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ON ONE INEQUALITY IN WEIGHTED L_p SPACES CONNECTED WITH THE PROBLEM OF EXISTENCE OF TRACES ON HYPERPLANES

JAN KADLEC † (*)

1. Introduction.

Let it be given a hyperplane $\Pi, 0 \in \Pi$ in the Euclidean N-space R^N. If u is a function of variable $x \in R^N$ we can consider the function $Zu = u/\Pi$ defined on the hyperplane Π such that

$$Zu(x) = u(x) \qquad \forall\, x \in \Pi.$$

Let $\mathcal{L} = \{B^1, \ldots, B^N\}, B^i \in R^N$ be certain basis of $R^N, x = \sum_{i=1}^{N} x_i B^i$. Then we can treat the function u as a function $u_{\mathcal{L}}$ of N variables $x_1, \ldots, x_N$

$$u_{\mathcal{L}}(x_1, \ldots, x_N) = u(x).$$

Suppose that $\mathcal{L}' = \{B^1, \ldots, B^{N-1}\}$ is a basis of the hyperplane Π. Then

$$Zu(x) = (Zu)_{\mathcal{L}'}(x_1, \ldots, x_{N-1}) = u_{\mathcal{L}}(x_1, \ldots, x_{N-1}, 0).$$

So, the basis $\mathcal{L}$ is connected with Π.

In R^N let us have a fundamental basis $\mathcal{L}_0 = \{B^{1*}, \ldots, B^{N*}\}$. A relation between $\mathcal{L}$ and $\mathcal{L}_0$ is described by the $N \times N$-matrix

$$B = \begin{pmatrix} B_1^1, B_2^1, \ldots, B_N^1 \\ B_1^2, B_2^2, \ldots, B_N^2 \\ \cdots\cdots\cdots \\ B_1^N, B_2^N, \ldots, B_N^N \end{pmatrix}$$

$$\det B \neq 0$$

where $B^i = \sum_{j=1}^{N} B_j^i B^{j^*}$.

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The Editorial Committee deeply regrets the author's untimely death.

The space of all infinitely differentiable functions in R^N with compact support will be denoted by $\mathcal{D}(R^N)$.

We use this notation: If $\xi = (\xi_1, \dots, \xi_n)$, $x = (x_1, \dots, x_n)$ then

$$\langle x, \xi \rangle = \sum_{i=1}^{n} \xi_i x_i .$$

If $\xi = (\xi_1, \dots, \xi_N)$, put $\bar{\xi} = (\xi_1, \dots, \xi_{N-1})$, $\xi = (\bar{\xi}, \xi_N)$ and similary $x = (\bar{x}, x_N)$. If A is a matrix then the inverse and transpose of A we denote by A^{-1} and A' resp., ξA is product of the vector ξ and the matrix A.

Let $u(x) = u(x_1, \dots, x_N)$ be a function of N variables, $u \in \mathcal{D}(R^N)$. Put

$$\mathcal{F}u(\xi) = \int_{R^N} e^{-i \langle \xi, x \rangle} u(x)\, dx$$

the Fourier transform of u. If u is a function of points in R^N then put

$$\mathcal{F}_{\mathcal{L}} u(\xi) = \mathcal{F} u_{\mathcal{L}}(\xi)$$

and similary for functions of $N - 1$ variables.

Let $u \in \mathcal{D}(R^N)$. Then by lemma 2.3 one has

$$\mathcal{F}_{\mathcal{L}'} Zu(\bar{\xi}) = \frac{1}{2\pi} |\det B|^{-1} \int_{-\infty}^{\infty} \mathcal{F}_{\mathcal{L}_0} u(\xi B^{-1'})\, d\xi_N .$$

From properties of $\mathcal{F}_{\mathcal{L}_0} u$ we can deduce properties of $\mathcal{F}_{\mathcal{L}'} Zu$. So, in this paper we will study properties of the operator T given by

$$Tf(\bar{\xi}) = \frac{1}{2\pi} |\det B|^{-1} \int_{-\infty}^{\infty} f(\xi B^{-1'})\, d\xi_N ,$$

that is of $T = \mathcal{F}_{\mathcal{L}'} Z \mathcal{F}_{\mathcal{L}_0}$.

Properties of T are dependent on the position of hyperplane Π. Suppose

$$\Pi = \left\{ x = \sum_{i=1}^{N} x_i B^{i*}, \sum_{i=1}^{N} x_i a_i = 0 \right\} \tag{1.1}$$

where

(1.2) $$a_1 = a_2 = \dots = a_r = 0$$

$$a_{r+1} \neq 0, a_{r+2} \neq 0, \dots, a_{N-1} \neq 0, a_N = 1.$$

Theorem 2.5. gives us the possibility to take

(1.3) $$B = \begin{pmatrix} 1, 0, 0, \dots, 0, -a_1 \\ 0, 1, 0, \dots, 0, -a_2 \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ 0, 0, 0, \dots, 0, 1, -a_{N-1} \\ 0, 0, 0, \dots, 0, 0, 1 \end{pmatrix}$$

that is

(1.4) $$\xi B^{-1'} = (\xi_1, \xi_2, \dots, \xi_r, \xi_{r+1} + a_{r+1}\xi_N, \dots, \xi_{N-1} + a_N \xi_N, \xi_N).$$

In the following we shall study spaces $\mathcal{W}_p^{(\mathcal{K})}(R^N)$ given by a convex set $\mathcal{K}$ and by p real, $1 < p < \infty$.

We say $\mathcal{K} \in \mathfrak{p}$ if the set $\mathcal{K}^e$ of all extremal points of the bounded convex set $\mathcal{K}$ is finite. Put

$$\mathcal{P}_{\mathcal{K}}(\xi) = \max_{A \in \mathcal{K}^e} |\xi|^A = \max_{A \in \mathcal{K}} |\xi|^A$$

where $|\xi|^A = |\xi_1|^{A_1} |\xi_2|^{A_2} \dots |\xi_N|^{A_N}$.

Then $\mathcal{W}_p^{(\mathcal{K})}(R^N)$ is the space of all measurable functions f for which

$$|f|_{\mathcal{W}_p^{(\mathcal{K})}(R^N)} = |\mathcal{P}_{\mathcal{K}} f|_{L_p(R^N)}$$

is finite.

In this paper are given necessary and sufficient conditions for

(1.5) $$|Tf|_{L_p(R^{N-1})} \leqq C |\mathcal{P}_{\mathcal{K}} f|_{L_p(R^N)} \qquad (C < \infty).$$

Put $H_p^{(0)}(R^N) = \mathcal{F}^{-1} L_q(R^N)$, $H_p^{(\mathcal{K})}(R^N) = \mathcal{F}^{-1} \mathcal{W}_q^{(\mathcal{K})}(R^N)$, $1/p + 1/q = 1$

(for the precise sense of Fourier transform $\mathcal{F}$ see Lizorkin [9]; let us note only that $H_p^{(\mathcal{K})}(R^N)$ is not generally a subspace of temperated distributions S').

The inequality (1.5) can be rewritten in the form

$$| Z^* u |_{H_q^{(0)}(R^{N-1})} \leqq C | u |_{H_q^{(\mathcal{K})}},$$

where $Z^* = \mathcal{F}_{\mathcal{L}'}^{-1} T \mathcal{F}_{\mathcal{L}_0}$.

For $u \in \mathcal{D}(R^N) \cap H_q^{(\mathcal{K})}(R^N)$ one has $Z^* u = Zu$. So $Z^* u$ can be treated as trace of u on Π.

Validity of (1.5) depends on the mutual position of the set

$$q\mathcal{K} = \{X \in R^N, q^{-1} X \in \mathcal{K}\}, \qquad 1/q + 1/p = 1$$

and the $(N - r - 1)$-dimensional simplex $\mathcal{S}$ given by the coordinate vectors

$$I_1 = (\underbrace{0, \dots, 0}_{r}, 1, 0, \dots, 0, 0)$$

$$I_2 = (\underbrace{0, \dots, 0}_{r}, 0, 1, \dots, 0, 0)$$

$$\dots\dots\dots\dots$$

$$I_{N-r} = (0, \dots, 0, 0, 0, \dots, 0, 1).$$

Necessary and sufficient conditions for (1.5) are described in theorem 4.7 and theorem 6.7 (see remark 6.8). It must be $q\mathcal{K} \cap \mathcal{S} \neq \emptyset$ and the set $q\mathcal{K}$ must be in a certain sense « well distributed » with respect to $\mathcal{S}$.

In the following we also use this notation: if $x = (x_1, \dots, x_N) \in R^N$ then

$$x' = (x_1, \dots, x_r) \in R^r, x'' = (x_{r+1}, \dots, x_N) \in R^s, x = (x', x''), s = N - r.$$

Here the number r is given by (1.2).

2. Dual traces.

2.1 LEMMA. Let $u \in \mathcal{D}(R^N)$. Then

$$\mathcal{F}_{\mathcal{L}} u(\xi) = |\det B|^{-1} \mathcal{F}_{\mathcal{L}_0} u(\xi B^{-1'}). \tag{2.1}$$

Proof. Using the substitutions $xB = y$ and $u_{\mathcal{L}}(x) = u_{\mathcal{L}_0}(xB)$ one obtains

$$\mathcal{F}_{\mathcal{L}} u(\xi) = |\det B|^{-1} \int_{R^N} e^{-i \langle y, \xi B^{-1'} \rangle} u_{\mathcal{L}_0}(y)\, dy = |\det B|^{-1} \mathcal{F}_{\mathcal{L}_0}(\xi B^{-1'}).$$

By a similar argument one obtains

2.2 LEMMA. Let $u \in \mathcal{D}(R^N)$. Put $v(x) = u(x + x_0)$. Then

$$\mathcal{F}_{\mathcal{L}} v(\xi) = e^{i < x_0, \xi >} \mathcal{F}_{\mathcal{L}} u(\xi). \tag{2.2}$$

2.3 LEMMA. Let $u \in \mathcal{D}(R^N)$. Let us denote $v = Zu$ on Π, the trace of u on the hyperplane Π. Then

$$\mathcal{F}_{\mathcal{L}'} v(\bar{\xi}) = \frac{1}{2\pi} |\det B|^{-1} \int_{-\infty}^{\infty} \mathcal{F}_{\mathcal{L}_0} u(\xi B^{-1'}) \, d\xi_N . \tag{2.3}$$

Proof: It is known

$$\mathcal{F}_{\mathcal{L}'} v(\bar{\xi}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}_{\mathcal{L}} u(\xi) \, d\xi_N .$$

Using the lemma 2.1 one obtains (2.3).

If we put $u = \mathcal{F}_{\mathcal{L}_0}^{-1} f$ in (2.3) we have

$$Tf = \mathcal{F}_{\mathcal{L}'} Z \mathcal{F}_{\mathcal{L}_0}^{-1} f(\bar{\xi}) = \frac{1}{2\pi} |\det B|^{-1} \int_{-\infty}^{\infty} f(\xi B^{-1'}) \, d\xi_N .$$

2.4 DEFINITION. Let f be a measurable function in R^N such that the (Lebesgue) integral

$$\int_{-\infty}^{\infty} f(\xi B^{-1'}) \, d\xi_N$$

exists for a. e. $\bar{\xi} = (\xi_1, \dots, \xi_{N-1}) \in R^{N-1}$. Then the function $g = Tf$ of $N - 1$ variables given by

$$g(\bar{\xi}) = \frac{1}{2\pi} |\det B|^{-1} \int_{-\infty}^{\infty} f(\xi B^{-1'}) \, d\xi_N \tag{2.4}$$

is said to be the dual trace (in the basis $\mathcal{L}'$) of the (dual) function f on the hyperplane Π.

2.5 **THEOREM.** The dual trace of the function f is independent on the basis $\mathcal{L}$ in this sense: if $\mathcal{L}_* = \{B_*^1, \ldots, B_*^N\}$ is another basis which fulfils our conditions,

$$B_* = \begin{pmatrix} B_*^1 \\ \vdots \\ B_*^N \end{pmatrix},$$

$$\begin{pmatrix} B_*^1 \\ \vdots \\ B_*^{N-1} \end{pmatrix} = C \begin{pmatrix} B^1 \\ \vdots \\ B^{N-1} \end{pmatrix}, \tag{2.5}$$

where C is a regular $(N-1) \times (N-1)$-matrix and

$$g^* (\bar{\xi}) = \frac{1}{2\pi} \, | \det B_* |^{-1} \int_{-\infty}^{\infty} f(\xi B_*^{-1'}) \, d\xi_N$$

then

$$g(\bar{\xi}) = | \det C | \, g^* (\bar{\xi} C').$$

In other words: if g is the Fourier transform $\mathcal{F}_{\mathcal{L}'} v$ of some function v defined on Π then $g^* = \mathcal{F}_{\mathcal{L}'_*} v$.

Proof. Put

$$\mathcal{B} = \begin{pmatrix} B^1 \\ \vdots \\ B^{N-1} \end{pmatrix}, \quad \mathcal{B}_* = \begin{pmatrix} B_*^1 \\ \\ B_*^{N-1} \end{pmatrix}.$$

Then there is a vector $d = (\bar{d}, d_N)$; $d_N \neq 0$ such that for

$$\mathcal{C} = \left(\begin{array}{c|c} C & 0 \\ \hline \bar{d} & d_N \end{array} \right),$$

one has

$$B_* = \mathcal{C} B, \tag{2.6}$$

$$B^{-1'} = \mathcal{C}' B_*^{-1'}$$

$$| \det B_* | = | \det C | \, | d_N | \, | \det B |$$

$$(\bar{\xi}, \xi_N) \, \mathcal{C}' = \left(\bar{\xi} C', \sum_{i=1}^{N} d_i \xi_i \right).$$

By (2.6) we have

$$g(\bar{\xi}) = \frac{1}{2\pi} \mid \det B \mid^{-1} \int_{-\infty}^{\infty} f(\xi B^{-1'}) \, d\xi_N =$$

$$= (\text{using the substitution } \xi_N = d_N^{-1} \tau) =$$

$$= \frac{1}{2\pi} \mid d_N \mid^{-1} \mid \det B \mid^{-1} \int_{-\infty}^{\infty} f((\bar{\xi}, \tau / d_N) \, B^{-1'}) \, d\tau =$$

$$= \frac{1}{2\pi} \mid \det C \mid \mid \det B_* \mid^{-1} \int_{-\infty}^{\infty} f((\bar{\xi}, \tau / d_N) \, \mathcal{C}' \, B_*^{-1'}) \, d\tau =$$

$$= \mid \det C \mid \frac{1}{2\pi} \mid \det B_* \mid^{-1} \int_{-\infty}^{\infty} f\left(\left(\bar{\xi} \, C', \sum_{i=1}^{N-1} d_i \xi_i + \tau\right) B_*^{-1'}\right) d\tau =$$

$$= \left(\text{using the substitution } \sigma = \sum_{i=1}^{N-1} d_i \xi_i + \tau\right) =$$

$$= \mid \det C \mid g^* (\bar{\xi} \, C').$$

This completes the proof of the first part of theorem 2.5.
Let, now, $g = \mathcal{F}_{\mathcal{L}'} v$. Then by lemma 2.1 one has

$$\mathcal{F}_{\mathcal{L}'_*} v(\bar{\xi}) = \mid \det D \mid^{-1} \mathcal{F}_{\mathcal{L}'} v(\bar{\xi} \, D^{-1'}), \tag{2.7}$$

where D is the matrix of coordinates of vectors $B_*^1, \ldots, B_*^{N-1}$ in the basis $\mathcal{L}'$. We have $\mathcal{B}_* = C \mathcal{B}$ and so $D = C$. By (2.7) we have

$$g(\bar{\xi}) = \mathcal{F}_{\mathcal{L}'} v(\bar{\xi}) = \mid \det C \mid \mathcal{F}_{\mathcal{L}'_*} v(\bar{\xi} \, C') = \mid \det C \mid g^* (\xi \, C')$$

and so

$$g^* = \mathcal{F}_{\mathcal{L}'_*} v.$$

This completes the proof.

3. Conditions for continuity of the operator T.

3.1 THEOREM. The operator T is continuous from $\mathcal{W}_p^{(k)}$ into L_p $(p > 1)$ iff for $1/q = 1 - 1/p$ one has

$$|T| = \frac{1}{2\pi} |\det B|^{-1/q} \left(\operatorname*{supess}_{\bar{\xi} \in R^{N-1}} \int_{-\infty}^{\infty} [\mathcal{P}_{\mathcal{K}}(\xi B^{-1'})]^{-q} \right)^{1/q} < \infty. \tag{3.1}$$

Proof. Let $g = Tf$, $\tilde{h}(\xi) = h(\bar{\xi})$, $u = \mathcal{P}_{\mathcal{K}} f$. Then

$$|g|_{L_p} = \sup_{|h|_{L_q}=1} \int_{R^{N-1}} h(\bar{\xi})\, g(\bar{\xi})\, d\bar{\xi} =$$

$$= \sup_{|h|_{L_q}=1} \frac{1}{2\pi} |\det B|^{-1} \int_{R^N} \tilde{h}(\xi)\, f(\xi B^{-1'})\, d\xi$$

$$= \sup_{|h|_{L_q}=1} \frac{1}{2\pi} |\det B|^{-1} \int_{R^N} \tilde{h}(\xi) \frac{u(\xi B^{-1'})}{\mathcal{P}_{\mathcal{K}}(\xi B^{-1'})}\, d\xi$$

and

$$|T| = \sup_{|f|_{\mathcal{W}_p^{(k)}}=1} |Tf|_{L_p} =$$

$$= \frac{1}{2\pi} |\det B|^{-1} \sup_{|u|_{L_p}=1} \sup_{|h|_{L_q}=1} \int_{R^N} \tilde{h}(\xi) \frac{u(\xi B^{-1'})}{\mathcal{P}_{\mathcal{K}}(\xi B^{-1'})}\, d\xi =$$

$$= (\text{using the substitution } \xi = \eta B') =$$

$$= \frac{1}{2\pi} \sup_{|h|_{L_q}=1} \sup_{|u|_{L_p}=1} \int_{R^N} \frac{\tilde{h}(\eta B')}{\mathcal{P}_{\mathcal{K}}(\eta)} u(\eta)\, d\eta = \frac{1}{2\pi} \sup_{|h|_{L_q}=1} \left| \frac{\tilde{h}(\eta B')}{\mathcal{P}_{\mathcal{K}}(\eta)} \right|_{L_q} =$$

$$= \frac{1}{2\pi} \sup_{|h|_{L_q}=1} \left(\int_{R^N} \left[\frac{\tilde{h}(\eta B')}{\mathcal{P}_{\mathcal{K}}(\eta)} \right]^q d\eta \right)^{1/q} = (\text{using the substitution } \eta = \xi B^{-1'}) =$$

$$= \frac{1}{2\pi} \sup_{|h|_{L_q}=1} |\det B|^{-1/q} \left(\int_{R^{N-1}} |h(\bar{\xi})|^q \left(\int_{-\infty}^{\infty} |\mathcal{P}_{\mathcal{K}}(\xi B^{-1'})|^{-q}\, d\xi_N \right) d\bar{\xi} \right)^{1/q} =$$

$$= \frac{1}{2\pi} |\det B|^{-1/q} \left(\operatorname*{supess}_{\bar{\xi} \in R^{N-1}} \int_{-\infty}^{\infty} [\mathcal{P}_{\mathcal{K}}(\xi B^{-1'})]^{-q}\, d\xi_N \right)^{1/q}.$$

This completes the proof.

The main aim of this paper is to find necessary and sufficient conditions for K and B for (3.1) to hold, that is to estimate the integral

$$\int_{-\infty}^{\infty} [\mathcal{P}_{\mathcal{K}}(\xi B^{-1'})]^{-q}\, d\xi_N \,. \tag{3.2}$$

Without loss of generality we can suppose $q = 1$.

In the following, it is denoted by $I'' = (1, \dots, 1)$, $I_k'' = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 is on the place k, $I = (0, I'')$, $I_k = (0, I_k'')$.

3.2 Lemma. Let

$$B = \begin{bmatrix} 1, 0, \dots\dots, & 0 \\ 0, 1, \dots\dots, & 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ 0, \dots, 0, 1, 0, \dots, & 0 \\ 0, \dots, 0, 1, 0, \dots, & a_{r+1} \\ 0, \dots, 0, 1, 0, \dots, & a_{r-2} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ 0, \dots\dots, 0, 1, & a_{N-1} \\ 0, \dots\dots\dots, 0, & 1 \end{bmatrix} \left.\begin{matrix} \\ \\ \\ \\ \\ \end{matrix}\right\} s \tag{3.3}$$

where $a_{r+1} \neq 0, \dots, a_{N-1} \neq 0$. Then

$$\operatorname*{supess}_{\xi \in R^{N-1}} \int_{-\infty}^{\infty} [\mathcal{P}_{\mathcal{K}}(\xi\, B^{-1'})]^{-1}\, d\xi_N \tag{3.4}$$

is finite iff it is finite the number

$$\gamma(\mathcal{K}, s) = \operatorname*{supess}_{\xi \in R^N} \int_{-\infty}^{\infty} [\max_{A \in \mathcal{K}} |\,\xi'\,|^{A'}\, |\,\xi'' - \tau I''\,|^{A''}]^{-1}\, d\tau. \tag{3.5}$$

Proof. One has

$$\xi\, B^{-1'} = (\xi_1, \dots, \xi_r, \xi_{r+1} + a_{r+1}\,\xi_N, \dots, \xi_{N-1} + a_{N-1}\,\xi_N, \xi_N).$$

Using the substitutions

$$\xi_N = \eta_N - \tau, \; a_i \xi_N + \xi_i = \eta_i \qquad (i = r+1, \dots, N-1)$$

one has

$$(3.4) \qquad \operatorname*{supess}_{\eta \in R^N} \int_{-\infty}^{\infty} [\max_{A \in \mathcal{K}} |\eta'|^{A'} |a_{r+1}|^{A_{r+1}} \dots |a_{N-1}|^{A_{N-1}} |\eta'' - \tau I''|^{A''}]^{-1} \, d\tau$$

$$= [\max_{A \in \mathcal{K}} |a_{r+1}|^{A_{r+1}} \dots |a_{N-1}|^{A_{N-1}}]^{-1} \operatorname*{supess}_{\xi \in R^N} \int_{-\infty}^{\infty} [\mathcal{P}_{\mathcal{K}}((\xi', \xi'' - \tau I''))]^{-1} \, d\tau$$

and the proof is finished.

Take k, $1 \leqq k \leqq s$ and put

$$\tilde{\xi}_k = (|\xi_1|, \dots, |\xi_{r+k-1}|, |\xi_{r+k+1}|, \dots, |\xi_N|)$$

$$\xi_k(\tau) = (|\xi_1|, \dots, |\xi_{r+k-1}, \tau, |\xi_{r+k+1}|, \dots, |\xi_N|)$$

$$\eta^k(\tau) = (|\eta_1|, \dots, |\eta_r|, |\eta_{r+1} - \eta_k|, \dots, |\eta_{r+k-1} - \eta_k|, \tau,$$

$$|\eta_{r+k+1} - \eta_k|, \dots, |\eta_N - \eta_k|)$$

for $\xi = (\xi_1, \dots, \xi_N)$, $\eta = (\eta_1, \dots, \eta_N)$.

Further put $(i_1 < i_2 < \dots < i_t)$

$$P_{i_1, \dots, i_t} \xi = (\xi_1, \dots, \xi_{i_1-1}, \xi_{i_1+1}, \dots, \xi_{i_2-1}, \xi_{i_2+1}, \dots, \xi_{i_t-1},$$

$$\xi_{i_t+1}, \dots, \xi_N, \xi_{i_1} + \dots + \xi_{i_t}).$$

3.3 **Lemma.** Let $1 \leqq k \leqq s$. Then

$$(3.6) \qquad \operatorname*{supess}_{\eta \in R^N} \int_0^{\frac{1}{2} \min_{r < i \neq j \leqq N} |\eta_i - \eta_j|} [\mathcal{P}_{\mathcal{K}}(\eta^k(\tau))]^{-1} \, d\tau$$

is finite iff

$$(3.7) \qquad \gamma(\mathcal{K}, s, k) = \operatorname*{supess}_{\tilde{\xi}_k \in R^{N-1}} \int_0^{\min_{\substack{r < i \leqq N \\ i \neq k+r}} |\xi_i|} [\mathcal{P}_{\mathcal{K}}(\xi_k(\tau))]^{-1} \, d\tau < \infty .$$

Proof. To fix the ideas put $k = 1$. Obviously, (3.6) is less or equal to (3.7). Let $\tilde{\xi}_1$ be such that

$$\gamma(\mathcal{K}, s, 1) < 2 \int\limits_0^{\min\limits_{r+1<i\leqq N} |\xi_i|} [\mathcal{P}_{\mathcal{K}}(\xi_1(\tau))]^{-1} \, d\tau.$$

Without loss of generality, changing arrangement of indices, one can suppose

$$0 < |\xi_{r+2}| < \dots < |\xi_N| \, .$$

Put

$$\eta_i = \xi_i \qquad (i = 1, \dots, r)$$

$$\eta_{r+1} = 0$$

$$\eta_{r+i} = \sum_{j=r+2}^{r+i} |\xi_i| \qquad (i = 2, \dots, s).$$

Then

$$\min_{\substack{i \neq j \\ r<i,\, j\leqq N}} |\eta_i - \eta_j| = \min_{1+r<i\leqq N} |\xi_i|$$

and for $i \neq r + 1$, $r < i \leqq N$ one has

$$|\xi_i| \leqq |\eta_i - \eta_{r+1}| = |\eta_i| \leqq s\,|\xi_i| \tag{3.8}$$

it is

$$\gamma(\mathcal{K}, s, 1) \leqq 2 \int\limits_0^{\min\limits_{r<i\neq j\leqq N} |\eta_i - \eta_j|} (\mathcal{P}_{\mathcal{K}}(\xi_1(\tau))]^{-1} \, d\tau.$$

Using (3.8) and the substitution $\tau \to \frac{1}{2}\tau$ one has

$$\gamma(\mathcal{K}, s, 1) \leqq C \int\limits_0^{\frac{1}{2}\min\limits_{r<i\neq j\leqq N} |\eta_i - \eta_j|} [\mathcal{P}_{\mathcal{K}}(\eta^k(\tau))]^{-1} \, d\tau.$$

Now, we use this procedure for a set of $\tilde{\xi}_1$ of positive measure and finish the proof.

3.4 Lemma. $\gamma(\mathcal{K}, s) < \infty$ iff $\gamma(\mathcal{K}, s, i) < \infty$ $(i = 1, 2, \dots, s)$ and

$$\gamma(P_{i_1 i_2}\mathcal{K}, s - 1) < \infty \qquad (r + 1 \leqq i_1 < i_2 \leqq N).$$

Proof. Put

$$\delta = \frac{1}{2} \min_{r<i\neq j\leqq N} | \eta_i - \eta_j |$$

$$\mathcal{J}_k^+ = \langle \eta_{r+k}, \eta_{r+k} + \frac{\delta}{2} \rangle$$

$$\mathcal{J}_k^- = \langle \eta_{r+k} - \frac{\delta}{2}, \eta_{r+k} \rangle \qquad (k = 1, \dots, s)$$

$$\mathcal{J}_0 = (-\infty, \infty) - \bigcup_{k=1}^{s} \mathcal{J}_k^+ \cup \mathcal{J}_k^- .$$

Let $\delta = \frac{1}{2} | \eta_{i_1+r} - \eta_{i_2+r} |$, $r < i_1 < i_2 \leqq N$. Then for $\tau \in \mathcal{J}_0$ we have

$$| \tau - \eta_{i_1+r} | \leqq | \tau - \eta_{i_2+r} | + | \eta_{i_1+r} - \eta_{i_2+r} | =$$

$$= | \tau - \eta_{i_2+r} | + 2\delta \leqq 5 | \tau - \eta_{i_2+r} |$$

and so

$$\frac{1}{5} | \tau - \eta_{i_2+r} | \leqq | \tau - \eta_{i_1+r} | \leqq 5 | \tau - \eta_{i_2+r} | . \tag{3.9}$$

Let $\delta \neq 0$. Then for $\tau \in \mathcal{J}_j^+ \cup \mathcal{J}_j^-$, $j \neq k$ we have

$$| | \tau - \eta_{k+r} | - | \eta_{j+r} - \eta_{k+r} | | \leqq | \tau - \eta_{j+r} | \leqq$$

$$\leqq \frac{\delta}{2} \leqq \frac{1}{4} | \eta_{j+r} - \eta_{k+r} |$$

and so

$$\frac{3}{4} | \eta_{j+r} - \eta_{k+r} | \leqq | \tau - \eta_{k+r} | \leqq \frac{5}{4} | \eta_{j+r} - \eta_{k+r} | . \tag{3.10}$$

Now

$$\int_{-\infty}^{\infty} [\max_{A \in \mathcal{K}} | \eta' |^{A'} | \eta'' - \tau I'' |^{A''}]^{-1} \, d\tau = \int_{\mathcal{J}_0} + \sum_{k=1}^{s} \left(\int_{\mathcal{J}_k^+} + \int_{\mathcal{J}_k^-} \right) .$$

$\int_{-\infty}^{\infty}$ is a bounded function of η iff any of these integrals $\int_{\mathcal{J}_0}$, $\int_{\mathcal{J}_k^+}$, $\int_{\mathcal{J}_k^-}$ is a bounded function of η. In the integral $\int\limits_{\mathcal{T}_k^+}$ and $\int\limits_{\mathcal{T}_k^-}$ we can use (for $\delta \neq 0$) (3.10) and write $| \eta_{j+r} - \eta_{k+r} |$ instead of $| \tau - \eta_{k+r} |$. Using the substitution $\eta_{k+r} - \tau \to \tau$ and lemma 3.3 we finally obtain that $\int\limits_{\mathcal{T}_k^+}$ and $\int\limits_{\mathcal{T}_k^-}$ are bounded iff $\gamma(\mathcal{K}, s, k) < \infty$.

For $i_1 \neq i_2$, $r < i_1 < i_2 \leqq N$ put

$$\mathcal{M}_{i_1, i_2} = \{\eta,\ 0 \neq \min_{r < i \neq j \leqq N} | \eta_i - \eta_j | = | \eta_{i_1} - \eta_{i_2} | \}.$$

Then $\bigcup\limits_{i_1 \neq i_2} \mathcal{M}_{i_1, i_2} = R^N - \mathcal{M}$, where $\mathcal{M}$ is a set of measure zero.

Let $\eta \in \mathcal{M}_{i_1, i_2}$. Then, using (3.9), we can write $| \tau - \eta_{i_1} |$ instead of $| \tau - \eta_{i_2} |$ in the integral $\int\limits_{\mathcal{T}_0}$. So $\int\limits_{\mathcal{T}_0} \leqq C\gamma (P_{i_1, i_2} \mathcal{K},\ s - 1)$.

On the other hand if

$$\operatorname*{supess}_{\eta \in R^N} \int\limits_{\mathcal{T}_0} [\max_{A \in \mathcal{K}} | \eta' |^{A'} | \eta'' - \tau I'' |^{A''}]^{-1} d\tau < \infty,$$

then taking $\eta_{i_1} \to \eta_{i_2}$ we have

$$\gamma (P_{i_1, i_2} \mathcal{K},\ s - 1) < \infty.$$

4. Sufficient conditions for $\gamma (\mathcal{K}, s, i) < \infty$.

4.1 Lemma. Let $\mathcal{K} \in \mathfrak{p}$ and $A^{(0)} \in \mathcal{K}$. Then for $\xi_i \neq 0$ we have

$$| \xi |^{A^{(0)}} \leqq \max_{A \in \mathcal{K}^e} | \xi |^A .$$

Proof. $A^{(0)} = \sum\limits_{A \in \mathcal{K}^e} \lambda_A A$, $\lambda_A \geqq 0$, $\sum\limits_{A \in \mathcal{K}^e} \lambda_A = 1$ and so

$$| \xi |^{A^{(0)}} = \prod_{A \in \mathcal{K}^e} (| \xi |^A)^{\lambda_A} \leqq \prod_{A \in \mathcal{K}^e} (\max_{A \in \mathcal{K}^e} | \xi |^A)^{\lambda_A} =$$

$$= (\max_{A \in \mathcal{K}^e} | \xi |^A)^{\Sigma \lambda_A} = \max_{A \in \mathcal{K}^e} | \xi |^A .$$

4.2 LEMMA. If $\mathcal{K}_1 \subset \mathcal{K} \in \mathfrak{p}$ then

$$\mathcal{P}_{\mathcal{K}_1}(\xi) = \max_{A \in \mathcal{K}_1} |\xi|^A \leqq \max_{A \in \mathcal{K}} |\xi|^A = \max_{A \in \mathcal{K}^e} |\xi|^A = \mathcal{P}_{\mathcal{K}}(\xi). \tag{4.1}$$

Proof. We use lemma 4.1.

4.3 LEMMA. If $\mathcal{K}_1 \subset \mathcal{K} \in \mathfrak{p}$ then

$$\gamma(\mathcal{K}_1, s) \geqq \gamma(\mathcal{K}, s),$$

$$\gamma(\mathcal{K}_1, s, i) \geqq \gamma(\mathcal{K}, s, i) \qquad (i = 1, 2, \dots, s).$$

Proof. Lemma is immediate consequence of (4.1).

4.4 LEMMA. Let there be $A^{(0)} \in \mathcal{K} \in \mathfrak{p}$ such that

$$A_i^{(0)} = 0 \ (i = 1, \dots, r),\ 0 \leqq A_i^{(0)} \ (i = r + 1, \dots, N),\ A_{k+r}^{(0)} < 1 \text{ and } \sum_{i=r+1}^{N} A_i^{(0)} = 1.$$

Then $$\gamma(\mathcal{K}, s, k) < \infty.$$

Proof. For example, let $k = s$. Then

$$\gamma(\mathcal{K}, s, s) \leqq \gamma(\{A^{(0)}\}, s, s) = \operatorname*{supess}_{\bar{\xi} \in R^{N-1}} \int_0^{\min_{r<i<N} |\xi_i|} |\xi_s(t)|^{-A^{(0)}} d\tau =$$

$$= \operatorname*{supess}_{\bar{\xi} \in R^{N-1}} \int_0^{\min_{r<i<N} |\xi_i|} |\bar{\xi}|^{-\overline{A^{(0)}}} \tau^{-A_N^{(0)}} d\tau \leqq$$

$$\leqq \operatorname*{supess}_{\bar{\xi} \in R^{N-1}} |\bar{\xi}|^{-\overline{A^{(0)}}} (1 - A_N^{(0)})^{-1} (\min_{r<i<N} |\xi_i|)^{1-A_N^{(0)}} \leqq$$

$$\leqq \operatorname*{supess}_{\bar{\xi} \in R^{N-1}} (\min_{r<i<N} |\xi_i|)^{-\sum_{i=1}^{N-1} A_i^{(0)}} (1 - A_N^{(0)})^{-1} (\min_{r<i<N} |\xi_i|)^{1-A_N^{(0)}} \leqq \frac{1}{1 - A_N^{(0)}} < \infty.$$

4.5 Lemma. Let $\mathcal{K}_1 \subset \mathcal{K} \in \mathfrak{p}$ and let $\mathcal{K}_1$ be a segment $\overline{A^{(1)} A^{(2)}}$, where $A^{(1)} = I_k + A^{(0)}$, $A^{(2)} = I_k - A^{(0)}$, $A^{(0)}_{r+k} > 0$. Then

$$\gamma(\mathcal{K}, s, k) < \infty.$$

Proof. Suppose $k = s$. Then

$$\frac{1}{2}\gamma(\mathcal{K}, s, s) \leqq \frac{1}{2}\gamma(\mathcal{K}_1, s, s) \leqq$$

$$\leqq \operatorname*{supess}_{\bar{\xi} \in R^{N-1}} \int_0^{\min\limits_{r<i<N} |\xi_i|} (|\bar{\xi}|^{\overline{A^{(0)}}} \tau^{A_N^{(0)}} + |\bar{\xi}|^{-\overline{A^{(0)}}} \tau^{-A_N^{(0)}})^{-1} \tau^{-1} \, d\tau \leqq$$

$$\leqq \operatorname*{supess}_{\bar{\xi} \in R^{N-1}} \int_0^{\infty} \leqq \operatorname*{supess}_{\bar{\xi} \in R^{N-1}} \left(|\bar{\xi}|^{\overline{A^{(0)}}} \int_0^{|\bar{\xi}|^{-\overline{A^{(0)}}/A_N^{(0)}}} \tau^{A_N^{(0)}-1} \, d\tau + |\bar{\xi}|^{-\overline{A^{(0)}}} \int_{|\bar{\xi}|^{-\overline{A^{(0)}}/A_N^{(0)}}}^{\infty} \tau^{-A_N^{(0)}-1} \, d\tau \right) \leqq$$

$$\leqq \frac{2}{A_N^{(0)}} < \infty.$$

4.6 Definition. The convex hull of the set $\{I_1, \dots, I_s\}$ is denoted by $\mathcal{S}$. We say that $\mathcal{K}$ regularly penetrates the hyperplane $X_{r+k} = 1$ if there is a segment $\overline{A^{(1)} A^{(2)}}$ such that $I_k \in \overline{A^{(1)} A^{(2)}}$, $A^{(1)}_{r+k} < 1 < A^{(2)}_{r+k}$, $A^{(1)} \in \mathcal{K}$, $A^{(2)} \in \mathcal{K}$.

The mapping

$$P_t = P_{i_1^{(t)}, i_2^{(t)}} P_{i_1^{(t-1)}, i_2^{(t-1)}} \dots P_{i_1^{(1)}, i_2^{(1)}}$$

(that is

$$P_t X = P_{i_1^{(t)}, i_2^{(t)}} (P_{i_1^{(t-1)}, i_2^{(t-1)}} (\dots (P_{i_1^{(1)}, i_2^{(1)}} X) \dots)) \,)$$

is said to be the admissible projection of order t $(1 \leqq t \leqq s-1)$ if $r+1 \leqq$ $\leqq i_1^{(k)} < i_2^{(k)} \leqq N - k + 1$. P_0 is defined as the identity.

4.7 Theorem. Let $\mathcal{K} \in \mathfrak{p}$ and

1) $\mathcal{K} \cap \mathcal{S} \neq \emptyset$

2) if P_t is the admissible projection of order t and

$$P_t \mathcal{K} \cap P_t \mathcal{S} = \{(\underbrace{0, \dots, 0}_{r}, \underbrace{0, \dots, 0, 1}_{k}, \underbrace{0, \dots, 0}_{s-t-k})\}$$

then $P_t \mathcal{K}$ regularly penetrates the hyperplane $X_{r+k} = 1$ (in R^{N-t}).

Then $\gamma(\mathcal{K}, s) < \infty$ (1).

Proof. We use mathematical induction. We prove that for any admissible projection of order $t\,(0 \leqq t \leqq s-1)$ is

$$\gamma(P_t \mathcal{K},\ s-t) < \infty.$$

First, if $t = s-1$ then $P_t = P_{1,2,\dots,s}$. It is clear that $P_{s-1}\mathcal{S} = \{(0, \dots, 0, 1)\}$ and $P_{s-1}\mathcal{K} \cap P_{s-1}\mathcal{S} \neq \emptyset$. So, $P_{s-1}\mathcal{K} \cap P_{s-1}\mathcal{S} = \{(\underbrace{0, \dots, 0}_{r}, 1)\}$. By condition 2) and lemma 4.5 we have $\gamma(P_{s-1}\mathcal{K}, 1) = \gamma(P_{s-1}, 1, 1) < \infty$.

Let $\gamma(P_t \mathcal{K},\ s-t) < \infty$ for any admissible projection P_t of order $t > t_0$. Suppose P_{t_0} is an admissible projection of order t_0. Then

$$\gamma(P_{i_1, i_2} P_{t_0} \mathcal{K},\ s - t_0 - 1) < \infty \tag{4.2}$$

for any $r < i_1 < i_2 \leqq N - t_0$. On the other hand $P_{i_1, i_2} P_{t_0} \mathcal{K} = P_{i_1, i_2}(P_{t_0}\mathcal{K})$. Using condition 1) we have $P_{t_0}\mathcal{K} \cap P_{t_0}\mathcal{S} \neq \emptyset$. If

$$P_{t_0}\mathcal{K} \cap P_{t_0}\mathcal{S} \neq \{(\underbrace{0, \dots, 0}_{r},\ \underbrace{0, \dots, 0, 1}_{k},\ \underbrace{0, \dots, 0}_{s-t_0-k})\}$$

then using lemma 4.4 one has

$$\gamma(P_{t_0}\mathcal{K},\ s - t_0,\ k) < \infty. \tag{4.3}$$

If

$$P_{t_0}\mathcal{K} \cap P_{t_0}\mathcal{S} = \{(\underbrace{0, \dots, 0}_{r},\ \underbrace{0, \dots, 0, 1}_{k},\ \underbrace{0, \dots, 0}_{s-t_0-k})\},$$

using condition 2) and lemma 4.5, one obtains (4.3). It follows from (4.2), (4.3) and lemma 3.4 that

$$\gamma(P_{t_0}\mathcal{K},\ s - t_0) < \infty.$$

So we can conclude (for $t_0 = 0$)

$$\gamma(\mathcal{K},\ s) < \infty.$$

and the proof is finished.

---

(1) For $t = 0$ the condition 2) takes the form : $\mathcal{K} \cap \mathcal{S} = \{I_k\} \Longrightarrow \mathcal{K}$ regularly penetrates the hyperplane $X_{r+k} = 1$.

In the following we will prove that sufficient conditions 1, 2) of theorem 4.7 are also necessary for $\gamma(\mathcal{K}, s) < \infty$.

5. Some geometrical lemmas.

5.1 LEMMA (Helly, cfr. [5], [19], [20]). Let $\mathcal{M}_i (i = 1, \dots, n)$ be convex sets in R^N. Let be for any $j_1, j_2, \dots, j_{N+1}$

$$\bigcap_{i=1}^{N+1} \mathcal{M}_{ji} \neq \emptyset.$$

Then

$$\bigcap_{i=1}^{n} \mathcal{M}_i \neq \emptyset.$$

Proof. Obviously, for $N = 1$ the lemma is true. Let the lemma take place in any Euclidean space of dimension $< N$.

First suppose $n = N + 2$. Put $\mathcal{N}_i = \bigcap_{k=1, k \neq i}^{N+2} \mathcal{M}_k \neq \emptyset$. Then there is $X^{(i)} \in \mathcal{N}_i$. The convex hull of $X^{(1)}, \dots, X^{(i-1)}, X^{(i+1)}, \dots, X^{(N+2)}$ is denoted by Δ_i; the convex hull of $X^{(1)}, \dots, X^{(N+2)}$ is denoted by Δ. Obviously

$$\Delta_i \subset \mathcal{M}_i, \qquad X^{(i)} \in \bigcap_{\substack{k \neq 1 \\ k=1, 2, \dots, N+2}} \Delta_i.$$

If the dimension of Δ is $< N$ then $\bigcap_{i=1}^{N+2} \Delta_i \neq \emptyset$ and so $\bigcap_{i=1}^{N+2} \mathcal{M}_i \neq \emptyset$.

If the dimension of Δ is N then there is at least one Δ_i whose dimension is N. Suppose that this is for Δ_{N+2}. Then

$$X^{(N+2)} = \sum_{i=1}^{N+1} \lambda_i X^{(i)}, \quad \sum_{i=1}^{N+1} \lambda_i = 1.$$

Without loss of generality we can suppose

$$\lambda_1 \geqq 0, \dots, \lambda_k \geqq 0, \lambda_{k+1} < 0, \dots, \lambda_{N+1} < 0.$$

Put

$$\mu_i = \mu_{N+2} \lambda_i \qquad (i = 1, \dots, k)$$

$$\mu_i = - \mu_{N+2} \lambda_i \qquad (i = k + 1, \dots, N + 1)$$

$$\mu_{N+2} = \left(\sum_{i=1}^{k} \lambda_i \right)^{-1}.$$

Then

$$\mu_i \geqq 0, \ \sum_{i=1}^{k} \mu_i = \sum_{i=k+1}^{N+2} \mu_i = 1$$

and

$$X^* = \sum_{i=1}^{k} \mu_i X^{(i)} = \sum_{i=k+1}^{N+2} \mu_i X^{(i)}.$$

So $X^* \in \Delta_i (i = 1, 2, \dots, N+2)$ and $X^* \in \bigcap_{i=1}^{N+2} \Delta_i \subset \bigcap_{i=1}^{N+2} \mathcal{M}_i \neq \emptyset$. So, for $n = N + 2$, the assertion of the lemma is true.

Now suppose that the lemma takes place for any $n < n_0$. Let $\mathcal{M}_1, \dots, \mathcal{M}_{n_0}$ be convex sets such that for any $j_1, \dots, j_{N+1}$ one has $\bigcap_{i=1}^{N+1} \mathcal{M}_{j_i} \neq \emptyset$. By the first part of the proof we know $\bigcap_{i=1}^{N+2} \mathcal{M}_{j_i} \neq \emptyset \ \forall j_1, j_2, \dots, j_{N+2}$. Put $\mathcal{M}_i^* = \mathcal{M}_i \cap \mathcal{M}_{n_0} (i = 1, \dots, n_0 - 1)$. The number of convex sets $\mathcal{M}_i^*$ is $n_0 - 1$ and intersection of any $N + 1$ sets $\mathcal{M}_i^*$ is not empty. So $\bigcap_{i=1}^{n_0-1} \mathcal{M}_i^* = \bigcap_{i=1}^{n_0} \mathcal{M}_i \neq \emptyset$ and the proof is finished.

5.2 LEMMA. Let $f_i (i = 1, 2, \dots, n), f$ be linear functionals on R^N such that

$$f_i(X) \leqq 0 \ \forall\ i = 1, \dots n \Longrightarrow f(X) \leqq 0.$$

Then there are $\lambda_i \geqq 0$ such that $f = \sum_{i=1}^{n} \lambda_i f_i$.

Proof. Let this assertion be true for any Euclidean space of dimension $< N$. Put

$$\mathcal{M}_i = \{X \in R^N;\ f_i(X) \leqq 0\}$$
$$\mathcal{M} = \{X \in R^N;\ f\ (X) > 0\}.$$

The sets $\mathcal{M}_i, \mathcal{M}$ are convex and $\bigcap_{i=1}^{n} \mathcal{M}_i \cap \mathcal{M} = \emptyset$. On the other hand $0 \in \bigcap_{i=1}^{n} \mathcal{M}_i \neq \emptyset$ and, using lemma 5.1, there are sets $\mathcal{M}_i, \dots, \mathcal{M}_{i_N}$ such that $\bigcap_{j=1}^{n} \mathcal{M}_{i_j} \cap \mathcal{M} = \emptyset$, it is

$$f_{i_j}(X) \leqq 0 \ \ (j = 1, \dots, N) \Longrightarrow f(X) \leqq 0.$$

So, in the following, we can can consider only the case $n = N$. Obviously

$$f = \sum_{i=1}^{N} \lambda_i f_i. \tag{5.1}$$

Put $R = \{X \in R^N;\ f_i(X) = 0 \ \forall\ i = 1, \dots, N\}$.

If f_i are linearly dependent then the dimension of R^N/R is $< N$. For $X \in \tilde{X} \in R^N/R$ put $f_i^*(\tilde{X}) = f_i(X), f^*(\tilde{X}) = f(X)$. Then $f_j^*(\tilde{X}) \leqq 0 \Longrightarrow f^*(\tilde{X}) \leqq$ $\leqq 0$ and there are $\lambda_i \geqq 0$ such that $f^* = \sum_{i=1}^{N} \lambda_i f_i^*$ and so $f = \sum_{i=1}^{N} \lambda_i f_i \ (\lambda_i \geqq 0)$.

If f_i are linearly independent then there are $X^{(1)}, \dots, X^{(N)}$ such that $f_i(X^{(j)}) = -\delta_{ij}$ (it is $= 0$ for $i \neq j$, $= -1$ for $i = j$). Obviously $X^{(1)}, \dots, X^{(N)} \in$ $\in \mathfrak{M}_i \ (i = 1, \dots, N)$ and so $f(X^{(i)}) \leqq 0 \quad (i = 1, \dots, N)$. Using (5.1) we have

$$f(X^{(j)}) = \sum_{i=1}^{N} \lambda_i f_i(X^{(j)}) = -\lambda_j \leqq 0$$

and so $\lambda_j \geqq 0$.

In case $\lambda \mathfrak{M} = \mathfrak{M}$ for $\lambda > 0$ the convex set $\mathfrak{M}$ is said to be a cone. So the cones need not be closed.

5.3 Lemma. Let $\mathfrak{M}$ be a cone in R^N, $\mathfrak{M} \neq R^N$. Then there is a linear functional $f \neq 0$ such that for $X \in \mathfrak{M}$ it is $f(X) \geqq 0$ and in any inner point X of $\mathfrak{M}$ it is $f(X) > 0$.

Proof. There are $P \neq 0$ on the boundary of $\mathfrak{M}$, a constant C and a functional $f \neq 0$ such that for $X \in \mathfrak{M}$ one has $f(X) + C \geqq 0$ and $f(P) + C = 0$. The points $\lambda P \ (\lambda > 0)$ are boundary points of $\mathfrak{M}$ and so

$$\lambda f(P) + C = f(\lambda P) + C \geqq 0 \qquad (\lambda > 0). \tag{5.2}$$

From (5.2) it follows $f(P) = C = 0$ and so for $X \in \mathfrak{M}$ one has $f(X) \geqq 0$. The rest of assertion is obvious.

5.4 Lemma Let $\mathfrak{M}, \mathfrak{N}$ be cones, $\mathfrak{M} \neq R^N \neq \mathfrak{N}$ and $\mathfrak{M} \cap \mathfrak{N} \subset \{0\}$. Then there is a linear functional $f \neq 0$ such that

$$f(X) \geqq 0 \text{ for } X \in \mathfrak{M}$$

$$f(X) \leqq 0 \text{ for } X \in \mathfrak{N}$$

and the inequalities are sharp inside of $\mathfrak{M}$ and $\mathfrak{N}$.

Proof. $\mathfrak{M} \cap (-\mathfrak{M})$ is a convex cone and so

$$\dim (\mathfrak{M} \cap -\mathfrak{M}) = N \longrightarrow \mathfrak{M} \cap -\mathfrak{M} = R^N \Longrightarrow \mathfrak{M} = R^N.$$

So under the hypotheses of the lemma

$$\dim (\mathcal{M} \cap - \mathcal{M}) < N, \ \dim (\mathcal{N} \cap - \mathcal{N}) < N$$

and there is $Z \notin (\mathcal{M} \cap - \mathcal{M}) \cup (\mathcal{N} \cap - \mathcal{N}) \cup \{0\}$.
The convex hull of $\mathcal{M} \cap (- \mathcal{N})$ will be denoted by $\mathcal{K}$ and suppose $Z \in \mathcal{K}$, $- Z \in \mathcal{K}$. Then there are $\lambda, \mu, 0 \leqq \lambda \leqq 1, 0 \leqq \mu \leqq 1, X_1 \in \mathcal{M}, Y_1 \in \mathcal{M}, X_2 \in \mathcal{N}$, $Y_2 \in \mathcal{N}$ such that

$$Z = \lambda X_1 + (1 - \lambda) X_2 = - \mu Y_1 - (1 - \mu) Y_2$$

and so

$$\lambda X_1 + \mu Y_1 + (1 - \lambda) X_2 + (1 - \mu) Y_2 = 0.$$

If either $\lambda = \mu = 0$ or $\lambda = \mu = 1$ then $Z \in \mathcal{M} \cap \mathcal{N}$ and this is in contradiction with the fact that $\mathcal{M} \cap \mathcal{N} \subset \{0\}$. So $\lambda + \mu \neq 0 \neq 2 - (\lambda + \mu)$. Put

$$Z_1 = \frac{\lambda X_1 + \mu Y_1}{\lambda + \mu}, \ Z_2 = \frac{(1 - \lambda) X_2 + (1 - \mu) Y_2}{2 - (\lambda + \mu)}.$$

Then

$$Z_1 = - \frac{2 - (\lambda + \mu)}{\lambda + \mu} Z_2 .$$

On the other hand $Z_1 \in \mathcal{M}, Z_2 \in - \mathcal{N}$ and so $Z_1 \in \mathcal{M} \cap \mathcal{N}$ which is in contradiction with $\mathcal{M} \cap \mathcal{N} \subset \{0\}$ and $Z \neq 0$.

So we have proved that either $Z \notin \mathcal{K}$ or $- Z \notin \mathcal{K}$ $(Z \neq 0)$ and so $\mathcal{K} \neq R^N$. Using lemma 5.3 we can find a linear functional $f \neq 0$ such that for $X \in \mathcal{K} \supset \supset \mathcal{M} \cap (- \mathcal{N})$

$$f(X) \geqq 0$$

and the proof is finished.

5.5 LEMMA. Let $\mathcal{K} \in \mathfrak{p}$ and Π be a hyperplane in R^N. Suppose that there is at most one extremal point of $\mathcal{K}$ in Π. Let $X \in \mathcal{K} \cap \Pi$ and X be not extremal point of $\mathcal{K}$. Then there is $B \in R^N$ such that $X + B \in \mathcal{K} - \Pi$, $X - B \in \in X - \Pi$ (that is the segment $\overline{X + B, X - B}$ penetrates the hyperplane Π).

Proof. Assertion of this lemma is sufficiently obvious.

5.6 LEMMA Let $\mathcal{K} \in \mathfrak{p}$. Let

$$\mathcal{M}(\mathcal{K}) = \{X \in R^N; \langle A, X \rangle \leqq 0 \ \forall A \in \mathcal{K}\}$$

have empty interior. Then $0 \in \mathcal{K} - \mathcal{K}^e$.

Proof. Let $0 \in (R^N - \mathcal{K}) \cup \mathcal{K}^e$. Then there is a hyperplane Π such that $0 \in \Pi$, $\Pi \cap \mathcal{K} \subset \{0\}$ and $\mathcal{K}$ lies in one of the semispaces R_1, R_2 defined by Π, $R^N = \Pi \cup R_1 \cup R_2$. Suppose $\mathcal{K} \subset R_1 \cup \Pi$. There is a point $S \in R_2$, $S \neq 0$, $\langle S, X \rangle = 0 \ \forall \ X \in \Pi$. Then there is certain neighborhood $\mathcal{U}$ of S such that $\mathcal{U} \subset \mathcal{M}(\mathcal{K})$ and this is a contradiction. So $0 \in \mathcal{K} - \mathcal{K}^e$.

5.7 Lemma. Let $\mathcal{K} \in \mathfrak{p}$ and

$$\mathcal{M}(\mathcal{K}) = \{0\}.$$

Then $\mathcal{K}$ is a neighbourhood of 0.

Proof. By lemma 5.6, $0 \in \mathcal{K}$. If 0 is boundary point of $\mathcal{K}$ then there is a linear functional $f \neq 0, f(X) \leqq 0 \ \forall \ X \in \mathcal{K}$. Let $f(X) = \langle A, X \rangle$. Then, using lemma 5.2, one has

$$A = \sum_{X \in \mathcal{K}^e} \lambda_x X, \qquad \lambda_x \geqq 0.$$

Obviously $\lambda = \sum_{X \in \mathcal{K}^e} \lambda_x > 0$ and $\lambda^{-1} A \in \mathcal{K}$. Further $f(\lambda^{-1} A) = \langle A, \lambda^{-1} A \rangle =$ $= \lambda^{-1} \langle A, A \rangle > 0$ and this is the contradiction.

6. Necessary conditions for $\gamma(\mathcal{K}, s, k) < \infty$.

Put

$$R_+^N = \{X \in R^N, \ X = (X_1, \dots, X_N); \ X_i > 0 \ \forall \ i\}.$$

For $X \in R_+^N$ put

$$lg\, X = (lg\, X_1, \dots, lg\, X_N) = (lg\, X', lg\, X'') = (lg\, \bar{X}, lg\, X_N).$$

For $X \in R^N$ put

$$e^x = (e^{X_1}, \dots, e^{X_N}).$$

If $\mathcal{M} \subset R_+^N$ then

$$lg\, \mathcal{M} = \{X \in R^N, \ e^X \in \mathcal{M}\}$$

If $\mathcal{M} \subset R^N$ then

$$e^{\mathcal{M}} = \{X \in R_+^N, \ lg\, X \in \mathcal{M}\}.$$

If $\mathcal{M} \in (0, \infty)$ put

$$\mu_{lg}\, \mathcal{M} = \int_{\mathcal{M}} \frac{d\tau}{\tau}.$$

6.1 LEMMA. Let $\mathcal{K} \in \mathfrak{p}$ and

$$\operatorname*{supess}_{\xi_k \in R_+^{N-1}} \int_0^\infty [P_{\mathcal{K}}(\xi_k(\tau))]^{-1}\, d\tau < \infty .$$

Then the interior of the set

$$\mathcal{M}_k(\mathcal{K}) = \{X \in R^N, \langle A - I_k, X \rangle \leqq 0 \ \forall\, A \in \mathcal{K}\}$$

is empty.

Proof. Suppose $k = s$, that is $\xi_k(\tau) = (\bar{\xi}, \tau)$. Put for $\bar{\xi} \in R_+^N$

$$m(\bar{\xi}) = \{\tau, \tau \in (0, \infty);\ |\xi_s(\tau)|^{A - I_s} \leqq 1 \ \forall\, A \in \mathcal{K}\}.$$

Then

$$\tau \in m(\bar{\xi}) \Longleftrightarrow \frac{P_{\mathcal{K}}(\xi_s(\tau))}{\tau} \leqq 1 \Longleftrightarrow$$

$$\Longleftrightarrow \langle A - I_s, lg\, \xi_s(\tau) \rangle \leqq 0 \ \forall\, A \in \mathcal{K} \Longleftrightarrow lg\, \xi_s(\tau) \in \mathcal{M}_s(\mathcal{K}).$$

One has

$$\int_0^\infty \frac{d\tau}{\mathcal{P}_{\mathcal{K}}(\xi_s(\tau))} \geqq \int_{m(\bar{\xi})} \frac{\tau}{\mathcal{P}_{\mathcal{K}}(\xi_s(\tau))} \frac{d\tau}{\tau} \geqq \mu_{lg}\, m(\bar{\xi}).$$

Put $X(T) = lg(\xi(\tau)) = (\bar{X}, T)$. For $\bar{X} = lg\, \bar{\xi}$ one has

$$\inf m(\bar{\xi}) = \inf_{\langle A - I_s, X(T) \rangle \leqq 0 \ \forall A \in \mathcal{K}} e^T = \inf_{X(T) \in \mathcal{M}_s(\mathcal{K})} e^T ,$$

$$\sup m(\bar{\xi}) = \sup_{X(T) \in \mathcal{M}_s(\mathcal{K})} e^T$$

and $m(\bar{\xi})$ is a segment (with ends $\inf m(\bar{\xi})$ and $\sup m(\bar{\xi})$) or it is empty. On the other hand

$$\mu_{lg}\, m(\bar{\xi}) \begin{cases} = 0 & \text{if} \quad m(\bar{\xi}) = \emptyset \\ = lg \sup m(\bar{\xi}) - lg \inf m(\bar{\xi}) = \end{cases} \tag{6.1}$$

$$= \sup_{X(T) \in \mathcal{M}_s(\mathcal{K})} T - \inf_{X(T) \in \mathcal{M}_s(\mathcal{K})} T \quad \text{if } m(\bar{\xi}) \neq \emptyset .$$

The set $\mathcal{M}_s(\mathcal{K})$ is a cone. If $\mathcal{M}_s(\mathcal{K})$ has at least one inner point then there is a ball $\mathcal{U} \subset \mathcal{M}_s(\mathcal{K})$ such that $\lambda\,\mathcal{U} \subset \mathcal{M}_s(\mathcal{K})$ $(\lambda > 0)$. Then, obviously, using (6.1), one has

$$\infty = \underset{\bar{\xi} \in R_+^{N-1}}{\text{supess}} \quad \mu_{lg}\, m(\bar{\xi}) \leqq \underset{\bar{\xi} \in R_+^{N-1}}{\text{supess}} \int_0^\infty \frac{d\tau}{\mathcal{P}_{\mathcal{K}}(\xi_s(\tau))}$$

which is the contradiction. So the interior of $\mathcal{M}_s(\mathcal{K})$ must be empty.

6.2 LEMMA. Under the hypotheses of lemma 6.1 is $I_k \in \mathcal{K} - \mathcal{K}^e$. If there is at most one extremal point of $\mathcal{K}$ on the hyperplane $X_{r+k} = 1$ then $\mathcal{K}$ regularly penetrates the hyperplane $X_{r+k} = 1$.

Proof. Using the lemma 5.6 one has $I_k \in \mathcal{K} - \mathcal{K}^e$. Then one can use the lemma 5.5.

6.3 LEMMA. Let $\mathcal{K} \in \mathfrak{p}$ and

$$\underset{\tilde{\xi}_k \in R^{N-1}}{\text{supess}} \int_0^\infty \frac{d\tau}{\mathcal{P}_{\mathcal{K}}(\xi_k(\tau))} < \infty . \tag{6.2}$$

Then $\mathcal{K}$ regularly penetrates the hyperplane $X_{r+k} = 1$.

Proof. Suppose $k = s$. Let us denote $\mathcal{K}_\alpha^e = \mathcal{K}^e - \{X \in R^N,\, X_N = 1\}$ and by $\mathcal{K}_\alpha$ the convex hull of $\mathcal{K}_\alpha^e$, $\mathcal{K}_\beta^e = \mathcal{K}^e - \mathcal{K}_\alpha^e$, $\mathcal{K}_\beta$ the convex hull of $\mathcal{K}_\beta^e$. Then

$$d = \text{dist}(\mathcal{K}_\alpha^e, \{X \in R^N, X_N = 1\}) > 0.$$

For $\bar{\xi} \in R_+^{N-1}$ put

$$m(\bar{\xi}) = \{\tau \in (0, \infty);\, |\xi_s(\tau)|^{A - I_s} \leqq 1 \;\forall\, A \in \mathcal{K}_\alpha^e\} =$$

$$= \{\tau \in (0, \infty);\, |\xi_s(\tau)|^{A - I_s} \leqq 1 \;\forall\, A \in \mathcal{K}_\alpha\}.$$

Then

$$\int_0^\infty [\mathcal{P}_{\mathcal{K}}(\xi_s(\tau))]^{-1}\, d\tau \geqq \int_{m(\bar{\xi})} \tau\, [\mathcal{P}_{\mathcal{K}}(\xi_s(\tau))]^{-1} \frac{d\tau}{\tau} \geqq$$

$$\geqq \int_{m(\bar{\xi})} \frac{1}{\max(1, \tau^{-1}\mathcal{P}_{\mathcal{K}_\beta}(\xi_s(\tau)))} \frac{d\tau}{\tau} \geqq [\max(1, \mathcal{P}_{\mathcal{K}_\beta \dot{-} I_s}(\xi_s(\tau)))]^{-1}\, \mu_{lg}\, m(\bar{\xi}). \text{ (}^2\text{)}$$

(2) $\mathcal{M} \dot{-} A = \{X \in R^N,\, X + A \in \mathcal{M}\}$.

Put $X(T) = lg\,\xi(\tau)$,

$$\mathcal{M} = \{X \in \mathcal{R}^N, \langle A - I_s, X\rangle \leqq 0 \ \forall\ A \in \mathcal{K}_\alpha\}$$

$$\mathcal{N} = \{X \in \mathcal{R}^N, \langle A - I_s, X\rangle \leqq 0 \ \forall\ A \in \mathcal{K}_\beta\}$$

$$\mu(\mathcal{M}, \bar{X}) = \mu\{T \in (-\infty, \infty); (\bar{X}, T) \in \mathcal{M}\}.$$

By a similar argument as in the proof of lemma 6.1 one has

$$\mu_{lg}\, m(\bar{\xi}) = \mu(\mathcal{M}, \bar{X}).$$

Further

$$\mathcal{P}_{\mathcal{K}_\beta \dot{-} I_s}(\xi_s(\tau)) = e^{\max\limits_{A \in \mathcal{K}^e_\beta} \langle A - I_s, X\rangle} = e^{\max\limits_{A \in \mathcal{K}^e_\beta} \langle \bar{A}, \bar{X}\rangle}$$

By (6.2) one has

$$\text{(6.3)} \qquad \operatorname*{supess}_{\bar{X} \in R^{N-1}} \frac{\mu(\mathcal{M}, \bar{X})}{e^{\max\limits_{A \in \mathcal{K}^e_\beta} (\langle \bar{A}, \bar{X}\rangle, 0)}} < \infty\,.$$

The interior of $\mathcal{M}$ will be denoted by $\mathcal{M}^0$ and the projection $(\bar{X}, X_N) \to \bar{X}$ will be denoted by P.

If $\bar{X} \in P\mathcal{N} \cap P\mathcal{M}^0, \bar{X} \neq 0$ then for $\lambda > 0$ one has $\lambda \bar{X} \in P\mathcal{N} \cap P\mathcal{M}^0$, $\langle \bar{A}, \lambda\bar{X}\rangle \leqq 0 \ \forall\ A \in \mathcal{K}^e_\beta$ and

$$g(\lambda) = \frac{\mu(\mathcal{M}, \lambda\bar{X})}{e^{\max\limits_{A \in \mathcal{K}^e_\beta} (\langle \bar{A}, \lambda\bar{X}\rangle, 0)}} \geqq \mu(\mathcal{M}, \lambda\bar{X}) = \lambda\mu(\mathcal{M}, \bar{X}).$$

From $\bar{X} \in P\mathcal{M}^0$ it follows $\mu(\mathcal{M}, \bar{X}) > 0$ and so

$$\lim_{\lambda \to \infty} g(\lambda) = \infty.$$

On the other hand

$$f(\bar{X}) = \frac{\mu(\mathcal{M}, \bar{X})}{e^{\max\limits_{A \in \mathcal{K}^e_\beta} (\langle \bar{A}, \bar{X}\rangle, 0)}}$$

is a continuous function of $\bar{X}$ (because $d > 0$, that is $\mu(\mathcal{M}, \bar{X})$ is a continuous function of $\bar{X}$) and so $\sup_{\bar{X} \in R^{N-1}} f(\bar{X}) = \infty$; this is the contradiction. So one has

$$(6.4) \qquad P\mathcal{N} \cap P\mathcal{M}^0 \subset \{0\}.$$

Evidently, $P\mathcal{N}$ is closed and $P\mathcal{M}^0$ is open.

There are three possibilities:

1) $P\mathcal{N} = \{0\}$

2) $\mathcal{M}^0 = \emptyset$

3) $\mathcal{M}^0 \neq \emptyset$ and there is $\bar{X} \in P\mathcal{N}$, $\bar{X} \neq 0$.

Ad 1) $P\mathcal{N} = \{0\}$. Then $\mathcal{K}_\beta$ is a neighbourhood of I_s in the hyperplane $X_N = 1$ and by (6.3) there is at least one point $\bar{X} \neq 0$ such that $\mu(\mathcal{M}, \bar{X}) < \infty$. So, there are $X^{(1)} \in \mathcal{K}$, $X^{(2)} \in \mathcal{K}$ such that $X_N^{(1)} < 1$, $X_N^{(2)} > 1$. Then the convex hull K_γ of $\mathcal{K}_\beta \cup \{X^{(1)}, X^{(2)}\}$ is a neighbourhood of I_s in R^N, $K_\gamma \subset \mathcal{K}$ and the assertion of lemma 6.3 is true.

Ad 2) $\mathcal{M}^0 = \emptyset$. Then by lemma 5.6 $I_s \in \mathcal{K}_\alpha$. There is no extremal point of $\mathcal{K}_\alpha$ in the hyperplane $X_N = 1$. By lemma 5.5 $\mathcal{K}_\alpha$ regularly penetrates the hyperplane $X_N = 1$ and the assertion of lemma 6.3 is true too.

Ad 3) $\mathcal{M}^0 \neq \emptyset$ and there is $\bar{X} \in P\mathcal{N}$, $\bar{X} \neq 0$. Then $\bar{X} \in P\mathcal{M}^0$ and so $P\mathcal{M}^0 \neq R^{N-1} \cdot \mathcal{M}^0 \neq \emptyset$ implies $P\mathcal{M}^0 \neq \emptyset$; there is $\bar{Y} \in P\mathcal{M}^0$, $\bar{Y} \neq 0$. So $P\mathcal{N} \neq R^{N-1}$. By lemma 5.4 there is $\bar{Z} \in R^{N-1}$, $\bar{Z} \neq 0$ such that for $X \in P\mathcal{M}^0$ one has $\langle \bar{Z}, \bar{X} \rangle > 0$ and for $\bar{X} \in P\mathcal{N}$ one has $\langle \bar{Z}, \bar{X} \rangle \leqq 0$. Then

$$\langle (\bar{Z}, 0), X \rangle > 0 \quad \text{for} \quad X \in \mathcal{M}^0$$

$$\langle (\bar{Z}, 0), X \rangle \leqq 0 \quad \text{for} \quad X \in \mathcal{N}.$$

By lemma 5.2

$$\bar{Z} = \sum_{X \in K_\beta^\varepsilon} \lambda_X \bar{X}, \qquad \lambda_X \geqq 0$$

and obviously $\lambda = \Sigma \lambda_X > 0$ and $\lambda^{-1} \bar{Z} \in P\mathcal{K}_\beta$. Therefore $B = I_s + \lambda^{-1}(\bar{Z}, 0) \in \mathcal{K}_\beta$. Further $d > 0$ and so there is $C > 0$ such that

$$(6.4) \qquad \mu(\mathcal{M}, \bar{X}) = 0 \quad \text{if} \quad \langle \bar{Z}, \bar{X} \rangle \leqq 0$$

$$\mu(\mathcal{M}, \bar{X}) \leqq C \langle \bar{Z}, \bar{X} \rangle \quad \text{if} \quad \langle \bar{Z}, \bar{X} \rangle \geqq 0.$$

(6.4) implies

$$\mu(\mathcal{M}, \bar{X})\, e^{-\langle \bar{Z}, \bar{X}\rangle} = 0 \quad \text{if} \quad \langle \bar{Z}, \bar{X}\rangle \leqq 0$$

$$\mu(\mathcal{M}, \bar{X})\, e^{-\langle \bar{Z}, \bar{X}\rangle} \leqq C \langle \bar{Z}, \bar{X}\rangle\, e^{-\langle \bar{Z}, \bar{X}\rangle} \quad \text{if} \quad \langle \bar{Z}, \bar{X}\rangle \geqq 0.$$

The convex hull of $\mathcal{K}_\alpha^e \cup \{B\}$ will be denoted by $\mathcal{K}_\delta$. Then

$$\int_{m(\bar{\xi})} \frac{d\tau}{\mathcal{P}_{K_\delta}(\xi_s(\tau))} \leqq \int_{m(\bar{\xi})} \frac{1}{|\xi_s(\tau)|^{B-I_s}} \frac{d\tau}{\tau} = \frac{\mu(\mathcal{M}, \bar{X})}{e^{\langle \bar{Z}, \bar{X}\rangle}} \leqq C_1 < \infty. \tag{6.5}$$

The set $m(\bar{\xi})$ is either empty or a segment. Put

$$m_n = \max\left(0, \inf m(\bar{\xi}) - \frac{1}{n}\right)$$

$$M_n = \sup m(\bar{\xi}) + \frac{1}{n}.$$

Then

$$\int_{(0,\infty)-m(\bar{\xi})} \frac{d\tau}{\mathcal{P}_{K_\delta}(\xi_s(\tau))} = \lim_{n\to\infty} \int_{(0,\infty)-(m_n, M_n)} \frac{d\tau}{\mathcal{P}_{K_\delta}(\xi_s(\tau))}.$$

Let $m_n \neq 0$. Then there is $A^{(1)} \in K_\alpha^e$ such that $|\xi_s(\tau)|^{A^{(1)}-I_s} \geqq 1$ for $\tau \in (0, m_n)$. That is $A_N^{(1)} < 1$ and $m_n \leqq |\bar{\xi}|^{\overline{A^{(1)}}/(1-A_N^{(1)})}$. Then

$$\int_0^{m_n} \frac{d\tau}{\mathcal{P}_{K_\delta}(\xi_s(\tau))} \leqq \int_0^{|\bar{\xi}|^{\overline{A^{(1)}}/(1-A_N^{(1)})}} \frac{d\tau}{|\bar{\xi}|^{\overline{A^{(1)}}}\, \tau^{A_N^{(1)}}} = \frac{1}{1-A_N^{(1)}}. \tag{6.6}$$

Let $M_n \neq \infty$. Then there is $A^{(2)} \in \mathcal{K}_\alpha^e$ such that $|\xi_s(\tau)|^{A^{(2)}-I_s} \geqq 1$ for $\tau \in (M_n, \infty)$, that is $A_N^{(2)} > 1$ and $M_n \geqq |\bar{\xi}|^{\overline{A^{(2)}}/(1-A_N^{(2)})}$. Then

$$\int_{M_n}^{\infty} \frac{d\tau}{\mathcal{P}_{K_\delta}(\xi_s(\tau))} \leqq \int_{|\bar{\xi}|^{\overline{A^{(1)}}/(1-A_N^{(2)})}}^{\infty} \frac{d\tau}{|\bar{\xi}|^{\overline{A^{(2)}}}\, \tau^{A_N^{(2)}}} = \frac{1}{A_N^{(2)}-1}. \tag{6.7}$$

Using (6.5), (6.6), (6.7) one has

$$\operatorname*{supess}_{\bar\xi \in R_+^{N-1}} \int_0^\infty [\mathcal{P}_{\mathcal{K}_\delta}(\xi_s(\tau))]^{-1}\, d\tau < \infty .$$

By lemma 6.2 $\mathcal{K}_\delta$ regularly penetrates the hyperplane $X_N = 1$, $\mathcal{K}_\delta \subset \mathcal{K}$ and the proof is finished.

Let $\mathcal{K} \in \mathfrak{p}$. The convex hull of the set $\mathcal{K} \cup \bigcup_{\substack{i \neq k\\ i=1,2,\dots,s}} \{2I_k - I_i\}$ will be denoted by $\mathcal{K}^{(k)}$.

6.4 Lemma

$$\operatorname*{supess}_{\bar\xi_k \in R_+^{N-1}} \int_0^\infty \frac{d\tau}{\mathcal{P}_{\mathcal{K}^{(k)}}(\xi_k(\tau))} \leqq \gamma(\mathcal{K}, s, k) + 1. \tag{6.8}$$

Proof. Let $\min\limits_{\substack{r<i\leqq N\\ i\neq k+r}} |\xi_i| = |\xi_j|$ $(j \neq k + r)$. Then

$$\int_{\substack{\min |\xi_i|\\ r<i\leqq N\\ i\neq k+r}}^\infty \frac{d\tau}{\mathcal{P}_{\mathcal{K}^{(k)}}(\xi_k(\tau))} \leqq \int_{|\xi_j|}^\infty \frac{d\tau}{\tau^2 |\xi_j|^{-1}} = 1$$

and

$$\int_0^{\substack{\min |\xi_i|\\ r<i\leqq N\\ i\neq k+r}} \frac{d\tau}{\mathcal{P}_{\mathcal{K}^{(k)}}(\xi_k(\tau))} \leqq \int_0^{\substack{\min |\xi_i|\\ r<i\leqq N\\ i\neq k+r}} \frac{d\tau}{\mathcal{P}_{\mathcal{K}}(\xi_k(\tau))} = \gamma(\mathcal{K}, s, k).$$

6.5 Lemma. Let $\mathcal{K} \in \mathfrak{p}$ and $\gamma(\mathcal{K}, s, k) < \infty$ for $k = 1, 2, \dots, s$. Then $\mathcal{K} \cap \mathcal{S} \neq \emptyset$.

Proof. By lemmas 6.4, 6.2 $I_k \in \mathcal{K}^{(k)}$. Put

$$\pi_k = \left\{ X = \sum_{i=1}^s \lambda_i I_i,\ \lambda_k \geqq 0,\ \sum_{i=1}^s \lambda_i = 1 \right\}.$$

Then $\mathcal{S} = \bigcap_{i=1}^s \pi_k$. Put $\mathcal{K}^* = \mathcal{K} \cap \bigcup_{i=1}^s \pi_k$. Then $I_k \in \mathcal{K}^{(k)}$ implies $\bigcap_{i\neq k} \pi_i \cap \mathcal{K}^* \neq \emptyset$. By lemma 5.1 $\bigcap_{i=1}^s \pi_k \cap \mathcal{K}^* \neq \emptyset$ and so $\mathcal{K} \cap \mathcal{S} \neq \emptyset$.

6.6 LEMMA. Let $\mathcal{K} \in \mathfrak{p}$, $\mathcal{K} \cap \mathcal{S} = \{I_k\}$ and

$$\operatorname*{supess}_{\tilde{\xi}_k \in R_+^{N-1}} \int_0^\infty \frac{d\tau}{\mathcal{P}_{\mathcal{K}^{(k)}}(\xi_k(\tau))} < \infty .$$

Then $\mathcal{K}$ regularly penetrates the hyperplane $X_{r+k} = 1$.

Proof. The convex hull of $\bigcup_{i \neq k} \{2I_k - I_i\}$ will be denoted by $\mathcal{S}^{(k)}$. By lemma 6.3 there is B such that $I_k + B \in \mathcal{K}^{(k)}$, $I_k - B \in \mathcal{K}^{(k)}$, that is there are λ, μ, $0 \leqq \lambda \leqq 1$, $0 \leqq \mu \leqq 1$; $X^{(1)}, X^{(2)} \in \mathcal{S}^{(k)}$; $Y^{(1)}, Y^{(2)} \in \mathcal{K}$ such that

$$I_k + B = \lambda X^{(1)} + (1-\lambda) Y^{(1)}$$

$$I_k - B = \mu X^{(2)} + (1-\mu) Y^{(2)} .$$

Obviously $\lambda + \mu \neq 2$. Suppose $\lambda + \mu \neq 0$. Then

$$2I_k = (\lambda + \mu) X^{(3)} + (2 - \lambda - \mu) Y^{(3)}$$

where

$$X^{(3)} = \frac{\lambda X^{(1)} + \mu X^{(2)}}{\lambda + \mu} \in \mathcal{S}^{(k)}, \qquad Y^{(3)} = \frac{(1-\lambda) Y^{(1)} + (1-\mu) Y^{(2)}}{2 - \lambda - \mu} \in \mathcal{K}.$$

Put

$$\alpha = (2 - \lambda - \mu)/(\lambda + \mu) > 0.$$

Then

$$2I_k - X^{(3)} = \alpha Y^{(3)} + (1 - \alpha) I_k$$

and

$$Y^{(3)} = I_k + \alpha^{-1} (I_k - X^{(3)}).$$

Obviusly $2I_k - X^{(3)} \in \mathcal{S}$, $2I_k - X^{(3)} \neq I_k$.

Let $\alpha \leqq 1$. Then $I_k \in \mathcal{K}$, $Y^{(3)} \in \mathcal{K}$ implies

$$2I_k - X^{(3)} = (1 - \alpha) I_k + \alpha Y^{(3)} \in \mathcal{K}$$

and so $I_k \neq 2I_k - X^{(3)} \in \mathcal{K} \cap \mathcal{S}$ which is the contradiction.

Let $\alpha^{-1} < 1$. Then $I_k \in \mathcal{S}$, $2I_k - X^{(3)} \in \mathcal{S}$ and so

$$Y^{(3)} = (1 - \alpha^{-1}) I_k + \alpha^{-1} (2I_k - X^{(3)}) \in \mathcal{S}.$$

Therefore

$$I_k \neq Y^{(3)} \in \mathcal{K} \cap \mathcal{S}$$

and it is a contradiction too.

Assumption $\lambda + \mu \neq 0$ is false and so $\lambda = \mu = 0$ and $I_k + B \in \mathcal{K}$, $I_k - B \in \mathcal{K}$.

6.7 THEOREM. Let $\mathcal{K} \in \mathfrak{p}$. Then the conditions 1), 2) of theorem 4.7 are sufficient and necessary for $\gamma(\mathcal{K}, s) < \infty$.

Proof. The assertion is an immediate consequence of theorem 4.7, lemma 6.5, lemma 6.4, lemma 6.6 and lemma 3.4.

6.8 REMARK. Validity of the inequality (1.5) is equivalent to $\gamma(q\mathcal{K}, s) < \infty$ where $1/q = 1 - 1/p$, $1 < p < \infty$ (see theorem 3.1 and lemma 3.2).

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