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# FRACTIONAL POWERS OF ELLIPTIC DIFFERENTIAL OPERATORS (*)

by TAMAR BURAK

## 1. Introduction and notations.

We investigate in this work fractional powers of elliptic differential operators on a compact $C^\infty$ $n$ dimensional manifold $X$, without a boundary. Let $A$ be an elliptic differential operator of order $m$ with coefficients in $C^\infty(X)$. We define fractional powers of $A$ under the assumptions that the range of the symbol $\sigma_m(A)$ of $A$ is disjoint to a ray in the complex plane and that zero is a regular point or a pole of the first order of the resolvent of $\overline{A}$, the closure of $A$ in $H_0(X)$. We show that $\overline{A}^S$, restricted to $C^\infty(X)$, in an elliptic pseudodifferential operator of order $ms$.

Our proofs are based on the possibility of expanding $(q^m - \overline{A})^{-k}$, $k=1, 2, \ldots$, considered as a family of operators depending on a parameter $q$ in the angle $\theta_1 \leq \arg q \leq \theta_2$ into an asymptotic sum of canonical families which are defined in the same angle. This sort of expansion does not carry over to the more general case of $(q^m - \overline{A})^{-k}$ with $A$ an elliptic pseudodifferential operator. In particular we obtain as a consequence of the validity of such an expansion a pointwise asymptotic expansion in an angle for the diagonal values of the kernels of $(\lambda - \overline{A})^{-[n/m]-1}$ with the aid of the powers $|\lambda|^{-[n/m]-1+(n-j)/m}$ of $|\lambda|$.

Recently R. T. Seeley (11) has proved that also if $A$ is an elliptic invertible pseudodifferential operator with the range of $\sigma_m(A)$ disjoint to a ray $A^s$ is an elliptic pseudodifferential operator of order $ms$. Furthermore

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he has investigated the kernels $K(xys)$ of $A^s$ with $s < -n/m$ and proved that $K(xxs)$ has an extension to a meromorphic function of $s$ with simple poles at $(j-n)/m$, $j = 0, 1, 2, \ldots$ and $j \neq n$.

In case $A$ is restricted to be a differential operator it is convenient to investigate properties of $K(xys)$ using the above mentioned expansion of $[\lambda - A]^{-[n/m]-1}$. In particular we obtain that the poles of $K(xxs)$ are situated at $(j-n)/m$, $j = 0, 1, 2, \ldots$ and $(j-n)/m$ different from a non negative integer. The results thus obtained generalize results in Minakshisundaram and Pleijel (9) where in order to obtain the distribution of the eigenvalues of a Laplacian of a Riemannian manifold the series $\Sigma\, \lambda_n^s\, \Phi_n(x)\, \overline{\Phi_n(y)}$ is investigated. Here $\lambda_n$ and $\Phi_n$ are the eigenvalues and the eigenfunctions of the Laplacian. In this case the poles of $\Sigma\, \lambda_n^s\, \Phi_n(x)\, \overline{\Phi_n(x)}$ are situated at $\frac{1}{2} n + j$, $j = 0, 1, 2, \ldots$ when $n$ is odd and at $\frac{1}{2} n$, $\frac{1}{2} n + 1, \ldots$ 2, 1 when $n$ is even.

We add that the above methods can be adjusted to investigate interior properties of $A^s$ or of resolvents of realizations of $A$ where $A$ is an elliptic differential operator in a bounded domain in $R^n$. The results on $A^s$ thus obtained generalize results in Kotaké-Narasimhan (8). It is shown there that if $A$ has a positive self adjoint realization $A^s$ has a very regular kernel.

In an appendix we mention results concerning the existence of rays of minimal growth of the resolvent of $A$, with $A$ an elliptic pseudodifferential operator on $X$ and concerning the completeness of the generalized eigenfunctions of $A$.

We refer to Kohn-Nirenberg (7) and to R. S. Palais (10) for the definitions and basic properties of pseudodifferential operators on $R^n$ and on a manifold respectively.

We mention also that the results in section 2 here are proved with the aid of a technique similar to that of Kohn-Nirenberg (7) and R. S. Palais (10) and we do not repeat the proofs here. Main details of the proofs are given in (4).

We introduce now the following notations:

Let $R^n$ be the $n$ dimensional Euclidean space. Let $xy$ be the scalar product in $R^n$. For a multi index $\alpha = (\alpha_1, \ldots, \alpha_n)$ let $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1! \ldots \alpha_n!$ and $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$ when $\xi \in R^n$. We put $D_j = -i \frac{\partial}{\partial x_j}$, $D = (D_1 \ldots D_n)$ and $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$. Similarly $\partial_j = \frac{\partial}{\partial \xi_j}$, $\partial = (\partial_1 \ldots \partial_n)$ and $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$.

As usual let $S$ be the space of $C^\infty$ complex functions defined in $R^n$

which together with all their derivatives die down faster than any power of $|x|$ at infinity. In $S$ we denote $\tilde{u}(\xi) = (2\pi)^{-n/2} \int e^{-ix\xi} u(x)\, dx$.

For $u \in S$ and real $s$ we denote by $\| u \|_s$ the $s$ norm of $u$ given by

1.1 $$\| u \|_s^2 = \int (1 + |\xi|^2)^s \, |\tilde{u}(\xi)|^2 \, d\xi .$$

Hs is then the completion of $S$ in the $s$ norm. For every real $s$ and complex $q$ we introduce in $S$ an $s, q$ norm which is equivalent to the $s$ norm and is given by

1.2 $$\| u \|_{s,q}^2 = \int (1 + |\xi|^2 + |q|^2)^s \, |\tilde{u}(\xi)|^2 \, d\xi .$$

Let $A(q)$ be a family of linear operators from $S$ to $S$ defined in a truncated angle $\Delta_M = \{q, \theta_1 \leq \arg q \leq \theta_2, \; |q| \geq M\}$, with $\theta_1 \leq \theta_2$ and $M > 0$. Let $\sigma$ be a real number. We say that $A(q)$ is of order $\sigma$ in $\Delta_M$ if for every $s$ there is a $C$ such that if $q \in \Delta_M$ and $u \in S$

1.3 $$\| A(q)\, u \|_{s,q} < C \, \| u \|_{s+\sigma,q} .$$

A family as above is of order $-\infty$ in $\Delta_M$ if it is of order $\sigma$ for every $\sigma$. It is clear that if $A(q)$ is of order $-\infty$ in $\Delta_M$, for every real $s$ $\| A(q)\, u \|_s$ dies down as $q \to \infty$ in $\Delta_M$ faster than any power of $|q|$. If $M = 0$ we put $\Delta_M = \Delta$.

Let $X$ be an $n$ dimensional compact $C^\infty$ manifold without a boundary. We fix on $X$ a finite complete set $\chi_j$, $j = 1, 2, \ldots K$, of coordinates and denote by $O_j$ the open set where $\chi_j$ is defined. Let $\Phi_j, j = 1 \ldots K$, be a partition of unity on $X$ subordinate to $\{O_j\}$.

For $\Phi \in C^\infty(X)$ we denote by $M_\Phi$ the transformation from $C^\infty(X)$ to $C^\infty(X)$ given by $M_\Phi u = \Phi \cdot u$. Let $\chi$ be a set of coordinates defined in $C^\infty(0)$. We denote by $\chi^*$ the transformation from $C^\infty \chi(0)$ to $C^\infty(0)$ given by $(\chi^* u)(x) = u(\chi(x))$ and we denote by $\chi_*$ the transformation from $C^\infty(0)$ to $C^\infty \chi(0)$ given by $(\chi_* u)(x) = u(\chi^{-1}(x))$.

As usual for every real $s$ and $u \in C^\infty(X)$ we put

1.4 $$\| u \|_s^2 = \Sigma \, \| x_{j*} \, \Phi_j u \|^2 s, R^n$$

and denote by $H_s = H_s(X)$ the completion of $C^\infty(X)$ in the $s$ norm.

For $u \in H_s$ and $V \in H - s$ we put

1.5 $$(u, v) = \Sigma (\chi_{j*} \Phi_j u, \chi_{j*} \Phi_j v)_{R^n} .$$

with this scalar product $H_0$ is a Hilbert space.

In $C^\infty(X)$ we introduce the $s, q$ norm by

1.6 $$\| u \|^2_{s,q} = \sum_j \| \chi_{j*} \Phi_j u \|^2_{s,q,R^n} .$$

Again we say that a family $A(q)$, $q \in \Delta_M$, of linear operators from $C^\infty(X)$ to $C^\infty(X)$ is of order $\sigma$ in $\Delta_M$ if for every $s$ there is a $C$ such that 1.3 holds.

We remind that a pseudodifferential operator of order $s$, on $X$, is elliptic if $\sigma_s(A) \neq 0$ on $T^*(x)$. Here $\sigma_s(A)$ is the symbol of $A$ which is a complex valued function defined on $T^*(x)$: the bundle obtained from the cotangent bundle of $X$ by deleting zero from each fiber.

We abbreviate in the following « pseudo differential operator » by P.D.O.

## 2. Expansible families of pseudodifferential operators.

We consider in this section one parametric families $A(q)$, $q \in \Delta$, of P.D.O. on $X$ that have an asymptotic expansion, in a sense made precise in definition 3 below, into a sum of canonical families defined as follows:

DEFINITION 1. A family $A(q)$, $q \in \Delta$, of linear operators from $S$ to $S$ is a canonical family of real degree $\sigma$ if

2.1 $$\widetilde{A(q)\, u}(\xi) = (2\pi)^{-n/2} \int_{R^n} e^{-ix\xi}\, a(x\xi q)\, \psi(\xi q)\, u(x)\, dx)$$

where $a(x\xi q) = (|\xi|^2 + |q|^2)^{\sigma/2} a_0(x\xi q)$ and $a_0(x\xi q)$ is defined for $x \in R^n$, $\xi \in R^n$ and $q \in \Delta$, that satisfy $|\xi|^2 + |q|^2 \neq 0$, and has the following properties: For every $x \in R^n$ $a_0(x\xi q)$ is positive homogeneous of degree zero in $(\xi q)$, $a_0(x\xi q)$ is infinitely differentiable in $x, \xi, q_1 = \mathrm{Re} \cdot q$ and $q_2 = I_m, q$. There exists $\lim_{X \to \infty} a_0(x\xi q)$. Let $a_0(\infty\, \xi\, q) = \lim_{X \to \infty} a_0(x\xi q)$ then

2.2 $$(1 + |x|^p)\, D^\alpha\, \partial^\beta \frac{\partial^{m_1}}{\partial q_1^{m_1}} \frac{\partial^{m_2}}{\partial q_2^{m_2}} (a_0(x\xi q) - a_0(\infty\, \xi q) \longrightarrow 0$$

as $|x| \longrightarrow \infty$ uniformly in $(\xi, q)$ such that $|\xi|^2 + |q|^2 = 1$.

The function $\psi(\xi q)$ is defined for $\xi \in R^n$ and complex $q$, is infinitely differentiable in $\xi$, $q_1$ and $q_2$ and satisfies: $0 \leq \psi(\xi q) \leq 1$, $\psi(\xi q) = 0$ when $|\xi|^2 + |q|^2 \leq \frac{1}{2}$ and $\psi(\xi q) = 1$ when $|\xi|^2 + |q|^2 \geq 1$, $a(x\xi q)$ is called the symbol of $A(q)$.

Let $A(q)$, $q \in \Delta$, be a canonical family of degree $\sigma$. For every $q \in \Delta$ $A(q)$ is a P.D.O. of true order less than or equal to $\sigma$. If $\{re^{i\theta}; r \geq 0\}$ if a fixed ray in $\Delta$ the symbol of the operator $A(re^{i\theta})$ is given by

$$\Sigma \frac{r^n}{n!} \frac{\partial^n}{r^n} a(x\xi\, re^{i\theta})_{r=0}\,.$$

DEFINITION 2. A family $A'(q)$, $q \in \Delta$, of linear operators from $S$ to $S$ is canonical of degree $\sigma$ and of second kind if

$$A'(q)\, u(x) = (2\pi)^{-n/2} \int e^{-ix\eta}\, a(x\eta q)\, \psi(\eta q)\, \tilde{u}(\eta)\, d\eta \tag{2.3}$$

with $a(x\eta q)$ and $\psi(\eta q)$ as in definition 3.1.

We have

LEMMA 1. Let $A(q)$, $q \in \Delta$, be a canonical family of degree $\sigma$. $A(q)$ is of order $\sigma$ in $\Delta$ in the sense of 1.3.

We define now

DEFINITION 3. A family $A(q)$, $q \in \Delta$, of linear operators from $S$ to $S$ is called expansible in $\Delta$, if there exists a sequence $A_j(q)$, $q \in \Delta$, $j = 1, 2, \ldots$ of canonical families af respective degrees $r_j$ with the following properties: $r_j$ is monotonically decreasing to $-\infty$ and for every $n$ the order of $A(q) - \sum^{n} A_j(q)$ in $\Delta$ is less than $r_n$.

Let $a(x\xi q)$ be the symbol of $A_j(q)$. The formal sum $\Sigma\, a_j(x\xi q)$ is called a symbol of $A(q)$. $A(q)$ is called of degree $r_1$, $a_1(x\xi q)$ is the top order symbol of $A(q)$ and we put $\sigma_{r_1}[A(q)](x\xi q) = a_1(x\xi q)$. Lemma 2 below as well as lemma 1 above are proved in analogy to proofs in Kohn-Nirenberg (7). Details of the proofs are given in (4).

LEMMA 2. a) Let $A(q)$ $q \in \Delta$, and $B(q)$, $q \in \Delta$, be canonical families of respective degree $s$ and $\sigma$. Let $a(x\xi q)$ be the symbol of $A(q)$ and let $b(x\xi q)$ be the symbol of $B(q)$. The family $A(q)\, B(q)$, $q \in \Delta$, is expansible of degree

$\sigma + s$. A symbol of $A(q) \cdot B(q)$ is given by

2.4 $$\sum_{r=0}^{\infty} \sum_{|\alpha| = r} \frac{1}{\alpha!} (-D)^{\alpha} a(x\xi q)\, \partial^{\alpha} b(x\xi q).$$

*b*) Let $A'(q)$, $q \in \Delta$, be the canonical family of second kind with symbol $a(x\xi q)$ $A'(q)$ is expansible in $\Delta$ and a symbol of $A'(q)$ is given by

2.5 $$\sum_{r=0}^{\infty} \sum_{|\alpha| = r} (-1)^r \frac{1}{\alpha!} D^{\alpha} \partial^{\alpha} a(x\xi q).$$

The following theorem ensures the uniqueness of the expansion of an expansible family:

THEOREM 1. Let $A(q)$, $q \in \Delta$, be a canonical family of degree $\sigma$ with symbol $a(x\xi q)$. If the order of $A(q)$ in $\Delta$ is less than $\sigma$ $a(x\xi q) \equiv 0$.

PROOF: We rely in the proof on the following theorem in S. Agmon (3): Let $T$ be a bounded linear operator in $L_2(\Omega)$, $\Omega$ an open set in $R^n$ possessing the cone property. Suppose that the range of $T$ and the range of its adjoint $T^*$ are contained in $H_m(\Omega)$ for some $m > n$. ($m$ not necessarily an integer if $\Omega = R^n$). Then $T$ is an integral operator, $Tf = \int_{\Omega} K(xy) f(y)\, dy$, $f \in L_2(\Omega)$, with a continuous and bounded kernel $k(xy)$ satisfying

2.6 $$|K(xy)| < \gamma \left( \| T \|_m + \| T^* \|_m \right)^{n/m} \| T \|_0^{1-n/m}$$

where $\gamma$ is a constant depending only on $m$, $n$ and on the dimension of the cone in the cone property of $\Omega$.

It is easily seen that it is sufficient to prove the theorem for $\sigma < -n$. We suppose first that $a(x\xi q) \geq 0$. Let $\sigma - \delta$ with $\delta > 0$ be the order of $A(q)$ in $\Delta$. Let $A'(q)$ be the canonical family of second kind with symbol $a(x\xi q)$. For every $u, v \in S$ $(A(q)u, v) = (u, A'(q)v)$. As a result also $A'(q)$ is of order $\sigma - \delta$ in $\Delta$. It follows from 1.3 and 1.2 that there exists a constant $c$ such that

2.7 $$\| A(q) u \|_0 < C |q|^{-|\sigma| - \delta}, \quad \| A(q) u \|_{|\sigma| + \delta} < C, \quad \| A'(q) u \|_{|\sigma| + \delta} < C.$$

Let $K(xyq)$ be the kernel of $A(q)$. From 2.6 it follows by 2.7 that

2.8 $$|K(xyq)| < C |q|^{-|\sigma| - \delta + n}.$$

But from definition 1 of canonical families it follows that

2.9 $$K(xxq) = (2\pi)^{-n} \, |q|^{-|\sigma|+n} \int_{R^n} a\left(x\xi \frac{q}{|q|}\right) d\xi.$$

Combining 2.8 with 2.9 we obtain that if $\{re^{i\theta}; r \geq 0\}$ is a ray in $\Delta$ and $q = re^{i\theta}$ 2.9 holds only if $\int_{R^n} a(x\xi e^{i\theta})\, d\xi = 0$ for every $x \in R^n$. It follows then from the assumption $a(x\xi q) \geq 0$ that $a(x\xi q) \equiv 0$.

In the general case where we do not assume $a(x\xi q) \geq 0$, let $C(q)$, $q \in \Delta$ be the canonical family of degree $2\sigma$ and symbol $|a(x\xi q)|^2$. As $|a(x\xi q)|^2 \geq 0$, to prove that $a(x\xi q) \equiv 0$ it is sufficient to show that the order of $C(q)$ in $\Delta$ is less than $2\sigma$. Let $\sigma - \delta$ with $\delta > 0$ be the order of $A(q)$ in $\Delta$. Let $A'(q)$, $q \in \Delta$, be the canonical family of second kind with symbol $\overline{a(x\xi q)}$. Again $A'(q)$ is of order $\sigma - \delta$ in $\Delta$ and $A(q)\, A'(q)$ is of order $2\sigma - 2\delta$ in $\Delta$. By lemma 2, *a* and *b*, $A(q)\, A'(q)$ differs from $C(q)$ by a family of order less than $2\sigma$ in $\Delta$. Thus the order of $C(q)$ in $\Delta$ is also less that $2\sigma$.

We mention the following properties of canonical families which we need in the following sections:

THEOREM 2. *a*) Let $A(q)$ $q \in \Delta$, and $B(q)$ $q \in \Delta$, be expansible families of respective degrees $\sigma$ and $s$ $A(q)\, B(q)$ is expansible of degree $\sigma + s$ in $\Delta$. The symbol of $A(q)\, B(q)$ is given by

2.10 $$\sum_{jk\alpha} (-D)^\alpha a_j(x\xi q)\, \partial^\alpha b_k(x\xi q)$$

where $\Sigma\, a_j(x\xi q)$ is the symbol of $A(q)$ and $\Sigma\, b_j(x\xi q)$ is the symbol of $B(q)$.

*b*) Let $a_j(x\xi q)$, $q \in \Delta$, $j = 1, 2, \ldots$ be a sequence of symbols of canonical families of respective degrees $r_j$. Suppose $r_j$ is monotonically decreasing to $-\infty$. There exists a family $A(q)$, expansible in $\Delta$, with symbol $\Sigma\, a_j(x\xi q)$.

*c*) Let $A(q)$ be expansible in $\Delta$, there exist families $A^*(q)$ and $A'(q)$ expansible in $\Delta$ such that $A(q) - A'(q)$ is of order $-\infty$ in $\Delta$ and such that for every $u, v \in S$ $(A'(q)\, u, v) = (u, A^*(q)\, v)$. Let $\Sigma\, a_j(x\xi q)$ be the symbol of $A(q)$. The symbol of $A^*(q)$ is given by

2.11 $$\sum_{\alpha j} \frac{(-1)^{|\alpha|}}{\alpha\,!} D^\alpha\, \partial^\alpha\, \overline{a_j(x\xi q)}.$$

*d*) Let $A(q)$ be expansible in $\Delta$. Let $\Omega_i$, $i = 1, 2$, be open sets in $R^n$ with positive distance. For every real $s$ there is a $C$ such that if $u_i \in S$

$i = 1, 2$ and has its support in $\Omega_i$ then for every $q \in \Delta$

2.12 $$\| (A(q) u_1, u_2) \| < C \| u_1 \|_{s,q} | u_2 |_{s,q}.$$

*e*) Let $\Omega$ be an open set in $R^n$. Let $\chi$ be a $C^\infty$ diffeomorphism from $\Omega$ into $R^n$. Let $\Phi_i(x) \in C_0^\infty(\Omega)$ $i = 1, 2, .$ Let $A(q)$ be an expansible family in $\Delta$ of degree $s$. Let $a(x\xi q)$ be the top order symbol of $A(q)$. Let $B(q)$ be the family defined in $\Delta$ by $B(q) = M_{\Phi_1} \chi^* A(q) \chi_* M_{\Phi_2}$. $B(q)$ is expansible in $\Delta$ and of degree $s$. Let $b(x\xi q)$ be the top order symbol of $B(q)$

2.13 $$b(x\xi q) = \Phi_1(x)\, a(\chi(x), J^{-1}(x)\xi, q)\, \Phi_2(x)$$

where $J(x)$ is the adjoint of $\dfrac{d\chi}{dx}$.

Theorem 2 is proved in analogy to proofs in Kohn-Nirenberg (7) and R. S. Palais (10). Details of the proofs are given in (4).

Let $X$ be a compact $n$ dimensional $C^\infty$ manifold without a boundary. We introduce families $A(q)$ $q \in \Delta$ of linear operators from $C^\infty(X)$ to $C^\infty(X)$ which are expansible in $\Delta$ in the sense of the following definition :

DEFINITION 4. Let $A(q)$ $q \in \Delta$ be a family of linear operators from $C^\infty(X)$ to $C^\infty(X)$. $A(q)$ is expansible in $\Delta$ if it satisfies:

1) Let $\Phi_i$ $i = 1, 2$ be functions in $C^\infty(X)$ with disjoint supports. The family $M_{\Phi_1} A(q) M_{\Phi_2}$ is of order $-\infty$ in $\Delta$.

2) Let $O$ be an open set in $X$. Let $\chi$ be a set of local coordinates defined in $O$. Let $\Phi_1$ and $\Phi_2$ be in $C^\infty(X)$ with supports in $O$. The family $\chi_* M_{\Phi_1} A(q) M_{\Phi_2} \chi^*$ is expansible in $\Delta$ in the sense of definition 3.

If for every $O$, $\Phi_i$ and $\chi$ as above $\chi_* M_{\Phi_1} A(q) M_{\Phi_2} \chi^*$ is of degree, $s$, $A(q)$ is called of degree $s$.

The validity of definition 4 follows from properties (*d*) and (*e*) in theorem 2.

Let $A(q)$, $q \in \Delta$, be an expansible family as in definition 4 of degree $s$. It follows from property (*e*) in theorem 2 that there exists a function $\sigma_s[A(q)]$ defined in a subset of the cartesian product of the cotangent bundle of $X$ and $\Delta$ and such that

2.14 $$\Phi_1(x)\Phi_2(x)\sigma_s[A(q)](\Sigma\xi_j d\chi_j(x), q) = \sigma_s[\chi_* M_{\Phi_1} A(q) M_{\Phi_2} \chi^*](\chi(x), \xi, q)$$

for every $\Sigma \xi_j d\chi_j(x)$ in the cotangent bundle of $X$ and $q \in \Delta$ that satisfy $\Sigma |\xi_j|^2 + |q|^2 \neq 0$. $\sigma_s[A(q)]$ is called the symbol of $A(q)$.

We define also

DEFINITION 5. Let $M > 0$. A family $A(q)$, $q \in \Delta_M$, of linear operators from $C^\infty(X)$ to $C^\infty(X)$ is expansible in $\Delta_M$ if there exists a family $A'(q)$ expansible in $\Delta$ such that $A(q) - A'(q)$ is of order $-\infty$ in $\Delta_M$.

## 3. Expansible families with non vanishing symbols.

### The asymptotic expansion of the diagonal values of the kernels $(\lambda - A)^{-[n/m]-1}$.

THEOREM 1. Let $A(q)$, $q \in \Delta$, be an expansible family of degree $s$. Let $\sigma_s[A(q)](\Sigma \xi_j d\chi_j(x), q) \neq 0$ when $q \in \Delta$ and $\Sigma \xi_j^2 + |q|^2 \neq 0$. There exists a family $B(q)$ expansible of degree $-s$ in $\Delta$ such that $A(q) B(q) - I$ and $B(q) A(q) - I$ are of order $-\infty$ in $\Delta$.

3.1 $$\sigma_s[A(q)]\, \sigma_{-s}[B(q)] = 1.$$

PROOF. Let $A(q)$ be an expansible family of P.D.O. in $R^n$ of degree $r$. Let $a(x\xi q) = \sigma_r[A(q)](x\xi q)$ and let $a(x\xi q) \neq 0$ when $x \in \Omega$, $\xi \in R^n$, $q \in \Delta$ and $|\xi|^2 + |q|^2 \neq 0$. Let $\theta$ and $\Phi$ be in $C_0^\infty(\Omega)$ such that $\theta\Phi = \Phi$. There exist families $B_1(q)$ and $B_2(q)$, of P.D.O. in $R^n$, expansible in $\Delta$ and of degree $-r$ such that $M_\Phi A(q) B_1(q) M_\theta - M_\Phi$ and $M_\Phi B_2(q) A(q) M_\theta - M_\Phi$ are of order $-\infty$ in $\Delta$. In fact let $\psi \in C_0^\infty(\Omega)$ and let $\psi\theta = \theta$. $\psi(x) a(x\xi q)^{-1}$ is a symbol of a canonical family $W(q)$ of degree $-r$ in $\Delta$. If follows from theorem 2 (*a*) in 2 that $S_1(q) = -A(q) W(q) + M\psi$ and $S_2(q) = -W(q) A(q) + M\psi$ are expansible and of negative order in $\Delta$.

Let $S(q)$, $q \in \Delta$, be an expansible family of P.D.O. in $R^n$ of negative order $-\varrho$. There exists then an expansible family $T(q)$, $q \in \Delta$, such that $(I - S(q)) T(q) - I$ and $T(q)(I - S(q)) - I$ are of order $-\infty$ in $\Delta$. In fact $T(q)$ is a family corresponding by theorem 2, (*b*), 2 to the symbol whose terms of degrees bigger than $-(n+1)\varrho$ coincide with the terms of degrees bigger than $-(n+1)\varrho$ in the expansion of $\sum_{i=0}^{n} S(q)^i$.

Let $T_i(q)$ $i = 1, 2$ $q \in \Delta$ be expansible families such that $(I - S_1(q)) T_1(q) - I$ and $T_2(q)(I - S_2(q)) - I$ are of order $-\infty$ in $\Delta$. Let $B_1(q) = W(q) T_1(q)$ and $B_2(q) = T_2(q) W(q)$. It is easily seen that $B_i(q)$ are families as required and that $\sigma_{-r}[B_i(q)](x\xi q) = \sigma_r[A(q)](x\xi q)^{-1}$ for evey $x$ such that $\theta(x) = 1$.

Let $A(q)$ be as in the statement of this theorem. With the aid of a suitable choice of partitions of unity on $X$ and the above local result we obtain families $B_1(q)$ and $B_2(q)$ of P.D.O. on $X$, expansible in $\Delta$ and of degree $-s$ such that $A(q)\,B_1(q)-I$ and $B_2(q)\,A(q)-I$ are of order $-\infty$ in $\Delta$, and $\sigma_{-s}[B_i(q)]\,\sigma_s[A(q)]=1$, $i=1,2$. As $A(q)\,B_1(q)-I$ and $B_2(q)A(q)-I$ are of order $-\infty$ in $\Delta$ also $B_1(q)-B_2(q)$ is of order $-\infty$ there. Thus $B(q)=B_1(q)$ satisfies the theorem.

It follows from the proof of this theorem and from the proof of theorem 2 (*b*) 2 that $B(q)$ can be chosen so that for every $u \in C^\infty(X)$ $B(q)\,u$ is a continuous function from $\Delta$ to $H_0(X)$.

Let $A$ be an elliptic differential operator of order $m$ with coefficients in $C^\infty(X)$. Let the range of $\sigma_m(A)$ be disjoint to $\Gamma=\{\lambda\,;\lambda=q^m\; q\in\Delta\}$. The family $A-q^m$, $q\in\Delta$, satisfies the assumptions of theorem 1 here. It is well known, and it follows also from theorem 1 here, that there exist constants $M$ and $C$ such that $\{\lambda\,;\lambda=q^m\; q\in\Delta_M\}\subset\varrho(\bar{A})$, with $\varrho(\bar{A})$ the resolvent set of $\bar{A}$, and in $\Delta_M$ $\|(q^m-\bar{A})^{-1}\|<C\,|q|^{-m}$. Comparing $(q^m-\bar{A})^{-1}$, $q\in\Delta_M$, with the family $B(q)$ given by theorem 1 we obtain easily

THEOREM 2. Let $A$ be an elliptic differential operator of order $m$ with coefficients in $C^\infty(X)$. Let the range of $\sigma_m(A)$ be disjoint to $\Gamma=\{\lambda\,;\lambda==q^m\; q\in\Delta\}$. Let $M>0$ be such that $\{\lambda\,;\lambda=q^m\; q\in\Delta_M\}\subset\varrho(\bar{A})$. $(q^m-\bar{A})^{-1}$, restricted to $C^\infty(X)$, is an expansible family in $\Delta_M$ of degree $-m$.

It follows from theorem 2 that for every natural $k$, $(q^m-\bar{A})^{-k}$ restricted to $C^\infty(X)$, is an expansible family of degree $-mk$ in $\Delta_M$. In particular for every real $\sigma$ there is $C$ such that if $u\in C^\infty(X)$

$$\|(q^m-\bar{A})^{-K}u\|\; mk+\sigma, q < C\,\|U\|\,\sigma, q. \tag{3.2}$$

We mention also the following lemma which we need in section 4.

LEMMA 1. Let $A$ satisfy the assumptions of theorem 2. Let zero be a regular point or a pole of first order of the resolvent of $\bar{A}$. Let $\lambda\in\varrho(\bar{A})$ if $\lambda=q^m$, $\lambda\neq0$ and $q\in\Delta$. Let $B(q)$, $q\in\Delta$, be the family corresponding to $A(q)=q^m-A$ by theorem 1 here. Let $M>0$. For every real $s$ and $\varrho$ there is a $C$ such that if $q\in\Delta$, $|q|<M$, $q\neq0$ and $u\in C^\infty(X)$

$$\|((q^m-\bar{A})^{-1}-B(q))\,Au\|\; s < C\;|\,U\,\|\,\varrho \tag{3.3}$$

with the aid of theorem 2 here and theorem 3.1 in S. Agmon (3) which we already mentioned in the proof of theorem 1.2 we obtain the following result:

THEOREM 3. Let $A$ be an elliptic differential operator of order $m$ with coefficients in $C^\infty(X)$. Let the range of $\sigma_m(A)$ be disjoint to $\Gamma=\{\lambda\,;\lambda=$

$= q^m \,,\; q \in \Delta\}$. Let $M > 1$ be such that $\{\lambda\,;\,\lambda = q^m\; q \in \Delta_M\} \subset \varrho(\bar{A})$. Let $\chi$ be a set of local coordinates defined in $\Omega \subset X$. Let $O \subset \bar{O} \subset \Omega$. Let $\theta \in C^\infty(X)$ with support in $\Omega$ such that $\theta = 1$ in $0$. Let $x' = \chi(x)$. Let $R(x'\, y'\, \lambda)$, $\lambda = q^m\; q \in \Delta_M$ be the kernel of $\chi_* M_\theta (\lambda - \bar{A})^{-[n/m]-1} M_\theta \chi^*$ Let $a_j(x'\, \xi q)$ be the symbol of the canonical family of degree $-m([n/m]+1)-j$ in the expansion of $\chi_* M_\theta (q^m - \bar{A})^{-[n/m]-1} M_\theta \chi^*$. For every $k$ there is a $C_k$ such that if $x \in 0$ and $\lambda = q^m$ with $q = |q|\, e^{i\theta}$.

$$3.4 \qquad \left| R(x'\, x'\, \lambda) - (2\pi)^{-n} \sum_{j=0}^{k} |\lambda|^{-[n/m]-1+(n-j)/m} \int_{R^n} a_j(x'\, \xi e^{i\theta})\, d\xi \right| < C_k\, |\lambda|^{-[n/m]-1+(n-k-1)/m}$$

$a_j(x'\, \xi e^{i\theta})$ is determined recursively by the coefficients of $A$.

PROOF: To justify the statement of the theorem we notice that for every $\lambda \in \varrho(\bar{A})$, $(\lambda - \bar{A})^{-[n/m]-1}$ and $(\lambda - \bar{A})^{-[n/m]-1*}$ are bounded operators from $H_0(X)$ to $H_m([n/m]+1)(X)$ and as a result $(\lambda - \bar{A})^{-[n/m]-1}$ has a continuous kernel. Let $A_j(q)$ be the canonical family with symbol $a_j(x'\, \xi q)$. For every $u \in S$ and $q \in \Delta$ such that $|q| > 1$ we have

$$3.5 \qquad \widetilde{A_j(q)\, u}(\xi) = (2\pi)^{-n/2} \int e^{-ix\xi} a_j(x'\, \xi q)\, u(x')\, dx'.$$

As a result the diagonal values $\alpha_j(x'\, x'\, q)$ of the kernel, $\alpha_j(x', y'\, q)$, of $A_j(q)$ satisfy

$$3.6 \qquad \alpha_j(x'\, x'\, q) = (2\pi)^{-n}\, |q|^{-([n/m]+1)m+n-j} \int_{R^n} a_j(x'\, \xi e^{i\theta})\, d\xi\,.$$

It follows from definition 3.2 of an expansible family of P.D.O. in $R^n$, and from the above mentioned theorem in S. Agmon (3) that for every $k$ there is a $C_k$ and an $N(k)$ such that for $x \in 0$ we have

$$3.7 \qquad \left| (R(x'\, x'\, \lambda)) - \sum_{j=0}^{N(k)} \alpha_j(x'\, x'\, \lambda^{1/m}) \right| < C_k\, |\lambda|^{-([n/m]+1)+(n-k-1)/m}\,.$$

Combining 3.6 with 3.7 we obtain 3.4.

It follows from theorem 2(a).2 and from the relation $(q^m - \bar{A})^{-[n/m]-1} \cdot (q^m - A)^{[n/m]+1} = I$ that $a_j(x'\, \xi e^{i\theta})$, with $x \in 0$, are determined recursively by the coefficients of $A$.

We mention also the following result of theorem 2.

THEOREM 4. Let the assumptions of theorem 3 be satisfied. Let $R(xy\lambda)$ be the kernel of $[\lambda - A]^{-[n/m]-1}$. Let $x \neq y$. For every $\alpha, \beta$ $\partial_x^\alpha \partial_y^\beta R(xy\lambda)$ tends rapidly to zero as $\lambda \to \infty$ in $\Gamma$, uniformly in $(xy) \in \Omega_1 \times \Omega_2$ where $\Omega_1$ and $\Omega_2$ are disjoint sets in $X$ with positive distance.

PROOF: To justify the statement of the theorem we notice that for every $\lambda \in \varrho(\bar{A})$, $(\lambda - \bar{A})^{-[n/m]-1}$ is a P.D.O. on $X$ and as a result the kernel of $(\lambda - A)^{-[n/m]-1}$ is infinitely differentiable in the complement of the diagonal of $X x X$. The statement of the theorem follows easily from theorem 2 here, from property 1 in the definition 4.2 of an expansible family, from theorem 13.9 in S. Agmon (2) and from Sobolev's theorems.

## 4. Fractional powers of elliptic differential operators.

Let $A$ be an elliptic differential operator of order $m$ with coefficients in $C^\infty(X)$. We define fractional powers of $A$ under the assumption that the range of $\sigma_m(A)$ is disjoint to a fixed ray in the complex plane. It follows then that a whole angle, $\{\lambda ; \lambda \neq 0, \ \theta_1 \leq \arg \lambda \leq \theta_2\}$ is disjoint to the range of $\sigma_m(A)$. We assume also that zero is a regular point or a pole of first order of the resolvent of $\bar{A}$. It follows from the discreteness of the spectrum of $\bar{A}$ and from the known behavior of $(\lambda - \bar{A})^{-1}$ at infinity that a subangle, $\Gamma$, of the above angle belongs to $\varrho(\bar{A})$ and that there is a $C$ such that if $\lambda \in \Gamma$ $\|\lambda(\lambda - \bar{A})^{-1}\| < C$. For simplicity we have assumed that $\Gamma$ contains the negative real axis. We have the following theorem:

THEOREM 1: Let $A$ be as above. Let $0 < \alpha < 1$. Let $A_\alpha$ be the operator defined in $C^\infty(X)$ by

4.1 $$A_\alpha u = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1} (\lambda + \bar{A})^{-1} \bar{A} u \, d\lambda.$$

$A_\alpha$ is an elliptic P.D.O of order $m\alpha$. $\sigma_{m\alpha}(A_\alpha) = \sigma_m(A)^\alpha$.

PROOF: It follows form the assumption on $A$ that, in an angle $\Delta$ that, contains the positive axis, $A + q^m$ satisfies the assumptions of theorem 1.3 Let $B(q)$, $q \in \Delta$, be a family expansible in $\Delta$ such that $(A + q^m) B(q) - I$ and $B(q)(A + q^m) - I$ are of order $-\infty$ in $\Delta$, chosen so that for every $u \in C^\infty(X)$ $B(q)\,u$ is a continuous function from $\Delta$ to $H_0(X)$. It follows form theorem 2.3 and lemma 1.3 that the operator $S$ defined in $C^\infty(X)$ by

4.2
$$Su = m \int_0^\infty q^{m(\alpha-1)} q^{m-1}((\bar{A} + q^m)^{-1} - B(q))\, Au\,dq$$

is of order $-\infty$ in $\Delta$. Thus it is sufficient to show that the operator $B_\alpha$ defined in $C^\infty(X)$ by

4.3
$$B_\alpha u = \frac{\sin \pi\alpha}{\pi}\, m \int_0^\infty q^{m(\alpha-1)} q^{m-1}\, B(q)\, u\,dq,$$

is an elliptic P.D.O. of order $m(\alpha - 1)$ with $\sigma_{m(\alpha-1)}(B_\alpha)\cdot\sigma_m(A)^{1-\alpha} = 1$. It follows easily from the definition of an expansible family of P.D.O on $X$ that to prove this result it is sufficient to check that if $A(q)$ is a canonical family of P.D.O. in $R^n$, defined in $\Delta$, of degree $\sigma < -m$ and with symbol $a(x\xi q)$ the operator $P$ defined in $S$ by

4.4
$$Pu = m \int_0^\infty q^{-1+m\alpha} A(q)\, u\,dq$$

differs by an operatar of order $-\infty$ from the canonical operator with symbol $m \int q^{-1+m\alpha} a(x\xi q)\,dq$. Let $A'(q)$, $q \in \Delta$, be the family of linear operators from $S$ to $S$ given by

4.5
$$\widetilde{A'(q)u}(\xi) = 2\pi^{-2/n} \int e^{-ix\xi} \zeta(\xi)\, a(x\xi q)\,dx$$

where as in the definition of a canonical P.D.O. in $R^n$, $\zeta(\xi) \in C^\infty(R^n)$ $0 \leq \zeta(\xi) \leq 1$, $\zeta(\xi) = 0$ when $|\xi| \leq \frac{1}{2}$ and $\zeta(\xi) = 1$ when $|\xi| \geq 1$. Let $P'$ be defined in $S$ by

4.6
$$P' u = m \int_0^\infty q^{-1+m\alpha} A'(q)\, u\,dq.$$

It follows from definition 1.2 of canonical families, from 4.4, 4.5 and 4.6 $P - P'$ is of order $-\infty$. Let $u, v \in S$ then

4.7
$$(P' u, v) = m \int_0^\infty q^{-1+m\alpha} (A'(q)\, u, v)\,dq.$$

It follows from Parseval's equality that

$$4.8 \qquad \widetilde{(P' \cdot u}, \widetilde{v}) = m \int_0^\infty q^{-1+m\alpha} (\widetilde{A'(q) u}, \widetilde{v})\, dq.$$

It follows from Fubini's theorem with the aid of 4.5 that

$$4.9 \quad (\widetilde{P' u}, \widetilde{v}) = \int (2\pi)^{-n/2} \left( \int_{R^n} e^{-ix\xi} \zeta(\xi) \left( m \int_0^\infty q^{-1+m\alpha} a(x\xi q)\, dq \right) u(x)\, dx \right) \widetilde{v}(\xi)\, d\xi$$

and as a result $P'$ coincides with the canonical P.D.O. determined by the symbol $m \int_0^\infty q^{-1+m\alpha} a(x\xi q)\, dq$.

Let $\bar{A}_\alpha$ be the closure of $A_\alpha$ in $H_0(X)$. It follows from the ellipticity of $A_\alpha$ that the domain of definition of $\bar{A}_\alpha$ is $H_{m\alpha}(X)$.

As $\bar{A}$ is a closed operator, densely defined in $H_0(X)$, such that $\varrho(\bar{A})$ contains a whole angle $\theta_1 \leq \arg \lambda \leq \theta_2$, except possibly the origin, and an estimate $\| \lambda (\lambda - \bar{A})^{-1} \| < C$ holds in this angle, a definition for the power $\bar{A}^\alpha$ is given by Kato (6). $\bar{A}_\alpha$ coincides with $\bar{A}^\alpha$. In fact the closed operator $\bar{A}^\alpha$ is defined in Kato (6) indirectly by

$$4.10 \quad (\lambda + \bar{A}^\alpha)^{-1} = \frac{\sin \pi \alpha}{\pi} \int_0^\infty \frac{\mu^\alpha}{\lambda^2 + 2\lambda\, \mu^2 \cos \pi\alpha + \mu^{2\alpha}} (\mu + \bar{A})^{-1}\, d\mu.$$

which holds for $\lambda > 0$ (and for $\lambda \geq 0$ if $0 \in \varrho(\bar{A})$).

In our case where zero is at most a pole of the resolvent of $\bar{A}$ if follows easily from 4.10 that $D(\bar{A}) \subset D(\bar{A}^\alpha)$ and in $D(\bar{A})$

$$4.11 \qquad \bar{A}^\alpha u = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1} (\lambda + \bar{A})^{-1} \bar{A} u\, d\lambda.$$

So that $\bar{A}^\alpha$ is closed extension of $A_\alpha$ with $\{\lambda ; \lambda > 0\} \subset \varrho(\bar{A}^\alpha)$. But it follows from the ellipticity of $A_\alpha$, from $\sigma_{m\alpha}(A_\alpha) = \sigma_m(A)^\alpha$ and from the remarks in the appendix 5 that if $\lambda > 0$ is sufficiently large also $\lambda \in \varrho(\bar{A}_\alpha)$ so that $\bar{A}_\alpha = \bar{A}^\alpha$.

As usual we put for $s = n + \alpha$ with $n$ natural and $0 < \alpha < 1$ $\bar{A}^s = \bar{A}^n \bar{A}^\alpha$. Again it follows from the ellipticity of $A$ and $A_\alpha$ that the domain of definition of $\bar{A}^{s}$ is $H_{ms}(X)$ and $\bar{A}^{s}$ is the closure in $H_0(x)$ of an elliptic P.D.O. of order $ms$. Also if $0 \in \varrho(\bar{A})$ and as usual $\bar{A}^{-s}$ with $s > 0$ is defined by. $\bar{A}^{-s} = (\bar{A}^{s})^{-1}$, in $C^\infty(X)$, $\bar{A}^{-s}$ is an elliptic P.D.O. of order $-ms$.

We investigate now properties of the kernels $\bar{A}^{s}$, with $s < -n/m$, or more generally properties of the kernels of the operators $K_s$, $s < -n/m$, given by

$$4.12 \quad K_s = \frac{\sin \pi s}{\pi} \int_1^\infty \lambda^s \left((\lambda + \bar{A})^{-1} - \frac{E_0}{\lambda}\right) d\lambda + \frac{1}{2\pi i} \int_{|\lambda|=1} \lambda^s \left((\lambda - \bar{A})^{-1} - \frac{E_0}{\lambda}\right) d\lambda,$$

where $E_0$ is the spectral projection on the null space of $\bar{A}$.

In case $0 \in \varrho(\bar{A})$ $K_s = \bar{A}^{s}$. $K_s$ satisfies the following lemma:

**Lemma 1.** Let $s < -n/m$ and let $K_s$ be given by 4.12 $K_s$ has a continuous kernel $K(xys)$.

**Proof**: In case $m \leq n$ we consider the following expression for $K_s$:

$$4.13 \quad K_s = \frac{\sin \pi s}{\pi} \frac{[n/m]!}{(s+1)\dots(s+[n/m])} \int_1^\infty \lambda^{s+[n/m]} (\lambda + \bar{A})^{-[n/m]-1} d\lambda + \frac{\sin \pi s}{\pi} \frac{E_0}{s}$$

$$+ \frac{1}{2\pi i} \frac{[n/m]!}{(s+1)\dots(s+[n/m])} \int_{|\lambda|=1} \lambda^{s+[n/m]} \left((\lambda - \bar{A})^{-[n/m]-1} - \frac{E_0}{\lambda^{[n/m]+1}}\right) d\lambda.$$

From 3.2 it follows

$$4.14 \qquad \|(\lambda + \bar{A})^{-[n/m]-1}\|_{[n/m]+1} \leq C; \quad \|(\lambda + \bar{A})^{-[n/m]-1}\|_0 \leq C |\lambda|^{-[n/m]-1}$$

$$4.15 \qquad \|(\lambda + \bar{A})^{-[n/m]-1*}\|_{[n/m]+1} \leq C.$$

4.15 is obtained by an application of 3.2 to $(\lambda + \bar{A})^{-[n/m]-1*}$ which coincides with $(\lambda + \bar{A}^*)^{-[n/m]-1}$ where $A^*$ is the formal adjoint of $A$ in $C^\infty(X)$. It follows from 4.14, 4.15 and 2.6 that the continuous kernel $S(xy\lambda)$ of $(\lambda + \bar{A})^{-[n/m]-1}$ satisfies

$$4.16 \qquad |S(xy\lambda)| \leq C |\lambda|^{-[n/m]-1+n/m}.$$

Also $S(xy\lambda)$ is a continuous function of $(xy\lambda)$ for $(xy) \in XxX$ and $\lambda \geq 1$.

As a result if $s < -n/m$ $K_s$ has a continous kernel given by

4 17
$$\frac{\sin \pi s}{\pi} \frac{[n/m]!}{(s+1)\dots(s+[n/m])} \int_1^\infty \lambda^{s+[n/m]} s(xy\lambda)\, d\lambda + \frac{\sin \pi s}{\pi} \frac{E_0(xy)}{s} +$$

$$\frac{1}{2\pi i} \frac{[n/m]!}{(s+1)\dots(s+[n/m])} \int_{|\lambda|=1} \lambda^{s+[n/m]} \left(R(xy\lambda) - \frac{E_0(xy)}{\lambda^{[n/m]+1}}\right) d\lambda.$$

Here $R(xy\lambda)$ is the kernel of $(\lambda - \bar{A})^{-[n/m]-1}$ and $E_0(xy)$ is the $C^\infty$ kernel of $E_0$

If $m > n$ we replace $\frac{[n/m]!}{(s+1)\dots(s+[n/m])}$ in 4.17 by 1.

**THEOREM 2.** Let $s < -n/m$. Let $x \neq y$. $K(xys)$ has an extension to an entire analytic function of $s$ which vanishes at the positive integers and also at zero if $0 \in \varrho(\bar{A})$. The extension $K(xys)$ thus obtained belongs to $C^\infty$ in the complement of the diagonal of $XxX$.

**PROOF.** Let $\theta$ and $\psi$ be functions in $C^\infty(X)$ with disjoint supports. Let $S(xy\lambda)$ be the infinitely differentiable kernel of $M_\theta(\lambda + \bar{A})^{-1} M_\psi$. It follows from theorem 4.3 that for every $\alpha\beta$ $\partial_x^\alpha \partial_y^\beta S(xy\lambda)$ tends rapidly to zero as $\lambda \to \infty$, uniformly in $(xy)$. Also $\partial_x^\alpha \partial_y^\beta S(xy\lambda)$ is a continuous function of $(xy\lambda)$. As a result the operator

4.18
$$\frac{\sin \pi s}{\pi} \int_1^\infty \lambda^s M_\theta(\lambda + \bar{A})^{-1} M_\psi\, d\lambda$$

with $s < n/m$ has the infinitely differentiable kernel given by

4.19
$$L(xys) = \frac{\sin \pi s}{\pi} \int_1^\infty \lambda^s S(xy\lambda)\, d\lambda$$

and for every $(xy)$ $L(xys)$ has an extension to an entire analytic function of $s$ which vanishes at the integers. The extension $L(xys)$ thus obtained is in $C^\infty(XxX)$. Similarly the operator

4.20
$$\frac{1}{2\pi i} \int_{|\lambda|=1} \lambda^s M_\theta \left((\lambda - \bar{A})^{-1} - \frac{E_0}{\lambda}\right) M_\psi\, d\lambda$$

with $s < -n/m$ has the infinitely differentiable kernel given by

4.21 $$M(xys) = \frac{1}{2\pi i} \int_{|\lambda|=1} \lambda^s (R(xy\lambda) - \theta(x) E_0(xy) \psi(y))\, d\lambda.$$

Here $R(xy\lambda)$ is the kernel of $M_\theta(\lambda - \bar{A})^{-1} M_\psi$ and $E_0(xy)$ is the kernel of $E_0$. For every $(xy) M(xys)$ has an extension to an entire analytic function of $s$ which vanishes at the non negative integers as a result of the regularity of $R(xy\lambda) - \dfrac{\theta(x) E_0(xy) \psi(y)}{\lambda}$ in the closed unit circle. The extension $M(xys)$ thus obtained is in $C^\infty(XxX)$.

To complete the proof we notice that if $E_0 \neq 0$ and $s < -n/m$

4.22 $$\frac{\sin \pi s}{\pi} M_\theta \int_1^\infty \lambda^{s-1} E_0\, d_\lambda M_\psi$$

has the infinitely differentiable kernel

4.23 $$\frac{\sin \pi s}{\pi s} \theta(x) E_0(xy) \psi(y).$$

For $x = y$ we have the following result:

THEOREM 3: Let $K(xys)$ be the kernel of $K_s$ with $s < -n/m$ every $x \in X$ $K(xxs)$ has an extension to a meromorphic function of $s$ with simple poles at the points $(j-n)/m$, $j = 0, 1, 2, \ldots$, that differ from the naturals and from zero.

Let $\chi$ be a set of local coordinates defined in $\Omega \subset X$. Let $x' = \chi(x)$. Let $0$ be an open set such that $0 \subset \bar{0} \subset \Omega$. Let $\theta$ be in $C^\infty(X)$ with support in $\Omega$ and let $\theta(x) = 1$ in $0$. Let $K'(x'x's)$ be the kernel of $\chi_* M_\theta K_s M_\theta \chi^*$. Let $a_j(x'\xi q)$ be the symbol of the canonical family of degree $-m([n/m]+1)-j$ in the expansion of $\chi_* M_\theta (q^m + \bar{A})^{-[n/m]-1} M_\theta \chi^*$. Let $i$ be a natural number

4.24 $$K'(x'x'i) = -\frac{(2\pi)^{-n} [n/m]!}{(i+1)\ldots(i+[n/m])} \int a_{mi+n}(x'\xi 1)\, d\xi.$$

If $E_0 = 0$ 5.24 holds also for $i = 0$.

If $E_0 \neq 0$, let $E_0'(x'y')$ be the kernel of $\chi_* M_\theta E_0 M_\theta \chi^*$, then

4.25 $$K'(x'x'0) = -(2\pi)^{-n} \int a_n(x'\xi 1)\, d\xi + E_0'(x'x').$$

The residues of $(K'(x'x's)$ are as follows:

Let $i$ be a negative integer with $i \geq -n/m$. The residue of $K'(x'x's)$ at $s = i$ is given by

4.26 $$-(2\pi)^{-n} \frac{[n/m]!}{\pi_{\substack{j=1 \\ j\neq -i}}^{[n/m]} (j+i)} \int_{R^n} a_{mi+n}(x'\xi 1)\, d\xi.$$

Let $(j-n)/m$, $j = 0, 1, 2, \ldots$, be different from an integer. The residues of $K'(x'x's)$ at $s = (j-n)/m$ are given by

4.27 $$-(2\pi)^{-n} \frac{\sin(\pi(j-n)/m)}{\pi} \frac{[n/m]!}{\left(\frac{j-n}{m}+1\right)\cdots\left(\frac{j-n}{m}+[n/m]\right)} \int a_j(x'\xi 1)\, d\xi.$$

PROOF: We consider the expression 4.13 for $K_s$. Let $s < -n/m$. It is easily seen that $\frac{1}{2\pi i} \frac{[n/m]!}{(s+1)\ldots(s+[n/m])} \int_{|\lambda|=1} \lambda^{s+[n/m]} \left((\lambda - \bar{A})^{-[n/m]-1} - \right.$ $\left. - \frac{E_0}{\lambda^{[n/m]+1}}\right) d\lambda$ has a continuous kernel $M(xys)$ and for every $(xy)$ $M(xys)$ has an extension to an entire analytic function of $s$ which vanishes at the bigger from $-n/m$ integers. Thus to prove the theorem it is sufficient to show that if $N(xys)$ wilh $s < -n/m$ is the kernel of

4.28 $$N_s = \frac{\sin \pi s}{\pi} \frac{[n/m]!}{(s+1)\ldots(s+[n/m])} \int_1^\infty \lambda^{s+[n/m]} (\lambda + \bar{A})^{-[n/m]-1}\, d\lambda$$

$N(xxs)$ has an extension to a meromorphic function of $x$ with simple poles at $(j-n)/m$, $j = 0, 1, 2, \ldots$, with $(j-n)/m$ different from the non negative integers, and to determine $N(xxi)$ and the residues of $N(xxs)$.

An application of theorem 3.3 to $(\lambda + A)^{-[n/m]-1}$, with $\lambda \geq 1$, yields the asymptotic expansion

4.29 $$(2\pi)^{-n} \sum_{j=0}^{\infty} \lambda^{-[n/m]-1+(n-j)/m} \int_{R^n} a_j(x'\xi 1)\, d\xi$$

for the diagonal values of the kernel of $\chi_* M_\theta (q^m + \bar{A})^{-[n/m]-1} M_\theta \chi^*$ where $a_j(x'\xi 1)$ are as in the statement of the theorem.

Let $N'(x'x's)$ be the kernel of $\chi_* M_\theta N_s M_\theta \chi^*$. The location of the poles of $N'(x'x's)$ follows from 4.28, 4.29 and from

4.30 $$\int_1^\infty \lambda^{s+[n/m]} \lambda^{-[n/m]-1+(n-j)/m} \, d\lambda = - \frac{1}{s - \frac{j-n}{m}} .$$

The values of $K'(x'x'i)$ and the residues of $K'(x'x's)$ are given by 4.24-4.27 as a result of 4.28, 4.29, 4.30 and the relation of $N'(x'x's)$ to $K'(x'x's)$.

## 5. Appendix.

We add here some remarks concerning the existence of rays of minimal growth of the resolvent of an elliptic P. D. O. on $X$ and concerning the completeness of the generalized eigenfunctions of $A$.

We prove in (4) the following theorems which we do not prove here:

THEOREM 1: Let $A$ be an elliptic P. D. O of positive order $s$. Let the range of $\sigma_s(A)$ be disjoint to the angle $\Gamma = \{\lambda ; \theta_1 \leq \arg \lambda \leq \theta_2)$ with $\theta_1 \leq \theta_2$. Let $\bar{A}$ be the closure of $A$ in $H_0(X)$. There exist constants $M$ and $C$ such that if $|\lambda| > M$ and $\lambda \in \Gamma$ $\lambda \in \varrho(\bar{A})$ and $\| \lambda (\lambda - \bar{A})^{-1} \| < C$.

We have also the following:

THEOREM 2: Let $A$ be an elliptic P. D. O. of positive order $s$. Let $\lambda_0 \in \varrho(\bar{A})$. For every $\varepsilon > 0$ $(\lambda_0 - \bar{A})^{-1} \in C_{n/s+\varepsilon}$.

The proof of this theorem is based on $(\lambda_0 - \bar{A})^{-1}$, restricted to $C^\infty(X)$, being an elliptic P. D. O. of order $-m$ and on the following lemma in S. Agmon (1):

Let $T$ be a compact operator in $H_0^\#$ which carries $H_0^\#$ into $H_0^\#$ with $\sigma > 0$. Let $\{\lambda_j\}$ be the sequence of eigenvalues of $T$ each repeated according to the multiplicity. For every $\varepsilon > 0$ $\Sigma |\lambda_j|^{n/\sigma+\varepsilon} < \infty$.

From theorems 2 and 3 it follows with the aid of (XI.9.31) in Dunford-Schwartz (5):

THEOREM 3. Let $A$ be an elliptic P. D. O. of positive order $s$. Let the range of $\sigma_s(A)$ be contained in an angle $\Gamma$: $\{\lambda ; \theta_1 < \arg \lambda < \theta_2\}$ with $\theta_2 - \theta_1$ less than $\pi s/n$. The generalized eigenfunctions of $\bar{A}$ are complete in $H_0(X)$.

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