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**A priori estimates for hypoelliptic differential equations in a half-space**

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# A PRIORI ESTIMATES FOR HYPOELLIPTIC DIFFERENTIAL EQUATIONS IN A HALF-SPACE

by LEIF ARKERYD

## 0. Introduction.

Our aim is to show that every distribution solution  $u$  of a formally hypoelliptic partial differential equation

$$\mathcal{A}u = f \text{ in } R_+^n,$$

satisfying Dirichlet's boundary conditions

$$D_1^j u = 0, j = 0, \dots, l \text{ on } R^{n-1},$$

does belong to  $C^\infty$ , if  $f$  does. In analogy with the elliptic case (cf. Arkeryd [1]), it is natural to try to obtain à priori estimates

$$(0.1) \quad N_1(u) \leq CN_2(\mathcal{A}u) + N_3(u)$$

with suitable norms  $N_1, N_2, N_3$ , with in particular  $N_3$  « weaker » than  $N_1$ . These estimates are proved in two steps:

1°. The inequality (0.1) is established for operators with constant coefficients.

2°. For operators

$$\mathcal{A} = A + \sum a_j Q_j,$$

where  $A$  and  $Q_j$  have constant coefficients,  $Q_j$  is weaker than  $A$  and  $a_j \in C^\infty$ , the inequality (0.1) can be obtained from the constant coefficient case 1° if

$$N_2(aQ_j u) \leq C \sup |a| N_1(u) + C' N_3(u).$$

In Peetre [8] (see also Schechter [9] and Matsuzawa [7])

$$N_2(u) = \left( \int_{R_+^n} |u|^2 dx \right)^{1/2}$$

is considered, but then (0.1) is not true for all formally hypoelliptic operators; the second step does not always work. Here we use instead

$$N_2(u) = \inf \left( \int_{R^n} |A_-^{-1} \tilde{u}|^2 dx \right)^{1/2},$$

if  $A = A_+ \cdot A_-$  is the « canonical » decomposition of  $A$ , with inf taken over all  $\tilde{u} \in S'(R^n)$ , satisfying  $\tilde{u} = u$  in  $R_+^n$ . In the same way we take

$$N_1(u) = \inf \left( \int_{R^n} |A_+ \tilde{u}|^2 dx \right)^{1/2}.$$

Then step 1<sup>0</sup> is immediate (cf. [8], [11]) and the main difficulty is to prove 2<sup>0</sup>. This can be done by use of a commutator lemma analogous to Friedrich's lemma, which follows from the basic estimate

$$\left| \frac{\partial A_-}{\partial \xi_j} \right| \leq C |A_-| |\xi'|^{-\epsilon}, \quad \xi \in R^n, |\xi'| \geq M.$$

Let us mention that Hörmander [4] has proved a regularity theorem for operators with constant coefficients and general boundary conditions. He does not, however, use *a priori* estimates, but explicit formulas for the corresponding Green and Poisson kernels.

The plan of the paper is as follows. Section 1 contains some preliminaries concerning the distribution spaces involved. Section 2 contains the proof of the basic estimate of the Friedrich's type mentioned above. In Section 3 and Section 4 the applications to differential equations are given. Since they are rather routine, we have cut down the exposition to a minimum.

## 1. Spaces $H_{B_+}^+, s$ and $H_{B_-}^+, s$ .

The Fourier transform of an element  $f$  in one of the Schwartz classes  $S$  or  $S'$  (see [10]) is denoted by  $Ff$ , the inverse transform by  $\bar{F}f$ ,  $\bar{F}Ff = f$ .

We take formally

$$Ff(\xi) = \int_{R^n} e^{-ix\xi} f(x) dx$$

and use the notation

$$P(D)f = \bar{F}P Ff,$$

where  $P$  is a function on  $R^n$ . The following functions will often be used;

$$P(\xi) = A(\xi) = \xi_1 + i \left( 1 + \sum_2^n \xi_j^2 \right)^{1/2},$$

$$P(\xi) = A_1(\xi) = \left( 1 + \sum_2^n \xi_j^2 \right)^{1/2}.$$

By

$$A = A(D) = A(D_1, D') = \sum a_\alpha D^\alpha, \quad D^\alpha = (i)^{-|\alpha|} \frac{\partial}{\partial x_{\alpha_1}} \dots \frac{\partial}{\partial x_{\alpha_n}}$$

we denote a hypoelliptic differential operator with constant coefficients and write

$$(1.1) \quad A(\xi) = A(\xi_1, \xi') = a \prod_1^{m_+} (\xi_1 - \varrho_j^+(\xi')) \prod_1^{m_-} (\xi_1 - \varrho_j^-(\xi')) = aA_+ A_-$$

with  $a = a_1 \dots a_n$ . Here  $m_+$  is the number of roots  $\varrho_j^+$  with positive and  $m_-$  the number of roots  $\varrho_j^-$  with negative imaginary part. We require, that  $A$  satisfies the root condition, i. e. that  $m_+$  and  $m_-$  are independent of  $\xi'$  for  $|\xi'| \geq M$ . It is no restriction to take  $a = 1$ . We set

$$(1.2) \quad B_{(\pm)} = \begin{cases} A_{\pm}(\xi) & \text{if } |\xi'| \geq M_1 \\ (\xi_1 \mp i)^{m_{\pm}} & \text{if } |\xi'| < M_1, \end{cases}$$

where the value of  $M_1 \geq M$  will be defined in Section 2. The following norms are used;

$$\|u\| = \left( \int_{R^n} |u(x)|^2 dx \right)^{1/2}, \quad \|u\|_P = \|P(D)u\|, \quad \|u\|_P^\pm = \inf \|\tilde{u}\|_P,$$

where inf is taken over all  $\tilde{u} \in S'$ , whose restrictions to

$$R_+^n = \{x; x_1 > 0\}$$

are equal to  $u$ , and such that

$$P(D)\tilde{u} \in L^2.$$

The notation  $\tilde{u}$  is used below in this sense. Particular norms of this type are

$$\|u\|_{B_+,s}^+ = \|u\|_{A_1^s B_+}^+,$$

$$\|u\|_{B_-^{-1},s}^+ = \|u\|_{A_1^s B_-^{-1}}^+,$$

$$\|u\|_{r,s}^+ = \|u\|_{A_1^r \cdot A_1^s}^+.$$

The corresponding spaces are denoted by

$$H_{B_+,s}^+, H_{B_-^{-1},s}^+ \text{ and } H_{r,s}^+.$$

The space corresponding to  $\|\cdot\|_P$  is denoted by  $H_P$ . Paley-Wiener's theorem gives

$$(1.3) \quad \|u\|_{B_+,s}^+ \approx \left( \int A_1^{2s} (\|F_{x'} u(\cdot, \xi')\|_{B_+(\cdot, \xi')}^+)^2 d\xi' \right)^{1/2},$$

$$(1.4) \quad \|u\|_{B_-^{-1},s}^+ \approx \left( \int A_1^{2s} (\|F_{x'} u(\cdot, \xi')\|_{B_-^{-1}(\cdot, \xi')}^+)^2 d\xi' \right)^{1/2}.$$

The local spaces (cf. 2.5 in [4])

$$(H_{B_+,s}^+)^{\text{loc}}, (H_{B_-^{-1},s}^+)^{\text{loc}} \text{ and } (H_{r,s}^+)^{\text{loc}}$$

correspond to the above spaces. About  $H_{0,s}^+$  we need the following fact, which goes back to Hörmander and Lions [6].

LEMMA 1.1. Let  $c \in C_0^\infty(\overline{K_+^n})$ . Then

$$\|cv\|_{0,s}^+ \leq \sup |c| \|v\|_{0,s}^+ + K_s \|v\|_{0,s-1}^+$$

for all  $v \in H_{0,s}^+$ , and with the constant  $K_s$  independent of  $v$ .

Next we state some lemmas in  $H_{A_1^s B_+}^+$ .

LEMMA 1.2.  $C_0^\infty$  is dense in  $H_{A_1^s B_+}$ .

PROOF: We prove in Section 2, that (2.4)

$$|B_-(\xi) - B_-(\xi + \eta)| \leq C(1 + |\eta|^2)^{k/2} |B_-(\xi)|,$$

for all  $\xi, \eta \in R^n$ . Here and below constants are written  $C$  and  $K$ , sometimes with index. As the same inequality holds for  $B_+$ , it follows that

$$|B_+(\xi + \eta)| \leq C'(1 + |\eta|)^{k/2} |B_+(\xi)|,$$

and consequently

$$|A_1^s(\xi' + \eta') B_+(\xi + \eta)| \leq C'(1 + |\eta|)^{|s|+k/2} |A_1^s(\xi') B_+(\xi)|.$$

But from this inequality follows that  $C_0^\infty$  is dense in  $H_{A_1^s B_+}$ . See [5] Remark p. 36 and Theorem 2.2.1).

We now use Lemma 1.2 to approximate elements of  $H_{A_1^s B_+}$  with support in a half-space.

LEMMA 1.3. Let  $u \in H_{A_1^s B_+}$ ,  $\text{supp } u \subset \overline{R_+^n}$ . Then  $u$  is the limit in  $H_{A_1^s B_+}$  of a sequence  $(u_j)_1^\infty$  of functions

$$u_j \in C_0^\infty(R^n), \text{supp } u_j \subset R_+^n.$$

PROOF. Denote by  $\tau_h$  translation by  $h$  along the  $x_1$ -axis. Then

$$\begin{aligned} \|\tau_h u - u\|_{A_1^s B_+} &= \left( \int |A_1^s B_+|^2 |e^{ih\xi_1} - 1|^2 |Fu|^2 d\xi \right)^{1/2} \leq \\ &\leq 2 \left( \int_{\dot{\mathcal{E}}} |B_+ A_1^s Fu|^2 d\xi \right)^{1/2} + \sup_{\mathcal{E}} |e^{ih\xi_1} - 1| \|u\|_{A_1^s B_+}, \end{aligned}$$

which can be made arbitrarily small by a suitable choice of  $\mathcal{E}$  and  $h$ . As the statement of the lemma is already established implicitly for  $\tau_h u$  by Lemma 1.2, this ends the proof.

REMARK. In Lemma 1.3,  $B_+$  can be replaced by  $B_-^{-1}$  and  $\overline{R_+^n}$  by

$$\overline{R_-^n} = \{x; x_1 \leq 0\}.$$

By definition, that a function  $u \in H_{B_+, s}^+$  has the boundary values

$$(1.5) \quad D_1^j u(0, x') = 0, \quad j = 0, \dots, m_+ - 1,$$

means, that there is a  $\tilde{u} \in H_{A_1^s B_+}$  with

$$\tilde{u} = 0 \text{ for } x_1 < 0, \quad \tilde{u} = u \text{ for } x_1 > 0.$$

Finally we need

**LEMMA 1.4.** A function  $u$  satisfying (1.5) is in  $H_{B_+, s}^+$  if and only if it is in  $H_{B_+, s-1}^+$  and

$$\frac{u(x_1, x' + h') - u(x_1, x')}{|h'|}$$

is bounded in  $H_{B_+, s-1}^+$  independently of  $h' = (h_2, \dots, h_n)$

**PROOF:** The proof is immediate if we notice that with

$$\tilde{u} = u \text{ for } x_1 > 0, \quad \tilde{u} = 0 \text{ for } x_1 < 0,$$

there is a characterization of  $B_+ \tilde{u}$  in  $H_{A_1^s}$  by the same kind of difference quotients.

## 2. A version of Friedrich's lemma.

The derivation of the *a priori* inequality mentioned in Section 0, is for  $m_- > 0$  based on a commutator lemma analogous to Friedrich's lemma (see e. g. [2]), which is established in this section. The proof depends on a number of lemmas, for which we need the following estimates of hypoelliptic polynomials;

$$(2.1) \quad \begin{aligned} |A^\alpha(\xi)/A(\xi)| &\leq C_\alpha |\xi|^{-c|\alpha|} \quad \text{if } \xi \in R^n, |\xi'| \geq M, \\ \left| \frac{\partial A(\xi)}{\partial \xi_j} / A(\xi) \right| &\leq C |\xi|^{-c} \quad \text{if } |\operatorname{Im} \xi_1| \leq C' |\xi'|^c, \xi' \in R^{n-1}, |\xi'| \geq M, \\ |\operatorname{Im} \varrho_j(\xi')| &> C' |\xi'|^c \quad \text{if } |\xi'| \geq M \end{aligned}$$

for some  $c > 0$  and with  $A^a(\xi) = D^a A(\xi)$  (see Hörmander [5]).

LEMMA 2.1. If  $\xi$  belongs to the cylinder  $|\xi'| \geq M$ , then for all  $\nu$

$$\left| \frac{\partial A_-(\xi)}{\partial \xi_\nu} \right| A_-(\xi) \leq K |\xi'|^{-b}.$$

Here  $C$  is independent of  $\xi$  and  $c^2 > b > 0$ .

PROOF. As the coefficients of  $A_-(\xi)$  are analytic in  $|\xi'| \geq M$  (see [3] p. 289-290), the derivatives  $\frac{\partial A_-}{\partial \xi_\nu}$  exist. Cauchy's formula gives

$$\frac{\partial A_-(\xi)}{\partial \xi_\nu} \Big| A_-(\xi) = \frac{1}{2\pi i} \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{\partial}{\partial \xi_\nu} A(\xi_1 - \tau, \xi') \frac{d\tau}{A(\xi_1 - \tau, \xi')\tau}, \quad \min_{1 \leq j \leq m} \text{Im } \rho_j^+ > \varepsilon > 0.$$

Take  $q$  and  $p$  such that

$$qc > 1, \quad \frac{1}{q} + \frac{1}{p} = 1.$$

Then by (2.1) and with  $\varepsilon = C' |\xi'|^c$  we obtain

$$\begin{aligned} \left| \frac{\partial A_-(\xi)}{\partial \xi_\nu} \right| A_-(\xi) &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{C d\sigma}{(|\xi_1 - \sigma| + |\xi'|)^c |\sigma + iC' |\xi'|^c|} \leq \\ &\leq \frac{C}{2\pi} \left( \int_{-\infty}^{\infty} \frac{d\sigma}{(|\xi_1 - \sigma| + |\xi'|)^{cq}} \right)^{1/q} \cdot \left( \int_{-\infty}^{\infty} \frac{d\sigma}{|\sigma + iC' |\xi'|^c|^p} \right)^{1/p} \leq \\ &\leq K |\xi'|^{-c/q}. \end{aligned}$$

The next lemma compares  $A_-(\xi)$  with  $A_-(\xi + \eta)$  for small real  $\eta$ . For technical reasons, we only make that comparison in a cylinder

$$|\xi'| \geq M_1 \geq 2M,$$

with  $M_1$  so large that

$$|\xi'| \leq 2|\xi' + \eta'| \leq 4|\xi'| \quad \text{if } |\eta| \leq |\xi'|^b.$$

This is the constant  $M_1$  mentioned in formula (1.2).



LEMMA 2.2. Take  $\xi$  with  $|\xi'| \geq M_1$  and  $|\eta| \leq |\xi'|^b$ . Then

$$(2.2) \quad |A_-(\xi + \eta)| \leq K' |A_-(\xi)|,$$

$$(2.3) \quad |A_-(\xi + \eta) - A_-(\xi)| \leq C |\eta| |\xi'|^{-b} |A_-(\xi)|,$$

wich  $K'$  and  $C$  independent of  $\xi$  and  $\eta$ .

PROOF. We write

$$\log \frac{A_-(\xi + \eta)}{A_-(\xi)} = \int_0^1 \sum_1^n \eta_j A_-^{-1}(\xi + t\eta) \frac{\partial}{\partial \xi_j} A_-(\xi + t\eta) dt.$$

The integrand can be estimated by Lemma 2.1. The restrictions on  $\eta$  and  $M_1$  then give

$$\begin{aligned} \left| \int_0^1 \eta_j A_-^{-1}(\xi + t\eta) \frac{\partial}{\partial \xi_j} A_-(\xi + t\eta) dt \right| &\leq \\ &\leq K \int_0^1 |\xi'|^b |\xi' + t\eta'|^{-b} dt \leq K2^b, \end{aligned}$$

and so

$$|A_-(\xi + \eta)| \leq |A_-(\xi)| e^{nK2^b} = K' |A_-(\xi)|.$$

The inequality (2.3) follows from

$$\begin{aligned} |A_-(\xi + \eta) - A_-(\xi)| &= \left| \int_0^1 \sum \eta_j \frac{\partial}{\partial \xi_j} A_-(\xi + t\eta) dt \right| \leq \\ &\leq K |\eta| n \int_0^1 |A_-(\xi + t\eta)| |\xi + t\eta|^{-b} dt \leq \\ &\leq K |\eta| \cdot n K' |A_-(\xi)| 2^b |\xi'|^{-b} = C |\eta| |\xi'|^{-b} |A_-(\xi)|. \end{aligned}$$

The estimate that corresponds to (2.3) for  $|\eta| > |\xi'|^b$ , is much more easily obtained.

LEMMA 2.3. If  $|\xi'| \geq M_1$ ,  $|\xi' + \eta'| \geq M_1$  and  $|\eta| \geq |\xi'|^b$ , then

$$|A_-(\xi) - A_-(\xi + \eta)| \leq C |\eta|^d |\xi'|^{-c} |A_-(\xi)|,$$

where  $d$  is independent of  $\xi$  and  $\eta$ .

PROOF. Under the given restrictions on  $\xi$  and  $\eta$ , for some  $a > 0$  the following inequalities hold;

$$|\varrho_j(\xi')| \leq |\xi'|^a \leq |\eta|^{a/b}.$$

$$|\varrho_j(\xi' + \eta')| \leq |\xi' + \eta'|^a \leq C_a |\eta|^{a/b},$$

$$|\operatorname{Im} \varrho_j(\xi')| \geq C |\xi'|^c.$$

Hence

$$\left| \frac{\varrho_j(\xi') - \varrho_j(\xi' + \eta')}{\xi_1 - \varrho_j(\xi')} \right| \leq C \frac{(1 + |\eta|)^{a/b}}{|\xi'|^c},$$

which gives the desired estimate, when inserted into

$$\begin{aligned} \frac{A_-(\xi) - A_-(\xi + \eta)}{A_-(\xi)} &= \frac{\prod_1^{m_-} (\xi_1 - \varrho_j(\xi')) - \prod_1^{m_-} (\xi_1 + \eta_1 - \varrho_j(\xi' + \eta'))}{\prod_1^{m_-} (\xi_1 - \varrho_j(\xi'))} = \\ &= \sum_{j=1}^{m_-} \frac{-\eta_1 - \varrho_j(\xi') + \varrho_j(\xi' + \eta')}{\xi_1 - \varrho_j(\xi')} \prod_{r>j} \frac{\xi_1 + \eta_1 - \varrho_r(\xi' + \eta')}{\xi_1 - \varrho_r(\xi')}. \end{aligned}$$

Recalling from (1.2) that

$$B_-(\xi) = \begin{cases} A_-(\xi), & |\xi'| \geq M_1 \\ (\xi_1 + i)^{m_-}, & |\xi'| < M_1, \end{cases}$$

and using Lemmas 2.2 and 2.3, the main step in the proof of our commutator lemma easily follows.

LEMMA 2.4. There are constants  $k$  and  $C$  independent of  $\xi, \eta \in R^n$ , such that

$$(2.4) \quad \left| \frac{B_-(\xi) - B_-(\xi + \eta)}{B_-(\xi)} \right| \leq C \frac{(1 + |\eta|^2)^{k/2}}{(1 + |\xi' + \eta'|^2)^{b/2}}.$$

PROOF. The points  $\xi$  and  $\xi + \eta$  can be situated inside or outside the cylinder  $|\xi'| = M_1$ . This gives four cases, which are treated separately.

1<sup>o</sup>. By Lemmas 2.2 and 2.3, the inequality (2.4) is fulfilled for  $|\xi'| \geq M_1$  and  $|\xi' + \eta'| \geq M_1$ .

2<sup>o</sup>. For  $|\xi'| \geq M_1$  and  $|\xi' + \eta'| < M_1$ , write the left-hand side of (2.4) as in the proof of Lemma 2.3.

$$\frac{B_-(\xi) - B_-(\xi + \eta)}{B_-(\xi)} = \sum_{j=1}^{m-} \frac{-\eta_1 - \varrho_j(\xi') - i}{\xi_1 - \varrho_j(\xi')} \prod_{\nu > j} \frac{\xi_1 + \eta_1 + i}{\xi_1 - \varrho_\nu(\xi')}.$$

Each factor can be estimated by

$$C' \frac{(1 + |\eta|^2)^{k'}}{(1 + |\xi' + \eta'|^2)^{e/2}}$$

for some  $k'$  and  $C'$ , which obviously implies (2.4).

3<sup>o</sup>. The case  $|\xi'| < M_1$ ,  $|\xi' + \eta'| \geq M_1$  is treated analogously.

4<sup>o</sup>. If  $|\xi'| < M_1$  and  $|\xi' + \eta'| < M_1$  the inequality is well-known.

When  $Q$  is weaker than  $A$ , we have

$$\begin{aligned} \| B^{-1} A_1^s a Q u \| &\leq \\ &\leq \left\| B^{-1} A_1^s (a B_- - B_- a) \frac{Q}{B} B_+ u \right\| + \left\| A_1^s a \frac{Q}{B} B_+ u \right\|. \end{aligned}$$

Because  $A$  is hypoelliptic, we have

$$\left| \frac{Q(\xi)}{B(\xi)} \right| \leq C \quad \text{for } \xi \in R^n$$

(see [5] p. 102). Then the first term on the right side can be estimated by Lemma 2.4 as follows;

$$\begin{aligned} &\left\| B^{-1} A_1^s (a B_- - B_- a) \frac{Q}{B} B_+ u \right\| = \\ &= \left\| B^{-1}(\xi) A_1^s(\xi') \int F a(\eta) (B_-(\xi - \eta) - B_-(\xi)) \frac{Q(\xi - \eta)}{B(\xi - \eta)} B_+(\xi - \eta) F u(\xi - \eta) d\eta \right\| \\ &\leq C \left\| A_1^s(\xi') \int F a(\eta) \frac{A^k(\eta)}{A_1^b(\xi')} B_+(\xi - \eta) F u(\xi - \eta) d\eta \right\| \leq \\ &\leq C \left\| \int F a(\eta) A^k(\eta) A_1^{|s-b|}(\eta) A_1^{s-b}(\xi' - \eta') B_+(\xi - \eta) F u(\xi - \eta) d\eta \right\| \leq \\ &\leq C \int |F a A^k A_1^{|s-b|}| d\eta \| A_1^{s-b} B_+ u \|. \end{aligned}$$

This estimate, together with the inequality

$$\|u\|_{B_+, s-b}^+ \leq \varepsilon \|u\|_{B_+, s}^+ + C_\varepsilon \|u\|_{B_+, s-1}^+$$

of Ehrling-Nirenberg type, gives

**THEOREM 2.1.** *Let  $a \in C_0^\infty(\overline{R_+^n})$ . Then for  $\varepsilon > 0$ ,*

$$\|(aB_- - B_- a)QB_-^{-1}u\|_{B_-^{-1}, s}^+ \leq \varepsilon \|u\|_{B_+, s}^+ + C_\varepsilon \|u\|_{B_+, s-1}^+,$$

with  $C_\varepsilon$  independent of  $u \in H_{B_+, s}^+$ .

### 3. A priori inequalities for hypoelliptic operators.

**THEOREM 3.1.** *Let*

$$\mathcal{L}(x, D) = A(D) + \sum_1^m a_j(x) Q_j(D),$$

where  $A(D)$  is hypoelliptic and  $Q_1(D), \dots, Q_m(D)$  are weaker than  $A(D)$ . If  $a_1(x), \dots, a_m(x) \in C_0^\infty(\overline{R_+^n})$  and

$$\sum_j \sup |a_j(x)| < \varepsilon$$

for some sufficiently small  $\varepsilon > 0$ , then

$$(3.1) \quad \|u\|_{B_+, s}^+ \leq C(\|\mathcal{L}u\|_{B_-^{-1}, s}^+ + \|u\|_{B_+, s-1}^+)$$

for all  $u \in H_{B_+, s}^+$ , satisfying the boundary conditions (1.5).

**PROOF.** We prove the theorem for  $m_- > 0$ . The modifications in the simpler case  $m_- = 0$  are obvious. As Lemma 1.3 shows, it is sufficient to prove the theorem for  $u \in C_0^\infty(\overline{R_+^n})$ . According to a theorem by Peetre ([8], Lemma 4)

$$(3.2) \quad \|F_{x'} u(\cdot, \xi')\|_{A_+(\cdot, \xi)}^+ \leq \|A(\cdot, \xi') F_{x'} u(\cdot, \xi')\|_{A_-^{-1}(\cdot, \xi)}^+.$$

if  $|\xi'| \geq M$  and if  $u \in C_0^\infty(\mathbb{R}_+^n)$ . The proof is based on the Paley-Wiener theorem. We multiply (3.2) by  $A_1^s$  and integrate in  $\xi'$  (cf. (1.3), (1.4)) getting

$$\|u\|_{B_{+,s}}^+ \leq \|Au\|_{B_{-,s}^-}^+ + \sqrt{1+M^2} \|u\|_{B_{+,s-1}}^+.$$

It follows that

$$(3.3) \quad \|u\|_{B_{+,s}}^+ \leq \left\| \left( A + \sum_1^m a_j Q_j \right) u \right\|_{B_{-,s}^-}^+ + \sum_1^m \|a_j Q_j u\|_{B_{-,s}^-}^+ + C \|u\|_{B_{+,s-1}}^+.$$

But

$$\begin{aligned} \|a_j Q_j u\|_{B_{-,s}^-}^+ &\leq \|B_- a_j Q_j B_-^{-1} u\|_{B_{-,s}^-}^+ + \\ &+ \|(a_j B_- - B_- a_j) Q_j B_-^{-1} u\|_{B_{-,s}^-}^+. \end{aligned}$$

The last term can be estimated by use of Theorem 2.1, and in view of Lemma 1.1, we can estimate the first term in the following way;

$$\begin{aligned} \|B_- a_j Q_j B_-^{-1} u\|_{B_{-,s}^-}^+ &\leq \|a_j Q_j B_-^{-1} u\|_{0,s}^+ \leq \\ \sup |a_j| \|Q_j B_-^{-1} u\|_{0,s}^+ + K \|Q_j B_-^{-1} u\|_{0,s-1}^+ &\leq \\ \leq C_j (\sup |a_j| \|u\|_{B_{+,s}}^+ + K \|u\|_{B_{+,s-1}}^+). \end{aligned}$$

Here  $C_j$  is independent of  $u$  and  $a_j$ . We have now proved that

$$\|a_j Q_j u\|_{B_{-,s}^-}^+ \leq (C_j \sup |a_j| + \varepsilon) \|u\|_{B_{+,s}}^+ + C \|u\|_{B_{+,s-1}}^+,$$

which together with (3.3) gives the desired estimate (3.1), if we assume for instance, that  $m\varepsilon + \sum_{j=1}^m C_j \sup |a_j| < 1/2$ .

#### 4. Regularity.

In this section

$$\mathcal{A} = A + \sum a_j Q_j$$

is formally hypoelliptic. Before the main regularity theorem we formulate a result on regularity in the  $x'$ -directions.

**THEOREM 4.1.** *Let  $u \in H_{B_+, r}^+$  for some  $r$  and let  $u$  satisfy (1.5). Define  $\mathcal{A}(x, D)$  as in Theorem 3.1 Then*

$$u \in H_{B_+, s}^+ \text{ if } \mathcal{A}u \in H_{B_-, s}^+.$$

**PROOF.** It is always possible to choose  $r$ , so that  $r = s - \nu$  for some integer  $\nu$ . If  $r \leq s - 1$  then the quotient

$$\frac{u(x_1, x' + h) - u(x_1, x')}{|h|}$$

is bounded in  $H_{B_+, r}^+$  by Theorem 3.1. Then by Lemma 1.4,  $u \in H_{B_+, r+1}^+$ . By iteration, this proves the theorem.

**THEOREM 4.2.** *Let  $u \in D'(\overline{R_+^n})$  and satisfy (1.5) Then*

$$u \in (H_{B_+, s}^+)^{\text{loc}} \text{ if } \mathcal{A}u \in (H_{B_-, s}^+)^{\text{loc}}.$$

**PROOF.** The theorem means that  $\psi u \in H_{B_+, s}^+$  if  $\psi \in C_0^\infty(\overline{R_+^n})$ . It is no restriction to take all  $Q_j$  hypoelliptic and  $\psi$  with «small» support. For each such function  $\psi$ , we take another  $\Phi$  of the same type with  $\Phi = 1$  in a neighbourhood of  $\text{supp } \psi$ . We first show that  $\Phi u \in H_{B_+, r}^+$  for some  $r$ , when  $\text{supp } \Phi$  is small enough for  $\mathcal{A}$  to fulfil the conditions of Theorem 3.1 in some open set  $\omega \supset \text{supp } \Phi$ . From the fact that

$$|B_-| \leq K A^{m_-} A_1^{m_0}$$

for some  $m_0$ , it follows

$$\mathcal{A}u \in (H_{B_-, s-m_0}^+)^{\text{loc}}$$

if

$$\mathcal{A}u \in (H_{B_-, s}^+)^{\text{loc}}.$$

As the  $Q_j$ 's are hypoelliptic, there is a  $d > 0$  such that for large  $\xi'$

$$|Q_j^\alpha / Q_j| \leq |\xi'|^{-d|\alpha|}, \quad |A^\alpha / A| \leq |\xi'|^{-d|\alpha|}.$$

Take

$$\Phi_0 \in C_0^\infty(R_+^n)$$

with  $\Phi_0 = 1$  in a neighbourhood of  $\text{supp } \Phi$  and  $\text{supp } \Phi_0 \subset \omega$ . As  $u \in D'$ , we have

$$\Phi_0 u \in H_{\sigma}^{+ \tau}$$

for some integer  $\sigma$  and real  $\tau$ . If  $\sigma < m_+$  we construct a sequence of  $C_0^\infty(\overline{R_+^n})$ -functions

$$\Phi_0, \Phi_1, \dots, \Phi_\mu = \Phi, \mu = m_+ - \sigma$$

with  $\Phi_{j-1} = 1$  in a neighbourhood of  $\text{supp } \Phi_j$ . Let  $m_1 = m_+ + m_-$  be the order of the derivative  $D_1$  in  $\mathcal{A}$  and  $m'$  the total order. As

$$(D^\alpha \Phi_1)(A^\alpha + \sum a_j Q_j^\alpha) \Phi_0 u \in H_{\sigma-m_1+1, \tau-m'+1}^+ \text{ when } \alpha \neq 0$$

and

$$\Phi_1(A + \sum a_j Q_j) u \in H_{-m_-, s-m_0}^+,$$

Leibniz' formula shows that

$$\begin{aligned} \Phi_1(A + \sum a_j Q_j) u + \sum_{|\alpha| \neq 0} D_\alpha \Phi_1(A^\alpha + \sum a_j Q_j^\alpha) \Phi_0 u &= \\ &= (A + \sum a_j Q_j) \Phi_1 u \in H_{\sigma-m_1+1, \min(\tau-m'+1, s-m_0)}^+. \end{aligned}$$

Then by partial regularity (see e. g. [5]), for some  $\tau'$

$$\Phi_1 u \in H_{\sigma+1, \tau'}^+$$

and so, by iteration, for some  $r'$

$$\Phi u \in H_{m_+, r'}^+.$$

For some  $r$  this will give

$$(4.1) \quad \Phi u \in H_{B_+, r}^+.$$

Take  $r$  so that with  $q = c/b$

$$r = \frac{(s-r)q}{d}$$

is an integer. Let  $(\psi_j)_0^\vee$  be a sequence analogous to  $(\Phi_j)_0^\mu$ , and with

$$\psi_0 = \Phi, \psi_r = \psi.$$

The terms in

$$(A + \sum a_j Q_j) \psi_1 u = \psi_1 (A + \sum a_j Q_j) u + \sum_{|\alpha| \neq 0} D^\alpha \psi_1 (A^\alpha + a_j Q_j^\alpha) \psi_0 u$$

can be estimated as in the proof of Theorem 2.1 and Theorem 3.1. With our choice of  $d$  this gives

$$\| a_j D^\alpha \psi_1 Q_j^\alpha \psi_0 u \|_{B_{-1, r+d/q}^+} \leq K \| \psi_0 u \|_{B_{+, r}^+},$$

$$\| D^\alpha \psi_1 A^\alpha \psi_0 u \|_{B_{-1, r+d/q}^+} \leq K \| \psi_0 u \|_{B_{+, r}^+},$$

and so

$$(A + \sum a_j Q_j) \psi_1 u \in H_{B_{-1, r+d/q}^+}^+.$$

Then by Theorem 4.1

$$\psi_1 u \in H_{B_{+, r+d/q}^+}^+.$$

Repeating this  $\nu$  times gives

$$\Phi u \in H_{B_{+, s}^+}^+,$$

and so

$$u \in (H_{B_{+, s}^+}^+)^{\text{loc}}.$$

CORR. 4.1. If  $\mathcal{L} u \in C^\infty(\overline{R_+^n})$  and  $u \in \mathcal{D}'(\overline{R_+^n})$  satisfies (1.5), then  $u \in C^\infty(\overline{R_+^n})$ .

PROOF. This follows by partial regularity from Theorem 4.2.



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