOF: From Theorem 6.1 we have that $D(L) = \Pi H^{2m-m_j-\frac{1}{2}}(\Gamma)$, $D(L^*) = \Pi H^{\mu_j+\frac{1}{2}}(\Gamma)$, alg. and top. The boundedness of Γ implies that the imbedding of $\Pi H^{2m-m_j-\frac{1}{2}}(\Gamma)$ into $\Pi H^{-m_j-\frac{1}{2}}(\Gamma)$ is compact. It then follows by a standard theorem (see e.g. Beals [6] p. 348) that $Z(L)$ is finite dimensional and $R(L)$ is closed. Similarly, the imbedding of $\Pi H^{\mu_j+\frac{1}{2}}(\Gamma)$ into $\Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$ is compact, so that $Z(L^*)$ is finite dimensional and $R(L^*)$ is closed. Altogether L has closed range, and $Z(L)$ and $R(L)^\perp = Z(L^*)$ are finite dimensional, so L is a Fredholm operator; similarly L^* is a Fredholm operator. By Corollary 2.4 it follows that $\widetilde{A}$ and $\widetilde{A}^*$ are Fredholm operators.

LEMMA 6.5. *Let K be an operator in $(\mathcal{D}'(\Gamma))^m$, for which there exists $t>0$ such that*

(i) *K maps $\Pi H^{s+t-m_j-\frac{1}{2}}(\Gamma)$ continuously into $\Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$ for all $s \in [-t, 2m]$.*

(ii) *For all $s \in [0, 2m]$ one has:*

$$(6.14) \qquad \varphi \in \Pi H^{-m_j-\frac{1}{2}}(\Gamma),\ \ K\varphi \in \Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma) \ \textit{imply} \ \varphi \in \Pi H^{s+t-m_j-\frac{1}{2}}(\Gamma)$$

Then L, defined by (6.2), *is closed and satisfies*

$$D(L) = \Pi H^{2m+t-m_j-\frac{1}{2}} \quad \textit{alg. and top.}$$

PROOF: Using (i) for $s = -t$, we obtain from Lemma 6.1 that L is closed. For $s = 2m$ we see that K maps $\Pi H^{2m+t-m_j-\frac{1}{2}}(\Gamma)$ continuously into $\Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$; then since P maps $\Pi H^{2m+t-m_j-\frac{1}{2}}(\Gamma)$ continuously into $\Pi H^{2m+t-\mu_j-\frac{1}{2}}(\Gamma)$ which is continuously imbedded in $\Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$, we find by consideration of the definition of L that

$$\Pi H^{2m+t-m_j-\frac{1}{2}}(\Gamma) \subset D(L) \text{ alg. and top.} \tag{6.15}$$

Now let $\varphi \in D(L)$. If for some $0 \leq r < 2m$

$$P\varphi \in \Pi H^{r-\mu_j-\frac{1}{2}}(\Gamma), \quad P\varphi \notin \Pi H^{r+\varepsilon-\mu_j-\frac{1}{2}}(\Gamma), \quad \text{all } \varepsilon > 0, \tag{6.16}$$

then the same is true for $K\varphi$, since $K\varphi - P\varphi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$. By (6.14), this implies that $\varphi \in \Pi H^{r+t-m_j-\frac{1}{2}}(\Gamma)$. Then $P\varphi \in \Pi^{r+t-\mu_j-\frac{1}{2}}(\Gamma)$, which contradicts (6.16) since $t > 0$. Thus $P\varphi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$. Since $\varphi \in D(L)$, this implies $K\varphi \in \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$. Using (6.14) again, we conclude that $\varphi \in \Pi H^{2m+t-m_j-\frac{1}{2}}(\Gamma)$. We have then proved that

$$D(L) \subset \Pi H^{2m+t-m_j-\frac{1}{2}}(\Gamma) \text{ alg.} \tag{6.17}$$

By the closed graph theorem, (6.15) and (6.17) together imply that

$$D(L) = \Pi H^{2m+t-m_j-\frac{1}{2}}(\Gamma) \text{ alg. and top.}$$

LEMMA 6.6. *Assumptions of Lemma* 6.5. *Let K' be the adjoint of K, defined according to Lemma* 6.2; *it maps $\Pi H^{s-2m+\mu_j+\frac{1}{2}}(\Gamma)$ continuously into $\Pi H^{s-t-2m+m_j+\frac{1}{2}}(\Gamma)$ for all $s \in [0, 2m+t]$. Let $L : \Pi H^{-m_j-\frac{1}{2}}(\Gamma) \to$ $\to \Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$ be defined by* (6.2). *Then the adjoint $L^*: \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma) \to$ $\to \Pi H^{m_j+\frac{1}{2}}(\Gamma)$ is exactly the operator defined by* (6.9).

PROOF: The proof is similar to that of Lemma 6.4, so we will only describe it very briefly.

One has for all $\psi \in \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma)$, all $\varphi \in D(L) = \Pi H^{2m+t-m_j-\frac{1}{2}}(\Gamma)$;

$$\langle L\varphi, \psi \rangle_{\{2m-\mu_j-\frac{1}{2}\}\{-2m+\mu_j+\frac{1}{2}\}} = \langle K\varphi, \psi \rangle_{\{2m-\mu_j-\frac{1}{2}\}\{-2m+\mu_j-\frac{1}{2}\}} - \langle P\varphi, \psi \rangle_{\{2m-\mu_j-\frac{1}{2}\}\{-2m+\mu_j+\frac{1}{2}\}}$$

$$= \langle \varphi, K'\psi \rangle_{\{2m+t-m_j-\frac{1}{2}\}\{-2m-t+m_j+\frac{1}{2}\}} - \langle \varphi, P'\psi \rangle_{\{2m-m_j-\frac{1}{2}\}\{-2m+m_j+\frac{1}{2}\}}$$

since K_{2m+t} and K_0' are adjoints, and P_{2m} and P_0' are adjoints. Thus

$$(6.18) \quad \langle L\varphi, \psi \rangle_{\{2m-\mu_j-\frac{1}{2}\}\{-2m+\mu_j+\frac{1}{2}\}} = \langle \varphi, K'\psi - P'\psi \rangle_{\{2m+t-m_j-\frac{1}{2}\}\{-2m-t+m_j+\frac{1}{2}\}} \text{ all } \varphi \in D(L),$$

$$\psi \in \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma).$$

L^* is defined by (6.10), then (6.18) shows that L^* is the operator satisfying

$$D(L^*) = \{\psi \in \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma) \mid K'\psi - P'\psi \in \Pi H^{m_j+\frac{1}{2}}(\Gamma)\}$$

$$L^*\psi = K'\psi - P'\psi \text{ for } \psi \in D(L^*).$$

In the following theorem we strengthen the hypothesis on K to get the results for $\tilde{A}^*$ as well as those for $\tilde{A}$.

THEOREM 6.2. *Let K be an operator in $(\mathcal{D}'(\Gamma))^m$ for wich there exists $t > 0$ such that* (i) - (iii) (4) *are satisfied:*

(i) *K maps $\Pi H^{s+t-m_j-\frac{1}{2}}(\Gamma)$ continuously into $\Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$ for all $s \in [-t, 2m+t]$*

(ii) *For all $s \in [0, 2m]$ one has*

$$(6.19) \quad \varphi \in \Pi H^{-m_j-\frac{1}{2}}(\Gamma), \quad K\varphi \in \Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma) \text{ imply } \varphi \in \Pi H^{s+t-m_j-\frac{1}{2}}(\Gamma).$$

---

(4) The assumptions (i)-(iii) are satisfied if K is an isomorphism of $\Pi H^{s+t-m-\frac{1}{2}}(\Gamma)$ onto $\Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$ for $s \in [-t, 2m+t]$.

(iii) *Let K' denote the adjoint defined acording to Lemma* 6.2 *; it maps* $\Pi H^{s-2m+\mu_j+\frac{1}{2}}(\Gamma)$ *continuously into* $\Pi H^{s-t-2m+m_j+\frac{1}{2}}(\Gamma)$ *for all* $s\in[-t,2m+t]$. *Assume that K' satisfies*

$$(6.20)\qquad \varphi\in \Pi H^{-2m+\mu_j+\frac{1}{2}}(\Gamma),\quad K'\varphi\in \Pi H^{s-2m+m_j+\frac{1}{2}}(\Gamma)\ \textit{imply}$$

$$\varphi\in \Pi H^{s+t-2m+\mu_j+\frac{1}{2}}(\Gamma)$$

for all $s\in[0,2m]$.

Then the realization $\widetilde{A}$ of A which represents the boundary condition $Cu=KBu$ is closed and satisfies

$$(6.21)\qquad D(\widetilde{A})\subset H^{2m}(\Omega)\ \textit{alg. and top.},$$

and

$$(6.22)\qquad \{u\in D(\widetilde{A})\mid Au\in H^{t}(\Omega)\}\subset H^{2m+t}(\Omega)\ \textit{alg. and top.},$$

where $\{u\in D(\widetilde{A})\mid Au\in H^{t}(\Omega)\}$ *is provided with the norm* $(|u|_0^2+|Au|_t^2)^{\frac{1}{2}}$.

The adjoint $\widetilde{A}^*$ is exactly the realization of A' which represents the pure condition $C'u=K'B'u$, and it satisfies (6.21) *and* (6.22) *with A replaced by A'.*

PROOF: Follows from Lemma 6.5 and 6.6 and Theorem 2.1 and 3.2, in a way similar to the proof of Theorem 6.1.

COROLLARY 6.2. *Assumptions of Theorem* 6.2. *The boundedness of Γ implies that $\widetilde{A}$ and $\widetilde{A}^*$ are Fredholm operators.*

PROOF: Analogous to the proof of Corollary 6.1.

For the last theorem we operate with a fixed set of norms in the spaces $H^r(\Gamma)$ (r real).

LEMMA 6.7. *Assume that all four systems* $\{A,B\}$, $\{A',B'\}$, $\{A,C\}$ *and* $\{A',C'\}$ *satisfy the hypotheses* ($\mathcal{E}$) *and* ($\mathcal{U}$) *of Chapter I. Then there exist constants $c_s>0$ such that P satisfies*

$$|P\varphi|_{\{s-\mu_j-\frac{1}{2}\}}\geq c_s|\varphi|_{\{s-m_j-\frac{1}{2}\}},\qquad \textit{all}\ \varphi\in \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$$

for all real s (Theorem 5.2).

Let K be an operator in $(\mathcal{D}'(\Gamma))^m$ for which there exist constants $\varepsilon_s > 0$ such that K satisfies

$$|K\varphi|_{\{s-\mu_j-\frac{1}{2}\}} \leq (c_s - \varepsilon_s)\,|\varphi|_{\{s-m_j-\frac{1}{2}\}}, \qquad \textit{all } \varphi \in \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$$

for $s = 0$ and $s = 2m$. Then

$$D(L) = \Pi H^{2m-m_j-\frac{1}{2}}(\Gamma) \textit{ alg. and top.},$$

and L maps $D(L)$ $1-1$ onto $\Pi H^{2m-\mu_j-\frac{1}{2}}(\Gamma)$.

Let the adjoint K' be defined as in Lemma 6.2. Then L^* is determined by (6.9) and satisfies

$$D(L^*) = \Pi H^{\mu_j+\frac{1}{2}}(\Gamma) \textit{ alg. and top.},$$

it maps $D(L^*)$ $1-1$ onto $\Pi H^{m_j+\frac{1}{2}}(\Gamma)$.

PROOF: It follows from the assumptions that

$$|(K-P)\varphi|_{\{s-\mu_j-\frac{1}{2}\}} \geq \varepsilon_s |\varphi|_{\{s-m_j-\frac{1}{2}\}}, \qquad \text{all } \varphi \in \Pi H^{s-m_j-\frac{1}{2}}(\Gamma),$$

for $s = 0$ and $s = 2m$, so that $K - P$ is an isomorphism of $\Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ onto $\Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$ for $s = 0$ and $s = 2m$. By interpolation (Theorem I. 1.1) the same holds for all $s \in [0, 2m]$.

Let L be defined by (6.2). Lemma 6.1 shows that L is closed. From the fact that $K - P$ maps $\Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$ isomorphically onto $\Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$ for all $s \in [0, 2m]$ follows immediately that

$$D(L) = \Pi H^{2m-m_j-\frac{1}{2}}(\Gamma) \quad \text{alg. and top.}$$

and that L maps $D(L)$ $1-1$ onto $\Pi H^{2m-m_j-\frac{1}{2}}(\Gamma)$.

To find L^* we use that the adjoint of $(K-P)_s : \Pi H^{s-m_j-\frac{1}{2}}(\Gamma) \to \Pi H^{s-\mu_j-\frac{1}{2}}(\Gamma)$ is exactly $(K' \dot{-} P')_{2m-s} : \Pi H^{-s+\mu_j+\frac{1}{2}} \to \Pi H^{-s+m_j+\frac{1}{2}}(\Gamma)$ for $s \in [0, 2m]$, again isomorphisms. The rest of the lemma is then easily shown.

THEOREM 6.3. *Assume that all four systems $\{A, B\}$, $\{A', B'\}$, $\{A, C\}$ and $\{A', C'\}$ satisfy the hypotheses $(\mathcal{C})$ and $(\mathcal{U})$. Let*

$$c_s = \inf \{ \mid P\varphi \mid_{\{s-\mu_j - \frac{1}{2}\}} \mid \varphi \in \Pi H^{s-m_j-\frac{1}{2}}(\Gamma), \mid \varphi \mid_{\{s-m_j-\frac{1}{2}\}} = 1 \},$$

and let K be an operator in $(\mathcal{D}'(\Gamma))^m$ for which there exist constants $\varepsilon_s > 0$ such that

$$\mid K\varphi \mid_{\{s-\mu_j-\frac{1}{2}\}} \leq (c_s - \varepsilon_s) \mid \varphi \mid_{\{s-m_j-\frac{1}{2}\}} \qquad \text{all } \varphi \in \Pi H^{s-m_j-\frac{1}{2}}(\Gamma)$$

for $s = 0$, $s = 2m$. Let K' be the adjoint of K according to Lemma 6.2.

Then the realization $\widetilde{A}$ of A which represents the boundary condition $Cu = KBu$ satisfies

$$D(\widetilde{A}) \subset H^{2m}(\Omega) \quad \textit{alg. and top.} \tag{6.23}$$

and $0 \in \varrho(\widetilde{A})$.

Moreover, $\widetilde{A}^*$ is the realization of A' which represents the boundary condition $C'u = K'B'u$,

$$D(\widetilde{A}^*) \subset H^{2m}(\Omega) \quad \textit{alg. and top.} \tag{6.24}$$

and $0 \in \varrho(\widetilde{A}^*)$.

PROOF: Follows easily from Lemma 6.7, using Theorems 2.1, 2.4 and 3.2.

COROLLARY 6.3. *Assumptions of Theorem 6.3. The boundedness of Ω implies that the inverses $\widetilde{A}^{-1}$ and $(\widetilde{A}^*)^{-1}$ are compact operators.*

PROOF: Follows from (6.23), using that the imbedding of $H^{2m}(\Omega)$ into $H^0(\Omega)$ is compact.

APPENDIX. Preliminaries for Chapter II.

All operators considered are linear.

Let S be an operator with domain $D(S)$ in Hilbert space K (norm $\mid u \mid_K$) and range $R(S)$ in a Hilbert space H (norm $\mid u \mid_H$). We say that S in an operator from K into H and write in short: $S: K \rightarrow H$.

The nullspace $Z(S)$ of S is defined by

$$Z(S) = \{ u \in D(S) \mid Su = 0 \}.$$

The graph $G(S)$ of S is defined as the subset of $K \times H$ determined by

$$G(S) = \{[u, Su] \mid u \in D(S)\}.$$

$K \times H$ is a Hilbert space with the hermitian norm

$$| [u, v] |_{K \times H} = (| u |_K^2 + | v |_H^2)^{\frac{1}{2}} .$$

If $G(S)$ is closed in $K \times H$ we say that S is closed; if $\overline{G(S)}$ is the graph of an extension of S we say that S is closable, the extension is then called the closure of S and is denoted by $\bar{S}$.

The norm $| u |_S = (| u |_K^2 + | Su |_H^2)^{\frac{1}{2}}$, defined in $D(S)$, is called the graphnorm (with respect to S). When S is closed, $D(S)$ is a Hilbert space under the graphnorm.

Recall that the adjoint $S^* \colon H \longrightarrow K$ can be defined if and only if $\overline{D(S)} = K$. Moreover, when $\overline{D(S)} = K$, S is closable if and only if $\overline{D(S^*)} = H$, and in the affirmative case $(\bar{S})^* = S^*$, and $S^{**} \colon K \longrightarrow H$ exists and equals $\bar{S}$.

When K is a closed subspace of H, S can be considered as operator in H. We then define

$$m(S) = \inf \{ \operatorname{Re} (Su, u) \mid u \in D(S), | u | = 1 \} \geq -\infty. \tag{A.1}$$

If $m(S)$ is $> 0, \geq 0, > -\infty$ or $= -\infty$, we will say that S is positive, nonnegative, lower bounded or unbounded below, respectively. (Note that in this definition, (Su, u) is not required to be real for all $u \in D(S)$). With this definition, S is dissipative in the sense of Phillips [25] if and only if $-S$ is nonnegative.

We also define the numerical range $\nu(S)$ of S.

$$\nu(S) = \{ (Su, u) \mid u \in D(S), | u | = 1 \}. \tag{A.2}$$

The closure of the numerical range $\nu(S)$ is denoted by $\bar{\nu}(S)$.

When $K = H$, the resolvent set $\varrho(S)$ of S is defined by

$$\varrho(S) = \{ \lambda \in \mathbb{C} \mid (\lambda - S)^{-1} \text{ exists and is bounded, everywhere defined in } H \},$$

and the spectrum $\sigma(S)$ of S is

$$\sigma(S) = \mathbb{C} \setminus \varrho(S), \text{ the complement of } \varrho(S) \text{ in } \mathbb{C}.$$

DEFINITION A.1. *An operator S in H will be said to be maximal non-negative if it is non-negative and has no proper nonnegative extension.*

This means that $-S$ is maximal dissipative, following Phillips [25]; we have chosen the present terminology in order to define maximal positive and maximal lower bounded operators as well (Definition A.2 later).

Closed maximal dissipative operators are of special interest because they constitute exactly the class of infinitesimal generators of strongly continuous semigroups of contraction operators (see Phillips]25]). It was proved by Hille, Yosida and others (see Hille-Phillips [16]), that a necessary and sufficient condition for an operator T in H to be the infinitesimal generator of a strongly continuous semigroup of contractions is that it satisfies

(i) T is densely defined and closed

(ii) $\sigma(T)$ is contained in the halfplane $\{\operatorname{Re}\lambda \leq 0\}$

(iii) $\|(\lambda - T)^{-1}\| \leq (\operatorname{Re}\lambda)^{-1}$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$.

Phillips proved in [25] the following statements for dissipative operators :

1º When T is maximal dissipative then T is closed if and only if T is densely defined.

2º If T and T^* are adjoints then T is maximal dissipative if and only if T^* is maximal dissipative.

3º If T and T^* are adjoints and both are dissipative then both are maximal dissipative.

Transforming the preceding characterizations into our terminology, we get

PROPOSITION A.1. *Let S be a closed operator in H. Then the following statements are equivalent:*

(i) *S is maximal nonnegative*

(ii) *S is densely defined and S^* is maximal nonnegative*

(iii) *S is densely defined and S and S^* are nonnegative*

(iv) $\sigma(S) \subset \{\operatorname{Re}\lambda \geq 0\}$, $\|(\lambda - S)^{-1}\| \leq |\operatorname{Re}\lambda|^{-1}$ *for all* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda < 0$.

In the description of maximal nonnegative operators, the numerical range $\nu(S)$ (defined by (A. 2)) can be useful. $\nu(S)$ was investigated by Stone ([30], Chapter IV). He proved the following statements :

LEMMA A.1.

(i) *For any linear operator* S, $\nu(S)$ *is convex.*

(ii) *If the distance between* λ *and* $\nu(S)$ *equals* $d > 0$, *then* $\lambda - S$ *has an inverse with* $\|(\lambda - S)^{-1}\| \leq d^{-1}$.

(iii) *Let* S *be closed, and let* $\lambda_0 \in \varrho(S)$ *with* $\|(\lambda_0 - S)^{-1}\| \leq d_0^{-1}$ *for some* $d_0 > 0$. *Then every point in the circle* $|\lambda - \lambda_0| < d_0$ *belongs to* $\varrho(S)$.

Since $\overline{\nu}(S)$ is a convex subset of $\mathbb{C}$, $\mathbb{C} \setminus \overline{\nu}(S)$ has 0, 1 or 2 components (in the case of two components, $\overline{\nu}(S)$ is a parallelstrip). By successive applications of Lemma A.1, one obtains

LEMMA A.2. *Let* S *be closed. If one point of a component of* $\mathbb{C} \setminus \overline{\nu}(S)$ *belongs to* $\varrho(S)$, *then all of that component is in* $\varrho(S)$.

We can now add the following two descriptions to Proposition A.1, noting that an operator S is nonnegative if and only if $\nu(S) \subset \{\mathrm{Re}\,\lambda \geq 0\}$:

PROPOSITION A.2. *Let* S *be closed. The following statements are equivalent with the statements* (i) - (iv) *of Proposition* A.1 :

(v) $\sigma(S)$ *and* $\nu(S)$ *are contained in* $\{\mathrm{Re}\,\lambda \geq 0\}$

(vi) $\nu(S)$ *is contained in* $\{\mathrm{Re}\,\lambda \geq 0\}$ *and one point of* $\{\mathrm{Re}\,\lambda < 0\}$ *is in* $\varrho(S)$.

PROOF: Follows easily from Proposition A.1, Lemma A.1 and Lemma A.2.

COROLLARY A.2. *A closed operator* S *is maximal symmetric if and only if* iS *and* $-iS$ *are nonnegative, and one of them is maximal nonnegative.*

PROOF: Uses Proposition A.2 (vi) and the fact that S is maximal symmetric if and only if S is symmetric and either i or $-i$ belongs to $\varrho(S)$.

REMARK A.1. If S is maximal normal, $\overline{\nu}(S)$ *equals* the closed convex hull of $\sigma(S)$. (The definition of maximal normal operators and the proof of this theorem, are given in Stone [30]. Such operators are by some authors just called normal). Then S is maximal nonnegative if and only if $\sigma(S) \subset \{\mathrm{Re}\,\lambda \geq 0\}$.

We can now prove

LEMMA A.3. *Let* S *and* S^* *be adjoints. Let* $\lambda \in \mathbb{C} \setminus \overline{\nu}(S)$. *Then* $\lambda \in \varrho(S)$ *if and only if* $\overline{\lambda} \in \mathbb{C} \setminus \overline{\nu}(S^*)$.

PROOF: Since $\overline{\nu}(S)$ is convex, λ has a positive distance from a closed halfplane containing $\overline{\nu}(S)$. By a rotation $e^{i\theta}$ we can carry the halfplane into a « positive halfplane » $\{\operatorname{Re} \xi \geq \xi_0\}$ containing $\overline{\nu}(e^{i\theta} S)$ such that $\operatorname{Re} e^{i\theta} \lambda < \xi_0$.

1) If $\lambda \in \varrho(S)$ then $\operatorname{Re} e^{i\theta} \lambda \in \varrho(e^{i\theta} S)$, so that, by Proposition A. 2 (vi), $e^{i\theta} S - \xi_0$ is maximal nonnegative. By Proposition A. 1 (ii) the adjoint $e^{-i\theta} S^* - \xi_0$ is nonnegative, which implies that $\overline{\nu}(e^{-i\theta} S^*) \subset \{\operatorname{Re} \xi \geq \xi_0\}$. Since $\operatorname{Re} e^{-i\theta} \overline{\lambda} = \operatorname{Re} e^{i\theta} \lambda < \xi_0$, $e^{-i\theta} \overline{\lambda} \in \mathbb{C} \setminus \overline{\nu}(e^{-i\theta} S^*)$. Thus $\overline{\lambda} \in \mathbb{C} \setminus \overline{\nu}(S^*)$.

2) If $\overline{\lambda} \in \mathbb{C} \setminus \overline{\nu}(S^*)$, both $\lambda - S$ and $\overline{\lambda} - S^*$ have bounded inverses (Lemma A.1 (ii)). Since S is closed, $R(\lambda - S)$ is closed. Therefore

$$H = R(\lambda - S) \oplus Z(\overline{\lambda} - S^*),$$

which equals $R(\lambda - S)$, since $Z(\overline{\lambda} - S^*) = \{0\}$.

Thus $\lambda \in \varrho(S)$.

If $\varkappa$ is a subset of $\mathbb{C}$ we will denote the conjugate set $\{\overline{\lambda} \mid \lambda \in \varkappa\}$ by $\varkappa'$.

PROPOSITION A.3. *Let S and S^* be adjoints. Let $\varkappa$ be a component of $\mathbb{C} \setminus \overline{\nu}(S)$, form which $\varkappa \cap \varrho(S) \neq \Phi$. Then $\varkappa \subset \varrho(S)$, and $\varkappa' = \{\overline{\lambda} \mid \lambda \in \varkappa\}$ satisfies: $\varkappa'$ is a component of $\mathbb{C} \setminus \overline{\nu}(S)$, and $\varkappa' \subset \varrho(S^*)$.*

PROOF: By Lemma A.2, $\varkappa \subset \varrho(S)$. Applying Lemma A.3 to every point of $\varkappa$ we obtain that $\varkappa' \subset \mathbb{C} \setminus \overline{\nu}(S^*)$, and that $\varkappa' \subset \varrho(S^*)$. Let $\varkappa_1$ be the component of $\mathbb{C} \setminus \overline{\nu}(S^*)$ containing $\varkappa'$. We can now apply an argumentation to $\varkappa_1$ similar to the above, to obtain that $\varkappa_1' \subset \varkappa$. Thus altogether $\varkappa' = \varkappa_1$.

COROLLARY A.3. *Let S and S^* be adjoints, and let $\overline{\nu}(S) \neq \mathbb{C}$, $\overline{\nu}(S)$ not equal to a halfplane. If $\sigma(S) \subset \overline{\nu}(S)$ (this holds if merely one point in each component of $\mathbb{C} \setminus \overline{\nu}(S)$ is in $\varrho(S)$) then*

$$\sigma(S^*) \subset \overline{\nu}(S^*) = \overline{\nu}(S)' \; (= \{\overline{\lambda} \mid \lambda \in \overline{\nu}(S)\}).$$

PROOF: The corollary follows by consideration of the possible convex sets in the plane: If $\overline{\nu}(S)$ is not a halfplane there are the two possibilities: 1) $\overline{\nu}(S)$ is parallel strip, 2) $\overline{\nu}(S)$ contains no parallel-strip.

In the first case $\mathbb{C} \setminus \overline{\nu}(S)$ has two components which are both in $\varrho(S)$. Each of these are by conjugation carried into a component of $\mathbb{C} \setminus \overline{\nu}(S^*)$ by Proposition A.3; then since $\mathbb{C} \setminus \overline{\nu}(S^*)$ has at most two components, $\mathbb{C} \setminus \overline{\nu}(S)' = \mathbb{C} \setminus \overline{\nu}(S^*)$. Proposition A.3 now also implies that $\mathbb{C} \setminus \overline{\nu}(S^*) \subset \varrho(S^*)$, so $\sigma(S^*) \subset \overline{\nu}(S^*)$.

In the second case, there is no convex subset of $\bar{\nu}(S)'$ whose complement has two components, so the component $\mathbb{C} \setminus \bar{\nu}(S)'$ of $\mathbb{C} \setminus \bar{\nu}(S^*)$ must equal the whole set $\mathbb{C} \setminus \bar{\nu}(S^*)$. Again we obtain $\mathbb{C} \setminus \bar{\nu}(S^*) \subset \varrho(S^*)$, so $\sigma(S^*) \subset \bar{\nu}(S^*)$.

REMARK A.2. To illustrate the assumption in Corollary A.3 that $\bar{\nu}(S)$ is not a halfplane, we mention the following: If S is the adjoint of a maximal symmetric operator, and $\bar{\nu}(S)$ is equal to a halfplane, then $\sigma(S) \subset \bar{\nu}(S)$, but $\mathbb{C} \setminus \bar{\nu}(S^*) \neq \mathbb{C} \setminus \bar{\nu}(S)'$.

DEFINITION A.2. *An operator S in H will be said to be maximal lower bounded if it is lower bounded and has no proper lower bounded extension.*

An operator S in H will be said to be maximal positive if it is positive and has no proper positive extension.

For closed operators one has the following equivalent description which makes it easier to verify that an operator satisfies Definition A. 2 (which is most in accordance with the concept of maximality):

PROPOSITION A.4. *Let S be closed.*

(i) *S is maximal lower bounded if and only if there exists $c \in \mathbb{R}$ so that $S - c$ is maximal nonnegative.*

(ii) *S is maximal positive if and only if there exists $c > 0$ so that $S - c$ is maximal nonnegative.*

In the affirmative cases in (i) *and* (ii), *c can be chosen equal to $m(S)$.*

PROOF :

(i) Let S be maximal lower bounded. Then $S - m(S)$ is nonnegative and has no proper lower bounded extension ; thus it is in particular maximal nonnegative.

Conversely, let c be a real number for which $S - c$ is maximal nonnegative. By Proposition A.1 (iv), $\lambda - (S - c)$ maps $D(S)$ $1 - 1$ onto H for all $\lambda > 0$. Then if S_1 is a proper extension of S, $\lambda - (S_1 - c)$ is not $1 - 1$ for $\lambda > 0$. Therefore S_1 cannot be lower bounded, for in that case $\mu + m(S_1) - S_1$ would be $1 - 1$ for $\mu > 0$. This shows that S is maximal lower bounded.

(ii) Let S be maximal positive. Then $m(S) > 0$. Let $T = S - \frac{1}{2} m(S)$; T is positive and has no proper extension T_1 with $m(T_1) > -\frac{1}{2} m(S)$; thus in particular T is maximal nonnegative. It follows from (i) that T is maximal lower bounded, and then from (i) that $T - m(T)$ is maximal nonnegative. Using that $T = S - \frac{1}{2} m(S)$ we now get that S is maximal lower bounded, and

$$S - \frac{1}{2} m(S) - m\left(S - \frac{1}{2} m(S)\right) = S - \frac{1}{2} m(S) - \frac{1}{2} m(S) = S - m(S)$$

is maximal nonnegative.

Conversely, let $S - c$ be maximal nonnegative for some $c > 0$. Since $S - c$ is nonnegative, S is positive. By (i), S is maximal lower bounded, thus it has in particular no proper positive extension, i. e., S is maximal positive.

COROLLARY A.4. *Let S be closed. S is maximal nonnegative [maximal positive] if and only if S is maximal lower bounded and $m(S) \geq 0$ $[m(S) > 0]$.*

COROLLARY A.5. *Let S be closed. Then S is maximal positive / maximal nonnegative / maximal lower bounded, if and only if S is densely defined and S and S^* are both positive / nonnegative / lower bounded, respectively.*

PROOF: Follows from Proposition A.1 (iii) and Proposition A.4.

BIBLIOGRAPHY

[1] S. AGMON, A. DOUGLIS and L. NIRENBERG : *Estimates near the boundary of solutions of elliptic partial differential equations satisfying general boundary conditions I.* Comm. Pure Appl. Math. **12** (1959), 623-727.

[2] M. S. AGRANOVICH and A. S. DYNIN : *General boundary value problems for elliptic systems in an n-dimensional domain.* Dokl. Akad. Nauk SSSR **146** (1962), 511-514; Soviet Math. **3** (1962), 1323-1327.

[3] N. ARONSZAJN and A. N. MILGRAM : *Differential operators on Riemannian manifolds.* Rend. Circolo Mat. Palermo **2** (1952), 1-61.

[4] W. G. BADE and R. S. FREEMAN : *Closed extensions of the Laplace operator determined by a general class of boundary conditions.* Pacific J. Math. **12** 1962, 395-410.

[5] R. W. BEALS : *Non-local boundary value problems for elliptic partial differential operators.* Dissertation, Yale University 1964.

[6] R. W. BEALS : *Non-local boundary value problems for elliptic operators.* Amer. J. Math. **87** (1965), 315-362.

[7] M. S. BIRMAN : *On the theory of self-adjoint extensions of positive definite operators.* Mat. Sbornik **38** (80) (1956), 431-450.

[8] M. S. BIRMAN : *Perturbations of the continuous spectrum of a singular elliptic operator by varying the boundary and the boundary conditions.* Vestnik Leningrad Univ. **17**, no. 1 (1962), 22-55.

[9] J. W. CALKIN : *Symmetric transformations in Hilbert space.* Duke Math. Journal **7** (1940), 504-508.

[10] J. ERCOLANO and M. SCHECHTER : *Spectral theory for operators generated by elliptic boundary problems with eigenvalue parameter in boundary conditions (I).* Comm. Pure Appl. Math. **18** (1965), 83-106.

[12] R. S. FREEMAN : *Closed operators and their adjoints associated with elliptic differential operators.* Pacific J. Math. **22** (1967). 71-97.

[13] K. O. FRIEDRICHS : *Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren.* Math. Ann. **109** (1934), 465-487.

[14] K. O. FRIEDRICHS : *On the differentiability of the solutions of linear elliptic differential operators.* Comm. Pure Appl. Math. **6**, (1953), 299-326.

[15] G. GRUBB : *A characterization of the non-local boundary value problems associated with an elliptic operator.* Dissertation, Stanford University, 1966.

[16] E. HILLE and R. S. PHILLIPS: *Functional Analysis and Semi Groups.* Revised edition, Amer. Math. Soc. Colloq. Publ. **31** (1957).

[17] L. HÖRMANDER : *Definitions of maximal differential operators.* Arkiv för Mat. **3** (1958), 501-404.

[18] L. HÖRMANDER : *Linear Partial Differential Operators.* Springer-Verlag 1963.

[19] L. HÖRMANDER : *Pseudo differential operators.* Comm. Pure Appl. Math. **18** (1965), 501-518.

[20] T. KATO : *Perturbations of continuous spectra by trace class operators.* Proc. Japan Acad. **33** (1957), 260-264.

[21] J. J. KOHN and L. NIRENBERG : *An algebra of pseudo differential operators.* Comm. Pure Appl. Math. **18** (1965), 269-308

[22] M. G. KREIN: *Theory of selfadjoint extensions of symmetric semi-bounded operators and applications (I).* Mat. Sbornik **20** (62) (1947), 431-495.

[23] J. L. LIONS: *Equations différentielles opérationnelles et problèmes aux limites.* Springer-Verlag 1961.

[24] J. L. LIONS and E. MAGENES: *Problemi ai limiti non omogenei*; (I), (III), (IV), (V): Annali Scuola Norm. Pisa, **14** (1960), 259-308; **15** (1961), 39-101, 311-326; **16** (1962), 1-44; - *Problèmes aux limites non homogènes*; (II): Annales Inst. Fourier (Grenoble) **11** (1961), 137-178; (VI): J. d'Analyse Math. **11** (1963), 165-188.

[24′] J. L. LIONS and E. MAGENES: *Problèmes aux limites non homogènes et applications, Vol. I* Ed. Dunod. Paris 1968.

[25] R. S. PHILLIPS: *Dissipative operators and hyperbolic systems of partial differential equations.* Trans. Amer. Math. Soc. **90** (1959), 193-254.

[26] F. RIESZ and B. SZ.-NAGY: *Lecons d'analyse fonctionnelle.* Académie des Sci. de Hongrie, Akadémiai Kradó, Budapest 1953.

[27] M, ROSENBLUM: *Perturbations of the continuous spectrum and unitary equivalence.* Pacific J. Math. **7** (1957), 997-1010.

[28] M. SCHECHTER: *General boundary value problems for elliptic differential equations.* Comm. Pure Appl. Math. **12** (1959), 457-486.

[29] M. SCHECHTER: *Nonlocal elliptic boundary value problems.* Ann. Scuola Norm. Sup. Pisa **20** (1966), 421-441.

[30] M. H. STONE: *Linear Transformations in Hilbert Space and their Applications to Analysis.* Amer. Math. Soc. Colloq. Publ. **15** (1932) (reprinted 1951).

[31] S. V. USPENSKII: *Properties of the classes W_p^r with a frational derivative on differential manifolds.* Dokl. Akad. Nauk **132** (1960), 60-62.

[32] I. M. VISHIK: *On general boundary value problems for elliptic differential equations.* Trudy Mosc. Mat. Obsv. **1** (1952), 187-246; Amer. Math. Soc. Transl. (2) **24**, (1963), 107-172

UNIVERSITY OF COPENHAGEN

The main part of the work was carried out while the author was supported by NSF Grant GP-5841.