ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

LAMBERTO CESARI

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 22, nº 2 (1968), p. 193-227

<http://www.numdam.org/item?id=ASNSP_1968_3_22_2_193_0>

SOBOLEV SPACES AND MULTIDIMENSIONAL LAGRANGE PROBLEMS OF OPTIMIZATION (*)

LAMBERTO CESARI

In [3] and [4] we have proved existence theorems for multidimensional Lagrange problems with control variables and unilateral constraints in a fixed bounded domain of a real Euclidean space E_ν, $\nu > 1$. These theorems are based on known weak compactness properties of Sobolev spaces $W_p^l(G)$ for $l \geq 1$, $p > 1$, and G a fixed bounded open subset of E_ν. In the present paper we shall prove a further existence theorem based on compactness properties of the same Sobolev spaces with $p = 1$. This theorem will extend to multidimensional Lagrange problems the well known Nagumo-Tonelli existence theorem for free problems of the calculus of variations. We have already extended this theorem to unidimensional Lagrange problems in [1, 2].

In the present paper we shall also consider multidimensional Lagrange problems in which the underlying domain is a fixed but unbounded domain in E_ν. We shall extend the previous existence theorems of [3] and [4] as well as the existence theorem of the present paper to cases where the underlying fixed domain is unbounded.

§ 1. A few remarks on weak compactness in L_p for $p \geq 1$.

Weak compactness theorems in spaces $L_p(G)$ with $p \geq 1$ and G a bounded subset of E_ν are well known. Here we state briefly one of these theorems together with some of its extensions to cases where $p = 1$ or G is unbounded. We denote by t the real vector variable $t = (t^1, \dots, t^\nu) \in E_\nu$.

If $z(t)$, $t \in G$, denotes a real-valued function defined on a subset G of E_ν by the notation $z \in L_p(G)$, $p > 1$, we shall mean, as usual, that z is measurable,

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Pervenuto alla Redazione il 14 Settembre 1967.

(*) Research partially supported by US-AFOSR Grant 942-65 at the University of Michigan.

and that $|z|^p$ is L_1-integrable in G. The following statement is well known:

(1.i) Let G be a measurable bounded subset of E_ν, and $z_k(t)$, $t \in G$, $k = 1, 2, \dots$, a sequence of real valued measurable functions such that (α) $z_k \in L_p(G)$, $k = 1, 2, \dots$; (β) $\int_G |z_k|^p \, dt \leq M$, $k = 1, 2, \dots$, for some constants $p > 1$, $M \geq 0$. Then, there is a measurable function $z(t)$, $t \in G$, and a subsequence $[z_{k_s}]$ such that

$$z \in L_r(G), \quad 1 \leq r \leq p; \tag{1}$$

$$\int_G |z|^r \, dt \leq \underline{\lim}_{s \to \infty} \int_G |z_{k_s}|^r \, dt \quad \text{for every} \quad 1 \leq r \leq p; \tag{2}$$

$$\int_G z \, \Phi \, dt = \lim_{s \to \infty} \int_G z_{k_s} \Phi \, dt \tag{3}$$

for every real-valued measurable function $\Phi(t)$, $t \in G$, with $\Phi \in L_q(G)$, $q^{-1} + p^{-1} = 1$. All integrals above are finite.

REMARK 1. This theorem is usually proved as a consequence of general statements of Functional Analysis. Indeed, the space $L_p(G)$ with the usual L_p-norm is a uniformly convex normed space and hence symmetric by remarks of J. A. Clarkson, and consequently any strongly bounded sequence is weakly compact by a theorem of L. Alaoglu concerning weak topologies in normed linear spaces (see E. Rothe, Pacific Math. Journal 3, 1953, 493-499)). Nevertheless, there are direct proofs of statement (1.i) which are based on the remark that hypothesis (β) implies that the functions z_k are equiabsolutely integrable in G, that is, given $\varepsilon > 0$, there is some $\delta = \delta(\varepsilon) > 0$ such that $H \subset G$, H measurable, meas $H < \delta$ implies $\int_H |z_k| \, dt < \varepsilon$, $k = 1, 2, \dots$, (indeed, by Holder's inequality

$$\int_H |z_k| \, dt \leq \left(\int_H 1^q \, dt \right)^{1/q} \left(\int_H |z_k|^p \, dt \right)^{1/p} \leq M^{1/p} \, (\text{meas } H)^{1/q}$$

and it is enough to assume $\delta = \varepsilon^q M^{-q/p}$).

REMARK 2. There is a statement underlying (1.i), namely that, for G bounded, the integrability of $|z|^p$, $p > 1$, implies the integrability of every power $|z|^r$, $1 \leq r \leq p$. This statement is not valid for G unbounded. Actually, for G unbounded, (1.i) itself is not true, as the following example shows. Take $\nu = 1$, $G = [1, \infty)$, $p = 2$, $z_k(t) = t^{-1}$ for $1 \leq t \leq k$, $z_k(t) = 0$ for $k < t < +\infty$, $k = 1, 2, \ldots$. Then, we can take $z(t) = t^{-1}$ for $1 \leq t < +\infty$, and all z_k, z_k^2, z^2 are L_1-integrable, but z is not. For G unbounded the following statement (1.i)′ analogous to (1.i) holds.

(1.i)′ Let G be any measurable subset of E_ν, and $z_k(t)$, $t \in G$, $k = 1, 2, \ldots$, a sequence of real-valued measurable functions such that

$$(\alpha) \qquad z_k \in L_p(G), \quad k = 1, 2, \ldots, \qquad\qquad (\beta) \quad \int_G |z_k|^p \, dt \leq M, \quad k = 1, 2, \ldots,$$

for some constant $p > 1$, $M \geq 0$. Then, there is a measurable function $z(t)$, $t \in G$, and a subsequence $[z_{k_s}]$ such that

$$(4) \qquad z \in L_p(G);$$

$$(5) \qquad \int_G |z|^p \, dt \leq \varliminf_{s \to \infty} \int_G |z_{k_s}|^p \, dt;$$

$$(6) \qquad \int_G z \, \Phi \, dt = \lim_{s \to \infty} \int_G z_{k_s} \Phi \, dt$$

for every real-valued measurable function $\Phi(t)$, $t \in G$, with $\Phi \in L_q(G)$, $q^{-1} + p^{-1} = 1$. All integrals above are finite.

REMARK 3. A proof of (1.i)′ can be obtained by first extending all functions z_k to all of E_ν by taking $z_k(t) = 0$ for $t \in E_\nu - G$, and then by representing E_ν as the countable union of nonoverlapping hypercubes of side length one. On each of these hypercubes (1.i) holds, and the final subsequence $[z_{k_s}]$ can be obtained by the diagonal process. Statement (1.i)′ can be completed by the remark that if, together with (α) and (β), also the following hypothesis (γ) holds: (γ) $z_k \in L_1(G)$ and $\int_G |z_k| \, dt \leq M'$, $k = 1, 2, \ldots$, for some constant $M' \geq 0$, then it is also true that $z \in L_1(G)$, that $z \in L_r(G)$ for all $1 \leq r \leq p$, and (2) holds for all $1 \leq r \leq p$. For G unbounded the following statement (1.i)″ represents also an extension of (1.i) in a different direction.

(1.i)″ Let G be any measurable subset of E_ν, and $z_k(t)$, $t \in G$, $k = 1, 2, \ldots$, a sequence of real-valued measurable functions such that $(\alpha)'$ $z_k \in L_p(G \cap R)$, $p > 1$, for every interval $R \subset E_\nu$ and z_k restricted to $G \cap R$; $(\beta)'$ for every $R \subset E_\nu$

there is a constant $M = M(R)$ such that $\int_{G \cap R} | z_k |^p \, dt \leq M(R)$, $k = 1, 2, \dots$. Then, there is a measurable function $z(t)$, $t \in G$, and a subsequence $[z_{k_s}]$ such that

(7) $\quad z \in L_p(G \cap R)$ for every interval $R \subset E_\nu$ and z restricted to $G \cap R$;

(8) $$\int_{G \cap R} | z |^r \, dt \leq \varliminf_{s \to \infty} \int_G | z_{k_s} |^r \, dt \text{ for every } R \subset E_\nu \text{ and } 1 \leq r \leq p;$$

(9) $$\int_G z \, \Phi \, dt = \lim_{s \to \infty} \int_G z_{k_s} \Phi \, dt$$

for every real-valued measurable function $\Phi(t)$, $t \in G$, $\Phi \in L_q(G)$, $q^{-1} + p^{-1} = 1$, Φ of compact support. All integrals above are finite.

REMARK 4. For $p = 1$ statement (1.i) is not true, as the following well known example proves. Take $\nu = 1$, $G = [0, 1]$, $z_k(t) = k$ for $0 \leq t \leq k^{-1}$, $z_k(t) = 0$ for $k^{-1} < t \leq 1$, $k = 1, 2, \dots$. Then we can take $z(t) = 0$ for all $0 \leq t \leq 1$, and now for $\Phi = 1$, $0 \leq t \leq 1$, we have $\int_0^1 z_k \, dt = 1$, $\int_0^1 z \, dt = 0$, and (3) is not valid. For $p = 1$ and G bounded, statement (1.i) can be replaced by the following statement (1.ii).

(1.ii) Let G be any measurable bounded subset of E_ν, and $z_k(t)$, $t \in G$, $k = 1, 2, \dots$, a sequence of real-valued measurable functions such that (δ) the functions z_k are equiabsolutely integrable in G. Then there is a measurable function $z(t)$, $t \in G$, and a subsequence $[z_{k_s}]$ such that

(10) $$z \in L_1(G);$$

(11) $$\int_G | z | \, dt \leq \varliminf_{s \to \infty} \int_G | z_{k_s} | \, dt;$$

(12) $$\int_G z \, \Phi \, dt = \lim_{s \to \infty} \int_G z_{k_s} \Phi \, dt$$

for every measurable bounded function $\Phi(t)$, $t \in G$. All integrals above are finite.

REMARK 5. Since G is bounded and has, therefore, finite measure, condition (δ) certainly implies $\int_G | z_k | \, dt \leq M'$, $k = 1, 2, \dots$, for some constant M', and thus a condition analogous to (β) of (1.i) is superfluous here.

PROOF OF (1.ii). For the convenience of the reader we sketch a proof of the essentially known statement (1.ii). It is not restrictive to assume that G is contained in the hypercube $0 \le t^i \le N$, $i = 1, \dots, \nu$, for some integer N.

Let us define each function z_k in E_ν by taking $z_k(t) = 0$ for $t \in E_\nu - G$. Then the functions z_k are L_1-integrable in every interval $R_0 \subset E_\nu$, and $\int_{R_0} | z_k(t) | \, dt \le M$, $k = 1, 2, \dots$. Given any interval $R = [a, b]$, $a = (a^1, \dots, a^\nu)$, $b = (b^1, \dots, b^\nu)$, by $\int_a^b z_k(t)\, dt$ we shall denote the integral of z_k in R with the usual conventions concerning signs. Let R_0 be the interval $[0, N]$, $0 = (0, \dots, 0)$, $N = (N, \dots, N)$, $N > 0$. For every $k = 1, 2, \dots$, let us consider the function $Z_k(t) = \int_0^t z_k(\tau)\, d\tau$, defined for every $t = (t^1, \dots, t^\nu) \in R_0$, and where the integral ranges over the interval $[0, t]$. Then, for every interval $R \subset R_0$ the interval functions

$$\Psi_k(R) = \int_R z_k(\tau)\, d\tau, \quad k = 1, 2, \dots, \quad R = [a, b] \subset R_0,$$

can be expressed in terms of the usual differences of order ν of the functions Z_k with respect to the 2^ν vertices of R, say

$$\Psi_k(R) = \Delta_R Z_k = Z_k(b) - Z_k(a) \quad \text{for } \nu = 1,$$

$$\Psi_k(R) = \Delta_R Z_k = Z_k(b^1, b^2) - Z_k(a^1, b^2) - Z_k(b^1, a^2) + Z_k(a^1, a^2) \text{ for } \nu = 2, \text{ etc.}$$

As a consequence of (δ) the interval functions $\Psi_k(R) = \Delta_R Z_k$, $k = 1, 2, \dots$, are equiabsolutely continuous in the usual sense, that is, given $\varepsilon > 0$, there is some $\delta = \delta(\varepsilon) > 0$ such that, for every finite system $R_1, \dots, R_J$ of nonoverlapping intervals $R_j \subset R_0$, $j = 1, \dots, J$, with Σ_j meas $R_j \le \delta$, we have $\Sigma_j | \Psi_k(R_j) | \le \varepsilon$.

If $t, t' \in R_0$, $t = (t^1, \dots, t^\nu)$, $t' = (t^{1'}, \dots, t^{\nu'})$, and $| t - t' | = d$, let p_i, $i = 0, 1, \dots, \nu$, be the $\nu + 1$ points $p_0 = t$, $p_\nu = t'$,

$$p_i = (t^1, \dots, t^{\nu-i}, t^{\nu-i+1'}, \dots, t^{\nu'}), \quad i = 1, \dots, \nu - 1.$$

Note that

$$Z_k(t') - Z_k(t) = \sum_{i=1}^{\nu} [Z_k(p_i) - Z_k(p_{i-1})] = \sum_{i=1}^{\nu} \left[\int_0^{p_i} - \int_0^{p_{i-1}} \right] z_k(\tau)\, d\tau,$$

and that the two intervals $[0, p_i]$, $[0, p_{i-1}]$ differ by the single interval $r_i = (q_i, p_i)$, where

$$q_i = (0, \dots, 0, t^{\nu-i+1}, 0, \dots, 0), \quad p_i = (t^1, \dots, t^{\nu-i}, t^{\nu-i+1'}, \dots, t^{\nu'}),$$

and hence

$$Z_k(t') - Z_k(t) = \sum_{i=1}^{\nu} \int_{q_i}^{p_i} z_k(\tau)\, d\tau,$$

where meas $r_i \leq N^{\nu-1} \mid t^{\nu-i+1} - t^{\nu-i+1'} \mid \leq N^{\nu-1} \mid t - t' \mid = N^{\nu-1} d$. Thus, given $\varepsilon > 0$, whenever $\mid t - t' \mid = d < \varepsilon / N^{\nu-1}$ we have $\mid Z_k(t) - Z_k(t') \mid \leq N^{\nu-1} (\varepsilon / N^{\nu-1}) = \varepsilon$ for every $k = 1, 2, \dots$. This shows that the functions $Z_k(t)$, $t \in R_0$, $k = 1, 2, \dots$, are equicontinuous in R_0. Since $Z_k(0) = 0$, the same functions are also equibounded in R_0. By Ascoli's theorem there is, therefore, a subsequence Z_{k_s}, $s = 1, 2, \dots$, with $k_s \to \infty$, which is uniformly convergent in R_0 toward a continuous function $Z(t)$, $t \in R_0$. Since $Z_k(t) = 0$ for every $t = (t^1, \dots, t^\nu)$ with $t \in R_0$ and with $t^1 t^2 \dots t^\nu = 0$, we deduce that $Z(t) = 0$, for the same t.

For any interval $R = [a, b] \subset R_0$ we hawe

$$\int_R z_{k_s}(\tau)\, d\tau = \Delta_R Z_{k_s} = \Sigma_j \pm Z_{k_s}(\alpha_j),$$

where Σ_j ranges over the 2^ν vertices α_j of R with the usual sign conventions as mentioned above. As $s \to \infty$ we deduce

$$\Psi(R) = \lim_{s \to \infty} \int_R z_{k_s}(\tau)\, d\tau = \Delta_R Z = \Sigma_j \pm Z(\alpha_j),$$

and the convergence is uniform with respect to $R \subset R_0$. Since the interval functions $\Psi_k(R) = \Delta_R Z_k$ are equiabsolutely continuous in R_0, then the interval function $\Psi(R) = \Delta_R Z$ has the same property.

By Banach's theorem there is a measurable and L_1-integrable function $z(t)$, $t \in R_0$, with

$$\Psi(R) = \Delta_R Z = \int_R z(\tau)\, d\tau$$

for every $R \subset R_0$, and

$$\lim_{s \to \infty} \int_R z_{k_s}(\tau)\, d\tau = \int_R z(\tau)\, d\tau.$$

This relation proves (12) for every function Φ which is the characteristic function of an interval.

Thus (12) is proved also for functions Φ which are characteristic functions of a finite union of nonoverlapping intervals. If E is any measurable set, we can approach E in measure by means of a sequence of finite unions of nonoverlapping intervals, and then (12) can be proved for functions Φ which are the characteristic functions of measurable sets. Then (12) is proved also for measurable step functions Φ. Finally, any measurable bounded function Φ can be approached by means of a sequence of measurable step functions with the same bound, and thus (12) can be proved in general. Relation (11) is now a consequence of (12).

REMARK 6. For G unbounded, statement (1.ii) is not true, as the following example shows. Take $\nu = 1$, $G = [0, +\infty)$, $z_k(t) = 1$, for $0 \leq t \leq k$, $z_k(t) = 0$ for $k < t < +\infty$, $k = 1, 2, \ldots$. Then we can take $z(t) = 1$ for all $0 \leq t < +\infty$, and (10) in false, and so is (12), though (12) still holds for every measurable bounded function Φ with compact support.

For $p = 1$ and G unbounded, statement (1. ii) can be replaced by the following two statements.

(1.ii)′ The same as (1.ii) with G any measurable subset of E_ν, and the additional hypothesis $(\varepsilon) \int_G |z_k(t)|\, dt \leq M$, $k = 1, 2, \ldots$, and some constant M.

(1.ii)″ The same as (1.ii) with G any measurable subset of E_ν, where (δ) is replaced by the weaker hypothesis : (δ') for every interval $R \subset E_\nu$ the functions z_k restricted to $G \cap R$ are equiabsolutely integrable ; and the conclusions (10), (11), (12) are replaced by the following weaker statements :

$$z \in L_1(G \cap R) \text{ for every interval } R \subset E_\nu, \tag{13}$$

$$\int_{G \cap R} |z|\, dt \leq \varliminf_{s \to \infty} \int_{G \cap R} |z_k|\, dt \text{ for every interval } R \subset E_\nu; \tag{14}$$

$$\int_G z\, \Phi\, dt = \lim_{s \to \infty} \int_G z_{k_s} \Phi\, dt \tag{15}$$

for every measurable bounded function $\Phi(t)$, $t \in G$, with compact support.

§ 2. Weak compactness in Sobolev spaces W_p^1 for $p \geq 1$.

Weak compactness theorems in spaces $W_p^1(G)$, G an open subset of E_ν, $p \geq 1$, are well known [7] for G bounded. Here we summarize some of the results, and add some remarks for the cases where $p = 1$ or G is unbounded.

As we know an element z of $W_p^1(G)$ is a real-valued function $z(t)$, $t \in G$, with $z \in L_p(G)$, possessing generalized first order partial derivatives

$D_i z = \partial z/\partial t^i$, $i = 1, \dots, \nu$, a.e. in G [8], with $D_i z \in L_p(G)$, $i = 1, \dots, \nu$. If Dz denotes the ν-vector $(D_i z, i = 1, \dots, \nu)$, then $D_p(z, G)$ and $\widetilde{D}_p(z, G)$ denote the nonnegative numbers

$$D_p(z, G) = \Big(\int_G |Dz(t)|^p\, dt\Big)^{1/p}, \quad \widetilde{D}_p(z, G) = \Big(\int_G |z(t)|^p\, dt\Big)^{1/p} + D_p(z, G).$$

We shall consider below a proof of compactness theorems in $W_p^1(G)$ for $p \geq 1$ in which the main tool is the following lemma.

(2.i) LEMMA. If $z(t)$ is of class $W_p^1(R)$, $p \geq 1$, in an hypercube $R = [a, b]$ of side-length h, then

$$\int_a^b \int_a^b |z(t) - z(\tau)|^p\, dt\, d\tau \leq [(2\nu - 1)h]^p h^\nu D_p(z, R), \tag{16}$$

$$\int_a^b |z(t) - z_R|^p\, dt \leq [(2\nu - 1)h]^p D_p(z, R), \tag{17}$$

where $z_R = h^{-\nu}\int_R z(\tau)\, d\tau$ is the mean value of z in the hypercube R.

This lemma is well known [7]. For the convenience of the reader we shall sketch here the proof of the lemma for $p = 1$.

PROOF OF (2.i) FOR $p = 1$. We can approach z strongly in R by means of functions of class C^1. Therefore, it is enough to prove (2.i) for functions $z(t)$ of this class. Also, it suffices to give here the proof for $\nu = 2$, the proof for $\nu > 2$ being analogous. To simplify notations we shall replace t^1, t^2, τ^1, τ^2 by x, y, ξ, η respectively, and we take $R = [a, b, a+h, b+h]$. Since

$$\int_b^{b+h} dy \int_a^{a+h} |z_x(x, y)|\, dx = \int_a^{a+h} \int_b^{b+h} |z_x(x, y)|\, dx\, dy \leq D_1(z, R) < +\infty, \tag{18}$$

there must be some $\bar{y}$, $b \leq \bar{y} \leq b + h$, such that

$$\int_a^{a+h} |z_x(x, \bar{y})|\, dx \leq h^{-1} I_x,$$

where I_x is the double integral in the second member of (18). We shall denote by I_y the analogous integral in terms of z_y. For all (x, y), $(\xi, \eta) \in R$, we have now

$$z(x, y) - z(\xi, \eta) = \int_{\eta}^{\bar{y}} z_y(\xi, t)\, dt + \int_{\xi}^{x} z_x(s, \bar{y})\, ds + \int_{\bar{y}}^{y} z_y(x, t)\, dt.$$

By integration with respect to x, ξ in $[a, a+h]$ and with respect to y, η in $[b, b+h]$, and taking absolute values, we obtain

$$\int_a^{a+h}\int_b^{b+h}\int_a^{a+h}\int_b^{b+h} |z(x,y) - z(\xi, \eta)|\, dx\, dy\, d\xi\, d\eta \leq$$

$$\leq \iint_R \iint_R \left| \int_{\eta}^{\bar{y}} z_y(\xi, t)\, dt \right| dx\, dy\, d\xi\, d\eta + \iint_R \iint_R \left| \int_{\xi}^{x} z_x(s, \bar{y})\, ds \right| dx\, dy\, d\xi\, d\eta +$$

$$+ \iint_R \iint_R \left| \int_{\bar{y}}^{y} z_y(x, t)\, dt \right| dx\, dy\, d\xi\, d\eta$$

$$\leq \int_a^{a+h}\int_b^{b+h}\int_b^{b+h} dx\, dy\, d\eta \int_a^{a+h}\int_b^{b+h} |z_y(\xi, t)|\, d\xi\, dt +$$

$$+ \int_a^{a+h}\int_b^{b+h}\int_a^{a+h}\int_b^{b+h} dx\, dy\, d\xi\, d\eta \cdot h^{-1} \int_a^{a+h}\int_b^{b+h} |z_x(u, v)|\, du\, dv$$

$$+ \int_b^{b+h}\int_a^{a+h}\int_b^{b+h} dy\, d\xi\, d\eta \int_a^{a+h}\int_b^{b+h} |z_y(x, t)|\, dx\, dt$$

$$\leq h^3 \iint_R |z_y|\, dx\, dy + h^3 \iint_R |z_x|\, dx\, dy + h^3 \iint_R |z_y|\, dx\, dy \leq 3h^3 D_1(z, R).$$

This proves (16) for $p = 1$ and $\nu = 2$. We have now again for any $\nu \geq 2$ and $p = 1$ by using the notation $t = (t^1, \dots, t^\nu)$, $\tau = (\tau^1, \dots, \tau^\nu)$, and assuming (16) proved for every $\nu \geq 2$,

$$\int_R |z(t) - z_R|\, dt = \int_R \left| z(t) - h^{-\nu} \int_R z(\tau)\, d\tau \right| dt =$$

$$= h^{-\nu} \int_R \left| h^\nu z(t) - \int_R z(\tau)\, d\tau \right| dt$$

$$= h^{-\nu} \int_R \left| \int_R [z(t) - z(\tau)]\, d\tau \right| dt$$

$$\leq h^{-\nu} \int_R \int_R | z(t) - z(\tau) |\, dt\, d\tau$$

$$\leq h^{-\nu} \cdot (2\nu - 1)\, h^{\nu+1} D_1(z, R) = (2\nu - 1)\, h\, D_1(z, R).$$

This proves (17) for $p = 1$ and $\nu \geq 2$.

For functions of class $W_p^1(R)$, $p > 1$, in an interval $R = [a, b]$ a weak compactness theorem can be stated as follows.

(2.ii) Let $z_k(t)$, $t \in R = [a, b]$, $k = 1, 2, \dots$, be a sequence of real-valued functions of class $W_p^1(R)$, $p > 1$, with $\widetilde{D}_p(z_k, R) \leq M$, $k = 1, 2, \dots$, and some constant M. Then, there is a subsequence $[z_{k_s}]$ and a function $z(t)$, $t \in R$, of class $W_p^1(R)$ such that

$$\lim_{s \to \infty} \int_R | z(t) - z_{k_s}(t) |^p\, dt = 0,$$

$$\lim_{s \to \infty} \int_R D_i z_{k_s}(t)\, \Phi(t)\, dt = \int_R D_i z(t)\, \Phi(t)\, dt, \qquad i = 1, \dots, \nu,$$

for all $\Phi \in L_q(R)$, $q^{-1} + p^{-1} = 1$.

For $p = 1$ an analogous statement reads:

(2.iii) Let $z_k(t)$, $t \in R = [a, b]$, $k = 1, 2, \dots$, be a sequence of real valued functions of class $W_1^1(R)$ and with $\int_R | z_k(t) |\, dt \leq M$, $k = 1, 2, \dots$, for some constant M. Let us assume that the generalized first order partial derivatives $D_i z_k$, $k = 1, 2, \dots$, $i = 1, \dots, \nu$, are equiabsolutely integrable in R. Then, there is a subsequence $[z_{k_s}]$ and a function $z(t)$, $t \in R$, of class $W_1^1(R)$ such that

$$\lim_{s \to \infty} \int_R | z(t) - z_{k_s}(t) |\, dt = 0, \tag{19}$$

$$\lim_{s \to \infty} \int_R D_i z_{k_s}(t)\, \Phi(t)\, dt = \int_R D_i z(t)\, \Phi(t)\, dt, \qquad i = 1, \dots, \nu. \tag{20}$$

for every measurable bounded function $\Phi(t)$, $t \in R$.

For the convenience of the reader we give below the proof of (2.iii). For any $i = 1, \dots, \nu$, we shali denote by t_i' the set of variables $t^1, \dots, t^{i-1}, t^{i+1}, \dots, t^\nu$ and we shall often write (t_i', t^i) for t and dt_i' for $dt^1 \dots dt^{i-1} dt^{i+1} \dots dt^\nu$. We shall say that a property holds for almost all $R = [a, b] \subset G$ provided the property holds for all $R = [a, b]$ with $(a, b) \in G \times G$ with exception at most of a set of points $(a, b) \in G \times G$ of measure zero (in $E_\nu \times E_\nu$).

The generalized derivatives $D_i z(t) = \partial z / \partial t^i$, $i = 1, \dots, \nu$, of a function $z(t)$, $t \in G$, of class $W_1^1(G)$ can be characterized a. e. by the following important property: For almost all $R = [a, b] \subset G$, $a = (a^1, \dots, a^\nu)$, $b = (b^1, \dots, b^\nu)$, we have

$$\int_{a_i'}^{b_i'} [z(t_i', b^i) - z(t_i', a^i)\, dt_i' = \int_a^b D_i z(t)\, dt, \quad i = 1, \dots, \nu.$$

PROOF OF (2.iii). We may well assume that R is the interval $[0, b]$, or $0 \leq t^i \leq b^i$, $i = 1, \dots, \nu$. We can extend each function $z_k(t)$ in the interval $-b^i \leq t^i \leq b^i$, $i = 1, \dots, \nu$, by evenness of z_k in $t^1, \dots, t^\nu$, and then we can extend each function z_k in the whole of E_ν by periodicity of period $2b^i$ in t^i, $i = 1, \dots, \nu$. The functions z_k are now defined in E_ν and are of class W_1^1 in each fixed interval $R_0 \subset E_\nu$. The same functions z_k satisfy $\int_{R_0} |z_k|\, dt \leq M_0$ for a constant M_0 which does not depend on k but may depend on R_0, and the derivatives $D_i z_k$ are equiabsolutely integrable in R_0. We can take for R_0 an hypercube $0 \leq t^i \leq A$, $i = 1, \dots, \nu$, of side length $A > 0$, R_0 containing R in its interior. We shall prove (2.iii) for the hypercube R_0. As mentioned in Remark 5 of § 1, from the equiabsolute integrability of the derivatives $D_i z_k$ in R_0 we deduce that there is some constant M' such that

$$\int_{R_0} | D_i z_k(t) |\, dt \leq M', \quad k = 1, 2, \dots, i = 1, \dots, \nu. \tag{21}$$

Here M' does not depend on k but may depend on R_0.

For every integer $r = 0, 1, 2, \dots$, let us consider the subdivision of R_0 into $2^{\nu r}$ hypercubes R_{rj}, $j = 1, \dots, 2^{\nu r}$, of side length $2^{-r} A$, that we obtain by dividing R_0 by means of the hyperplanes $t^i = \alpha 2^{-r} A$, $\alpha = 1, \dots, 2^r$, $i = 1, \dots, \nu$. For every r let us denote by $z_{kr}(t)$, $t \in R_0$, the step function that we obtain by replacing $z_k(t)$ in R_{rj} by means of the mean value $(z_k)_{R_{rj}}$

of z_k in R_{rj}. For every r and for every $R = R_{rj}$ we have

$$| z_{kr}(t) | = \left| (2^{-r} A)^{-\nu} \int_R z_k(\tau)\, d\tau \right| \leq 2^{r\nu} A^{-\nu} \int_R | z_k(\tau) | \, d\tau \leq$$

$$\leq 2^{r\nu} A^{-\nu} M, \quad t \in R = R_{rj},$$

where the last member does not depend on k but depends on r. Thus, for every r, the sequence $z_{kr}(t)$, $t \in R_0$, $k = 1, 2, \dots$, is bounded in R_0.

For $r = 0$, $z_{k0}(t)$, $t \in R_0$, $k = 1, 2, \dots$, is a bounded sequence of constants, and thus there is a convergent subsequence, say still $z_{k0}(t)$, $t \in R_0$, $k = 1, 2, \dots$. We shall denote by $\bar{z}_0(t)$, $t \in R_0$, the limit function which is constant on R_0. For $r = 1$ the functions $z_{k1}(t)$, $t \in R_0$, $k = 1, 2, \dots$, are constant on each of the 2^ν subintervals R_{1j}, $j = 1, \dots, 2^\nu$, and thus there is a convergent subsequence, say still $z_{k2}(t)$, $t \in R_0$, $k = 1, 2, \dots$. We shall denote by $\bar{z}_2(t)$, $t \in R_0$, the limit function which is constant on each of the 2^ν intervals R_{1j}, $j = 1, \dots, 2^\nu$. By repeating this process of successive extractions we obtain, for every $r = 0, 1, 2, \dots$, a subsequence, say still $z_{kr}(t)$, $t \in R_0$, $k = 1, 2, \dots$, which is convergent as $k \to \infty$ toward a function $\bar{z}_r(t)$, $t \in R_0$, and all z_{kr} and $\bar{z}_r$ are constant on each of the $2^{\nu r}$ intervals R_{rj}, $j = 1, 2, \dots, 2^{\nu r}$. By the diagonal process we obtain now a subsequence of integers $[k_s]$ such that $z_{k_s}(t) \to \bar{z}_r(t)$ as $s \to \infty$ for all $t \in R_0$ and every $r = 0, 1, 2, \dots$. By a suitable reindexing we can always denote by $[k]$ the new sequence so that we have

$$\lim_{k \to \infty} z_{kr}(t) = \bar{z}_r(t) \tag{22}$$

for all $t \in R_0$ and every $r = 0, 1, 2, \dots$.

We shall now apply (17) of (2.i) to each function $z_k(t)$, $t \in R_0$, and each interval R_{rj}, $j = 1, \dots, 2^{\nu r}$. The mean values are now the values taken by the step functions $z_{kr}(t)$ on each R_{rj}, and hence

$$\int_{R_{rj}} | z_k(t) - z_{kr}(t) | \, dt \leq (2\nu - 1)\, 2^{-r} A\, D_1(z_k, R_{rj})$$

and, by summation over $j = 1, \dots, 2^{\nu r}$, also

$$\int_{R_0} | z_k(t) - z_{kr}(t) | \, dt \leq (2\nu - 1)\, 2^{-r} A D_1(z_k, R_0) \text{ for every } k = 1, 2, \dots. \tag{23}$$

For all r, s, $r < s$, we have now from (20) and (23),

$$\int_{R_0} | z_{kr}(t) - z_{ks}(t) | \, dt \leq 2\,(2\nu - 1)\, 2^{-r} A D_1(z_k, R_0) \leq 2\nu\,(2\nu - 1)\, 2^{-r} A M'. \tag{24}$$

For r, s fixed, $r < s$, we have $z_{kr} \to \bar{z}_r$, $z_{ks} \to \bar{z}_s$ as $k \to \infty$, uniformly in R_0 since, r and s being fixed, all functions z_{kr} are constant on each B_{rj} and all z_{k_s} are constant on each R_{sj}. Thus, as $k \to \infty$, from (23) we deduce

$$(25) \qquad \int_{R_0} |\bar{z}_r(t) - \bar{z}_s(t)|\, dt \leq 2\nu(2\nu - 1)\, 2^{-r} AM', \; r < s.$$

Also, for every r, we can determine an index $k = k_r$ such that

$$(26) \qquad \int_{R_0} |z_{k_r r}(t) - \bar{z}_r(t)|\, dt \leq 2\nu(2\nu - 1)\, 2^{-r} AM'.$$

Relation (25) shows that the sequence $\bar{z}_r(t)$, $t \in R_0$, $r = 0, 1, 2, \ldots$, converges in $L_1(R_0)$ toward a function $z(t)$, $t \in R_0$, of class $L_1(R_0)$, and that

$$(27) \qquad \int_{R_0} |\bar{z}_r(t) - z(t)|\, dt \leq 2\nu(2\nu - 1)\, 2^{-r} AM'.$$

By (23), (26), and (27), we deduce

$$\int_{R_0} |z_{k_r}(t) - z(t)|\, dt \leq$$

$$\leq \int_{R_0} |z_{k_r} - z_{k_r r}|\, dt + \int_{R_0} |z_{k_r r} - \bar{z}_r|\, dt + \int_{R_0} |\bar{z}_r - z|\, dt$$

$$\leq 5\nu(2\nu - 1)\, 2^{-r} AM',$$

that is, the sequence $z_{k_r}(t)$, $t \in R_0$, $r = 1, 2, \ldots$, converges in $L_1(R_0)$ toward a function $z(t), t \in R_0$, of class $L_1(R_0)$. Thus (19) of (2. iii) is proved.

Since the derivatives $D_i z_{k_r}(t)$, $t \in R_0$, $r = 1, 2, \ldots, i = 1, \ldots, \nu$, are equiabsolutely integrable in R_0, we can apply (1. ii) successively ν times, and extract a new subsequence, which we still indicate $[z_{k_r}]$ for the sake of simplicity, so that, together with relations (19), also the ν relations hold

$$(28) \qquad \lim_{r \to \infty} \int_{R_0} D_i z_{k_r}(t)\, \Phi(t)\, dt = \int_{R_0} p_i(t)\, \Phi(t)\, dt, \qquad i = 1, \ldots, \nu,$$

for some suitable functions $p_i(t)$, $t \in R_0$, of class $L_1(R_0)$, and for all measurable bounded functions $\Phi(t)$, $t \in R_0$. We have only to prove that these functions p_i are the generalized derivatives of the function z determined above.

For every $r = 0, 1, 2, \ldots$, and $i = 1, \ldots, \nu$, we know already that

$$(29) \qquad \int_{a_i'}^{b_i'} [z_{k_r}(t_i', b^i) - z_{k_r}(t_i', a^i)]\, dt_i' = \int_a^b D_i z_{k_r}(t)\, dt,$$

for almost all intervals $R = [a, b] \subset R_0$, $a = (a^1, \ldots, a^\nu)$, $b = (b^1, \ldots, b^\nu)$.

Then it is also true that for almost all intervals $R = [a, b] \subset R_0$ relations (29) holds for all $r = 0, 1, 2, \ldots$, and all $i = 1, \ldots, \nu$. Note that

$$\lim_{r \to \infty} \int_{R_0} | z_{k_r}(t) - z(t) |\, dt = 0,$$

where $R_0 = [0 \leq t^i \leq A, i = 1, \ldots, \nu]$, and we can write this relation in the form

$$\lim_{r \to \infty} \int_0^A dt^i \int_0^{A'} | z_{k_r}(t_i', t^i) - z(t_i', t^i) |\, dt_i' = 0,$$

where $[0, A']$ denotes the $(\nu - 1)$-dim. interval $0 \leq t^j \leq A$, $j \neq i$, $j = 1, \ldots, \nu$. This implies that, for almost all t^i of the 1-dim. interval $0 \leq t^i \leq A$, we have

$$\lim_{r \to \infty} \int_0^{A'} | z_{k_r}(t_i', t^i) - z(t_i', t^i) |\, dt_i' = 0.$$

Hence, for all $[a_i', b_i'] \subset [0, A']$ and almost all t^i, $0 \leq t^i \leq A$, we have

$$\lim_{r \to \infty} \int_{a_i'}^{b_i'} | z_{k_r}(t_i', t^i) - z(t_i', t^i) |\, dt_i' = 0.$$

Since

$$\left| \int_{a_i'}^{b_i'} z_{k_r}(t_i', t^i)\, dt_i' - \int_{a_i'}^{b_i'} z(t_i', t^i)\, dt_i' \right| \leq \int_{a_i'}^{b_i'} | z_{k_r}(t_i', t^i) - z(t_i', t^i) |\, dt_i',$$

we conclude that

$$\lim_{r \to \infty} \int_{a_i'}^{b_i'} z_{k_r}(t_i', t^i)\, dt_i' = \int_{a_i'}^{b_i'} z(t_i', t_i)\, dt_i'$$

for all $[a_i', b_i']$ and almost all t^i, $0 \leq t^i \leq A$. Thus, for almost all $[a, b] \subset R_0$ the first member of (29) converges, as $r \to \infty$, toward the first member of formula (30) below, and the second member of (29) toward the second member of (30) below. Thus, by (29), as $r \longrightarrow \infty$, we deduce

$$\int_{a_i'}^{b_i'} [z(t_i', b^i) - z(t_i', a^i)]\, dt_i' = \int_a^b p_i(t)\, dt, \tag{30}$$

for almost all $[a, b] \subset R_0$, $i = 1, \dots, \nu$. This proves that $p_i(t) = D_i z(t)$ almost everywhere in R_0, $i = 1, \dots, \nu$. Statement (2. iii) is thereby proved.

As in the previous papers [3,4] we shall assume for G and its boundary $S = \partial G$ a certain amount of regularity, and we shall say that a bounded open set G is of class K_{0l}, $l \geq 1$. To be precise, we may assume that $K = cl\, G = G \cup S$ is the union of finitely many nonoverlapping parts $K_1, \dots, K_J$, each $K_j = T_j(R)$ being the $1-1$ image of a rectangle R under a transformation T_j which is continuous with its inverse T^{-1} and has continuous partial derivatives up to the order $l, j = 1, \dots, J$. (A number of different assumptions can be made (See, for instance, S. L. Sobolev [8], Chap. I, § 10, p. 72, Remark). The further usual convention shall be made that the boundary S of G is the union of non overlapping parts λ_s, each λ_s being the image under T_j of one face l_j of R, or $\lambda_s = T_j(l_j)$ for only one j.

If G is unbounded, we shall say that G is of class K_{0l} if its closure $K = cl\, G$ is the countable union of nonoverlapping parts $K_1, K_2, \dots$, each $K_j = T_j(R)$ as before, and with the further assumptions that each set $V_N = \bigcup_{j=1}^N K_j = cl\, G_N$ is the closure of an open bounded set G_N of class K_{0l}, that every interval R of E_ν has a non-empty intersection with at most finitely many K_j, and that $G = \bigcup G_N$, $G_N \subset G_{N+1}$, that is, G is the union of the bounded open subsets G_N all of class K_{0l}.

Obviously, there are ∞-many decompositions as described of sets G of class K_{0l}, bounded or unbounded. Any such decomposition will be called a *typical representation* of the set G of class K_{0l}.

Statements (2. ii) and (2. iii) have now the following extensions to open sets G.

(2. iv) Let $z_k(t), t \in G, k = 1, 2, \dots$, be a sequence of functions of class $W_p^1(G), p > 1$, in a bounded open set G with $\widetilde{D}_p(z_k, G) \leq M$, $k = 1, 2, \dots$, for some constant M. Then, there is a subsequence $[z_{k_s}]$ and a function $z(t), t \in G$, of class $W_p^1(G)$ such that

$$\lim_{s \to \infty} \int_R |z_{k_s}(t) - z(t)|^p\, dt = 0,$$

$$\lim_{s \to \infty} \int_R D_i z_{k_s}(t)\, \Phi(t)\, dt = \int_R D_i z(t)\, \Phi(t)\, dt, \qquad i = 1, \dots, \nu,$$

for every $\Phi \in L_q(R)$, $q^{-1} + p^{-1} = 1$, and every interval $R \subset G$. If G is of class K_{01}, then we have also

$$\lim_{s \to \infty} \int_G | z_{k_s}(t) - z(t) |^p \, dt = 0,$$

$$\lim_{s \to \infty} \int_G D_i z_{k_s}(t) \, \Phi(t) \, dt = \int_G D_i z(t) \, \Phi(t) \, dt, \qquad i = 1, \dots, \nu,$$

for every $\Phi \in L_q(G)$.

(2. v) Let $z_k(t), t \in G, k = 1, 2, \dots$, be a sequence of functions of class $W_1^1(G)$ in a bounded open set G, with $\int_G | z_k | \, dt \leq M, k = 1, 2, \dots$, and some constant M, and whose generalized first order partial derivatives $D_i z_k$, $i = 1, \dots, \nu, k = 1, 2, \dots$, are equiabsolutely integrable in G. Then, there is a subsequence $[z_{k_s}]$ and a function $z(t), t \in G$, of class $W_1^1(G)$ such that

$$\lim_{s \to \infty} \int_R | z_{k_s}(t) - z(t) | \, dt = 0.$$

$$\lim_{s \to \infty} \int_R D_i z_{k_s}(t) \, \Phi(t) \, dt = \int_R D_i z(t) \, \Phi(t) \, dt, \qquad i = 1, \dots, \nu,$$

for every bounded measurable function $\Phi(t), t \in R$, and every interval $R \subset G$. If G is of class K_{01}, then we have also

$$\lim_{s \to \infty} \int_G | z_{k_s}(t) - z(t) | \, dt = 0,$$

$$\lim_{s \to \infty} \int_G D_i z_{k_s}(t) \, \Phi(t) \, dt = \int_G D_i z(t) \, \Phi(t) \, dt, \qquad i = 1, \dots, \nu,$$

for every bounded measurable function $\Phi(t), t \in G$.

REMARK 7. For G unbounded, (2.iv) and (2.v) are not true, as the following example shows. Take $z_k(t) = 0$ for all $t \in G = E_\nu$ with exception of the solid sphere of radius one and center $t_k = (k, \dots, k)$, where $z_k(t) = 1 - | t - t_k |$, $k = 1, 2, \dots$. Then $\widetilde{D}_p(z_k, G) = c_p$, $p \geq 1$, where c_p is a constant wich depends only on p, and the derivatives $D_i z_k$ are equiabsolutely integrable in G, but $\int_G | z_k - z |^p \, dt = C_p$ for all $k=1, 2, \dots$, where $C_p > 0$ is a constant which depends on p only.

For G unbounded (2.iv) is still valid under the same hypotheses, provided in the conclusion the strong convergence of z_{k_s} in the whole of G is replaced by

$$\int_G |z|^p\, dt \leq \underline{\lim}_{s \to \infty} \int_G |z_{k_s}|^p\, dt,$$

$$\int_G |D_i z|^p\, dt \leq \underline{\lim}_{s \to \infty} \int_G |D_i z_{k_s}|^p\, dt, \qquad i = 1, \ldots, \nu,$$

and we assume that Φ has compact support.

Analogously, for G unbounded (2.v) is still valid under the additional hypothesis $\int_G |D_i z_k|\, dt \leq M'$ for all $i = 1, \ldots, \nu$ and some constant M', provided in the conclusion the strong convergence of z_{k_s} in the whole of G is replaced by

$$\int_G |z|\, dt \leq \underline{\lim}_{s \to \infty} \int_G |z_{k_s}|\, dt,$$

$$\int_G |D_i z|\, dt \leq \underline{\lim}_{s \to \infty} \int_G |D_i z_{k_s}|\, dt, \qquad i = 1, \ldots, \nu,$$

and we assume that Φ has compact support.

For G unbounded, both theorems (2.iv) and (2.v) can be stated in weaker forms that can be deduced from the analogous theorems of § 1. To state these new theorems we shall always think of the unbounded open set G as the union of bounded open subsets, or $G = \bigcup G_N$, $G_N \subset G_{N+1}$, such that for any interval $R \subset G$ we have also $R \subset G_N$ for some N. If G is of class K_{01}, then we shall assume that the sets G_N are the open bounded subsets of G of class K_{01} described above relatively to any typical representation of the open set G of class K_{01}.

(2.iv)$'$ Let $z_k(t)$, $t \in G$, $k = 1, 2, \ldots$, be a sequence of measurable functions in the unbounded open set $G \subset E_\nu$, $G = \bigcup G_N$ as above. Assume that for every $N = 1, 2, \ldots$, we have $z_k \in W_p^1(G_N)$, $\widetilde{D}(z, G_N) \leq M_N$, $p > 1$, $k = 1, 2, \ldots$, for some constant M_N. Then, there is a subsequence $[z_{k_s}]$ and a function $z(t)$, $t \in G$, measurable in G, with $z \in W_p^1(G_N)$ for every N, such that

$$\lim_{s \to \infty} \int_R |z_{k_s}(t) - z(t)|^p\, dt = 0,$$

$$\lim_{s \to \infty} \int_R D_i z_{k_s}(t)\, \Phi(t)\, dt = \int_R D_i z(t)\, \Phi(t)\, dt, \qquad i = 1, \ldots, \nu,$$

for every interval $R \subset G$, and every function $\Phi(t), t \in R$, $\Phi \in L_q(R), q^{-1} + p^{-1} = 1$. If G is of class K_{01} and $G = \bigcup G_N, G_N \subset G_{N+1}$, is any typical representation of G, then we have also

$$\lim_{s \to \infty} \int_{G_N} | z_{k_s}(t) - z(t) |^p \, dt = 0,$$

$$\lim_{s \to \infty} \int_G D_i z_{k_s}(t) \, \Phi(t) \, dt = \int_G D_i z(t) \, \Phi(t) \, dt, \qquad i = 1, \dots, \nu,$$

for every $N = 1, 2, \dots$, and for every function $\Phi(t), t \in G$, $\Phi \in L_q(G), q^{-1} + p^{-1} = 1$, Φ of compact support.

(2.v)′ Let $z_k(t), t \in G, k = 1, 2, \dots$, be a sequence of measurable functions in the unbounded open set $G \subset E_\nu$, $G = \bigcup G_N$ as above. Assume that for every $N = 1, 2, \dots$, we have $z_k \in W_1^1(G_N), \int_{G_N} | z_k | \, dt \leq M_N$, $k = 1, 2, \dots$, for some constant M_N. Assume that, for every N the derivatives $D_i z_k$, $i = 1, 2, \dots, \nu$, $k = 1, 2, \dots$, are equiabsolutely integrable in G_N. Then, there is a subsequence $[z_{k_s}]$ and a function $z(t), t \in G$, measurable in G, with $z \in W_1^1(G_N)$ for every N, and such that

$$\lim_{s \to \infty} \int_R | z_{k_s}(t) - z(t) | \, dt = 0,$$

$$\lim_{s \to \infty} \int_R D_i z_{k_s}(t) \, \Phi(t) \, dt = \int_R D_i z(t) \, \Phi(t) \, dt, \qquad i = 1, \dots, \nu,$$

for every interval $R \subset G$, and every bounded measurable function $\Phi(t)$, $t \in R$. If G is of class K_{01}, and $G = \bigcup G_N, G_N \subset G_{N+1}$, is any typical representation of G, then we have also

$$\lim_{s \to \infty} \int_{G_N} | z_{k_s}(t) - z(t) | \, dt = 0,$$

$$\lim_{s \to \infty} \int_G D_i z_{k_s}(t) \, \Phi(t) \, dt = \int_G D_i z(t) \, \Phi(t) \, dt, \qquad i = 1, \dots, \nu,$$

for every $N = 1, 2, \dots$, and for every bounded measurable function $\Phi(t)$, $t \in G$, Φ of compact support.

REMARK 8. For functions $z(t), t \in G$, of any Sobolev space $W_p^l(G)$, $l \geq 1, p \geq 1$, all generalized derivatives $D^\alpha z(t), t \in G$, of all orders $|\alpha|$,

$0 \leq |\alpha| \leq l$, exist a. e. in G. Here $\alpha = (\alpha_1, \dots, \alpha_\nu)$ denotes an arbitrary system of integers $\alpha_i \geq 0$, with $|\alpha| = \alpha_1 + \dots + \alpha_\nu$. In addition $D^\alpha z \in L_p(G)$ for all α with $0 \leq |\alpha| \leq l$, and each derivative $D^\alpha z$ of order $0 \leq |\alpha| \leq l-1$ possesses boundary values Φ_i^α on the boundary $S = \partial G$ of G, provided G be, say, of class K_{0l}. If $D_h z(t)$ denotes the vector of all generalized partial derivatives of all orders $|\alpha| = h$, $\alpha = (\alpha_1, \dots, \alpha_\nu)$, $h = 0, 1, \dots, l$, (so that, for example, $D_0 z = z$ is a 1-vector), then let $D_p^l(z, G)$ and $\widetilde{D}_p^l(z, G)$ denote the nonnegative numbers

$$D_p^l(z, G) = \Big(\int_G |D_l z(t)|^p dt\Big)^{1/p},$$

$$\widetilde{D}_p^l(z. G) = \Sigma_{h=0}^l \Big(\int_G |D_h z(t)|^p dt\Big)^{1/p}.$$

The compactness theorems above in Sobolev spaces $W_p^1(G)$ can be extended without difficulties to Sobolev spaces $W_p^l(G)$, $l \geq 1$. All that is needed is to assure that the derivatives of maximum order $D^\alpha z(t)$, $t \in G$, $|\alpha| = l$, are equiabsolutely integrable, and that the other derivatives $D^\alpha z(t)$, $t \in G$, of lower orders, $0 \leq |\alpha| \leq l-1$, possess an integral bound.

§ 3 Preliminary notations for multidimensional Lagrange problems.

Let G be an open subset of the t-space E_ν, $t = (t^1, \dots, t^\nu)$, let $x = (x^1, \dots, x^n)$ denote a vector variable in E_n, and $u = (u^1, \dots, u^m)$ a vector variable in E_m. We shall denote the x^i, $i = 1, \dots, n$, as state variables, and the u^j, $j = 1, \dots, m$, as control variables. As usual, we denote by clG and by $bdG = \partial G$ the closure and the boundary of G. We also denote by $co\,H$ the convex hull of a set H, and thus $cl\,co\,H$ is the closure of the convex hull of H. For every $t \in cl\,G$, let $A(t)$ be a given nonempty subset of E_n, and let A be the set of all (t, x) with $t \in cl\,G$, $x \in A(t)$. For every $(t, x) \in A$ let $U(t, x)$ be a subset of E_m, and let M be the set of all (t, x, u) with $(t, x) \in A$, $u \in U(t, x)$. The set M defined above is a subset of $E_\nu \times E_n \times E_m$ and its projection on $E_\nu \times E_n$ is A. We assume G to be of class K_{0l} for some $l \geq 1$.

We shall consider vector functions $x(t) = (x^1, \dots, x^n)$, $u(t) = (u^1, \dots, u^m)$, $t \in G$, and, by analogy with the case $\nu = 1$, we may denote $x(t)$ as a trajectory, and $u(t)$ as a control function, or strategy. For every $i = 1, \dots, n$, we shall denote by $\{\alpha\}_i$ a given finite system of nonnegative integral indices $\alpha = (\alpha_1, \dots, \alpha_\nu)$, $0 \leq |\alpha| \leq l$, with $|\alpha| = \alpha_1 + \dots + \alpha_\nu$. We shall assume each component $x^i(t)$ of x to be locally L_{p_i} integrable in G and to possess the generalized partial derivatives $D^\alpha x^i(t)$, $\alpha \in \{\alpha\}_i$, all locally

L_{p_i} integrable in G for certain $p_i \geq 1$. In § 5 we shall assume all $p_i > 1$, in § 6 we may allow some p_i to be $= 1$. We shall assume that each component $u^j(t)$ of u is measurable in G.

Let N denote the total number of indices α contained in the n systems $\{\alpha\}_i, i = 1, \dots, n$, and let $f(t, x, u) = (f_{i\alpha})$ denote an N-vector function whose components are real-valued functions $f_{i\alpha}(t, x, u)$ defined on M. We shall consider the system of N partial differential equations in G:

$$D^\alpha x^i = f_{i\alpha}(t, x, u), \alpha \in \{\alpha\}_i, \qquad i = 1, \dots, n,$$

or briefly

$$Dx = f(t, x, u)$$

We are interested in pairs x, u of vector functions $x(t), u(t), t \in G$, as above, satisfying the constraints

$$(t, x(t)) \in A, u(t) \in U(t, x(t)) \text{ a.e. in } G, \tag{31}$$

which are often denoted as unilateral constraints, and the system of partial differential equations

$$D^\alpha x^i(t) = f_{i\alpha}(t, x(t), u(t)) \text{ a.e. in } G, \alpha \in \{\alpha\}_i, \qquad i = 1, \dots, n, \tag{32}$$

or in short

$$Dx(t) = f(t, x(t), u(t)), \text{ a.e. in } G.$$

Given $\delta > 0$ and a point $(t_0, x_0) \in A$, we shall mean by the closed neighborhood $N_\delta(t_0, x_0)$ of radius δ of (t_0, x_0) in A the set of all $(t, x) \in A$ at a distance $\leq \delta$ from (t_0, x_0). Also, we shall denote by U_ε the set of all points $u \in E_m$ at a distance $\leq \varepsilon$ from a given set U. We shall say that $U(t, x)$ is metrically upper semicontinuous at the point $(t_0, x_0) \in A$ provided that, given $\varepsilon > 0$, there is some $\delta = \delta(t_0, x_0; \varepsilon) > 0$ such that $U(t, x) \subset [U(t_0, x_0)]_\varepsilon$ for all $(t, x) \in N_\delta(t_0, x_0)$. We shall say that $U(t, x)$ is metrically upper semicontinuous in A provided $U(t, x)$ has this property at every point $(t_0, x_0) \in A$. This concept of metric upper semicontinuity is most often used when the sets $U(t, x)$ are compact and all contained in a bounded part of E_m. In general, the sets $U(t, x)$ are only closed and not compact, and in these cases it has been found that analogous concepts of upper semicontinuity, more topological in character, are needed. We shall denote these properties as properties (U) and (Q).

First, given $\delta > 0$ and a point $(t_0, x_0) \in A$, let us denote by $U(t_0, x_0; \delta)$ the set

$$U(t_0, x_0; \delta) = \bigcup U(t, x),$$

where $\bigcup$ ranges over all $(t, x) \in N_\delta(t_0, x_0)$. We shall say that $U(t, x)$ satisfies property (U) at a point $(t_0, x_0) \in A$ provided

$$U(t_0, x_0) = \bigcap_{\delta > 0} cl\, U(t_0, x_0; \delta),$$

that is,

$$U(t_0, x_0) = \bigcap_\delta cl \bigcup_{(t, x) \in N_\delta(t_0, x_0)} U(t, x).$$

We shall say that $U(t, x)$ satisfies property (U) in A if $U(t, x)$ satisfies property (U) at every point $(t_0, x_0) \in A$. A set $U(t, x)$ satisfying property (U) is necessarily closed as the intersection of closed sets. Property (U) is the so-called property of upper semicontinuity used by C. Kuratowski, E. Michael, and G. Choquet (see [4] for references).

Below we shall also consider the sets

$$Q(t, x) = f(t, x, U(t, x)) = [z \mid z = f(t, x, u), u \in U(t, x)] \subset E_N,$$

and other analogous ones, which we shall introduce as needed. We shall say that such a set $Q(t, x)$ satisfies property (Q) at a point $(t_0, x_0) \in A$ provided

$$Q(t_0, x_0) = \bigcap_{\delta > 0} cl\, co\, Q(t_0, x_0; \delta),$$

that is,

$$Q(t_0, x_0) = \bigcap_\delta cl\, co \bigcup_{(t, x) \in N_\delta(t_0, x_0)} Q(t, x).$$

We shall say that $Q(t, x)$ satisfies property (Q) in A if $Q(t, x)$ satisfies property (Q) at every point $(t_0, x_0) \in A$. A set $Q(t, x)$ satisfying property (Q) is necessarily closed and convex as the intersection of closed and convex subsets of E_N.

§ 4. Boundary conditions and the cost functional.

Beside the N-vector $f(t, x, u) = (f_{i\alpha})$, we shall consider a scalar function $f_0(t, x, u)$ defined on M, and we shall denote by $\tilde{f}(t, x, u)$ the $(N+1)$-vector function $\tilde{f}(t, x, u) = (f_0, f_{i\alpha})$, or $\tilde{f} = (f_0, f)$, defined on M. Concerning the n-vector function $x(t) = (x^1, \dots, x^n)$ we shall require that each function $x^i(t)$, $t \in G$, belongs to a Sobolev class $W_{p_i}^{l_i}(G)$ for given l_i and p_i, $1 \leq l_i \leq l$, $p_i \geq 1$, $i = 1, \dots, n$. By force of Sobolev's imbedding theorems [8], as well as by direct arguments, each function x^i and each of its derivatives $D^\alpha x^i$ for which $0 \leq |\alpha| \leq l_i - 1$ has boundary values Φ^i_α defined almost everywhere on the boundary $S = \partial G$ of G, each Φ^i_α being of class L_{p_i} on S.

We shall now require a set (B) of boundary conditions involving the boundary values of the functions x^i and of their generalized partial derivatives $D_\alpha x^i$, $0 \leq |\alpha| \leq l_i - 1$. On these boundary conditions (B) we assume the following closure property : (P_1) if $x(t)=(x^1,\dots,x^n)$, $x_k(t)=(x_k^1,\dots,x_k^n)$, $t \in G$, $k=1,2,\dots$, are vector functions whose components x^i, x_k^i belong to the Sobolev class $W_{p_i}^{l_i}(G)$, if $D_\beta x_k^i(t) \to D^\beta x^i(t)$ as $k \to \infty$ strongly in $L_{p_i}(G)$ for every β with $0 \leq |\beta| \leq l_i - 1$, if $D^\beta x_k^i(t) \to D^\beta x^i(t)$ as $k \to \infty$ weakly in $L_{p_i}(G)$ for every β with $|\beta| = l_i$, and if the boundary values $\Phi_{k\alpha}^i$ of $x_k^i(t)$, $i = 1, \dots, n$, on $S = \partial G$ satisfy boundary conditions (B), then the boundary values Φ_α^i of $x^i(t)$, $i = 1, \dots, n$, on $S = \partial G$ satisfy conditions (B).

For G unbounded but of class K_{0l} in a given representation $G = \bigcup G_N$, $G_N \subset G_{N+1}$ (see § 2) we shall replace (P_1) by the analogous condition, say still (P_1), where we assume only $x^i, x_k^i \in W_{p_i}^{l_i}(G_N)$ for every N, and we assume that $D^\beta x_k^i \to D^\beta x^i$ as $k \to \infty$ strongly in $L_{p_i}(G_N)$ for each N and $0 \leq |\beta| \leq l_i - 1$, and that $D^\beta x_k^i \to D^\beta x^i$ as $k \to \infty$ weakly in $L_{p_i}(G_N)$ for each N and $|\beta| = l_i$, $i = 1, \dots, n$.

For instance, if the boundary conditions (B) are defined by requiring that some of the boundary values coincide with preassigned continuous functions Φ_α^i on certain parts of ∂G, then, by force of Sobolev's imbedding theorems [8], as well as by direct argument, we know that property (P_1) is valid.

A pair $x(t) = (x^1, \dots, x^n)$, $u(t) = (u^1, \dots, u^m)$, $t \in G$, with $x^i \in W_{p_i}^{l_i}(G)$ ($x^i \in W_{p_i}^{l_i}(G_N)$ for every N if G is unbounded), u^j measurable in G, satisfying $(t, x(t)) \in A$, $u(t) \in U(t, x(t))$, $D^\alpha x^i(t) = f_{i\alpha}(t, u(t))$, $\alpha \in \{\alpha\}_i$, $i = 1, \dots, n$, a. e. in G, the boundary conditions (B), and $f_0(t, x(t), u(t)) \in L_1(G)$, is said to be admissible. A class Ω of admissible pairs is said to be complete if, for any sequence x_k, u_k, $k = 1, 2, \dots$ of pairs all in Ω, and any other admissible pair x, u such that $x_k \to x$ in the sense described under (P_1) (for G bounded or G unbounded), then the pair x, u belongs to Ω. The class of all admissible pairs is obviously complete.

The Lebesgue integral

$$I[x, u] = \int_G f_0(t, x(t), u(t))\, dt,$$

where x, u is any admissible pair, and where $dt = dt^1\, dt^2 \dots dt^\nu$, is said to be the cost functional or performance index.

We shall seek the minimum of $I[x, u]$ in classes Ω as above. Problems of maximum of I are equivalent to problems of minimum of $-I$, and corresponding existence theorems for a maximum can be obtained by existence theorems for a minimum by changing f_0 into $-f_0$, that is, changing the sense of the relative inequalities.

We shall often need below a second set of finite systems $\{\beta\}_i$, $i=1,\dots,n$, of indices $\beta=(\beta_1,\dots,\beta_\nu)$, $0\leq|\beta|\leq l_i$, and these systems $\{\beta\}_i$ may be different from the systems $\{\alpha\}_i$. As usual then, $\{\alpha\}_i\cup\{\beta\}_i$ and $\{\alpha\}_i\cap\{\beta\}_i$ shall denote the union and the intersection of the two systems $\{\alpha\}_i$ and $\{\beta\}_i$.

§ 5. Existence theorems for multidimensional Lagrange problems in Sobolev classes W_p^l, $p>1$.

EXISTENCE THEOREM 1. Let G be bounded, open and of class K_{0l}, let A be compact, let $U(t,x)$ be nonempty and compact for every $(t,x)\in A$, and let $U(t,x)$ be metrically upper semicontinuous on A. Let $\tilde{f}(t,x,u)=(f_0, f_{i\alpha}, \alpha\in\{\alpha\}_i, i=1,\dots,n)=(f_0,f)$ be continuous on M, and assume that the set $\tilde{\tilde{Q}}(t,x)$ of all $\tilde{z}=(z^0,z)\in E_{N+1}$ with $z^0\geq f_0(t,x,u)$, $z=f(t,x,u)$, $u\in U(t,x)$, is a convex subset of E_{N+1} for every $(t,x)\in A$. Let (B) be a system of boundary conditions satisfying property (P_1). Let Ω be a nonempty complete class of admissible pairs $x(t)=(x^1,\dots,x^n)$, $u(t)=(u^1,\dots,u^m)$, $t\in G$, $x^i\in W_{p_i}^{l_i}(G)$, $p_i>1$, $1\leq l_i\leq l$, $i=1,\dots,n$, u^j measurable, $j=1,\dots,m$, satisfying given inequalities

$$\int_G |D^\beta x^i|^{p_i}\,dt\leq N_{i\beta},\quad \beta\in\{\beta\}_i,\quad i=1,\dots n, \tag{33}$$

where the $N_{i\beta}$ are given constants, and the $\{\beta\}_i$ given finite systems of indices with $0\leq|\beta|\leq l_i$. Assume that $(x,u)\in\Omega$ and $I[x,u]\leq L_0$ implies $\int_G |D^\gamma x^i(t)|^{p_i}dt\leq$ $\leq L_{i\gamma}$ for some constants $L_{i\gamma}$ (which may depend on L_0, $N_{i\beta}$, (B), Ω, G) and all systems $\gamma=(\gamma_1,\dots,\gamma_\nu)$, $0\leq|\gamma|\leq l_i$, which are not in $\{\alpha\}_i\cup\{\beta\}_i$, $i=1,\dots,n$. Then, the cost functional $I[x,u]$ possesses an absolute minimum in Ω.

Now let G be unbounded, open, and of class K_{0l}, and let $G=\bigcup G_N$, $G_N\subset G_{N+1}$, be a typical representation of G. Let A be closed now instead of compact, but assume that for every closed interval R of E_ν the subset of all $(t,x)\in A$ with $t\in R\cap cl\,G$ is compact. Let us assume that $(*)\,f_0(t,x,u)\geq$ $\geq-\psi(t)$ for all $(t,x,u)\in M$, where $\psi(t)\geq0$ is a given L-integrable function defined on G. This condition is certainly satisfied if, say, $f_0\geq0$ on M. Since Ω is not empty and $f_0(t,x(t),u(t))$ is by hypothesis L-integrable on G for every pair x,u of the class Ω, then hypothesis $(*)$ assures that the infimum i of $I[x,u]$ in Ω is finite. Let us assume that for every pair x,u

of the class Ω we have

$$x^i \in W_{p_i}^{l_i}(G), \quad i = 1, \dots, n, \tag{34}$$

and that there are constants M_i' such that

$$\widetilde{D}_{p_i}^{l_i}(x^i, G) \leq M_i', \quad i = 1, \dots, n, \tag{35}$$

for every admissible pair x, u of the class Ω. Then Theorem 1 still holds, and for the minimizing admissible pairs x, u of the class Ω of which we assert the existence (at least one) we know that (34) and (35) hold.

Again, assume that G is unbounded, open, and of class K_{0l}, and let $G = \bigcup G_N$, $G_N \subset G_{N+1}$, be a typical representation of G. Let us assume again that A is closed but that for every closed interval $R \subset E_\nu$ the set of all $(t, x) \in A$ with $t \in R \cap cl\, G$ is compact, and let us assume that condition $(*)$ is satisfied. Theorem 1 holds also in a particularly weak form. Indeed, it is enough to know that the following conditions are fulfilled: For every pair x, u of the class Ω we have $x^i \in W_{p_i}^{l_i}(G_N)$, $N = 1, 2, \dots, i = 1, \dots, n$; relations (33) hold in the weak form

$$\int_{G_N} | D^\beta x^i (t) |^{p_i} \, dt \leq N_{i\beta}(N), \ \beta \in \{\beta\}_i, \ i = 1, \dots, n,$$

where the $N_{i\beta}(N)$ are given constants which may depend on N; $(x, u) \in \Omega$, $I[x, u] \leq L_0$ implies

$$\int_{G_N} | D^\gamma x^i (t) |^{p_i} \, dt \leq L_{i\gamma}(N)$$

for every N and all $\gamma = (\gamma_1, \dots, \gamma_\nu)$, $0 \leq |\gamma| \leq l_i$, which are not in $\{\alpha\}_i \cup \{\beta\}_i$, $i = 1, \dots, n$ — here the $L_{i\gamma}(N)$ are constants which may depend on N, L_0, $N_{i\beta}(N)$, (B), G, Ω. Under these weak assumptions, theorem 1 still holds, and for the miminizing admissible pairs (x, u) of the class Ω of which we assert the existence we know only that

$$\widetilde{D}_{p_i}^{l_i}(x^i, G_N) \leq M_{iN}', \ i = 1, \dots, n,$$

for every N, where the M_{iN}' are constants which may depend on N.

REMARK 9. It is not restrictive to consider only those pairs (x, u) of Ω satisfying a further constraint of the form $I[x, u] \leq L_0$ provided L_0 is large enough so that Ω so reduced is not empty. The minimizing pair will necessarily satisfy this further constraint. This remark holds also for theorems 2, 3, and 4, and will not be repeated. Theorem 1 for G bounded was proved in our previous paper [3]. For G unbounded the proof is the same with the use of the remarks on compactness theorems in §§ 1 and 2 of the present paper.

EXISTENCE THEOREM 2. Let G be bounded, open, and of class K_{0l}, let A be closed, let $U(t, x)$ be nonempty, closed, and satisfying property (U) in A. Let $\tilde{f}(t, x, u) = (f_0, f_{i\alpha}, \alpha \in \}\alpha\{_i, i = 1, \dots, n) = (f_0, f)$ be continuous on M, and assume that the set $\tilde{Q}(t, x) = \tilde{f}(t, x, U(t, x))$ is closed, convex, and satisfies property (Q) in A. Let (B) be a system of boundary conditions satisfying property (P_1). Let Ω be a nonempty, complete class of admissible pairs $x(t) = (x^1, \dots, x^n)$, $u(t) = (u^1, \dots, u^n)$, $t \in G$, $x^i \in W^{l_i}_{p_i}(G)$, $p_i > 1$, $1 \leq l_i \leq l$, $i = 1, \dots, n$, u^j measurable, $j = 1, \dots, m$, satisfying given inequalities

$$(36) \qquad \int_G | D^\beta x^i(t) |^{p_i} dt \leq N_{i\beta}, \ \beta \in \}\beta\{_i, \qquad i = 1, \dots, n,$$

$$\int_G | f_0(t, x(t), u(t)) |^{p_0} dt \leq N_0,$$

where $p_0 > 1$, $N_{i\beta} > 0$, $N_0 \geq 0$ are given constants, and $\}\beta\{_i$ is a given finite system of indices $\beta = (\beta_1, \dots, \beta_\nu)$ with $0 \leq |\beta| \leq l_i$. Assume that $(x, u) \in \Omega$ and $I[x, u] \leq L_0$ implies $\int_G | D^\gamma x^i(t) |^{p_i} dt \leq L_{i\gamma}$ for some constants $L_{i\gamma}$ (which may depend on L_0, $N_{i\beta}$, (B), Ω, G), and all systems $\gamma = (\gamma_1, \dots, \gamma_\nu)$, $0 \leq |\gamma| \leq l_i$, which are not in $\}\beta\{_i$, $i = 1, \dots, n$. Then, the cost functional $I[x, u]$ possesses an absolute minimum in Ω.

If G is unbounded, open, and of class K_{0l}, and $G = \bigcup G_N$, $G_N \subset G_{N+1}$, is a typical representation of G, let us assume that $(*)$ $f_0(t, x, u) \geq -\psi(t)$ for all $(t, x, u) \in M$, where $\psi(t) \geq 0$ is a given L-integrable function on G. If we know that for every pair x, u of the class Ω we have

$$(37) \qquad x^i \in W^{l_i}_{p_i}(G), \ i = 1, \dots, n, \ f_0(t, x(t), u(t)) \in L_1(G) \cap L_{p_0}(G),$$

and that there are constants M'_i, N_0, N'_0 such that

$$(38) \qquad \tilde{D}^{l_i}_{p_i}(x^i, G) \leq M'_i, \quad i = 1, \dots, n,$$

$$\int_G |f_0(t, x(t), u(t))|^{p_0} \, dt \leq N_0, \tag{39}$$

$$\int_G |f_0(t, x(t), u(t))| \, dt \leq N_0'$$

for every pair x, u of the class Ω, then theorem 2 still holds as above, and for the minimizing admissible pairs x, u of the class Ω of which we assert the existence we know that (37), (38), (39) hold.

For G unbounded, open, and of class K_{0l}, $G = \bigcup G_N$, $G_N \subset G_{N+1}$, a typical representation of G, Theorem 2 holds also in a particular weak form. Indeed, it is enough to know that the following hypotheses are satisfied: the condition (*) holds; for every pair x, u of the class Ω we have $x^i \in W_{p_i}^{l_i}(G_N)$, $i = 1, \dots, n$, for every N; relations (36) hold in the weak form

$$\int_{G_N} | D^\beta x^i(t) |^{p_i} \, dt \leq N_{i\beta}(N), \; \beta \in \{\beta\}_i, \; i = 1, \dots, n,$$

$$\int_{G_N} |f_0(t, x(t), u(t))|^{p_0} \, dt \leq N_0(N),$$

$$\int_G |f_0(t, x(t), u(t))| \, dt \leq N_0',$$

where N_0' is a constant and $N_{i\beta}(N)$, $N_0(N)$ are given constants which may depend on N; $(x, u) \in \Omega$, $I[x, u] \leq L_0$ implies

$$\int_{G_N} | D^\gamma x^i(t) |^{p_i} \, dt \leq L_{i\gamma}(N)$$

for all $N = 1, 2, \dots$ and all γ, $0 \leq |\gamma| \leq l_i$, which are not in $\{\beta\}_i$, $i = 1, \dots, n$ — here the $L_{i\gamma}(N)$ are constants which may depend on N as well as L_0, $N_{i\beta}(N)$, (B), $N_0(N)$, G, Ω as above). Under these weak assumptions, Theorem 2 still holds, and for the minimizing admissible pairs x, u of the class Ω of which we assert the existence we know only that

$$\widetilde{D}_{p.}^{l_i}(x^i, G_N) \leq M_{iN}', \; i = 1, \dots, n,$$

for every $N = 1, 2, \dots$, where the M_{iN}' are constants which may depend on N.

EXISTENCE THEOREM 3. Let G be bounded, open, and of class K_{0l}, let A be closed, let $U(t, x)$ be nonempty, closed, and satisfying property (U) in A. Let $\widetilde{f}(t, x, u) = (f_0, f_{i\alpha}, \alpha \in \{\alpha\}_i, i = 1, \dots, n) = (f_0, f)$ be continuous on M, and let us assume that the set $\widetilde{\widetilde{Q}}(t, x)$ of all $\widetilde{z} = (z^0, z) \in E_{N+1}$ with $z^0 \geq f_0(t, x, u)$, $z = f(t, x, u)$, $u \in U(t, x)$, is a convex closed subset of E_{N-1} satisfying property (Q) in A. Also, let us assume that $f_0(t, x, u) \geq -M_0$ for all $(t, x, u) \in M$ and some constant $M_0 \geq 0$. Let (B) be a system of boundary conditions satisfying property (P_1). Let Ω be a nonempty complete class of admissible pairs $x(t) = (x^1, \dots, x^n)$, $u(t) = (u^1, \dots, u^m)$, $t \in G$, $x^i \in W_{p_i}^{l_i}(G)$, $p_i > 1$, $1 \leq l_i \leq l$, $i = 1, \dots, n$, u^j measurable, $j = 1, \dots, m$, satisfying given inequalities

$$\int_G | D^\beta x^i(t) |^{p_i} \, dt \leq N_{i\beta}, \ \beta \in \{\beta\}_i, \ i = 1, \dots, n, \tag{40}$$

where the $N_{i\beta}$ are given constants, and the $\{\beta\}_i$ are given finite systems of indices $\beta = (\beta_1, \dots, \beta_\nu)$ with $0 \leq |\beta| \leq l_i$. Assume that $x, u \in \Omega$, $I[x, u] \leq L_0$, implies $\int_G | D^\gamma x^i(t) |^{p_i} \, dt \leq L_{i\gamma}$ for some constant $L_{i\gamma}$ (which may depend on L_0, $N_{i\beta}$, (B), Ω, G) and all systems $\gamma = (\gamma_1, \dots, \gamma_\nu)$, $0 \leq |\gamma| \leq l_i$ which are not in $\{\beta\}_i$, $i = 1, \dots, n$. Then, the cost functional $I[x, u]$ possesses an absolute minimun in Ω.

If G is unbounded, open, of class K_{0l}, and $G = \bigcup G_N$, $G_N \subset G_{N+1}$, is a typical representation of G, if we know that $f_0(t, x, u) \geq -\psi(t)$ where $\psi(t) \geq 0$ is of class $L_1(G)$, and if we know that for every pair x, u of the class Ω we have

$$x^i \in W_{p_i}^{l_i}(G), \qquad i = 1, \dots, n, \tag{41}$$

and that there are constants M_i', such that

$$\widetilde{D}_{p_i}^{l_i}(x^i, G) \leq M_i', \quad i = 1, \dots, n, \tag{42}$$

for every admissible pair x, u of the class Ω, then theorem 1 still holds as above, and for the minimizing admissible pair x, u of the class Ω of which we assert the existence we know that (41) and (42) hold.

For G unbounded, open, and of class K_{0l}, $G = \bigcup G_N$, $G_N \subset G_{N+1}$, a typical representation of G, Theorem 3 holds also in a particularly weak form. Indeed, it is enough to know that the following hypotheses are satisfied : $f_0(t, x, u) \geq -\psi(t)$ where $\psi(t) \geq 0$ is of class $L_1(G)$; for each pair x, u of the class Ω we have $x^i \in W_{p_i}^{l_i}(G_N)$, $i = 1, \dots, n$, for every N ; the relations

(40) hold in the weak form

$$\int_{G_N} | D^\beta x^i(t) |^{p_i} \, dt \leq N_{i\beta}(N), \ \beta \in \{\beta\}_i, \ i = 1, \dots, n'$$

where $N_{i\beta}(N)$ are given constants which may depend on N; $(x, u) \in \Omega$, $I[x, u] \leq L_0$, implies

$$\int_{G_N} | D^\gamma x^i(t) |^{p_i} \, dt \leq L_{i\gamma}(N)$$

for all $N = 1, 2, \dots$ and all γ, $0 \leq |\gamma| \leq l_i$, which are not in $\{\beta\}_i$, $i = 1, \dots, n$ — here the $L_{i\gamma}(N)$ are constants which may depend on N as well as L_0, $N_{i\beta}(N)$, (B), G, Ω as above. Under these weak assumptions Theorem 3 still holds, and for the minimizing admissible pairs x, u of the class Ω of which we assert the existence we know only that

$$\widetilde{D}_{p_i}^{l_i}(x^i, G_N) \leq M'_{iN}, \quad i = 1, \dots, n,$$

for every $N = 1, 2, \dots$, where the M'_{iN} are constants which may depend on N.

REMARK 10. Theorems 2 and 3 for G bounded have been proved in our previous paper [4]. For G unbounded the proofs are the same with the use of the remarks on compactness of §§ 1 and 2 of the present paper.

§ 6. An existence theorem for multidimensional Lagrange problems in Sobolev classes W_p^l, $p \geq 1$.

EXISTENCE THEOREM 4. Let G be bounded, open, and of class K_{0l}, let A be closed, let $U(t, x)$ be nonempty, closed, and satisfying property (U) in A. Let $\widetilde{f}(t, x, u) = (f_0, f_{i\alpha}, \alpha \in \{\alpha\}_i, \ i = 1, \dots, n) = (f_0, f)$ be continuous on M, and let us assume that the set $\overset{\approx}{Q}(t, x)$ of all $\widetilde{z} = (z^0, z) \in E_{N+1}$ with $z^0 \geq f_0(t, x, u)$, $z = f(t, x, u)$, $u \in U(t, x)$, is a convex closed subset of E_{N+1} satisfying property (Q) in A. Also, let us assume that (α) there is a continuous scalar function $\Phi(\xi)$, $0 \leq \xi < +\infty$, and two constants $C, D \geq 0$ such that $\Phi(\xi)/\xi \to +\infty$ as $\xi \to \infty$, and $f_0(t, x, u) \geq \Phi(|u|)$, $|f(t, x, u)| \leq C + D|u|$ for all $(t, x, u) \in M$. Let (B) be a system of boundary conditions satisfying property (P_1). Let Ω be a nonempty complete class of admissible pairs

$x(t) = (x^1, \dots, x^n)$, $u(t) = (u^1, \dots, u^m)$, $t \in G$, $x^i \in W_1^{l_i}(G)$, $1 \leq l_i \leq l$, $i = 1, \dots, n$, u^j measurable, $j = 1, \dots, m$, satisfying given inequalities

$$\int_G | D^\beta x^i(t) | \, dt \leq N_{i\beta}, \quad \beta \in \{\beta\}_i, \quad i = 1, \dots, n, \tag{43}$$

where the $N_{i\beta}$ are given constants, and the $\{\beta\}_i$ are given finite systems of indices $\beta = (\beta_1, \dots, \beta_\nu)$ with $0 \leq |\beta| \leq l_i$. Assume that $(x, u) \in \Omega$, $I[x, u] \leq L_0$ implies (1) $\int_G | D^\gamma x^i(t) | \, dt \leq L_{i\gamma}$ for some constants $L_{i\gamma}$ (which may depend on L_0, $N_{i\beta}$, (B), Ω, G) and all systems $\gamma = (\gamma_1, \dots, \gamma_\nu)$, $0 \leq |\gamma| \leq l_i - 1$, which are not in $\{\alpha\}_i \cup \{\beta\}_i$, $i = 1, \dots, n$; (2) $D^\gamma x^i(t)$, $t \in G$, are equiabsolutely integrable in G, $i = 1, \dots, n$, for all finite systems $\gamma = (\gamma_1, \dots, \gamma_\nu)$ with $|\gamma| = l_i$ which are not already in $\{\alpha\}_i$ (if any). Then, the cost functional $I[x, u]$ possesses an absolute minimum in Ω.

If G is unbounded, open, of class K_{0l}, and $G = \bigcup G_N$, $G_N \subset G_{N+1}$, is a typical representation of G, if we know that (i) $f_0(t, x, u) \geq -\psi(t)$ for all $(t, x, u) \in M$, where $\psi(t) \geq 0$ is of class $L_1(G)$; (ii) for every N there is a continuous function $\Phi_N(\xi)$, $0 \leq \xi < +\infty$, and two constants C_N, $D_N \geq 0$ such that $\Phi_N(\xi)/\xi \to +\infty$ as $\xi \to +\infty$, and $f_0(t, x, u) \geq \Phi_N(|u|)$, $|f(t, x, u)| \leq C_N + D_N |u|$ for all $(t, x, u) \in M$ with $t \in G_N$, if we know that for every pair (x, u) of the class Ω we have

$$x^i \in W_1^{l_i}(G), \quad i = 1, \dots, n, \tag{44}$$

and that there are constants M_i' such that

$$\tilde{D}_1^{l_i}(x^i, G) \leq M_i', \quad i = 1, \dots, n, \tag{45}$$

for every admissible pair x, u of the class Ω, and $(x, u) \in \Omega$, $I[x, u] \leq L_0$ implies (2) as above, then Theorem 4 still holds as above, and for the minimizing admissible pairs x, u of the class Ω of which we assert the existence we know that (44) and (45) hold.

For G unbounded, open, and of the class K_{0l}, $G = \bigcup G_N$, $G_N \subset G_{N+1}$, a typical representation of G, Theorem 4 holds also in a particulary weak form. Indeed, it is enough to know that the following hypotheses are satisfied: (i), (ii) above hold; for each pair x, u of the class Ω we have $x^i \in W_1^{l_i}(G_N$, $i = 1, \dots, n$, for every N; relations (43) hold in the weak form

$$\int_{G_N} | D^\beta x^i(t) | \, dt \leq N_{i\beta}(N), \quad \beta \in \{\beta\}_i, \quad i = 1, \dots, n,$$

where $N_{i\beta}(N)$ are given constants which may depend on N; $(x,u)\in\Omega$, $I[x,u]\leq L_0$ implies (1)$'$ $\int_{G_N} | D^\gamma x^i(t) | \, dt \leq L_{i\gamma}(N)$ for every N, where $L_{i\gamma}(N)$ are constants (which may depend on N, L_0, $N_{i\beta}(N)$, (B), Ω, G), and where $\gamma = (\gamma_1, \dots, \gamma_\nu)$ in any systems with $0 \leq |\gamma| \leq l_i - 1$, which is not in $\{\alpha\}_i \cup \{\beta\}_i$, $i = 1, \dots, n$; (2$'$) for every N, $D^\gamma x^i(t)$, $t \in G_N$, are equiabsolutely integrable in G_N, for $i=1,\dots,n$, and all finite systems $\gamma = (\gamma_1, \dots, \gamma_\nu)$ with $|\gamma| = l_i$ which are not in $\{\alpha\}_i$ (if any). Then, Theorem 4 still holds, and for the minimizing admissible pair x, u of the class Ω of which we assert the existence we know only that $\widetilde{D}_1^{l_i}(x^i, G_N) \leq M'_{iN}$, $i = 1, \dots, n$, for every $N = 1, 2, \dots$, where the M'_{iN} are constants which may depend on N.

PROOF. First let us consider the case G bounded. We have $\Phi(\xi) \geq -M_0$ for some number $M_0 \geq 0$, hence $\Phi(\xi) + M_0 \geq 0$ for all $\xi \geq 0$, and $f_0(t, x, u) + M_0 \geq 0$ for all $(t, x, u) \in M$. Therefore, for every pair x, u in Ω we have

$$I[x,u] = \int_G f_0 \, dt \geq \int_G \Phi(|u|) \, dt \geq -M_0 |G| > -\infty,$$

where $|G| = \text{meas } G$.

Let $i = \operatorname{Inf} I[x,u]$, where Inf is taken for all pairs $(x,u) \in \Omega$. Then i is finite.

Let $x_k(t)$, $u_k(t)$, $t \in G$, $k = 1, 2, \dots$, be a sequence of pairs all in Ω such that $I[x_k, u_k] \to i$ as $k \to \infty$. We may assume

$$i \leq I[x_k, u_k] = \int_G f_0(t, x_k(t), u_k(t)) \, dt \leq i + k^{-1} \leq i + 1, \quad k = 1, 2, \dots .$$

Let us prove that the functions $u_k^j(t)$, $D^\alpha x_k^i(t)$, $\alpha \in \{\alpha\}_i$, $i = 1, \dots, n$, $j = 1, \dots, m$, are equiabsolutely integrable in G. We shall use a well known argument. Let $\varepsilon > 0$ be any given number, and let $\sigma = 2^{-1}\varepsilon(|G| M_0 + |i| + 1)^{-1}$, where $|G| = \text{meas } G$.

Let $N > 0$ be a number such that $\Phi(\xi)/\xi > \sigma^{-1}$ for all $\xi \geq N$. Let E be any measurable subset of G with meas $E < \eta = \varepsilon/2N$. Let E_1 be the subset of all $t \in E$ where $u_k(t)$ is finite and $|u_k(t)| \leq N$, and let $E_2 = E - E_1$. Then $|u_k(t)| \leq N$ in E_1, and $\Phi(|u_k|)/|u_k| \geq 1/\sigma$, or $|u_k| \leq \sigma\Phi(|u_k|)$ a.e. in E_2. Hence

$$\int_E |u_k(t)| \, dt = \left(\int_{E_1} + \int_{E_2} \right) |u_k(t)| \, dt$$

$$\leq N \text{ meas } E_1 + \sigma \int_{E_2} \Phi(|u_k(t)|)\, dt$$

$$\leq N \text{ meas } E + \sigma \int_{E_2} [\Phi(|u_k(t)|) + M_0]\, dt$$

$$\leq N\eta + \sigma \int_G [\Phi(|u_k(t)|) + M_0]\, dt$$

$$\leq N\eta + \sigma \int_G [f_0(t, x_k(t), u_k(t)) + M_0]\, dt$$

$$\leq N\eta + \sigma(M_0 |G| + |i| + 1) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This proves that the vector functions $u_k(t)$, $t \in G$, $k = 1, 2, \dots$, are equiabsolutely integrable. From here we deduce

$$\int_E |D^\alpha x_k^i(t)|\, dt = \int_E |f_{i\alpha}(t, x_k(t), u_k(t))|\, dt \leq \int_E [C + D|u_k(t)|]\, dt$$

$$\leq C \text{ meas } E + D \int_E |u_k(t)|\, dt,$$

and this proves the equiabsolute integrability of the functions $D^\alpha x_k^i(t)$, $\alpha \in \{\alpha\}_i$, $i = 1, \dots, n$. Thus, all functions $D^\alpha x_k^i(t)$, $k = 1, 2, \dots$, $\alpha \in \{\alpha\}_i$, $i = 1, \dots, n$, are equiabsolutely integrable in G.

Since $|G| = \text{meas } G$ is finite, we conclude that the integrals

$$\int_G |D^\alpha x_k^i(t)|\, dt \leq N_{i\alpha}, \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, n, \quad k = 1, 2, \dots$$

admit of finite bounds $N_{i\alpha}$ which are independent of k. By the hypotheses of the theorem we conclude that all partial derivatives $D^\alpha x_k^i$ with $|\alpha| = l_i$, $i = 1, \dots, n$, are equiabsolutely integrable in G, and that for all derivatives $D^\alpha x_k^i$, $i = 1, \dots, n$, with $0 \leq |\alpha| \leq l_i - 1$, there are bounds

$$\int_G |D^\alpha x_k^i|\, dt \leq N_{i\alpha}, \text{ or } L_{i\alpha}, \quad 0 \leq |\alpha| \leq l_i - 1, \quad i = 1, \dots, n.$$

By application of the compactness theorems of § 2, we conclude that there are functions x^i with $x^i \in W_1^{l_i}(G)$, $i = 1, \dots, n$, and a suitable subsequence such that $x_k^i \to x^i$ in the sense described under property (P_1). The proof, from here on is identical with the one we have given in our previous paper [4].

For G unbounded the proof is the same with use of the remarks on compactness of §§ 1 and 2 of the present paper.

In theorem 4 we have taken all $p_i = 1$, $i = 1, \dots, n$. Combining the hypotheses and the arguments of theorems 3 and 4 it is possible to set up existence theorems with some exponents $p_i = 1$ and other exponents $p_i > 1$, $i = 1, \dots, n$.

REMARK 11. Property (Q) for the sets $Q(t, x)$ of existence theorem 4 is a consequence of growth condition (α) of the same theorem (or of (ii) if G is unbounded). Namely, property (Q) for G bounded can be shown to be a consequence of the continuity hypothesis, of property (U) for $U(t, x)$, and of the following milder growth condition: (β) Given $\varepsilon > 0$, there is a number $\bar{u}_\varepsilon \geq 0$ such that $1 \leq \varepsilon f_0(t, x, u)$, $|f(t, x, u)| \leq \varepsilon f_0(t, x, u)$ for all $(t, x, u) \in M$ with $|u| \geq \bar{u}_\varepsilon$. For G unbounded it is enough to know that a property (β) holds for every subset G_N. Furthermore, the continuity hypothesis of f_0 on M can be replaced by lower semicontinuity on M. [L. Cesari, Existence theorems for optimal controls of the Mayer type, SIAM J. Control, 6, no. 4, 1968].

REMARK 12. Growth condition (α) of existence theorem 4 for G bounded can be replaced by the following milder one: (ε) Given $\varepsilon > 0$, there is a nonnegative L-integrable function $\psi_\varepsilon(t)$, $t \in G$, which may depend on ε, such that $|f(t, x, u)| \leq \psi_\varepsilon(t) + \varepsilon f_0(t, x, u)$ for all $(t, x, u) \in M$. Indeed, first note that (ε) implies, for $\varepsilon = 1$, $0 \leq \psi_1(t) + f_0(t, x, u)$ for all $(t, x, u) \in M$, with $\psi_1(t) \geq 0$, and we shall denote by m the constant $m = \int_G \psi_1(t)\, dt < +\infty$. Then, given any $\varepsilon > 0$, take $\sigma = 2^{-1}\varepsilon(|i| + m + 1)^{-1}$, and let E denote any measurable subset of G. For $\alpha \in \{\alpha\}_i$, $i = 1, \dots, n$, we have

$$\int_E |D^\alpha x_k^i(t)|\, dt \leq \int_E [\psi_\sigma(t) + \sigma f_0(t, x_k(t), u_k(t))]\, dt$$

$$\leq \int_E \psi_\sigma\, dt + \sigma \int_G [f_0 + \psi_1]\, dt$$

$$\leq \int_E \psi_\sigma\, dt + \sigma[|i| + m + 1] \leq \varepsilon/2 + \int_E \psi_\sigma\, dt.$$

There is, therefore, some $\delta > 0$ such that meas $E < \delta$ implies $\int_E \psi_\sigma \, dt < \varepsilon/2$ and

$$\int_E | D^\alpha x_k^i(t) | \, dt \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

We have proved the equiabsolute integrability in G of all derivatives $D^\alpha x_k^i$, $\alpha \in \{\alpha\}_i$, $i = 1, \dots, n$, $k = 1, 2, \dots$. For G unbounded and the weak form of existence theorem 4 it is enough to know that a property (ε) holds in each set G_N.

§ 7. Weak solutions.

In many cases the sets $\tilde{Q}$, or $\approx{Q}$, are not convex, and examples show that an optimal solution may well fail to exist. It has been proposed, then, to replace the differential system and functional

$$D^\alpha x^i = f_{i\alpha}(t, x, u), \; \alpha \in \{\alpha\}_i, \; i = 1, \dots, n, \tag{47}$$

$$I = \int_G f_0(t, x, u) \, dt, \tag{48}$$

by a new analogous system of differential equations and a new functional, say,

$$D^\alpha x^i = g_{i\alpha}(t, x, v), \; \alpha \in \{\alpha\}_i, \; i = 1, \dots, n,$$

$$J = \int_G g_0(t, x, v) \, dt.$$

More precisely, we consider with R. V. Gamkrelidze [5] the differential system and functional defined by

$$D^\alpha x^i = \Sigma \lambda_j f_{i\alpha}(t, x, u^{(j)}), \; \alpha \in \{\alpha\}_i, \; i = 1, \dots, n, \tag{49}$$

$$J = \int_G \Sigma \lambda_j f_0(t, x, u^{(j)}) \, dt, \tag{50}$$

where Σ ranges from 1 to μ, where the new control variable v is now an $(m\mu + \mu)$-vector, say $u^{(1)}, \dots, u^{(\mu)}$, $\lambda_1, \dots, \lambda_\mu$, each $u^{(j)}$ being an m-vector subject to the usual constraint $u^{(j)} \in U(t, x)$, $j = 1, \dots, \mu$, and each λ_j being a scalar satisfying $\lambda_j \geq 0, j = 1, \dots, \mu$, $\Sigma \lambda_j = 1$. If we denote by λ the μ-vector $\lambda = (\lambda_1, \dots, \lambda_\mu)$

and by Γ the simplex defined by $\lambda_j \geq 0$, $j = 1, \dots, \mu$, $\Sigma \lambda_j = 1$, then for the new control variable $v = (u^{(1)}, \dots, u^{(m)}, \lambda)$ we have the constraint $v \in V = U^\mu \times \Gamma$, where $U^\mu = U \times \dots \times U$ (μ times). We shall then require that all $u^{(j)}$, λ are measurable functions on G, and that the new space variables x^i, $i = 1, \dots, n$, belong to the same Sobolev's spaces as before, $x^i \in W^{l_i}_{p_i}(G)$, $i = 1, \dots, n$.

Thus, there are still n state variables x^i as before, but they satisfy a more complex system of partial differential equations, and an analogous change has been made in the functional.

The variables λ_j can be thought of as probability distributions, and correspondingly the new (weak, or generalized) solutions can be thought of as generated by a probability distribution of the μ usual controls $u^{(j)}$ (acting contemporaneously).

Instead of the sets $\widetilde{Q}(t, x) = \widetilde{f}(t, x, U(t, x))$, or the sets $\stackrel{\approx}{Q}(t, x)$, we shall now consider the analogous sets $\widetilde{Q}^* = g(t, x, V(t, x))$, and $\stackrel{\approx}{Q}{}^*$, with

$$\widetilde{Q}^*(t, x) = \widetilde{g}(t, x, V)) = [\widetilde{\xi} \mid \widetilde{\xi} = \Sigma \lambda_j \widetilde{f}(t, x, u^{(j)}), \ (u^{(1)}, \dots, u^{(\mu)}, \lambda) \in U^\mu \times \Gamma].$$

Obviously $\widetilde{Q}^*$ is the set of all the points which can be thought of as convex combinations of μ points of $\widetilde{Q}$. Thus, for $\mu \geq N + 2$, the set $\widetilde{Q}^*$ is certainly convex and so will be the set $\stackrel{\approx}{Q}{}^*$.

Existence theorems 1, 2, and 3 can be repeated for the (weak, or generalized) problem above without any essential change. The same holds for theorem 4 by using growth condition (ε) of the last remark of § 6 (instead of condition (α)), since condition (ε) for f_0, f implies an analogous property for g_0, g. Indeed, if $|f(t, x, u)| \leq \psi_\varepsilon(t) + \varepsilon f_0(t, x, u)$ for all $(t, x) \in A$, $u \in U(t, x)$, then

$$\begin{aligned} |g(t, x, v)| &= |\Sigma_j \lambda_j f(t, x, u^{(j)})| \\ &\leq \Sigma_j \lambda_j [\psi_\varepsilon(t) + \varepsilon f_0(t, x, u^{(j)})] \\ &= \psi_\varepsilon(t) + \varepsilon g_0(t, x, v) \end{aligned}$$

for all $(t, x) \in A$, $v \in V = U^\mu \times \Gamma$.

REMARK 13. For $\nu = 1$ the introduction of weak, or generalized solutions has been associated to the remark that — under conditions often satisfied — the solutions of the modified equations (49) and the values of the corresponding integral J can be approached by means of usual solutions of the unmodified equations (47) and the values of the corresponding integral I. See, also for references, [2], II, nos. 14 and 15]). Such a question for $\nu > 1$ will be answered elsewhere.

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