NEIL S. TRUDINGER

Remarks concerning the conformal deformation of riemannian structures on compact manifolds

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 22, n° 2 (1968), p. 265-274

<http://www.numdam.org/item?id=ASNSP_1968_3_22_2_265_0>

REMARKS CONCERNING THE CONFORMAL DEFORMATION OF RIEMANNIAN STRUCTURES ON COMPACT MANIFOLDS

NEIL S. TRUDINGER (*)

§ 1 Introduction.

Yamabe [8] seeks to establish the following result:

THEOREM: *Any compact C^∞ Riemannian manifold of dimension $n \geq 3$ can be deformed conformally to a C^∞ Riemannian structure of constant scalar curvature.*

The author has recently noticed that the proof in [8] appears incomplete, causing the validity of Yamabe's theorem to be in doubt. The purpose of this paper, apart from indicating this error, is to establish the result itself under, however, some restriction on the curvature of the manifold (Section 3) and also to prove a related regularity result for arbitrary manifolds, namely that weak solutions of the deformation problem are necessarily smooth ones (Section 4).

The partial results of Section 3 are sufficient to show that, in more than 3 dimensions, a structure of constant positive scalar curvature may be topologically deformed into one of constant negative scalar curvature. This had originally been derived by Aubin [1] from Yamabe's theorem. Hopefully, a proof of Yamabe's theorem for arbitrary compact manifolds will be found.

We let M be an n dimensional, C^∞ Riemannian manifold, $n \geq 3$ and let g_{ij} denote its fundamental positive definite tensor. We consider an arbitrary

---

Pervenuto alla Redazione il 28 Nov. 1967.

(*) This research was partially supported by a Ford Foundation Postdoctoral grant at the Courant Institute of Mathematical Sciences, New York University.

conformal deformation of the g_{ij}, writing it as

$$\bar{g}_{ij} = u^{\frac{4}{n-2}} g_{ij} \tag{1}$$

where $u > 0$, $\in C^\infty(M)$.

Let R, $\bar{R}$ denote the scalar curvatures of the g_{ij}, $\bar{g}_{ij}$ respectively. Then according to [8], we have the equation

$$\frac{4(n-1)}{n-2} \Delta u - Ru = -\bar{R} u^{\frac{n+2}{n-2}} \tag{2}$$

where

$$\Delta = \frac{\partial}{\partial x^i}\left(g^{ij} \frac{\partial}{\partial x^i}\right)$$

denotes the Laplace-Beltrami operator corresponding to g_{ij}.

Thus the problem of conformally deforming the structure on M to one of constant scalar curvature is equivalent to proving the existence of a constant $\bar{R}$ and positive, $C^\infty(M)$ function u satisfying equation (2). We take up the study of this equation in the following sections.

The author is grateful to R. Gardner and J. Moser for many useful discussions concerning this problem.

§ 2 Yamabe's approach.

Yamabe [8] considers, instead of equation (2), the equations

$$\frac{4(n-1)}{n-2} \Delta u - Ru = -\lambda_q u^{q-1}, \quad q < \frac{2n}{n-2} = N \tag{3}$$

proving

THEOREM 1. (Yamabe) *For any $q < N$, there exists a constant λ_q and a positive C^∞ function u_q (normalized by $\int_M |u_q|^q dv = 1$)*(1) *satisfying equation* (3).

---

(1) We denote the volume element on M by dv and assume $\int_M dv = 1$.

The solutions λ_q, u_q are characterized by the variational problem of minimizing the functional

(4) $$F(u) = \int_M \left(\frac{2(n-1)}{n-2} \, |\nabla u|^2 + Ru^2 \right) dv \text{ where } |\nabla u|^2 = g^{ij} u_{x_i} u_{x_j}$$

over the class of functions in $W_2^1(M)$ (2) satisfying

$$\int_M |u|^q \, dv = 1.$$

We denote the above class by A_q so that

(5) $$\lambda_q = \inf_{A_q} F(u) = F(u_q).$$

It is readily seen that $|\lambda_q|$ is non-increasing in q and that as $q \rightarrow N$, $\lambda_q \rightarrow \lambda$ where

(6) $$\lambda = \inf_{A_N} F(u).$$

Since the proof in [8] of Theorem 1 (called Theorem B there) is somewhat unnecessarily involved, we insert here a simpler, more direct version. We let $\mu > 0$ denote the ellipticity constant of Δu so that

(7) $$g^{ij} \xi_i \xi_j \geq \mu \, |\xi|^2 \text{ for all } \xi \in E^n \text{ for all points of } M.$$

The proof is split into two stages:

(i) *Existence in* $W_2^1(M)$. We choose a « minimizing sequence », ie a sequence $u^{(n)} \in A_q$ with $F(u^{(n)}) \rightarrow \lambda_q$ as $n \rightarrow \infty$. Since $u^{(n)}$ is bounded in $W_2^1(M)$,

---

(2) The Sobolev spaces $W_p^k(M)$ for positive integer k, and $p \geq 1$ are defined by

$$W_p^k(M) = \{ u \, ; \, D^\alpha u \in L_p(M) \text{ for } |\alpha| \leq k \}$$

and

$$\| u \|_{W_p^k(M)} = \left(\sum_{|\alpha| \leq k} \int_M | D^\alpha u |^p \, dv \right)^{\frac{1}{p}}$$

where $\alpha = \alpha_1 \ldots \alpha_n$ is a multi-index and $D^\alpha u$ is to be understood in the sense of distribution theory.

it is weakly precompact. We show that $u^{(n)}$ is in fact precompact in $W_2^1(M)$. The argument is a standard one in the calculus of variations (cf. [6]). By the Sobolev imbedding theorem, a subsequence of the $u^{(n)}$, (which we immediately relabel as $u^{(n)}$ itself) converges in $L_q(M)$ to a function u_q in A_q. Hence for arbitrary $\varepsilon > 0$ and sufficiently large m, n depending on ε

$$\left\| \frac{u^{(n)} + u^{(m)}}{2} \right\|_{L_q(M)} \geq - \left\| \frac{u^{(n)} - u^{(m}}{2} \right\|_{L_q(M)} + \| u^{(n)} \|_{L_q(M)} \geq 1 - \varepsilon.$$

Thus

$$F\left(\frac{u^{(n)} - u^{(m)}}{2}\right) = - F\left(\frac{u^{(n)} + u^{(m)}}{2}\right) + \frac{1}{2}\left(F(u^{(n)}) + F(u^{(m)})\right)$$

$$\leq \lambda_q + \varepsilon - (1 - \varepsilon)\lambda_q = (1 + \lambda_q)\varepsilon$$

for sufficiently large m, n. It then follows (by (7)) that $u^{(n)}$ converges to u_q in $W_2^1(M)$. We then clearly have $F(u_q) = \lambda_q$ and hence u_q is a solution of the variational problem. Note that we may choose the $u^{(n)} \geq 0$ so that $u_q \geq 0$ a.e. Also the vanishing of the first variation yields the Euler-Lagrange equations for u_q, ie

$$\int\limits_M \left(\frac{4(n-1)}{n-2} g^{ij} u_{qx_i} \xi_j + R u_q \xi \right) dv = \lambda_q \int\limits_M u_q^{n-1} \xi \, dv \tag{8}$$

for all $\xi \in W_2^1(M)$ The function u_q is thus a weak solution of equation (3).

(ii) *Smoothness and positivety.* We quote a result frome lliptic theory.

LEMMA. *Let u be a weak $(W_2^1(M))$, non-negative, solution in M of the linear equation*

$$\Delta u + f u = 0 \tag{9}$$

where $f \in L_r(M)$, $r > n/2$. Then u is positive and bounded and we have the estimates

$$\sup_M |u|, \ |u|^{-1} \leq C(n, \|f\|_{L_r(M)}, \mu, \|u\|_{L_2(M)}).$$

The lemma may be proved in various ways. Yamabe [8] proves and uses similar statements. These are also consequences of Moser's work (see [3], [5], or [7]).

To apply the lemma to the present situation, we observe that u satisfies a linear equation of form (9) with

$$f(x) = \lambda_q u_q^{q-2} - R \in L_r \text{ where } r = N \,|\, (q-2) > n/2.$$

Thus u_q is bounded and positive. Elliptic regularity theory [2] then guarantees that u_q is C^∞.

To establish Yamabe's theorem from Theorem 1 it suffices to show that the functions u_q are equibounded and hence equicontinuous. Such is the purpose of the second part of Yamabe's work ([8], Theorem C). In his proof, however, the inequality (6.2) appears to be in error. The correct form should be the inequality

$$\| v^{(q)} \|_{q_n} \leq [C_5 \| v^{(q)} \|_{q_1}]^{(q-1)^{(n-1)}} \tag{10}$$

which is insufficient for the remainder of the proof to go through. More intuitively speaking, his proof of Theorem C cannot be expected to work since it does not distinguish at all two facts related to the problem:

(i) the presence of the term $-Ru$ in the equation and

(ii) the compactness of M.

Disregarding these factors, one would not expect uniform convergence of a subsequence of the u_q but rather merely convergence in $W^1_{2-\varepsilon}(M)$ for any $\varepsilon > 0$ to the trivial solution, as is the case with the problem

$$\Delta u = -\lambda u^{\frac{n+2}{n-2}}, \quad u \not\equiv 0 \in \overset{0}{W}{}_2^1(\Omega) \tag{11}$$

where Ω is a bounded domain in Euclidean n-space. See [4].

§ 3 Partial results.

We will show in the next section that a subsequence of the u_q converges in a certain sense to a *smooth* solution of equation (2). However the convergence is not strong enough to imply the non-triviality of the resulting solution. We demonstrate in this section that for a large class of manifolds, the convergence is sufficiently nice to guarantee a positive, smooth solution of (2).

THEOREM 2. *There exists a positive constant ε (depending on g^{ij}, R) such that if $\lambda < \varepsilon$, there exists a positive, C^∞ solution of equation (2) with $\bar{R} = \lambda$. Thus Yamabe's theorem is true under this assumption on the metric g^{ij}.*

PROOF. In the equation (8) consider a test function

$$\xi(x) = (u_q)^\beta \qquad \beta > 1.$$

The result is

$$\int_M \left(\frac{4\beta(n-1)}{n-2} g^{ij} u_{qx_i} u_{qx_j} (u_q)^{\beta-1} + R u_q \right) dv = \lambda_q \int_M (u_q)^{\beta-1+q} \, dv$$

and hence from (7)

$$(12) \qquad \int_M u_q^{\beta-1} |\nabla u_q|^2 \, dv \leq \frac{n-2}{4(n-1)\nu\beta} \int_M (\lambda_q u_q^{\beta-1+q} - R u_q^{\beta+1}) \, dv.$$

Writing $w = (u_q)^{(\beta+1)/2}$ the above inequality becomes

$$(13) \qquad \int_M |\nabla w|^2 \, dv \leq C(\nu, n, \beta) \int (\lambda_q w^2 (u_q)^{q-2} - R w^2) \, dv.$$

Let us suppose $\lambda > 0$ and apply the Sobolev and Hölder inequalities. We obtain, thus

$$\| w \|^2_{L_N(M)} \leq C_1(\nu, n, \beta) \lambda_q \| w \|^2_{L_N(M)} \| u_q \|^2_{L_N(M)} + C_2(\nu, n, \beta, \sup |R|) \int_M w^2 \, dv$$

$$\leq C_1 \lambda_q \| w \|^2_{L_N(M)} + C_2 \int_M w^2 \, dv$$

since $\| u_q \|_{L_N(M)}$ is bounded independently of q. Hence if $\lambda < C_1^{-1}$, $\lambda_q < C_1^{-1}$ for large enough q and we obtain

$$(14) \qquad \| w \|^2_{L_N(M)} \leq C \int_M w^2 \, dv.$$

Clearly from (13), this inequality continues to hold if $\lambda \leq 0$.

Now choose $\beta < N - 1$. Then we obtain

$$(15) \qquad \| w \|_{L_N(M)} \leq C(\mu, n, R)$$

and the u_q are subsequently equibounded by the Lemma. A subsequence therefore converges, with its derivatives, uniformly to a smooth solution of (2), which is also positive by the Lemma. Q.E.D.

Note that we have the following bound for λ,

$$\lambda \leq F(1) = \int_M R \, dv \tag{16}$$

and therefore the conclusion of Theorem 2 will hold if $\int_M R < \varepsilon$. As a special case we have then

COROLLARY 1. *Any compact C^∞ Riemannian manifold with non-positive mean* (3) *scalar curvature can be deformed conformally to a C^∞ Riemannian structure of constant scalar curvature.*

Corollary 1 may in fact be derived very simply from Theorem 1. For $\int_M R \, dv \leq 0$ implies $\lambda < 0$. Let the maximum value of u_q, say M_q, be taken on at a point $P \in M$. Then since $\Delta u_q(P) < 0$, we must have

$$-\lambda_q M_q^{q-1} \leq -R(P) M_q$$

and hence

$$M_q \leq (\sup |R| / -\lambda_q)^{\frac{1}{q-2}} \leq 1 + (\sup |R| / -\lambda)^{(n-2)/4} \tag{17}$$

and accordingly the u_q are equibounded.

Aubin [1] has shown that if $n \geq 3$ and R is a positive constant, then g_{ij} may be topologically transformed into a metric $\bar{g}_{ij}$ with scalar curvature $\bar{R}$ satisfying $\int \bar{R} < 0$. Hence as a further corollary we have

COROLLARY 2. *If $n \geq 3$, a structure of positive, constant scalar curvature may be deformed topologically into one of constant negative scalar curvature. Thus the sign of the scalar curvature has no topological significance in more than two dimensions.*

§ 4. A regularity Theorem.

We prove now the following result concerning weak solutions of equation (2).

THEOREM 3. *Let u be a $W_2^1(M)$ solution of an equation of the form* (2). *Then $u \in C^\infty(M)$.*

(3) The mean scalar curvature is the quantity $\int_M R \, dv$.

PROOF. The function u satisfies

$$\int_M \left(\frac{4(n-1)}{n-2} g^{ij} u_i \xi_j + Ru\,\xi\right) dv = \overline{R} \int_M |u|^{N-1} \xi\, dv \tag{17}$$

for all $\xi \in W_2^1(M)$. We choose an appropriate test function ξ similarly to a method of J. Serrin [5]. Define $\overline{u} = \sup(u, 0)$ and for a fixed $\beta > 1$ (to be eventually chosen as in Theorem 2) define the functions

$$G(\overline{u}) = \begin{cases} \overline{u}^{\beta} & \text{if } \overline{u} \leq l \\ l^{q-1}(q l^{q-1} \overline{u} - (q-1) l^q) & \text{if } \overline{u} > l \end{cases}$$

$$F(\overline{u}) = \begin{cases} \overline{u}^{q} & \text{if } \overline{u} \leq l \\ q l^{q-1} \overline{u} - (q-1) l^q & \text{if } \overline{u} > l. \end{cases} \tag{18}$$

where $2q = \beta + 1$.

The function $G(\overline{u})$ is a uniformly Lipshitz continuous function of u and hence belongs to $W_2^1(M)$. Likeuise $F(\overline{u})$. Observe also that G and F vanish for negative u and that

$$(F'(\overline{u}))^2 \leq q\, G'(\overline{u}), \qquad (F(\overline{u}))^2 \geq \overline{u}\, G(\overline{u}). \tag{19}$$

Let us now substitute in (17) test functions

$$\xi = \eta^2 G(\overline{u}) \tag{20}$$

where η is an arbitrary, non-negative $C^1(M)$ function. The result is, using (7),

$$\frac{4(n-1)\nu}{n-2} \int_M \eta_2\, G'(\overline{u})\, u_i^2\, dv \leq \int_M \left(\frac{4(n-1)}{n-2} \sup |g^{ij}|\, \eta\, |\eta_i| + \right.$$
$$\left. + \sup |R|\, u\, \eta^2 + \overline{R}\, u^{N-2} \eta^2 \right) G\, dv$$

and hence

$$\int_M \eta^2\, G'(\overline{u})\, u_i^2\, dv \leq C \int_M \left\{ (|\eta_i|^2 + \eta^2)(\overline{u})\, G(\overline{u}) + \eta^2\, \overline{u}^{N-2}\, \overline{u}\, G(\overline{u}) \right\} dv$$

where $C = C(\nu, n, \sup |g^{ij}, R|, \overline{R})$.

Using (19) we then obtain

$$\int_M \eta^2 F_i^2 \, dv \leq C q \int_M \{(|\eta_i|^2 + \eta^2) F^2 + \bar{u}^{N-2} \eta^2 F^2\} \, dv. \tag{21}$$

Let us take η now to have compact support in a coordinate patch of M. The integrals in (21) may then be replaced by integrals over a sphere S_R in E^n of radius R where $\eta = \eta(x) \in C_0^1(R)$. We choose R so that

$$\int_{S_R} |u|^N \, dv \leq (2Cq)^{-1}.$$

Then applying the Hölder and Sobolev inequalities to (21) we obtain

$$\|\eta F\|_{L_N(S)} \leq Cq \, \|(\eta + |\eta_i|) F\|_{L_2(S)} + \frac{1}{2} \|\eta F\|_{L_N(S)}$$

and hence

$$\|\eta F\|_{L_N(S)} \leq 2Cq \, \|(\eta + |\eta_i|) F\|_{L_2(S)}. \tag{22}$$

We choose β as in Theorem 2, ie $1 < \beta < (n+2)/(n-2)$ so that $2q < N$. Hence we may let $l \to \infty$ in (22) to obtain the estimate

$$\|\eta \bar{u}^q\|_{L_N(S)} \leq C \, \|(\eta + |\eta_i|) \bar{u}^q\|_{L_2(S)}. \tag{23}$$

Let $S_{R/2}$ denote the sphere concentric to S of radius $R/2$ and choose $\eta = 1$ on $S_{R/2}$, $|\eta_i| \leq 2/R$ on S_R. Then we obtain

$$\|\bar{u}^q\|_{L_N(S_{R/2})} \leq C(1 + R^{-1}). \tag{24}$$

Replacing u by $-u$, we obtain (24) also for the function u and employing a partition of unity clearly provides a global estimate

$$\|u\|_{L_r(M)} \leq C, \tag{25}$$

for some $r > N$ where C will also depend on the local L_N norms of u. The boundedness of u and subsequently its smoothness are now consequences of the Lemma. Q. E. D.

For the functions u_q defined in Section 2, we have a uniform estimate for the norms

$$\| u_q \|_{W^{2,\frac{2n}{n+2}}(M)}$$

and hence a subsequence converges in $W^1_{2-\varepsilon}(M)$ to a weak solution of (2). Hence we have

COROLLARY 3. *The functions u_q converge in $W^1_{2-\varepsilon}(M)$ for any $\varepsilon > 0$ to a non-negative $C^\infty(M)$ solution of equation* (2) *with $\bar{R} = \lambda$.*

Yamabe's theorem would thus follow if the non triviality of the above solution u could be demonstrated. This, of course, we have shown, in the preceding section but only under a restriction $\lambda < \varepsilon$ for certain positive ε.

Added in proof (May 1968). Since this paper was written, T. Aubin has found a proof of Yamabe's theorem by using a completely different variational approach.

REFERENCES

[1] T. AUBIN. *Sur la courbure scalaire des variétés riemanniennes compactes*. C. R. Acad. Sci. Paris, Vol. 262, 1966 pp. A 130-133.

[2] L. BERS, F. JOHN and M. SCHECHTER. *Partial differential equations*, Part. II. New York, 1964.

[3] J. MOSER. *On Harnack's theorem for elliptic differential equations*. Comm. Pure Appl. Math., Vol. 14, 1961, pp. 577-591.

[4] S. I. POHOZAEV. *Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$*, Dokl. Akad. Nauk. SSSR, Vol. 165, 1965 pp. 33-36.

[5] J. SERRIN. *Local behaviour of solutions of quasilinear equations*, Acta Math. Vol. 113, 1965 pp. 219-240.

[6] S. L. SOBOLEV. *Some applications of functional analysis in mathematical physics*. Transl. Math. Monographs, Vol. 7, Amer. Math. Soc., Providence, R. I., 1963.

[7] N. S. TRUDINGER. *On Harnack type inequalities and their application to quasilinear elliptic equations*. Comm. Pure Appl. Math., Vol. 20, 1967, pp. 721-747.

[8] H. YAMABE. *On a deformation of Riemannian structures on compact manifolds*. Osaka Math. J., Vol. 12, 1960 pp. 21-37.