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ON PERIOD RELATIONS FOR ABELIAN INTEGRALS ON ALGEBRAIC CURVES

by A. ANDREOTTI and A. L. MAYER (*)

In primo luogo non dovrà il Poeta *moderno* aver letti, né legger mai gli Autori antichi Latini o Greci. Imperocché nemeno gli antichi Greci o Latini hanno mai letti i *moderni*.

B. MARCELLO, *Il Teatro alla Moda*

Let H_g be the Siegel upper half plane of rank $g > 1$ and Γ be the modular group [26]. The space $V_g = H_g/\Gamma$ represents the space of moduli for principally polarised abelian varieties. The set of Jacobians, i. e. the moduli space of curves of genus g is open and dense in a $3g - 3$ dimensional analytic subspace M_g of V_g.

Let $\bar{J}$ be the counter image of M_g in H_g. Riemann raised the question of writing a set of equations for $\bar{J}$ by analytic functions on H_g. This problem is meaningful for any $g \geq 4$ because then $\bar{J}$ is a proper analytic subset of H_g.

For $g = 4$, $\bar{J}$ is of codimension one and Schottky [25] was able to write a polynomial in the « theta-nulls » non identically zero and vanishing on $\bar{J}$, so that $\bar{J}$ appears as an irreducible component of the set of zeros of that polynomial.

In this paper we consider the following problem:

Let X be a minimal positive polar divisor of the principally polarised abelian variety A, let $S(X)$ be the singular set of X and introduce the following invariant $r(A) =$ codimension of $S(X)$ in X

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Let V_g^r be the subset of V_g where $r(A) \leq r$. V_g^r is an analytic subset of V_g and V_g^{g-1} is of codimension one in V_g. We give here a proof of the fact that M_g is an irreducible component of V_g^3 (§ 1, 2 and 3).

By virtue of the geometric significance of the sets V_g^r we are able to write the equations of those sets (on the set over which X is irreducible) in terms of thetanulls and their derivatives. We thus obtain (when $r = 3$) a generalisation of Schottky's result for all values of $g \geq 4$ (§ 4).

At the end of the paper an outline of a procedure of Wirtinger is given which eventually may give more explicit computations and does not involve thetanulls' derivatives.

All the sets V_g^r (on the open subset where X is irreducible) are shown to be algebraic sets.

1. General remarks on algebraic curves.

1. *The canonical image.* *a*) Let C be a complete irreducible algebraic curve of genus g defined over an algebraically closed field k. Let Ω^1 be the sheaf of germs of holomorphic differentials on C and let $\omega_1, \dots, \omega_g$ be a basis for the vector space $H^0(C, \Omega^1)$. The canonical map

$$\Phi : C \longrightarrow P_{g-1}(k)$$

is defined by

$$\Phi(x) = (\omega_1(x), \dots, \omega_g(x)).$$

A change of basis in $H^0(C, \Omega^1)$ changes the map by an homography of $P_{g-1}(k)$ into itself. This map has the following properties (cf. [4]).

i) Φ is a morphism of C onto an algebraic non singular curve Γ of $P_{g-1}(k)$ not contained in any proper subspace of $P_{g-1}(k)$.

ii) If C is not hyperelliptic Φ is an isomorphism of C onto Γ. If C is hyperelliptic Φ is of degree 2 and Γ is a rational curve.

iii) If C is not hyperelliptic the hypersurfaces of order $l \geq 1$ of $P_{g-1}(k)$ cut out on Γ the complete linear series $|lK|$ where K is the canonical divisor on Γ.

b) We assume in the sequel that C is not hyperelliptic. From iii) and Riemann-Roch theorem one deduces that the linear system Σ of all quadrics containing Γ has projective dimension

$$d = \frac{1}{2} g(g+1) - (3g-3) - 1 = \frac{1}{2}(g-2)(g-3) - 1.$$

Let us represent a quadric $\Sigma a_{ij} x_i x_j = 0$ in $P_{g-1}(k)$ (the x_i's being homogeneous coordinates in that space) by the point of $P_\varrho(k)$, $\varrho = \frac{1}{2} g(g+1) - 1$, with homogeneous coordinates a_{ij}.

The linear group $GL(g, k)$ acts on $P_\varrho(k)$ by

$$(M, (a_{ij})) \longrightarrow {}^tM(a_{ij}) M$$

for any matrix $M \in GL(g, k)$ where (a_{ij}) denotes the matrix of the numbers a_{ij}.

The space $P_\varrho(k)$ decomposes then into g orbits W_r, $1 \leq r \leq g$, where

$$W_r = \{(a_{ij}) \in P_\varrho(k) \mid \text{rank}\,(a_{ij}) = r\}.$$

If Ω_r is the stabiliser of a point in W_r e. g.

$$\Omega_r = \left\{ M \in GL(g, k) \;\middle|\; {}^tM \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} M = \sigma_M \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} \text{ for some } \sigma_M \in k^* \right\}$$

then $W_r = GL(g, k)/\Omega_r$ and one easily computes then the dimension of W_r:

$$\dim W_r = gr - \frac{1}{2} r(r-1) - 1.$$

The Zariski-clousure of W_r is the algebraic variety $Y_r = \bigcup_{s \leq r} W_s$ and thus has the same dimension as W_r.

LEMMA 1. *In the linear system Σ there are no quadrics of rank ≤ 2. Moreover the quadrics of Σ of rank ≤ 4 are represented in $P_\varrho(k)$ by an algebraic variety of dimension $\geq g-4$ (non empty if $g \geq 4$).*

The quadrics of rank ≤ 2 being reducible, the first assertion follows from i). Since Σ is of codimension $3g-3$ in $P_\varrho(k)$ the variety $Y_4 \cap \Sigma$ is of codimension $\leq 3g-3$.

c) *Quadrics of Σ of rank* 4. Without loss of generality we may assume that the quadric Q of rank 4 has the equation $x_1 x_2 - x_3 x_4 = 0$. It is thus the projection from a projective space P_{g-5} (its singular set) of a non singular quadric of projective 3 space. This last is ruled by two distinct pencils of lines so that Q is ruled by two pencils $|P_{g-3}|$ and $|P'_{g-3}|$ of projective spaces of dimension $g-3$ ($x_1 = \lambda x_3$, $x_2 = \lambda x_4$ and $x_1 = \lambda x_4$, $x_3 = \lambda x_2$ for $\lambda \in k \cup \infty$).

Let $\widetilde{Q}$ be the monoidal transform of Q with center P_{g-5} and let $\pi : \widetilde{Q} \longrightarrow Q$ be the natural projection. The variety $\widetilde{Q}$ is non singular and $\pi^{-1}(P_{g-5}) = S$ is a divisor on $\widetilde{Q}$. The proper transforms of the two pencils

$|P_{g-3}|$ and $|P'_{g-3}|$ give two linear pencils $|D|$ and $|D'|$ on $\widetilde{Q}$. Let $\widetilde{\Gamma}$ be the proper transform of Γ; then $\pi|\widetilde{\Gamma}$ is an isomorphism of $\widetilde{\Gamma}$ onto Γ. We set, identifying Γ with $\widetilde{\Gamma}$:

$$G_0 = S\cdot\Gamma, \quad g_p^1 = |D|\cdot\Gamma, \quad g_q^1 = |D'|\cdot\Gamma$$

and remark that since $P_{g-3} + P'_{g-3}$ is a hyperplane section of Q we must have

$$|G_0 + g_p^1 + g_q^1| = |K|.$$

We have therefore the following

LEMMA 2. *Let Q be a quadric of Σ of rank 4. Then Q determines*

i) *two linear series g_p^1 and g_q^1 without fixed points cut out on Γ by the two ruling pencils $|P_{g-3}|$ and $|P'_{g-3}|$*

ii) *a divisor $G_0 \geq 0$ whose support is in the set where the vertex of Q meet Γ*

such that

$$|g_p^1 + g_q^1 + G_0| = |K|.$$

d) *Quadrics of Σ of rank* 3. We may assume $Q = \{x_1^2 + x_2 x_3 = 0\}$ so that Q is the projection of an irreducible plane conic from a vertex P_{g-4}. Thus Q is ruled by a single pencil $|P_{g-3}|$ of projective spaces of dimension $g-3$. One has the following

LEMMA 3. *Let Q be a quadric of Σ of rank 3. Then Q determines*

i) *a linear series g_p^1 without fixed points cut out on Γ by the ruling pencil $|P_{g-3}|$*

ii) *a divisor $G_0 \geq 0$ whose support is in the set where the vertex of Q meets Γ*

such that

$$|2g_p^1 + G_0| = |K|.$$

e) Conversely one has the following

LEMMA 4. *Let G_0 be a positive divisor on Γ and g_p^1, g_q^1 two linear series without fixed points such that*

$$|g_p^1 + g_q^1 + G_0| = |K|.$$

Then there is a quadric Q in Σ of rank ≤ 4 such that G_0, g_p^1, g_q^1 are determined by Q as in lemmas 2 and 3.

The rank of Q is 4 or 3 according as $g_p^1 \neq g_q^1$ or $g_p^1 = g_q^1$. If one of the series g_p^1 or g_q^1 is complete then the quadric Q is unique.

PROOF. Let G_p^1, G_p^2 (resp. G_q^1, G_q^2) be two distinct divisors of g_p^1 (resp. g_q^1). Their supports are disjoint since g_p^1 (resp. g_q^1) has no fixed point. Let $l_{\alpha\beta}(X) = 0$ be the unique hyperplane which cuts out the canonical divisor $G_p^\alpha + G_0 + G_q^\beta$ for $\alpha = 1, 2$, $\beta = 1, 2$. The rational function on Γ

$$h = l_{11}(x)\, l_{22}(x)\, l_{12}^{-1}(x)\, l_{21}^{-1}(x)$$

is not identically zero and well defined on Γ (lemma 1).

Moreover it has no zeros nor poles, thus it is a constant $c \neq 0$. The quadric

$$Q = l_{11}(x)\, l_{22}(x) - c l_{12}(x)\, l_{21}(x) = 0$$

satisfies the requirement of the lemma. It is the unique quadric of that sort if say g_p^1 is complete because then for any $G_q \in g_q^1$ the specialty index $i(G_q + G_0) = 2$ and thus $G_q + G_0$ determines uniquely the P'_{g-3} of the pencil $|P'_{g-3}|$ corresponding to that divisor. When G_q varies in g_q^1, P'_{g-3} describes $|P'_{g-3}|$ and thus Q being the set theoretic union of those P'_{g-3} is uniquely determined.

2. *Special curves of genus g a)* We assume now that the curve C carries a linear series g_h^1 of dimension 1 and degree h without fixed points. If $h \leq g - 1$ this linear series is special. Let D, D' be two distinct positive divisors of g_h^1. Since g_h^1 has no fixed points D and D' have disjoint supports and there exists a non-constant rational function f on C such that

$$(f) = \text{the divisor of } f = D' - D.$$

Let $E = H^0(C, \Omega^1)$ and let

$$F = \{\omega \in E \mid (\omega) \geq D\}.$$

We define a linear map

$$\lambda : F \longrightarrow E$$

by sending each element $\omega \in F$ into the element $f\omega$ of E. This map has the property that if H is any subspace of F such that $\lambda(H) \subset H$ then necessarily $H = 0$. In fact if $\omega \in H$ and $\omega \neq 0$, one has

$$(\omega) \geq D, (f\omega) \geq D, (f^2\omega) \geq D, \ldots, (f^h\omega) \geq D, \ldots .$$

This for large enough h (e. g. $h > 2g - 2$) is absurd.

b) We have to study the following situation. Let E be a finite dimensional vector space over a field k, F a subspace of E and

$$\lambda : F \longrightarrow E$$

a linear map of F into E. We say that λ is irreducible if for a linear subspace $H \subset F$ we have

$$\lambda(H) \subset H \Longrightarrow H = 0.$$

An irreducible map is certainly injective since for $H = \operatorname{Ker} \lambda$, $\lambda(H) = 0 \subset H$ and thus $\operatorname{Ker} \lambda = 0$. If a basis $e_1, \dots, e_r$ of F is completed into a basis $e_1, \dots, e_r, e_{r+1}, \dots, e_s$ of E the map λ is fully described by the action on the basis of F:

$$\begin{cases} \lambda(e_\alpha) = \sum_{\beta=1}^{s} \sigma_{\alpha\beta}\, e_\beta \\ 1 \leq \alpha \leq r \end{cases}$$

i. e. by the matrix $\sigma = (\sigma_{\alpha\beta})$.

Assume λ irreducible and consider the sequence of spaces

$$F,\ F \cap \lambda(F) = F_1,\ F_1 \cap \lambda(F_1) = F_2, \dots$$

One has $\dim F_1 < \dim F$ otherwise $\lambda(F) = F$ and $F = 0$, similarly $\dim F_2 < \dim F_1$ otherwise $\lambda(F_1) = F_1$ and $F_1 = 0, \dots$. There exists therefore an integer μ such that

$$F_\mu \neq 0, \quad F_\mu \cap \lambda(F_\mu) = 0.$$

LEMMA 5. *Under the specified assumptions one can find* $\mu + 1$ *subspaces* $E_1, \dots, E_{\mu+1}$ *in* F *such that*

i) $$F = \bigoplus_0^{\mu} \lambda^s(E_1) \bigoplus_0^{\mu-1} \lambda^s(E_2) \oplus \dots \oplus E_{\mu+1}$$

ii) $$F + \lambda(F) = \bigoplus_0^{\mu+1} \lambda^s(E_1) \bigoplus_0^{\mu} \lambda^s(E_2) \oplus \dots \bigoplus_0^{1} \lambda^s(E_{\mu+1}).$$

PROOF. If $\mu = 0$ we get

$$F + \lambda(F) = F \oplus \lambda(F).$$

By induction we may assume

$$F_\varrho = \bigoplus_0^{\mu'} \lambda^s (B_1) \bigoplus_0^{\mu'-1} \lambda^s (B_2) \oplus \dots \oplus B_{\mu'+1}$$

$$F_\varrho + \lambda (F_\varrho) = \bigoplus_0^{\mu'+1} \lambda^s (B_1) \bigoplus_0^{\mu'} \lambda^s (B_2) \oplus \dots \bigoplus_0^{1} \lambda^s (B_{\mu'+1})$$

where $\mu' = \mu - \varrho$.

Since $F_\varrho = F_{\varrho-1} \cap \lambda (F_{\varrho-1})$ both F_ϱ and $\lambda (F_\varrho)$ are contained in $\lambda (F_{\varrho-1})$. Let $B_{\mu'+2}$ be a complement in $\lambda (F_{\varrho-1})$ of $F_\varrho + \lambda (F_\varrho)$. We thus have

$$\lambda (F_{\varrho-1}) = \bigoplus_0^{\mu'+1} \lambda^s (B_1) \oplus \dots \bigoplus_0^{1} \lambda^s (B_{\mu'+1}) \oplus B_{\mu'+2} .$$

We set

$$E_1 = \lambda^{-1} (B_1), \dots , E_{\mu'+2} = \lambda^{-1} (B_{\mu'+2})$$

so that

$$F_{\varrho-1} = \bigoplus_0^{\mu'+1} \lambda^s (E_1) \oplus \dots \bigoplus_0^{1} \lambda^s (E_{\mu'+1}) \oplus E_{\mu'+2}$$

and we see that

$$F_{\varrho-1} + \lambda (F_{\varrho-1}) = \bigoplus_0^{\mu'+2} \lambda^s (E_2) \oplus \dots \bigoplus_0^{2} \lambda^s (E_{\mu'+1}) \bigoplus_0^{1} \lambda^s (E_{\mu'+2})$$

COROLLARY 1. *Choosing a proper basis in F and completing it to a proper basis of E the matrix σ of the map λ can be given the form*

$$\sigma = \begin{pmatrix} U_1 & 0 & 0 \\ \cdot & \cdot & \cdot \\ 0 & U_l & 0 \end{pmatrix}$$

where each U_α is a $r_\alpha \times s_\alpha$ rectangular matrix with $s_\alpha > r_\alpha$ of the form (0, I).

Note that $l \leq \dim E - \dim F$.

COROLLARY 2. *With respect to a choice of the basis as in the previous corollary the $\frac{1}{2} r (r + 1)$ elements*

$$Q_{\alpha\beta} \equiv e_\alpha \lambda (e_\beta) - e_\beta \lambda (e_\alpha) \; 1 \leq \alpha < \beta \leq r$$

of the symmetric tensor product of E with itself are linearly independent.

In fact if $\Sigma\, C_{\alpha\beta}\, Q_{\alpha\beta} = 0$, since e_1 occurs only in $Q_{1\beta}$ we must have $\sum\limits_{\beta>1} C_{1\beta}\, \lambda(e_\beta) = 0$. Since λ is injective $C_{1\beta} = 0\ \forall\ \beta$. Since e_2 occurs only in $Q_{2\beta}$ we must have $\sum\limits_{\beta>2} C_{2\beta}\, \lambda(e_\beta) = 0$ and thus $C_{2\beta} = 0\ \forall\ \beta$. Continuing in this way we get $C_{\alpha\beta} = 0\ \forall\ \alpha, \beta$.

c) Returning to the situation described at the beginning, by application of the previous lemma and corollaries we obtain the following

LEMMA 6 *Let Γ be non hyperelliptic canonical curve of genus $g \geq 4$. Let D and D' be two disjoint positive divisors on Γ, linearly equivalent and of degree $h \leq g-1$. Let f be a rational function on Γ with $(f) = D' - D$ and let $\{l_\alpha\}_{1\leq\alpha\leq r}$ be a basis of the linear forms on $P_{g-1}(k)$ with $(l_\alpha) \geq D$. If $\{l'_\alpha\}_{1\leq\alpha\leq r}$ is a basis of linear forms on $P_{g-1}(k)$ with $(l'_\alpha) \geq D'$ such that*

$$l'_\alpha = f l_\alpha \text{ on } \Gamma$$

then Γ lies in the algebraic variety Φ of $P_{g-1}(k)$ defined by

$$\operatorname{rank}\begin{pmatrix} l_1 & \cdots & l_r \\ l'_1 & & l'_r \end{pmatrix} \leq 1$$

and the $\frac{1}{2} r(r-1)$ quadratic forms $l_\alpha l'_\beta - l_\beta l'_\alpha$ $(\alpha < \beta)$ are linearly independent.

REMARK $r = \dim_k(F) =$ specialty index of D.

3. *Curves with a g^1_3*. *a*) If a non hyperelliptic curve C carries a linear series g^1_3 this must be complete and without fixed points (otherwise C would be hyperelliptic). If the characteristic of the groundfield k is $\neq 3$ one can construct a curve of this type for any value of $g \geq 3$. For instance for $g = 3k-1$ we can take for C the normalisation of the plane curve

$$z^{3k-2}\, y^3 = x^2 \prod_1^{3k-1} (x - a_i z).$$

For the other values of g we can take the normalisation of the plane curve

$$z^{r-3}\, y^3 = \prod_1^{r} (x - a_i z)$$

whose genus is $r-1$ or $r-2$ according to whether r is prime or not to 3.

PROPOSITION 1. *Let Γ be the canonical model of a non hyperelliptic curve C of genus $g \geq 4$ carrying a g^1_3.*

Then each divisor of the g^1_3 on Γ lies on a straight line and these lines describe a ruled surface Φ of degree $g-2$ whose equations by a suitable choice of coordinates can be put in the form

$$\operatorname{rank}\begin{pmatrix} x_0 \dots x_{m-1} & x_{m+1} \dots x_{g-2} \\ x_1 \dots x_m & x_{m+2} \dots x_{g-1} \end{pmatrix} \leq 1.$$

PROOF. Since the g^1_3 is complete the index of specialty of each divisor $D \in g^1_3$ is $i(D) = g-2$. This means that D lies on a straight line.

In this case the space F decribed in the previous section is of codimension 2 in $E = H^0(\Gamma, \Omega^1)$. Except in the case $g = 4$ where Γ lies on a cone, $F + \lambda F = E$, since $i(D+D') \leq g-4$ by Clifford's theorem [4]. Thus for the matrix σ in the canonical form of Corollary 1, we must have $l = 1$ or $l = 2$.

With the notations of lemma 6, the divisor of g^1_3 being given as ($f =$ const.), we recognize that the locus of these lines is the variety Φ. By the choice of the basis we see that in each case F can be written in the given form.

REMARK 1. For $g \geq 5$ one has $1 \leq m < g-2$ so that the surface Φ is non singular. For $g = 4$, Φ could very well be the cone $x_0 x_2 - x_1^2 = 0$ but there exists always a curve of genus 4 for which Φ is non singular.

Indeed if $m = g-2$, Φ is the cone of the straight lines joining the point $(0, \dots, 0, 1)$ to the points of a rational normal curve in $X_{g-1} = 0$. The order of the cone is $g-2$. Since Γ lies on this cone and is of order $2g-2$ the order of the cone is $\leq \frac{2g-2}{3}$ $\left(\text{a hyperplane though the vertex cuts the cone in at most } \frac{2g-2}{3} \text{ generators}\right)$. Hence $g \leq 4$.

For $g = 4$ any non singular curve which is the complete intersection of a quadric and a cubic in $P_3(k)$ is the canonical image of a curve of genus 4.

Except possibly for $g = 4$ we can assume $1 \leq m \leq \frac{1}{2}(g-2)$. This invariant was first discussed by Maroni [15].

REMARK 2. Each quadric through the curve Γ contains the surface Φ. The $\frac{1}{2}(g-2)(g-3)$ quadrics obtained from the second order minors of the matrix defining Φ are linearly independent, and thus span the full system Σ of all quadrics through Γ.

PROPOSITION 2. *Let C be a non hyperelliptic curve of genus $g \geq 4$ carrying a positive divisor D of degree 3 with $\dim |D| = 1$. Let E_0 be a divisor of degree $g-1$ with $\dim |E_0| = 1$. Then there exists a divisor $E \in |E_0|$ or a divisor $E' \in |K - E_0|$ such that either E or E' is $\geq D$.*

PROOF. Since C is not hyperelliptic we can identify C with its canonical image Γ.

Let $|E_0| = g^1_{g-1} = g^1_p + P_1 + \dots + P_{g-1-p}$ where $P_1 + \dots + P_{g-1-p}$ is the fixed part of $|E_0|$ so that g^1_p has no fixed points.

Since $\dim |E_0| = 1$ we have $i(E_0) = 2$ so that $\dim |K - E_0| = 1$.

Let analogously $|K - E_0| = g^1_q + Q_1 + \dots + Q_{g-1-q}$ where $Q_1 + \dots + Q_{g-1-q}$ is the fixed part of $|K - E_0|$ and g^1_q has no fixed points.

We construct the quadric $Q \in \Sigma$ corresponding to the data of the series g^1_p, g^1_q and of the divisor $G_0 = P_1 + \dots + Q_{g-1-q}$, as in lemma 4. Let V be the vertex of Q. This is a projective space of dimension $g-5$ or $g-4$ according as the rank of Q is 4 or 3. Let $\Delta \in |D|$ be a divisor consisting of 3 distinct points and disjoint from the finite set $V \cap \Gamma$. The 3 points of Δ lie on a straight line l not contained in V. Consider the projection with center V on a $P_3(k)$ or $P_2(k)$ (according to the dimension of V) not meeting V. The projection of l will be a line or a point in the image space. Therefore in any case l lies on a space P_{g-3} of one of the rulings of Q. This implies that either $\Delta \in g^1_p$ or $\Delta \in g^1_q$ and this proves our contention.

COROLLARY. *If C is a non hyperelliptic curve of genus $g \geq 4$ carrying a g^1_3, then for every complete linear series g^1_{g-1} of degree $g-1$ and dimension 1 one has*

either $$g^1_{g-1} = g^1_3 + P_1 + \dots + P_{g-4}$$

or $$|K - g^1_{g-1}| = \tilde{g}^1_{g-1} = g^1_3 + P_1 + \dots + P_{g-4}$$

where $P_1 + \dots + P_{g-4}$ is a fixed divisor of degree $g-4$.

For any integer $s \geq 1$ we denote by $(C)^{(s)}$ the s-fold symmetric product of C. This is a non singular algebraic variety [4].

PROPOSITION 3. *Under the same assumptions for C, there exists a proper subvariety $S \subset (C)^{(g-4)}$ such that for*

$$(P_1) + \dots + (P_{g-4}) \in (C)^{(g-4)} - S$$

the linear series

$$g^1_3 + P_1 + \dots + P_{g-4}$$

is a complete linear series of degree $g-1$ and dimension 1.

Proof. Let $D \in g_3^1$. The above condition on $P_1, \dots, P_{g-4}$ is equivalent to $i(D + P_1 + \dots + P_{g-4}) = 2$, i.e. the exceptional set is the set where $i(D + P_1 + \dots + P_{g-4}) > 2$.

First one can select $P_1, \dots, P_{g-4}$ such that $i(D + P_1 + \dots + P_{g-4}) = 2$. In fact $|K - g_3^1| = g_{2g-5}^{g-3}$. We can select P_1 outside the fixed divisor of this series. Then $P_2 \neq P_1$ outside the fixed divisor of $|g_{2g-5}^{g-3} - P_1| = g_{2g-6}^{g-4}$ and so on. We end up by selecting $P_1, P_2, \dots, P_{g-4}$ such that dim $|K - g_3^1 - P_1 - \dots - P_{g-4}| = 1$. This is what we wanted to prove. Secondly we remark that if $a = (a_1, \dots, a_g)$ and $b = (b_1, \dots, b_g)$ are two distinct points on the line containing D on the canonical curve Γ, the condition $i(D + P_1 + \dots + P_{g-4}) > 2$ is equivalent to the condition

$$\operatorname{rank} \begin{pmatrix} \omega_1(P_1), \dots, \omega_1(P_{g-4}) & a_1 & b_1 \\ \vdots & \vdots & \vdots \\ \omega_g(P_1), \dots, \omega_g(P_{g-4}) & a_g & b_g \end{pmatrix} \leq g - 3.$$

This condition on the cartesian product C^{g-4} defines a proper analytic set $\widetilde{S}$ invariant by the action of the symmetric group. Its image S in $(C)^{(g-4)}$ by the natural map $C^{g-4} \to (C)^{(g-4)}$ is a proper analytic subset of the space $(C)^{(g-4)}$.

To a complete linear series g_{g-1}^1 on the canonical curve Γ corresponds a unique quadric Q of rank ≤ 4 through the canonical curve. This quadric is described by the spaces P_{g-3} spanned by the divisors of g_{g-1}^1. In particular in the case under consideration we can consider the quadrics of rank ≤ 4 corresponding to the complete linear series of the form

$$g_3^1 + P_1 + \dots + P_{g-4}$$

where the P_i's are distinct generic points on Γ.

Proposition 4. *Under the same assumptions on C we can choose $\frac{1}{2}(g-2)(g-3)$ complete linear series of degree $g-1$ and dimension 1 of the form $g_3^1 + P_1 + \dots + P_{g-4}$, with distinct P_i's, such that the corresponding quadrics of rank ≤ 4 are linearly independent and thus span the full system of quadrics through the canonical curve.*

Proof. Consider the projection map

$$\lambda_{\alpha\beta} : C^{g-2} \to C^{g-4}$$

defined by $\lambda_{\alpha\beta}(P_1 \times \dots \times P_{g-2}) = P_1 \times \dots \times \widehat{P}_\alpha \times \dots \times \widehat{P}_\beta \times \dots \times P_{g-2}$. Let $\tilde{S}$ be the counter image in C^{g-4} of the set S defined in proposition 3. The set $\bigcup_{\alpha<\beta} \lambda_{\alpha\beta}^{-1}(S)$ is a proper analytic subset of C^{g-2} and thus we can select $P_1, \dots, P_{g-2}$ on Γ such that

i) the points $P_1, \dots, P_{g-2}$ are distinct

ii) for any choice of $\alpha < \beta$, $1 \leq \alpha, \beta \leq g-2$, the linear series $g_3^1 + P_1 + \dots + \widehat{P}_\alpha + \dots + \widehat{P}_\beta + \dots + P_{g-2}$ is complete of dimension 1.

iii) no divisor of the series $g_{2g-5}^{g-3} = |K - g_3^1|$ is $\geq P_1 + \dots + P_{g-2}$.

For any α, $1 \leq \alpha \leq g-2$ there is a unique divisor $H_\alpha \in g_{2g-5}^{g-3}$ such that $H_\alpha \geq P_1 + \dots + \widehat{P}_\alpha + \dots + P_{g-2}$. The $g-2$ divisors H_α are linearly independent divisors of g_{2g-5}^{g-3}. This can be seen as follows: we represent the divisors of g_{2g-5}^{g-3} by the points of a projective space P_{g-3}. The sets $E_\alpha = \{D \in g_{2g-5}^{g-3} \mid D \geq P_\alpha\}$ are represented by hyperplanes, and by iii) we have $\bigcap_{\alpha=1}^{g-2} E_\alpha = \emptyset$. Thus these hyperplanes are linearly independent and therefore the points $H_\alpha = \bigcap_{\beta \neq \alpha} E_\beta$ are also linearly independent. Let D, D' be two distinct divisors of g_3^1 on the canonical curve Γ. Then $D + H_\alpha$ is the divisor of a linear form l_α and $D' + H_\alpha$ is the divisor of another linear form l'_α. The linear forms l_α are linearly independent because the divisors H_α are linearly independent. The same is true for the forms l'_α.

Consider the rational functions

$$h_{\alpha\beta} = \frac{l_\alpha\, l'_\beta}{l_\beta\, l'_\alpha} \quad \text{for} \quad \alpha \neq \beta.$$

These, having no zeros or poles on Γ, are constants $\neq 0$ on Γ. Since $h_{\alpha\beta}\, h_{\beta\gamma}\, h_{\gamma\alpha} = 1$ and $h_{\alpha\beta} = h_{\beta\alpha}^{-1}$ we can find non zero constants C_α such that $h_{\alpha\beta} = C_\beta\, C_\alpha^{-1}$.

Replacing l_α by $C_\alpha\, l_\alpha$ we may thus assume $h_{\alpha\beta} = 1$ so that on Γ

$$\operatorname{rank} \begin{pmatrix} l_1, l_2, \dots, l_{g-2} \\ l'_1, l'_2, \dots, l'_{g-2} \end{pmatrix} = 1.$$

This is the equation of the ruled surface Φ (cf. lemma 6 and proposition 1). We know that the $\frac{1}{2}(g-2)(g-3)$ quadrics $Q_{\alpha\beta} = \det \begin{pmatrix} l_\alpha & l_\beta \\ l'_\alpha & l'_\beta \end{pmatrix} = 0$ for $\alpha < \beta$ $1 \leq \alpha, \beta \leq g-2$ are linearly independent. By construction the quadric $Q_{\alpha\beta} = 0$ corresponds to the linear series $g_3^1 + P_1 + \dots + \widehat{P}_\alpha + \dots + \widehat{P}_\beta + \dots + P_{g-2}$.

4. We end this section with a proposition which shows that the complete special series g^r_{g-1} of dimension > 1 are limiting cases of complete linear series g^1_{g-1} of dimension 1.

PROPOSITION 5. *Let g^r_{g-1} be a complete linear series of degree $g-1$ and dimension $r \geq 1$ on the curve C. We can choose a divisor $D_0 \in g^r_{g-1}$ such that in any neighborhood of (D_0) in $(C)^{(g-1)}$ there is a divisor D of $g-1$ distinct points with $\dim |D| = 1$.*

PROOF. Choose $P_1, \dots, P_{r-1}$ on C distinct and such that $g^1_{g-r} = |g^r_{g-1} - P_1 - \dots - P_{r-1}|$ is of dimension 1. Let Δ_0 be the fixed divisor of g^1_{g-r} so that $g^1_{g-r} = \Delta_0 + g^1_l$, $l = g - r - s$, $s =$ degree of Δ_0. Let $D_0 \in g^1_l$ be a divisor of l distinct points and consider the map

$$\lambda : (C)^{(r+s-1)} \longrightarrow (C)^{(g-1)}$$

defined by $\lambda((Q_1) + \dots + (Q_{r+s-1})) = (D_0) + (Q_1) + \dots + (Q_{r+s-1})$. The image of λ is an irreducible subvariety of $(C)^{(g-1)}$ containing the divisor $D = D_0 + \Delta_0 + P_1 + \dots + P_{r-1} \in g^r_{g-1}$.

If $(Q_1) + \dots + (Q_{r+s-1})$ is generic, (see proposition 3), we see that

$$|D_0 + Q_1 + \dots + Q_{r+s-1}| = g^1_l + Q_1 + \dots + Q_{r+s-1} .$$

Given any neighborhood of (D) in $(C)^{(g-1)}$, we can find $Q_1, \dots, Q_{r+s-1}$ such that i) $D_0 + Q_1 + \dots + Q_{r+s-1}$ consists of distinct points, ii) the complete series $|D_0 + Q_1 + \dots + Q_{r+s-1}|$ has dimension 1, iii) $\lambda((Q_1) + \dots + (Q_{r+s-1})) \in U$.

5. *Hyperelliptic curves.* In this case the canonical image Γ of C is a rational twisted curve of $P_{g-1}(k)$ of degree $g-1$. Every complete linear series g^1_{g-1} is of the form $g^1_2 + P_1 + \dots + P_{g-3}$, and conversely if $P_1, \dots, P_{g-3}$ are generic this series is complete. To each one of these series corresponds a quadric of rank 3 through Γ which is the projection of Γ from the space of dimension $g-4$ spanned by the images of the points $P_1, \dots, P_{g-3}$. Choosing $g-1$ linearly independent points $P_1, \dots, P_{g-1}$ on Γ the $\frac{1}{2}(g-1)(g-2)$ quadrics of rank 3 projecting Γ from the space spanned by $P_1, \dots, \widehat{P_\alpha}, \widehat{P_\beta}, \dots, P_{g-1}$ for $1 \leq \alpha, \beta \leq g-1$ $\alpha < \beta$, are linearly independent and span the full system of quadrics through Γ.

2. Theta functions and theta divisors.

6. *The theta function.* a) *By* H_g we denote the Siegel space of rank g, i. e. $H_g = \{z = z^{(g,g)} \mid {}^t z = z, \operatorname{Im} z > 0\}$, $z^{(g,g)}$ denoting a $g \times g$ matrix with

complex elements. Let $u = {}^t(u_1, \dots, u_g)$ be coordinates in $\mathbf{C}^g$. For every $z \in H_g$ the theta function is the following Fourier series:

$$\theta(u, z) = \sum_{m \in \mathbf{Z}^g} \exp \pi i \, ({}^t mzm + 2 \, {}^t mu).$$

This series has the following properties (cf [6] [12] [13] [27] [29])

i) it is uniformly convergent on any compact subset of $\mathbf{C}^g \times H^g$

ii) it has the following periodicity properties

$$\theta(u + In + zm, z) = e^{-\pi i ({}^t mzm + 2mu)} \, \theta(u, z)$$

for any $n, m \in \mathbf{Z}^g$.

iii) it satisfies the « heat equations »

$$\frac{\partial \theta}{\partial z_{\alpha\beta}} = 2 \pi i \, (1 + \delta_{\alpha\beta}) \frac{\partial^2 \theta}{\partial u_\alpha \, \partial u_\beta}$$

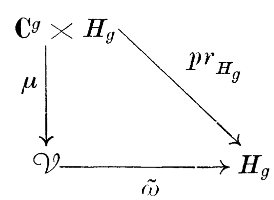
where $\delta_{\alpha\beta}$ is the Kronecker δ, for $1 \leq \alpha \leq \beta \leq g$.

iv) for any $z_0 \in H_g$, $\theta(u, z_0)$ is not identically zero but vanishes somewhere in $\mathbf{C}^g$.

We set $\Omega(z) = (I, z)$ and we consider the following representation ϱ of $\mathbf{Z}^{2g}$ as a group of automorphisms of $\mathbf{C}^g \times H_g$ associating to every vector $\gamma \in \mathbf{Z}^{2g}$ the map

$$\varrho(\gamma) \equiv \begin{cases} u \longrightarrow u + \Omega(z)\gamma \\ z \longrightarrow z. \end{cases}$$

The quotient space $\mathcal{V} = (\mathbf{C}^g \times H_g)/\varrho(\mathbf{Z}^{2g})$ is a complex manifold and we have a commutative diagram of holomorphic maps:



Since H_g is topologically a cell, $\mathbf{C}^g \times H_g$ is the universal covering space of $\mathcal{V}$. For every $z \in H_g$ let Λ_z be the discrete subgroup (of maximal rank) of $\mathbf{C}^g$ generated by the column vectors of $\Omega(z)$. Then $\tilde{\omega}$ is a proper map, and for each $z \in H_g$, $\tilde{\omega}^{-1}(z)$ is the complex torus $\mathbf{C}^g/\Lambda_z$.

b) We define

$$A(\theta) = \{(u, z) \in \mathbf{C}^g \times H_g \mid \theta(u, z) = 0\}$$

$$B(\theta) = \left\{(u, z) \in A(\theta) \mid \frac{\partial \theta}{\partial u_\alpha}(u, z) = 0 \text{ for } 1 \leq \alpha \leq g\right\}.$$

Let $\tau = pr_{H_g} \mid B(\theta)$, for any integer s with $0 \leq s \leq g-2$ we define

$$B_s(\theta) = \{x \in B(\theta) \mid \dim \tau^{-1}\tau(x) \geq s\}$$

so that $B_0(\theta) = B(\theta)$.

The sets $A(\theta)$ and $B(\theta)$ are analytic subsets of $\mathbf{C}^g \times H_g$.

LEMMA 7. *For any s, $0 \leq s \leq g-2$, the sets $B_s(\theta)$ are analytic subsets of $\mathbf{C}^g \times H_g$.*

This is almost an immediate consequence of a theorem of Remmert ([24] Satz 17). We remark that the sets $A(\theta)$ and $B_s(\theta)$ are invariant under the action of $\varrho(\mathbf{Z}^{2g})$.

PROPOSITION 6. i) *The set $A(\theta)$ is of pure dimension $\frac{1}{2}g(g+1)+g-1$ and $pr_H(A(\theta)) = H_g$.*

ii) *for any s, $0 \leq s \leq g-2$, the sets $pr_H(B_s(\theta))$ are analytic and $pr_H(B_0(\theta))$ is a proper analytic subset of H_g.*

PROOF. The first part of the proposition is a consequence of the property iv) of the θ-function. Moreover $A(\theta)$ being non void and the set of zeros of a holomorphic function in $\mathbf{C}^g \times H_g$ is of codimension one in that space.

To prove the second part we first remark that μ being a local isomorphism $\mu(B_s(\theta))$ is analytic. Since $\tilde{\omega}$ is proper it follows that $\tilde{\omega}\,\mu(B_s(\theta))$ is an analytic subset of H_g. It remains to prove that $pr_H(B_0(\theta)) \neq H_g$. This is a straightforward consequence of the following two lemmas:

LEMMA 8. *Let $\mathcal{V}$ be a complex space with countable topology and let $\tilde{\omega}: \mathcal{V} \to U$ be a proper holomorphic surjective map of $\mathcal{V}$ onto an open subset $U \subset \mathbf{C}^n$. Then there exists an open subset $V \subset U$ and a holomorphic section $s: V \to \mathcal{V}$* (i. e. $\tilde{\omega} \cdot s(u) = u \; \forall\, u \in V$, $V \neq \emptyset$).

PROOF. We may assume U connected; also since $\tilde{\omega}$ is proper we may assume $\mathcal{V}$ irreducible. If $S(\mathcal{V})$ is the singular set of $\mathcal{V}$ either $\tilde{\omega}(S(\mathcal{V})) = U$ or it is a proper analytic subset of U; in this case we may replace U with

$V - \tilde{\omega}(S(\mathcal{V}))$ and $\mathcal{V}$ with $\mathcal{V} - \tilde{\omega}^{-1}\tilde{\omega}(S(\mathcal{V}))$. By this procedure we see that we can assume that U is connected and $\mathcal{V}$ is a connected manifold. Also by analogous procedure we see that it is not restrictive to assume that the rank of the map $\tilde{\omega}$ is constant in $\mathcal{V}$. Thus we need only show that under these conditions the rank of $\tilde{\omega}$ equals n in $\mathcal{V}$. If the rank of $\tilde{\omega}$ is strictly $< n$ then each point $x \in \mathcal{V}$ has a neighborhood $N(x)$ such that $U - \tilde{\omega}(N(x))$ is nowhere dense by virtue of a lemma of Remmert ([24], p. 348-350). Cover $\mathcal{V}$ with a countable union $\{N(x_i)\}_{i \in \mathbf{N}}$ of such neighborhoods then $\tilde{\omega}(\mathcal{V}) = \bigcup_{i \in \mathbf{N}} \tilde{\omega}(N(x_i))$. This is absurd by a well known theorem of Baire.

The second lemma is a unicity theorem for the Cauchy problem of the heat equations.

Let U be open and connected in $\mathbf{C}^g$ and V be open and connected in $\mathbf{C}^{\frac{1}{2}g(g+1)}$; let $u = (u_1, \dots, u_g)$ be holomorphic coordinates in $\mathbf{C}^g$ and let $z = (z_{\alpha\beta})$, $1 \leq \alpha \leq g$, $1 \leq \beta \leq g$, $z = {}^t z$, be holomorphic coordinates in $\mathbf{C}^{\frac{1}{2}g(g+1)}$. The « heat equations » in $U \times V$ are a special case of a system of partial differential equations in the unknown function v of the form

$$\text{(I)} \qquad \begin{cases} \dfrac{\partial^2 v}{\partial u_j \, \partial u_k} = \Sigma\, a_{jk}^{\alpha\beta}(u, z) \dfrac{\partial v}{\partial z_{\alpha\beta}} + \Sigma\, b_{jk}^{\varrho}(u, z) \dfrac{\partial v}{\partial u_\varrho} + c_{jk}(u, z)\, v \\ 1 \leq j \leq k \leq g \end{cases}$$

where the a's, b's, c's are holomorphic functions in $U \times V$.

LEMMA 9. *Let $u = s(z)$ be a holomophic section of $U \times V \to V$ and let $v = v(u, z)$ be a holomorphic solution of* (I) *in $U \times V$.* If

$$v(s(z), z) = 0, \qquad \frac{\partial v}{\partial u_i}(s(z), z) = 0, \; 1 \leq i \leq g$$

then v is identically zero.

PROOF. In $\mathbf{C}^g \times V$ we can perform the change of coordinates

$$\begin{cases} u' = u - s(z) \\ z' = z. \end{cases}$$

Then the system (I) is changed into a system of the same type. In the new system of coordinates the section s is reduced to $u = 0$. Without loss of generality we may thus assume $s(z) = 0$. Let

$$v(u, z) = \sum_{\alpha \in \mathbb{N}^g} a_\alpha(z) u^\alpha$$

be the Taylor expansion of $v(u, z)$ near $u = 0$.

Substituting in (I) we get:

$$\text{(II)} \quad \begin{aligned} \sum a_\alpha(z) \frac{\partial^2 u^\alpha}{\partial u_j \partial u_k} &= \sum a^\varrho_{jk}(u, z) \sum \frac{\partial a_\alpha(z)}{\partial z_\varrho} u^\alpha + \\ &\sum b^l_{jk}(u, z) \sum a_\alpha(z) \frac{\partial u^\alpha}{\partial u_l} + c_{jk}(u, z) \sum a_\alpha(z) u^\alpha . \end{aligned}$$

Let $\mathfrak{g}$ be the ideal generated by $u_1, \dots, u_g$ in the ring of formal power series $\mathbf{C}\{u_1, \dots, u_g\}$. By the assumption, for any $z_0 \in V$ we have $v(z_0, u) \in \mathfrak{g}^2$. Equations (II) imply that if $v(z_0, u) \in \mathfrak{g}^k$ for $k \geq 2$, then $v(z_0, u) \in \mathfrak{g}^{k+1}$. Hence for any $z_0 \in V$, $v(z, u) \in \bigcap_{k>1} \mathfrak{g}^k = 0$.

The sets $N_s = pr_{H_g}(B_s(\theta))$ will be called the *ramification sets of order* s in H_g, $0 \leq s \leq g-2$. There are the obvious inclusions

$$N_{g-2} \subset \dots \subset N_1 \subset N_0 \subset H_g .$$

We note that this filtration of H_g by analytic sets is invariant under the action of the modular group. This follows from the theory of transformation of theta functions (cf. n. 15)

7. *The theta divisor on a Jacobian variety. a)* Let X be the Riemann surface of the algebraic curve C and let $\gamma_1 \dots, \gamma_{2g}$ be a basis of $H^1(X, \mathbf{Z})$ with the intersection matrix

$$((\gamma_i, \gamma_j)) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Let $\omega_1, \dots, \omega_g$ be a basis of $H^0(C, \Omega^1)$ so normalised that the period matrix $\left(\int_{\gamma_i} \omega_j\right)$ has the form (I, z). As is well known z is a $g \times g$ matrix with complex element such that ${}^t z = z$ and $\operatorname{Im}(z) > 0$. The matrix z represents thus a point of the Siegel space H_g. We denote by $J(C)$ the complex torus $\mathbf{C}^g / \Lambda_z$. If D is a divisor of degree 0 on C and σ is a differentiable 1-chain on X such that $D = \partial \sigma$ then the point $\lambda(D) = \left(\int_\sigma \omega_1, \dots, \int_\sigma \omega_g\right) \in \mathbf{C}^g$ is well determined by D modulo the elements of Λ_z. The map λ defines therefore a map (that we will still denote by λ) of the group G_0 of all divisors on X of degree 0 on $J(C)$. Abel's theorem asserts that $\lambda(D)=0$ if and only if D is linearly equivalent to zero: $D \equiv 0$. Moreover the image of λ, $\lambda(G_0)$ is the whole torus $J(C)$ so that $J(C)$ is isomorphic to the group of classes of divisors (by linear equivalence) of degree zero. The torus $J(C)$ is thus the « Jacobian variety » of C.

b) Let P_0 be a fixed point on C. For every $P \in C$ we set

$$\psi(P) = \left(\int_{P_0}^{P} \omega_1, \dots, \int_{P_0}^{P} \omega_g \right) \bmod \Lambda_z .$$

In this way we define a holomorphic map $\psi: C \longrightarrow J(C)$. This map is one to one (if $g \geq 1$) and an isomorphism of C onto $\psi(C)$. If $\mathbf{C}^g$ represents the tangent space to $J(C)$ at the origin and $P_{g-1}(\mathbf{C})$ represents the set of lines of $\mathbf{C}^g$ through the origin, by associating to every point $p \in \psi(C)$ the tangent line to $\psi(C)$ at p translated to the origin of $J(C)$ we define a map $\sigma: \psi(C) \longrightarrow P_{g-1}(\mathbf{C})$ analogous to the « Gauss map ». It is worth noticing that the canonical map Φ described in section 1 is nothing else than the composition of the map ψ just defined and the map σ, $\Phi = \sigma \circ \psi$.

By linearity we can extend the map ψ to the whole G_0. In particular for every $h > 0$ we obtain a holomorphic map

$$\pi: (C)^{(h)} \longrightarrow J(C)$$

from the h-fold symmetric product of C into $J(C)$ given by

$$\pi(P_1 + \dots + P_n) = \left(\sum_{i=1}^{h} \int_{P_0}^{P_i} \omega_1, \dots, \sum_{i=1}^{h} \int_{P_0}^{P_i} \omega_g \right) \bmod \Lambda_z .$$

If K is the canonical divisor of C, $\pi(K) = \varkappa \in J(C)$ is a well determined point of $J(C)$ which depends only on the choice of P_0.

c) The function of u $\theta(u, z)$ can be viewed as a holomorphic section of a line bundle over $J(C)$. Its divisor (θ) is thus defined on $J(C)$; it is a non empty holomorphic divisor. We recall the following theorem due essentially to Riemann.

THEOREM OF RIEMANN. i) *The map* $\pi: (C)^{(g)} \longrightarrow J(C)$ *is surjective.*

ii) *The image of the map* $\pi: (C)^{(g-1)} \longrightarrow J(C)$ *up to a translation by a point* $a \in J(C)$ *with* $2a = \varkappa$ *is the theta divisor* (θ).

iii) *The section* $\theta(u, z)$ *vanishes of order one on* (θ).

For the proof of this theorem one can see [13] [30].

Only the last condition iii) is not explicitly proved in the literature. It can be seen as follows. The support of (θ) is an irreducible variety because of ii). If $\theta(u, z)$ vanishes on it of order μ then for any $c \in J(C)$ the induced section $\theta(\psi(x) - c)$ on C either is identically zero or vanishes on each one of its zeros of order $\geq \mu$. But if $P_1, \dots, P_{g-1}$ are distinct and chosen in such a way that $P_0 + \dots + P_{g-1}$ is non special then for $c = \pi(P_1 + \dots + P_{g-1}) + a$ the section $\theta(\psi(x) - c)$ is not identically zero and vanishes to the first order at the points $P_0, \dots, P_{g-1}$ [13]. This implies that $\mu = 1$. We can now translate by a the map π so that $\pi((C)^{(g-1)}) = (\theta)$. Since $\theta(-u, z) = \theta(u, z)$ the involution $x \to -x$ on $J(C)$ changes (θ) into itself. If D is a divisor on C of degree $g-1$ then by this choice of the map π we obtain

$$\pi(|D|) = -\pi(|K - D|)$$

i.e. the involution $x \to -x$ on (θ) corresponds to the involution on the set of complete linear series of degree $g-1$ which associates to each such series the residual series with respect to the canonical one.

Given a point $P_1 + \dots + P_{g-1} \in (C)^{(g-1)}$ we denote by $i(P_1 + \dots + P_{g-1})$ the specialty index of the divisor $P_1 + \dots + P_{g-1}$.

PROPOSITION 7. (cf. [16]) *The subset of* $(C)^{(g-1)}$:

$$S_r = \{P_1 + \dots + P_{g-1} \in (C)^{(g-1)} \mid i(P_1 + \dots + P_{g-1}) \geq r\}$$

is the subset of $(C)^{(g-1)}$ *where the jacobian of the map* π *has rank* $\leq g - r$.

PROOF. Let $P_1 + \dots + P_{g-1} \in (C)^{(g-1)}$ and let V be an open set on C such that $P_i \in V$ for $1 \leq i \leq g-1$ and on which there exists a holomorphic function t with the following properties

i) at each point $a \in V$, $t - t(a)$ is a local parameter at a

ii) for $a, b \in V$, $a \neq b$, $t(a) \neq t(b)$.

The open set $(U)^{(g-1)}$ is a neighborhood of $P_1 + \dots + P_{g-1}$ in $(C)^{(g-1)}$. Let t_i denote the lifting to the Cartesian product U^{g-1} of the function t on the (i)-th factor. The elementary symmetric functions of the t_i's $\Phi_1 = t_1 + \dots + t_{g-1}, \dots, \Phi_{g-1} = t_1 \dots t_{g-1}$ can be taken as local coordinates on the set $(U)^{(g-1)}$. One has

$$d\Phi_1 \wedge \dots \wedge d\Phi_{g-1} = \prod_{i<j} (t_i - t_j)\, dt_1 \wedge \dots \wedge dt_{g-1}.$$

If w_α $1 \leq \alpha \leq g$ are coordinates on the universal covering $\mathbf{C}^g$ of $J(C)$ then the map π is given by

$$\pi(a_1 + \dots + a_{g-1}) = \left(w_\alpha = \int_{P_0}^{a_1} \omega_\alpha + \dots + \int_{P_0}^{a_{g-1}} \omega_\alpha + \text{const} \right)_{1 \leq \alpha \leq g}.$$

If $\omega_\alpha = A_\alpha(t)\, dt$ on U, the jacobian matrix $\dfrac{\partial(w_1, \dots, w_g)}{\partial(\Phi_1, \dots, \Phi_{g-1})}$ is easily computed in terms of the functions t_i and given by the matrix

$$\{ \prod_{i<j} (t_i - t_j) \}^{-1} (A_k(t_i))_{\substack{1 \leq k \leq g \\ 1 \leq i \leq g-1}},$$

at a point where $t_i \neq t_j$ $\forall$ i, j $i \neq j$. At a particular point $a_1 + \dots + a_{g-1} = n_1 P_1 + \dots + n_r P_r$ ($n_i \geq 1$, $\Sigma n_i = g - 1$) with $P_i \neq P_j$ if $i \neq j$, if we set $\tau_i = t(P_i)$, the rank of the Jacobian matrix $\partial(w)/\partial(\Phi)$ is the same as the rank of the matrix

$$\frac{1}{\prod\limits_{1<} (\tau_i - \tau_j)} \begin{pmatrix} A_1(P_1), \dots, \dfrac{d^{n_1-1} A_1(P_1)}{dt^{n_1-1}}, \dots, A_1(P_r), \dots, \dfrac{d^{n_r-1} A_1(P_r)}{dt^{n_r-1}} \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ A_g(P_1), \dots, \dfrac{d^{n_1-1} A_g(P_1)}{dt^{n_1-1}}, \dots, A_g(P_r), \dots, \dfrac{d^{n_r-1} A_g(P_r)}{dt^{n_r-1}} \end{pmatrix}.$$

The rank of this matrix is thus equal to $g - i(a_1 + \dots + a_{g-1})$.

In particular it follows from the above proposition that the sets S_r are all algebraic subsets of $(C)^{(g-1)}$. One has $S_1 = (C)^{(g-1)}$ and S_2 is a proper algebraic algebraic subset of $(C)^{(g-1)}$. From lemmas 1-4 and proposition 5 if $g \geq 4$ S_2 is non empty and contains always a point $D = P_1 + \dots + P_{g-1}$ with distinct P_i's and with $\dim |D| = 1$.

LEMMA 7. *Let C be non hyperelliptic of genus $g \geq 4$. Let $D = P_1 + \dots$ $\dots + P_{g-1}$ be a point of S_2 with the properties*

i) *the points P_i are distinct*

ii) *$\dim |D| = 1$.*

In a neighborhood of D, S_2 is of pure dimension $g - 3$.

PROOF. Let $h : C^{g-1} \to (C)^{(g-1)}$ be the natural map from the cartesian to the symmetric product of C. Since h has finite fibers it is enough to prove the statement for the set $\Sigma_2 = h^{-1}(S_2)$.

Let V_i be mutually disjoint neighborhoods of the points P_i and let $U = \Pi V_i$. Let $t = t_i$ be local parameters in V_i and let $\omega_\alpha = A_\alpha(t_i)\,dt_i$ on V_i. We have

$$\Sigma_2 \cap U = \{t_1 \times \dots \times t_{g-1} \in U \mid \operatorname{rank}(A_\alpha(t_i))_{\substack{1\leq\alpha\leq g\\ 1\leq i\leq g-1}} \leq g-2\}.$$

Since in the space of $g \times (g-1)$ matrices those of rank $\leq g-2$ are a subset of codimension 2, it follows that at each point of $\Sigma_2 \cap U$ the dimension of Σ_2 is $\geq g-3$. Suppose, if possible, that one irreducible component of $\Sigma_2 \cap U$ has dimension $g-2$. At a non singular point $t_1^0 \times \dots \times t_{g-1}^0$ of it, by renumbering the coordinates, it will have an equation of the form $t_{g-1} = g(t_1, \dots, t_{g-2})$ where g is holomorphic in a neighborhood of $t_1^0 \times \dots \times t_{g-2}^0$. Now the rank of the matrix $(A_\alpha(t_i))_{\substack{1\leq\alpha\leq g\\ 1\leq i\leq g-2}}$ is $g-2$ since $g-2$ generic points on the canonical curve are linearly independent. Moreover since C is not hyperelliptic $(A_\alpha(t_{g-1})) \neq (A_\alpha(t_i))$ for $1 \leq i \leq g-2$. By the definition of Σ_2 there will exist $g-2$ holomorphic functions $k_\sigma(t_1, \dots, t_{g-2})$, $1 \leq \sigma \leq$ $\leq g-2$, such

$$A_\alpha(t_{g-1}) = \sum_{\sigma=1}^{g-2} k_\sigma(t_1, \dots, t_{g-2})\, A_\alpha(t_\sigma) \quad \text{for} \quad t_{g-1} = g(t_1, \dots, t_{g-2}).$$

Two at least of these k_σ must be $\neq 0$ in a neighborhood of $t_1^0 \times \dots \times t_{g-2}^0$, for instance k_1 and k_2.

Taking derivatives of the above relation with respect to t_1 and t_2 one sees that the space spanned by the points $(A_\alpha(t_1)), \dots, (A_\alpha(t_{g-1})), \left(\frac{dA_\alpha}{dt_{g-1}}(t_{g-1})\right)$ contains also the points $\left(\frac{dA_\alpha(t_1)}{dt_1}\right)$ and $\left(\frac{dA_\alpha(t_2)}{dt_2}\right)$. It follows that for any choice of $t_1, \dots, t_{g-2}$ in small neighborhoods of $t_1^0, \dots, t_{g-2}^0$ respectively the corresponding points $Q_1, Q_2, \dots, Q_{g-2}$ on C have the property $i(2Q_1 +$ $+ 2Q_2 + Q_3 + \dots + Q_{g-2}) \geq 1$. This is impossible since $Q_1, \dots, Q_{g-2}$ are generic, as one sees, for example, by specializing the Q_j to a common non-Weierstrass point.

PROPOSITION 8. *Let C be non hyperelliptic of genus $g \geq 4$. Then*

a) *the set $\pi(S_2)$ is the singular set of (θ), and is of pure dimension $g-4$,*

b) *the points $\pi(D)$, where $D = P_1 + \dots + P_{g-1}$ is such that*

$$P_i \neq P_j \quad \text{if} \quad i \neq j, \qquad \dim|D| = 1,$$

are dense in $\pi(S_2)$,

c) *at each one of these points the multiplicity of* (θ) *is* 2 *and the quadratic equation*

$$\Sigma \frac{\partial^2\theta}{\partial u_i\, \partial u_j} (\pi(D))\, X_i X_j = 0$$

is the equation in $P_{g-1}(\mathbf{C})$ *of the quadric of rank* ≤ 4 *through the canonical curve corresponding to the series* $|D|$.

PROOF. From proposition 5 follows that the points $\pi(D)$ are dense in $\pi(S_2)$. From the previous lemma it follows that at a point D the local rank (in the sense of Remmert [24]) of the map $\pi \mid S_2$ is $g-4$. Thus $\pi(S_2)$ is of pure dimension $g-4$. From proposition 7 we deduce that the singular set of (θ) is contained in $\pi(S_2)$. It is also known (see [17]) that at each point $\pi(D)$ (θ) has a singular point of multiplicity 2. This statement will be reobtained in the course of the present proof. We lift the map π to the cartesian product C^{g-1} and use the same notations as in proposition 7 and lemma 7.

In a neighborhood $U = \Pi U_i$ of $D \in C^{g-1}$ the map π will have equations of the form

$$w_\alpha = w_\alpha(t_1, \dots, t_{g-1}) = \int_0^{t_1} \omega_\alpha + \dots + \int_0^{t_{g-1}} \omega_\alpha + \text{const}, 1 \leq \alpha \leq g.$$

Since $\pi(C^{g-1}) = (\theta)$ we get the identity $\theta(w(t_1, \dots, t_{g-1})) \equiv 0$, and therefore at any point $c \in U$ we get the conditions

$$\theta(w(c)) = 0\,;\ \left(\frac{\partial\theta}{\partial t_i}\right)_c = 0,\ 1 \leq i \leq g-1\,;\ \left(\frac{\partial^2\theta}{\partial t_i\, \partial t_j}\right)_c = 0\ \ 1 \leq i,j \leq g.$$

The first condition restates the fact $\pi(C^{g-1}) = (\theta)$. The second reads explicitly as follows:

$$\begin{cases} \Sigma \dfrac{\partial\theta}{\partial u_\alpha}(w(c))\, A_\alpha(c_i) = 0 \\ 1 \leq i \leq g-1. \end{cases}$$

If we denote by $M_\alpha(c)$ up to sign the minor determinants of order $g-1$ extracted from the matrix $(A_\alpha(c_i))_{\substack{1\leq\alpha\leq g\\ 1\leq i\leq g-1}}$ by deleting the α-th column, we get a set of holomorphic functions on U. Moreover $\varrho(c) = \dfrac{\partial\theta}{\partial u_\alpha}(w(c))\, M_\alpha(c)^{-1}$ is meromorphic and independent of α. From lemma 7 and proposition 7 we deduce that (Σ_2 being of codimension 2) this function is holomorphic and different from zero outside of a set of codimension 2. It follows that $\varrho(c)$ is holomorphic and $\neq 0$ in the whole of U.

Now if $c = c^0 \in \Sigma_2$ at the point $D \in U$, $M_\alpha(c^0) = 0$ and thus the point $w(c^0) = \pi(D)$ is singular on (θ).

If all second derivatives of $\theta(u)$ vanish at $u = w(c^0)$ it would follow from $\dfrac{\partial\theta}{\partial u^\alpha}(w(c)) = \varrho(c) M_\alpha(c)$ that at c^0, $(dM_\alpha(c))_{c_0} = 0$ for $1 \leq \alpha \leq g$.

Let $|D| = g_h^1 + P_{h+1} + \dots + P_{g-1}$ where g_h^1 has no fixed points. From this condition follows that for $1 \leq l \leq h$ the points $P_1, \dots, \widehat{P_l}, \dots, P_h$ on the canonical curve span a space of dimension $h-2$ which contains also the point P_l. It follows that the points $P_1, \dots, \widehat{P_l}, \dots, P_h, \dots, P_{g-1}$ span a space of dimension $g-3$ containing P_l.

But from the conditions $(dM_\alpha)_{c^0} = 0$ we see that in this space of dimension $g-3$ is also contained the tangent line to the canonical curve at P_l.

This argument could be repeated for any choice of $P_1 + \dots + P_h \in g_h^1$ since the point $w(c^0)$ is not changed (Abel's theorem). Now when $P_1 + \dots + P_h$ describe g_h^1 the space spanned by $P_1, \dots, P_{g-1}$ is a space of dimension $g-3$ describing one of the rulings of a quadric of rank ≤ 4 through the canonical curve. The pencil of these rulings would cut, outside of a fixed divisor, the series of divisors $2(P_1 + \dots + P_h)$. This is impossible by Bertini's theorem. Therefore u is a double point.

Now at $w(c^0)$, since the first derivatives of θ vanish, we deduce that

$$\begin{cases} \displaystyle\sum_{\alpha,\beta=1}^{g} \frac{\partial^2\theta}{\partial u_\alpha\,\partial u_\beta}(w(c^0))\, A_\alpha(c_i^0)\, A_\beta(c_j^0) = 0 \\ 1 \leq i, j \leq g-1. \end{cases}$$

Note that the bilinear form $H(X, Y) = \Sigma \dfrac{\partial^2\theta}{\partial u_\alpha\,\partial u_\beta}(w(c^0))\, X_\alpha Y_\beta$ is not identically zero (since $w(c^0)$ is of multiplicity 2 on (θ)) and that the previous condition says

$$H(\Sigma\,\lambda_i A(c_i^0), \Sigma\,\lambda_j A(c_j^0)) = 0$$

for any choice of the λ's. This means that the space of dimension $g-3$ spanned by the points $P_1, \dots, P_{g-1}$ on Γ is contained in the quadric $H(X, X) = 0$. When the divisor $P_1 + \dots + P_{g-1}$ varies in $|D|$ the point $w(c^0)$ does not change while the space spanned by $P_1, \dots, P_{g-1}$ describes the rulings P_{g-3} of the quadric corresponding to the complete linear series $|D|$. This achieves the proof.

COROLLARY. *Let C be a non hyperelliptic curve containing a g_3^1, then the singular set of (θ) consists of two irreducible components*

a) *the set with generic point $\pi(g_3^1 + P_1 + \dots + P_{g-4})$ where $P_1, \dots, P_{g-4}$ are generic points on C*

b) *the image of the previous set by the involution $x \longrightarrow -x$ on $J(C)$ whose generic point is $\pi(|K - (g_3^1 + P_1 + \dots + P_{g-4})|)$. These two components are distinct if $g > 4$. For $g = 4$ every algebraic curve of genus 4 (non hyperelliptic) contains a g_3^1 and the two above components are reduced to two points which possibly may coincide.*

Moreover the tangent cones to (θ) at the singular points give a system of quadrics of rank ≤ 4 through the canonical curve Γ which spans the full system of quadrics through Γ.

PROOF. This is a direct consequence of the previous proposition, of propositions 3, 4, and 5 and the corollary of proposition 2. The only thing that remains to be seen is the fact that, if $g > 4$, the two components of the singular set are distinct. Unless $g = 4$ and $|2g_3^1| = |K|$, (which occurs only in the special case when the quadric through Γ is a cone), $\dim|2g_3^1| \leq 2$ by Clifford's theorem. Then if D and D' and two divisors in g_3^1, $l(D + D' + P_1 + \dots + P_{g-4}) \leq 3$ if the P_j are sufficiently generic (cf. Lemma 1 of [17]). So we cannot have

$$|D + P_1 + \dots + P_{g-4}| = |K - D' - Q_1 - \dots - Q_{g-4}|$$

for any choice of $Q_1, \dots, Q_{g-4}$. Thus $\pi(g_3^1 + P_1 + \dots + P_{g-4})$ lies only in component a).

d) In the case C is hyperelliptic the set S_2 is of pure dimension $g - 2$ since every generic complete g_{g-1}^1 is of the form $g_2^1 + P_1 + \dots + P_{g-3}$. Again $\pi(S_2)$ is the singular set of (θ). This is irreducible and of dimension $g - 3$. As before one proves that the tangent cone at the point $\pi(g_2^1 + P_1 + \dots + P_{g-3})$, for $P_1, \dots, + P_{g-3}$ generic on C, is the quadric of rank 3 that projects the canonical curve Γ from the points $P_1, \dots, P_{g-3}$. These quadrics generate the full system of quadrics through the canonical curve Γ.

3. The modular space of polarized Jacobians.

8. *Teichmüller space.* Let X_0 be a standard model of a topological oriented surface of genus g. A *Teichmüller surface* is the data of a Riemann surface X and a homotopy class of orientation preserving homeomorphisms $f: X_0 \longrightarrow X$. The set of Teichmüller surfaces is the Teichmüller space $\mathcal{T}$.

We fix on X_0 a basis for the fundamental group of X_0 made of $2g$ closed paths $\sigma_1, \dots, \sigma_{2g}$ with intersection matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Then for any Teichmüller surface we can take the cycles $f(\sigma_i) = \gamma_i$ as a basis for the first homology group as we did in n. 7. Choosing on X a normalised basis for the space $H^0(X, \Omega^1)$ we can compute the corresponding period's matrix (I, z) as explained in n. 7. We thus obtain a natural map

$$\lambda : \mathcal{T} \longrightarrow H_g$$

associating to the Teichmüller surface $(X; [f])$ the point $z \in H_g$.

From the theory of Teichmüller spaces we borrow the following facts. On the space $\mathcal{T}$ one can introduce a structure of a connected complex manifold of dimension $3g - 3$ such that λ is a holomorphic map with discrete fibers.

For the theory of Teichmüller spaces the reader is referred to L. Ahlfors, L. Bers and H. Rauch (cf. the bibliography at the end of this paper).

9. *The space of Jacobians.* Let $J = \lambda(\mathcal{T})$. This is the subset of H_g representing Jacobians of algebraic curves of genus g. This space will be called the *Jacobi space.* Let us consider on H_g the Zariski topology (the closed sets are the analytic subsets of H_g) and let $\bar{J}$ be the closure of the set J in the Zariski topology. Since H_g is a Stein manifold the Zariski closure $\bar{J}$ is the analytic set

$$\{z \in H_g \mid f(z) = 0 \ \forall f \text{ holomorphic on } H_g \text{ with } f(J) = 0\}.$$

Since $J = \lambda(\mathcal{T})$ and $\mathcal{T}$ is an irreducible manifold, it follows that $\bar{J}$ is an *irreducible* analytic subset of H_g. In n. 6 we introduced the analytic sets $B_s(\theta)$ in $\mathbf{C}^g \times H_g$ and the analytic sets $N_s = pr_{H_g}(B_s(\theta))$ in H_g. From proposition 8 one deduces the inclusion

$$\bar{J} \subset N_{g-4}.$$

We want to prove the following

THEOREM 1. *The Zariski closure of the Jacobi space J is an irreducible analytic set of dimension $3g - 3$. It coincides with the unique irreducible component of the ramification set N_{g-4} containing J $(g \geq 4)$.*

PROOF. α) We first prove that $\dim \bar{J} \geq 3g - 3$. This is a consequence of the following form of the implicit function theorem (cf. [24]):

Let U be an open subset of $\mathbf{C}^n$ containing the origin and $f : U \to \mathbf{C}^p$ a holomorphic map, $f(0) = 0$. Suppose that the jacobian of f, $Df(x)$ has constant rank k for all $z \in U$. Then there are local biholomorphisms h and g of $\mathbf{C}^n$ and $\mathbf{C}^p$ respectively such that

$$g \circ f \circ h(z_1, \dots, z_n) = (z_1, \dots, z_k, 0, \dots, 0).$$

Indeed since λ has discrete fibers the rank of the jacobian of λ must be $3g - 3$ on an open (dense) subset of $\mathcal{T}$. Therefore the image $\lambda(\mathcal{T})$ cannot be contained in an analytic set of dimension $< 3g - 3$.

β) To complete the proof it is enough to show that at some point $z_0 \in J$ the dimension of the ramification set N_{g-4} is exactly $3g - 3$.

By definition $N_{g-4} = pr_{H_g}(B_{g-4}(\theta))$. Let M be an irreducible component of N_{g-4}. There exists an irreducible component Δ of $B_{g-4}(\theta)$ such that $pr_{H_g}(\Delta) = M$. In fact the counter image in $B_{g-4}(\theta)$ of M consists at most of denumerably many irreducible components of $B_{g-4}(\theta)$. The projection map being the product of a local isomorphism with a proper map, the projection of each one of these components is an analytic set. One at least of these sets must be M since by the Baire theorem M cannot be a countable union of proper analytic subsets. We need:

LEMMA 8. *At each simple point $z_0 \in M$ any tangent vector $\{dz_{\alpha\beta}\}$ to M satisfies the conditions*

$$\sum_{\alpha \leq \beta} \frac{\partial \theta}{\partial z_{\alpha\beta}}(u_0, z_0)\, dz_{\alpha\beta} = 0$$

for any $(u_0, z_0) \in pr_{H_g}^{-1}(z_0) \cap \Delta$.

PROOF OF THE LEMMA: Let $S(M)$ be the singular set of M and $S(\Delta)$ the singular set of Δ. The set

$$\Delta' = \Delta - pr_{H_g}^{-1}(S(M)) - S(\Delta)$$

is a connected manifold. The projection on the space H_g gives a holomorphic map of the connected manifold Δ' into the connected manifold $M - S(M)$. It is of maximal constant rank on an open dense subset Δ'' of Δ'. Moreover the projection of Δ'' is dense in M. If we prove the statement for the points $(u_0, z_0) \in \Delta''$, by continuity we deduce the statement at any other point (u_0, z_0) with $z_0 \in M - S(M)$ and $(u_0, z_0) \in pr_{H_g}^{-1}(z_0) \cap \Delta$.

Let $(u_0, z_0) \in \Delta''$. By the quoted implicit function theorem, we can find parametric equations of Δ'' in a neighborhood of (u_0, z_0) of the form

$$\begin{cases} u = u(y, t) \\ z = z(t) \end{cases}$$

where t is in a neighborhood of the origin in $\mathbf{C}^s$ ($s = \dim M$) and y in a neighborhood of the origin in $\mathbf{C}^r$ ($r = \dim \Delta - \operatorname{rank} pr_{H_g} | \Delta''$). Since $\Delta \subset B_{g-4}(\theta) \subset B_0(\theta)$ we must have

$$\theta(u(y, t), z(t)) \equiv 0, \qquad \frac{\partial \theta}{\partial u_i}(u(y, t), z(t)) \equiv 0 \text{ for } 1 \leq i \leq g.$$

Differentiating the first condition with respect to t and making use of the second set of conditions we get:

$$\sum_{\alpha \leq \beta} \frac{\partial \theta}{\partial z_{\alpha\beta}}(u(y, t), z(t)) \frac{\partial z_{\alpha\beta}}{\partial t_i} = 0.$$

For $y = 0$, $t = 0$ and $dz_{\alpha\beta} = \sum_1^s \lambda_i \left(\frac{\partial z_{\alpha\beta}}{\partial t_i}\right)_{t=0}$ we get the desired result since when the λ_i's vary, $\{dz_{\alpha\beta}\}$ describes the space of tangent vectors to M at z_0 because $z = z(t)$ are parametric equations of M in the neighborhood of z_0.

γ) To complete the proof it is enough to show that if M is any one of the irreducible components of N_{g-4} containing J at some point z_0, its dimension is $\leq 3g - 3$.

Let $z_0 \in J$ now be a point corresponding to a Jacobian variety of a non hyperelliptic curve C carrying a g_3^1. We do not know a priori if z_0 is simple.

Let A and $-A$ (possibly $A = -A$, if $g = 4$) be the two irreducible components of the singular set of $\{\theta(u, z_0) = 0\}$ on $J(C)$. Then Δ must contain one of the irreducible components of the counter image of A or $-A$ in $\mathbf{C}^g$, the universal covering $J(C)$. Call that one $\widetilde{A}$.

By virtue of the corollary of proposition 8 we can choose $\frac{1}{2}(g-2)(g-3)$ distinct points $(u_0^{(\alpha)}, z_0)$ for $1 \leq \alpha \leq \frac{1}{2}(g-2)(g-3)$ on $\widetilde{A}$ such that the corresponding quadrics of rank ≤ 4 through the canonical image Γ of C are linearly independent. By proposition 8 these quadrics have the equations

$\sum \frac{\partial^2 \theta}{\partial u_i \, \partial u_j}(u_0^{(\alpha)}, z_0) X_i X_j = 0$ or, equivalently by the « heat equations »

$$\sum_{\alpha \leq \beta} \frac{\partial \theta}{\partial z_{\alpha\beta}}(u_0^{(\alpha)}, z_0) X_\alpha X_\beta = 0.$$

δ) Let $\{z_\nu\}_{\nu \in \mathbf{N}}$ be a sequence of non singular points on M such that $\lim_{\nu \to \infty} z_\nu = z_0$.

We show now that the sequence $\{z_\nu\}_{\nu \in N}$ can be so chosen that we can lift it to sequences $(u_\nu^{(\alpha)}, z_\nu)_{\nu \in \mathrm{N}} \in \Delta$, for $1 \leq \alpha \leq \frac{1}{2}(g-2)(g-3)$, such that $\lim\limits_{\nu \to \infty} (u_\nu^{(\alpha)}, z_\nu) = (u_0^{(\alpha)}, z_0)$.

Let $\mu = pr_{H_g} | \Delta$. At each point $x \in \Delta$ one can consider the local rank $r_\mu(x)$ of this map according to Remmert and Stein, that is the integer $r_\mu(x) = \dim \Delta - \dim_x \mu^{-1}\mu(x)$. This function of x is lower semicontinuous on Δ (cf. [24] theorem 15). At $x_\alpha = (u_0^{(\alpha)}, z_0)$ we have $r_\mu(x_\alpha) = \dim \Delta - (g-4)$. By definition of $B_{g-4}(\theta)$ at each point $x \in \Delta$, $r_\mu(x) \leq \dim \Delta - (g-4)$. On an open neighborhood $U(x_\alpha)$ of x_α in Δ we must have $r_\mu(x) = r_\mu(x_\alpha) \; \forall x \in U(x_\alpha) \; \forall \alpha$. Now $\dim M = \sup\limits_{x \in \Delta} r_\mu(x) = r(x_\alpha)$ and $\mu: U(x_\alpha) \to M$ is a holomorphic map « without degeneracy », (i.e. of constant rank r_μ).

Let V be a neighborhood of z_0 in H_g such that $V \cap M = Y_1 \cup \ldots \cup Y_k$ decomposes in k irreducible components Y_i, $1 \leq i \leq k$, one for each irreducible germ of M at z_0. One of the irreducible components X of $\Delta \cap pr_{H_g}^{-1}(V)$ must contain the set $\tilde{\Delta}$. The projection of X on H_g, since it is an irreducible analytic set of dimension $= \dim M$, must coincide with one of the Y_i's. Call it Y.

Let $U'(x_\alpha)$ be the connected component containing x_α in $U(x_\alpha) \cap X$. Then

$$\mu: \quad U'(x_\alpha) \to Y$$

is a holomorphic map without degeneracy of constant rank r into a connected irreducible complex space Y of dimension r. By a theorem of Remmert ([24] theorem 28) μ is an open map.

This is true for any α, $1 \leq \alpha \leq \frac{1}{2}(g-2)(g-3)$, thus the set $\bigcap\limits_\alpha \mu(U'(x_\alpha))$ is an open neighborhood of z_0 in Y. Any non singular point of Y in that neighborhood can be lifted to a point in each one of the sets $U'(x_\alpha)$.

Since the neighborhoods $U(x_\alpha)$ can be chosen arbitrarily small our assertion follows.

(ε) Let $\varrho = \frac{1}{2}(g-2)(g-3)$. The $\varrho \times \frac{1}{2} g(g+1)$ matrix

$$\begin{pmatrix} \frac{\partial \theta}{\partial z_{11}}(u_\nu^{(1)}, z_\nu), \ldots, \frac{\partial \theta}{\partial z_{gg}}(u_\nu^{(1)}, z_\nu) \\ \frac{\partial \theta}{\partial z_{11}}(u_\nu^{(\varrho)}, z_\nu), \ldots, \frac{\partial \theta}{\partial z_{gg}}(u_\nu^{(\varrho)}, z_\nu) \end{pmatrix}$$

for $\nu \to \infty$ tends to a matrix of rank ϱ. For ν large enough the rank of this matrix will be ϱ. But then, for those ν's the conditions of the previous lemma define a space of tangent vectors of dimension $\leq \frac{1}{2} g(g+1) - \frac{1}{2}(g-2)(g-3) = 3g-3$. This shows that at those points the dimension of M is $\leq 3g-3$ and this concludes the proof.

REMARK. Let E be the subset of J representing hyperelliptic curves. One can easily show that the Zariski closure $\bar{E}$ of E is an irreducible analytic set.

By a similar and simpler argument one can prove that $\bar{E}$ is of dimension $2g-1$ and coincides with the unique irreducible component of the ramification set N_{g-3} containing E.

10. *a*) The group of automorphisms of X_0 (cf. n. 8) acts on the Teichmüller space $\mathcal{T}$ by

$$\alpha : \quad (x, [f]) \to (x, [f \circ \alpha]),$$

for $\alpha \in \mathrm{Aut}(X_0)$. If α is homotopic to the identity the action of α on $\mathcal{T}$ is the identity. If $N = \{\alpha \in \mathrm{Aut}(X_0) \mid \alpha$ homotopic to the identity$\}$ the action of $\mathrm{Aut}(X_0)$ on $\mathcal{T}$ reduces to the action of the group $\Lambda = \mathrm{Aut}(X_0)/N$ (note that N is a normal subgroup of $\mathrm{Aut}(X_0)$). Let H be the subgroup of $\mathrm{Aut}(X_0)$ defined by

$$H = \{\alpha \in \mathrm{Aut}(X_0) \mid \alpha : \quad H_1(X_0 \mathbf{Z}) \to H_1(X_0 \mathbf{Z}) \text{ is the identity}\}.$$

This is another normal subgroup of $\mathrm{Aut}(X_0)$. Setting $\Gamma = H/N \subset \Lambda$. One sees that Γ acts freely on $\mathcal{T}$ (cf. Rauch [21], lemma 2) and the manifold $T = \mathcal{T}/\Gamma$ represents the classes of « Torelli surfaces ». The natural map $\lambda : \mathcal{T} \to H_g$ can be factored through the natural map $\mathcal{T} \to \mathcal{T}/\Gamma = T$ and a map of degree 2 of T onto J. This last map (as it follows from Torelli's theorem) is obtained by dividing T by the action of the involutory automorphism τ of T corresponding to an orientation preserving diffeomorphisms of X_0 which changes the sign of the basis $\sigma_1, \ldots, \sigma_g$ (as homology basis). The fixed points of this automorphism of T are the points corresponding to hyperelliptic curves. These are known facts in the theory of the Teichmuller space. It follows that the Jacobi space J is in one to one correspondence with the normal space T/τ. In fact J is non singular at a point corresponding to a non hyperelliptic curve. In particular J is locally irreducible. From the theorem of Remmert ([24], theorem 28) it follows that *if* Y *is* *the Zariski closure of* J *in* H_g (i. e. the irreducible component of the ramification set N_{g-4} containing J) *then the map* $\lambda: \mathcal{T} \to Y$ *is an open map, i. e.* J *is an open subset of* Y.

b) From the proof of Theorem 1 we also deduce the following remark. *At a point* z_0 *of a Zariski open (non empty) subset of* J *the tangent space to* Y *at* z_0 *is a* $3g-3$ *dimensional space with the equations*

$$\sum_{\alpha \leq \beta} a_{\alpha\beta}\, dz_{\alpha\beta} = 0$$

where $\Sigma\, a_{\alpha\beta} X_\alpha X_\beta = 0$ *describe the full linear system of quadrics through the canonical curve corresponding to the point* z_0.

Proof. Let $S(N_{g-4})$ be the singular set of the ramification set N_{g-4}. The set $A = J - J \cap S(N_{g-4}) - E$ is Zariski open in J and non empty. Let $\mu = \frac{1}{2} g(g+1) - (3g-3)$ and let Δ_A be the part of Δ over A (with the notations of the proof of theorem 1). Let Δ_A^μ be the μ-th fibered product of Δ_A over A. For any point $(u^1, \dots, u^\mu;\ z_0) \in \Delta_A^\mu$ we consider the $\mu \times \frac{1}{2} g(g+1)$ matrix $\left(\frac{\partial \theta}{\partial z_{\alpha\beta}}(u^\sigma, z_0)\right)_{\substack{1\leq\sigma\leq\mu \\ 1\leq\alpha\leq\beta\leq g}}$. Let C be the analytic subset of Δ_A^μ where the rank of that matrix is $< \mu$. Let $\tau = pr_{H_g} \mid \Delta_A^{(\mu)}$. Consider the set $D = \{z_0 \in A \mid \tau^{-1}(z_0) \subset C\}$; for $z_0 \in A - D$ the desired requirements are satisfied. One has thus to prove that D is analytic. This is actually possible; however we can more simply remark that D is contained in the subset $D' = \{z_0 \in A \mid \dim \tau^{-1}|_C(z_0) \geq \mu(g-4)\}$ and that (a) D' is analytic as is proved by the usual arguments using Remmert's theory of holomorphic maps; (b) $D' \subsetneq A$ as it follows from the proof of theorem 1.

Remark. The above statement is to be considered as a weak form of a known theorem of Rauch [20] which says that at *any* point $z_0 \in J$ not representing a hyperelliptic curve, $3g-3$ of the local coordinates $z_{\alpha\beta}$ on H_g can be taken as local coordinates on J at z_0 provided the corresponding quadratic differentials $\omega_\alpha \omega_\beta$ are linearly independent.

4. The equations of the ramification sets.

11. *The Kummer variety.* a) We have remarked that the functional equations

$$\Phi(u + In + zm) = e^{-\pi i ({}^t m z m + 2\, {}^t m u)}\, \Phi(u)$$

for $m, n \in \mathbf{Z}^g$ where Φ is a holomorphic function on $\mathbf{C}^g \times H_g$ has a one dimensional space of solutions $k\theta(u, z)$, $k \in \mathbf{C}$, and that these solutions represent the holomorphic sections of the line bundle F on the torus $\mathbf{C}^g/\Lambda_z$ corsesponding to the factor of automorphy on $\mathbf{C}^g$:

$$J_\gamma(u, z) = e^{-\pi i ({}^t m z m + 2 {}^t m u)}$$

for $\gamma = (n, m) \in \mathbf{Z}^{2g}$. Replacing the line bundle F with F^l one is lead to consider the functional equations

$$\Phi(u + In + zm) = e^{-l\pi i ({}^t m z m + 2 {}^t m u)} \Phi(u)$$

These admit a l^g-dimensional space of solutions (the theta functions of order l). A basis for that space os solutions is given by the functions

$$(1)_l \qquad \theta_l[\mu](u, z) = \sum_{m \in \mathbf{Z}^g} e^{\pi i l \left\{ {}^t\left(m + \frac{\mu}{l}\right) z \left(m + \frac{\mu}{l}\right) + 2\, {}^t\left(m + \frac{\mu}{l}\right) u \right\}}$$

where μ describes a system of representatives of $\mathbf{Z}^g/l\mathbf{Z}^g$.

In particular for $l = 2$ one has a 2^g dimensioual space of theta functions of second order.

For any choice of $c \in \mathbf{C}^g$ the function

$$\Phi_c(u) = k\theta(u + c, z)\, \theta(u - c, z), \; k \in \mathbf{C},$$

is a theta function of second order and therefore a linear combination of the functions $\theta_2[\mu](u, z)$ of the basis considered above

An easy computation gives actually the useful identity

$$(2) \qquad \theta(u + c, z)\, \theta(u - c, z) = \sum_\mu \theta_2[\mu](c, z)\, \theta_2[\mu](u, z).$$

Let L be the vector space of theta functions of the second order in which we choose the basis given by the elements $\theta_2[\mu](u, z)$. The subset of L represented by the functions Φ_c has thus the parametric equations

$$(3) \qquad \lambda_\mu = k\theta_2[\mu](c, z), \; k \in \mathbf{C}, c \in \mathbf{C}^g.$$

Let $P_t(\mathbf{C})$, $t = 2^g - 1$, be the « projectification » of L in which the λ_μ's are accordingly taken as homogeneous coordinates. Then the the equations (3) for $k \neq 0$ and fixed $z \in H_g$ represent the general point of an algebraic subvariety of $P_t(\mathbf{C})$ which is called the Kummer (or Wirtinger) variety. This can be seen as follows.

First we remark that for fixed z and for any given $a \in \mathbf{C}^g$ we can find a point $c \in \mathbf{C}^g$ such that $\theta(a+c, z)\,\theta(a-c, z) \neq 0$. This is because $\theta(u, z)$ is not identically zero as function of u. Thus there always exists a theta function of second order which is different from 0 at a. This means that the map

$$\chi:\ \mathbf{C}^g \longrightarrow P_t(\mathbf{C})$$

defined by formulae (3) is a holomorphic map. By the periodicity conditions this map factors through a holomorphic map of $\mathbf{C}^g/\Lambda_z$ in $P_t(\mathbf{C})$. It follows that the image of χ is an irreducible compact analytic subset of $P_t(\mathbf{C})$, i.e. an irreducible algebraic variety.

b) Our preliminary object is the study of the map χ.

LEMMA 9. *For any given $z \in H_g$ the set of points $u_0 \in \mathbf{C}^g$ such that $\theta(u, z)$ and $\theta(u - 2u_0, z)$ are coprime at each point $u \in \mathbf{C}^g$ is everywhere dense in $\mathbf{C}^g$ At each one of these points χ is of rank g.*

PROOF. On the torus $\mathbf{C}^g/\Lambda_z$ let $A_1 \cup \dots \cup A_k$ be the decomposition into irreducible components of the set $\{\theta(u) = 0\}$. Select $p_i \in A_i$. Let $E' = \{\theta(p_1 + c) = 0\} \cup \dots \cup \{\theta(p_k + c) = 0\}$ and let E be its counter image in $\mathbf{C}^g$. If $-2u_0 \notin E$ then the sets $\{\theta(u) = 0\}$ and $\{\theta(u - 2u_0) = 0\}$ intersect in a set of codimension ≥ 2 and thus $\theta(u)$ and $\theta(u - 2u_0)$ are coprime everywhere. Moreover the set $\{u_0 \in \mathbf{C}^g \mid -2u_0 \notin E\}$ is everywhere dense.

Let u_0 be chosen as indicated and let us denote by $\theta^{(0)}, \dots, \theta^{(t)}$ the $t+1$ theta functions of second order of a basis for L. The rank of the jacobian of the map χ at u_0 equals g if and only if the matrix

$$J = \begin{pmatrix} \theta^{(0)}, & \dots, & \theta^{(t)} \\ \dfrac{\partial \theta^{(0)}}{\partial u_1}, & \dots, & \cdot \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \dots, & \dfrac{\partial \theta^{(t)}}{\partial u_g} \end{pmatrix}$$

is of rank $g+1$ at u_0 (cf. Conforto's book, pg. 144).

Suppose, if possible, that the rank of J at u_0 is $\leq g$ so that we have a relation of the form

$$a_0\, \theta^{(\alpha)}(u_0) = \sum_1^g a_i \frac{\partial \theta^{(\alpha)}}{\partial u_i}(u^0)$$

for all α, $0 \leq \alpha \leq t$, and with constants a_i not all zero. From (2) for any choice of c we get

$$a_0\, \Phi_c(u_0) = \sum_1^g a_i \frac{\partial \Phi_c}{\partial u_i}(u_0)$$

From this relation, dividing both sides by the non identically zero function in c, $\Phi_c(u_0)$, we get, setting $h(u) = \sum_1^g a_i \frac{\partial}{\partial u_i} (\log \theta(u))$,

$$a_0 = h(u^0 + c) + h(u^0 - c)$$

Now $h(u^0 + c)$, as a function of c, is holomorphic outside $\{\theta(u^0 + c) = 0\}$ while $h(u^0 - c)$ is holomorphic outside $\{\theta(u^0 - c) = 0\}$ thus $h(u^0 + c)$ and $h(u^0 - c)$, by the choice of u^0, are both holomorphic outside of a set of codimension 2 and thus holomorphic everywhere. Hence the function $h(u)$ is holomorphic. From the periodicity conditions for θ we get

$$(\alpha) \quad \{h(u + e_h) = h(u), \qquad (\beta) \quad \{h(u + ze_h) = h(u) - 2\pi i\, a \ , \ 1 \leq h \leq g.$$

when $e_h = {}^t(0 \dots, \underset{(h)}{1}, \dots 0)$. This implies that the first partial derivatives of $h(u)$ are periodic and therefore constants so that $h(u)$ must be a linear function of u:

$$h(u) = h_0 + h_1 u + \dots + h_g u_g .$$

From (α) we deduce then that $h_1 = \dots = h_g = 0$ and then by (β), that $a_1 = \dots = a_g = 0$. Hence $h = 0$ and therefore also $a_0 = 0$ (1).

LEMMA 10. *If $z \in H_g$ is such that $\theta(u, z)$ is irreducible then for any $v_0 \in \mathbf{C}^g$ which is not a period ($v_0 \notin \Lambda_z$) $\theta(u, z)$ and $\theta(u - v_0, z)$ are coprime.*

PROOF. If not $\theta(u)/\theta(u - v_0)$ must be a non vanishing holomorphic function of the form $\exp Q(u)$ where $Q(u)$ is a polynomial of first order in the u's. The periodicity conditions then lead to the conclusion $v_0 \in \Lambda_z$ (cf. for the detailed argument [12], pg. 196).

LEMMA 11. *The set*

$$\{z \in H_g \mid \theta(u, z) \text{ is reducible}\}$$

is an analytic set and is contained in the ramification set N_{g-2}

PROOF. Let D be a positive divisor on the torus $\mathbf{C}^g/\Lambda_z$ and let us denote by $C_1(D) = \frac{1}{2\pi i} {}^t du\, H\, d\bar{u}$ the unique (1,1) form with constant coefficients (i. e. harmonic) in the Chern class of the line bundle associated to D.

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(1) This proof follows the argument of C.L. Siegel given in Conforto's book [12], pg. 160.

From the theory of intermediary functions on the torus $T = \mathbf{C}^g/\Lambda_z$ we know that (cf. [29], ch. VI.)

(i) $C_1(D)$ has integral periods

(ii) The associated hermitian form ${}^t u H \bar{u}$ is non negative definite and if $\Phi(u)$ is an intermediary function with divisor D there exists an entire function $G(u)$ and a constant $C > 0$ such that

$$| \Phi(u) e^{G(u)} | \leq C e^{{}^t u H \bar{u}}$$

(iii) If D_1, D_2 are any two positive divisors then

$$\int_T C_1(D_1)^h C_1(D_2)^{g-h} = h!\,(g-h)!\, n_h$$

where n_h is an integer ≥ 0.

(iv) If D is the divisor of $\theta(u, z)$ on T then one has

$$\int_T C_1(D)^g = g!, \text{ and } H \text{ is positive definite.}$$

Suppose now that the divisor D of (θ) is reducible $D = D_1 + D_2$ with D_1 and D_2 holomorphic and non empty. Let us choose coordinates u (by a linear transformation in $\mathbf{C}^g$) such that the hermitian forms H and H_1 of D and D_1 respectively are in diagonal form. Since $H = H_1 + H_2$, H_2 being the hermitian form associated to D_2 we see that H_2 is also in diagonal form. From iii) and iv) we get $\Sigma n_h = 1$ so that for a certain l, $n_l = 1$, and $n_h = 0$ for $h \neq l$.

It follows then, by renumbering the coordinates, that one must have $H_1 = \mathrm{diag}(\varepsilon_1 \dots \varepsilon_l 0 \dots 0)$, $\varepsilon_i > 0$, and consequently $H_2 = \mathrm{diag}(0, \dots 0, \varepsilon_{l+1}, \dots, \varepsilon_g)$, $\varepsilon_j > 0$. From (ii) it then follows that D_1 is the divisor of an intermediary function $\psi_1(u) = \psi_1(u_1, \dots, u_l)$ independent of $u_{l+1}, \dots, u_g$, while D_2 is the divisor of an intermediary function $\psi_2(u) = \psi_2(u_{l+1}, \dots, u_g)$ independent of $u_1, \dots, u_l$.

In particular supp $D_1 \cap$ supp $D_2 \neq \emptyset$ so that the divisor of θ must have a singular set of dimension $\geq g-2$. Furthermore, in the spaces $\mathbf{C}^l$ and $\mathbf{C}^{g-l}$ given by the coordinates $u_1, \dots, u_l$ and $u_{l+1}, \dots, u_g$ the lattices $\Lambda_z \cap \mathbf{C}^l$ and $\Lambda_z \cap \mathbf{C}^{g-l}$ are of maximal rank, giving complex tori $T_1 = \mathbf{C}^l/(\Lambda_z \cap \mathbf{C}^l)$ and $T_2 = \mathbf{C}^{g-l}/(\Lambda_z \cap \mathbf{C}^{g-l})$. The restrictions of the forms H_1 and H_2 to T_1 and T_2 are positive definite and thus give polarizations of T_1 and T_2.

Now $(\Lambda_z \cap \mathbf{C}^l) \oplus (\Lambda_z \cap \mathbf{C}^{g-l})$ is of finite index r in Λ

$$r = \left(\int_{T_1} \frac{C_1 (D_1)^l}{l\,!} \right) \left(\int_{T_2} \frac{C_1 (D_2)^{g-l}}{(g-l)\,!} \right) = \text{order} \, (T_1 \cap T_2).$$

By Cor. 3 No. 5 VI of [29] we see that $T_1 \cap T_2$ contains only the identity, since $D = D_1 + D_2$ is left invariant by no translations other than the trivial one. Thus the polarizations induced by the forms H_1 and H_2 on the tori T_1 and T_2 are principal. So by choosing coordinates properly, one sees that the matrix z is equivalent under the modular group Γ_g to a matrix $\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$ with $z_1 \in H_l$ and $z_2 \in H_{g-l}$. Now for $1 \leq l \leq g-1$ we have embeddings $j_l \colon H_l \times H_{g-l} \to H_g$ by letting $j_l (z_1, z_2) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$. It is clear that $S = \bigcup_l (\Gamma \cdot \operatorname{Im} (j_l))$ is the set of all z for which $\{u \mid \theta (u, z) = 0\}$ is reducible. Now given $z \in \operatorname{Im} (j_l)$ there are only a finite number of cosets $M \Gamma_l \times \Gamma_{g-l}$ with $M \in \Gamma_g$, $Mz \in \operatorname{Im} (j_l)$ where Γ_l is the modular group of H_l. This follows from the finiteness of the isotropy subgroups of the modular groups, together with the fact that the divisor D may be uniquely decomposed into a finite number of irreducible components. Thus S is locally finite, and so is an analytic set, invariant under the modular group.

Let

$$\mathcal{E} = \{z \in H_g \mid \theta (u, z) \text{ is irreducible}\}.$$

We have seen that $\mathcal{E}$ contains the Zariski open subset of H_g $\mathcal{E}' = H_g - N_{g-2}$. Note that the Jacobi space J is contained in $\mathcal{E}'$. From lemmas 9 and 10 we get in particular the following

COROLLARY. *For any $z \in \mathcal{E}$ the jacobian of the map χ is of maximal rank ($= g$) at any point $u_0 \in \mathbf{C}^g$ which is not a half period $\left(u_0 \notin \frac{1}{2} \Lambda_z\right)$.*

LEMMA 12. *For any $z \in \mathcal{E}$, if $u, v \in \mathbf{C}^g$ have the same image under χ then either $u + v$ or $u - v$ is a period.*

PROOF. For some $\varrho \in \mathbf{C}^*$ we must have $\theta_2 [\mu] (u) = \varrho \theta_2 [\mu] (v) \; \forall \, \mu$. Thus $\Phi_c (u) = \varrho \Phi_c (v) \; \forall \, c \in \mathbf{C}^g$, i. e. $\theta (u + c) \, \theta (u - c) = \varrho \theta (v + c) \, \theta (v - c)$. Since $\theta (u + c)$ is irreducible it must divide either $\theta (v + c)$ or $\theta (v - c)$. In the first case by lemma 10 $u - v$ is a period, in the second $u + v$ is a period.

Let us denote for any $z \in H_g$ by $T_g(z)$ the torus $\mathbf{C}^g/\Lambda_z$ and by $K_g(z)$ the corresponding Kummer variety. From the previous lemmas one deduces easily the following

PROPOSITION 9. *For any $z \in \mathcal{E}$ the Kummer variety $K_g(z)$ is a holomorphic image of the torus $T_g(z)$ by a map of degree 2 which is of maximal rank everywhere except at the 2^{2g} points $u_0 \in T_g(z)$ of order 2 ($2u_0 = 0$); $K_g(z)$ has only 2^{2g} isolated singular points and its order (as a projective variety) is $g! \cdot 2^{g-1}$.*

For the last statement see [28] § 1. It is worth noticing that the degree of the map $\chi: T_g(z) \longrightarrow K_g(z)$ for $z \notin \mathcal{E}$ may be greater than 2, for instance for $g = 2$ and $z = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ that map is of degree 4.

12. *The equations of the Kummer variety.* To get a set of equations for the Kummer variety we give here a procedure which is inspired by a similar one given by C. L. Siegel in [28]. We assume $g \geq 2$.

Let us consider a generic projection of $K_g(z)$ onto a projective subspace P_{g+1} of $P_t(\mathbf{C})$ of dimension $g+1$. If $\lambda = {}^t(\lambda_0, \dots, \lambda_t)$ and $x = {}^t(x_0, \dots, x_{g+1})$ are homogeneous coordinates in $P_t(\mathbf{C})$ and P_{g+1} respectively, the projection is given by equations of the form

$$x = S\lambda$$

where S is a $(g+2) \times (t+1)$ matrix with elements s_{ij} that will be considered as indeterminates. The center of projection ($S\lambda = 0$) is a generic projective subspace of $P_t(\mathbf{C})$ of codimension $g+1$ and thus does not meet $K_g(z)$. The projection is therefore well defined on $K_g(z)$ and moreover it will be generally one to one. The image of $K_g(z)$ under this projection is therefore an irreducible hypersurface of P_{g+1} of degree $g! \cdot 2^{g-1}$ provided $z \in \mathcal{E}$. Over the field $\mathbf{C}(s)$ of rational functions in the s_{ij} the equation $f(x_0, \dots, x_{g+1}) = 0$ of that hypersurface is given by equating to zero a homogeneous polynomial in the x's of degree $g! \cdot 2^{g-1}$. This polynomial is uniquely determined up to a constant non zero factor by the condition

$$f(S\lambda(u, z)) = 0 \quad \forall\, u \in \mathbf{C}^g \tag{1}$$

where $\lambda(u, z)$ is given by $\lambda_\mu = \theta_2[\mu](u, z)$, i. e. the parametric equations of $K_g(z)$.

The equation $f(x) = 0$ over any purely transcendental extension of $\mathbf{C}(s)$ will be called a *normal equation* of $K_g(z)$.

Let us denote by $\alpha_0, \dots, \alpha_\varrho$ the coefficients of the generic homogeneous polynomial of degree $g! \cdot 2^{g-1}$ in the variables $x_0, \dots, x_{g+1}$.

Let

$$f(S\lambda(u,z)) = \sum_{\sigma \in \mathbf{N}^g} f_\sigma(\alpha, s, z)\, u^\sigma$$

be the Taylor expansion of $f(S\lambda)(u, z))$ as a function of the u's. The coefficients $f_\sigma(\alpha, s, z)$ are linear forms in the α's whose coefficients are polynomials with rational coefficients in the indeterminates s_{ij} and the « thetanulls »

$$C(r, \mu, z) = \left\{ \frac{\partial^{r_1+\dots+r_g}\, \theta_2[\mu](u, z)}{\partial u_1^{r_1} \dots \partial u_g^{r_g}} \right\}_{u=0}$$

for $r \in \mathbf{N}^g$, and μ ranging over a set of representatives of $\mathbf{Z}^g/2\mathbf{Z}^g$. Condition (1) is therefore equivalent to the system of homogeneous linear equations in the unknowns $\alpha_0, \dots, \alpha_\varrho$

$$\text{(2)} \qquad \begin{cases} f_\sigma(\alpha, s, z) = 0 \\ \sigma \in \mathbf{N}^g. \end{cases}$$

We now make use of the following

LEMMA 13. *Let* $z_0 \in H_g$ *and* $\Phi(u, z_0)$ *be a theta function of order* l. *If* $\Phi(u, z_0)$ *vanishes at* $u = 0$ *with all its partial derivatives of order* $\leq lg! \cdot 3^{g-1}$ *then* $\Phi(u, z_0)$ *is identically zero.*

PROOF. With the theta functions of order 3 one obtains a biregular projective imbedding of the torus $T_g(z_0)$ (cf. [12] pg. 159). The image manifold is an algebraic variety of order $g! \cdot 3^g$.

Suppose $\Phi(u, z_0)$ not identically zero. Then the positive divisor of $\Phi(u, z_0)$ is transformed by the projective imbedding into an algebraic positive cycle C of degree $lg! \cdot 3^{g-1}$. Let p be the image of $u = 0$. By the assumption p is a point of C of multiplicity $\geq lg! \cdot 3^{g-1} + 1$. This contradicts the theorem of Bézout.

As a consequence of this lemma we deduce the fact that in the system (2) one needs only to consider the system

$$\text{(3)} \qquad \begin{cases} f_\sigma(\alpha, s, z) = 0 \quad (\sigma = (\sigma_1, \dots, \sigma_g)\ |\sigma| = \Sigma\, \sigma_i) \\ |\sigma| \leq \sigma(g) \end{cases}$$

where $\sigma(g)$ is a bound depending only on g.

Since the system (3) has a unique solution up to a multiplicative factor $\neq 0$ the rank of the matrix of the coefficients must be ϱ. For any choice of ϱ rows, say for $\sigma = \sigma_1, \dots, \sigma_\varrho$, we can consider the determinants of the minors of order ϱ, $M_i^A(s, z)$, $0 \leq i \leq \varrho$, $A = (\sigma_1, \dots, \sigma_\varrho)$, extracted from the matrix of those ϱ rows. Choosing properly the sign for the determinants M_i^A we see that the coefficients α_i of the normal equation of $K_g(z)$ are proportional to these minors M_i^A. These minors are therefore either all zero of they can be taken as coefficients for the normal equation. For at least one system A of ϱ rows the latter is the case.

Introducing a set of new indeterminates η_A, $A = (\sigma_1, \dots, \sigma_\varrho)$, $|\sigma_i| \leq \sigma(g)$, we can set

$$\alpha_i = \Sigma\, \eta_A\, M_i^A(s, z)$$

for the coefficients of the normal equation of $K_g(z)$. *We thus get a normal equation for* $K_g(z)$, $z \in \mathcal{E}$

$$f(\eta, s, z\,; x) = 0 \tag{4}$$

in which the coefficients are polynomials over $\mathbf{Q}$ *in* η, s *and the thetanulls* $C(r, \mu, z)$.

From this fact one deduces the following

PROPOSITION 10. *There exists a finite set of homogeneous polynomials in the* λ_μ*'s* $\{g_h(x, \lambda)\}_{h \in H}$ *whose coefficients are polynomials over* $\mathbf{Q}$ *in the thetanulls* $C(r, \mu, z)$, *such that for any* $z \in \mathcal{E}$ *the set of equations*

$$\{g_h(z, \lambda) = 0\}$$

define in $P_t(\mathbf{C})$ *the variety* $K_g(z)$ *as the set of their common zeros.*

PROOF. We set $x = S\lambda$ in the normal equation (4) and rewrite the left hand side as a polynomial in the s_{ij} and η:

$$f(\eta, s, z, S\lambda) = \Sigma\, g_h(z, \lambda)\, \omega_h(s, \eta)$$

where $\omega_h(s, \eta)$ are the different monomials in the s_{ij} and the η's. The coefficients $g_h(z, \lambda)$ of those monomials satisfy the requirements of this proposition.

13. *The equations of the ramification sets.* We have defined in section 6 a sequence of analytic subsets N_s of H_g invariant under the action of the modular group for $0 \leq s \leq g-2$ giving a filtration of H_g:

$$N_{g-2} \subset N_{g-3} \subset \dots \subset N_1 \subset N_0 \subset H_g\,.$$

PROPOSITION 11. *For each* s *with* $0 \leq s \leq g-2$ *one can find a finite set of homogeneous polynomials with rational coefficients in the thetanulls*

$\{p_\nu(C(r, \mu, z))\}_{\nu \in I}$ *such that*

$$N_s \cap \mathcal{E} = \{z \in \mathcal{E} \mid p_\nu(C(r, \mu, z)) = 0 \quad \forall \nu \in I\}.$$

PROOF. We consider first the set of equations

$$K = \{g_h(z, \lambda) = 0\}$$

of the Kummer variety given by proposition 10. We add to this the system of equations

$$L \equiv \begin{cases} \Sigma \lambda_\mu \theta_2[\mu](0, z) = 0 \\ \Sigma \lambda_\mu \dfrac{\partial^2 \theta_2[\mu]}{\partial u_\alpha \partial u_\beta}(0, z) = 0 \quad \text{for} \quad 1 \leq \alpha \leq \beta \leq g \end{cases}$$

and the system

$$M = \left\{ \sum_{1 \leq i \leq s} w_\mu^{(i)} \lambda_\mu = 0 \right.$$

in which the coefficients $w_\mu^{(i)}$ are indeterminates.

Let z be fixed in $\mathcal{E}$ and let $\lambda^* = (\lambda_\mu^*)$ be a non trivial solution of K and L. From the system K we deduce the existence of a point $c^* \in \mathbf{C}^g$ such that

$$\lambda_\mu^* = \varrho \theta_2[\mu](c^*, z)$$

with some $\varrho \neq 0$. We thus get

$$\Sigma \lambda_\mu^* \theta_2[\mu](u, z) = \varrho \Phi_{c^*}(u) = \varrho \theta(u + c^*, z) \theta(u - c^*, z).$$

Since λ^* is a solution of L we then obtain $\theta^2(c^*, z) = 0$ and $\dfrac{\partial^2 \theta^2}{\partial u_\alpha \partial u_\beta}(c^*, z) = 0$, i. e.

$$\theta(c^*, z) = 0 \quad \text{and} \quad \frac{\partial \theta}{\partial u_i}(c^*, z) = 0.$$

This means that c^* is a singular point of $\theta(u, z) = 0$ in $\mathbf{C}^g$.

Eliminating λ from K, L, M by the Kronecker procedure we obtain a finite system of polynomials $(P_\alpha)_{\alpha \in A}$ homogeneous in the coefficients of the given equations and defined over $\mathbf{Q}$. These polynomials are thus rational polynomials in the thetanulls and the indeterminates $w_\mu^{(i)}$. If $z \in N_s \cap \mathcal{E}$ the P_α's vanish for any choice of the w's and conversely since the vanishing of them implies that the singular set of $\theta = 0$ is of dimension $\geq s$ (the map from $T_g(z)$ to $K_g(z)$, being of degree 2 with finite fibers, does not affect the dimensions).

To obtain the desired equations we have only to expand each P_a as a polynomial in the w's and take for p_ν the coefficients of these polynomials.

14. *a*) The equations K and L which by the elimination procedure described before give the equations over $\mathcal{C}$ of the ramification set, have for coefficients polynomials in the thetanulls $C(r, \mu, z)$. These as function of z are not, if $|r| > 0$, modular forms with respect to any subgroup G of finite index in the modular group Γ.

Our first goal is to show that the system L can be replaced by an equivalent one in which the coefficients are modular form with respect to a group G of finite index in Γ. We first prove the following

LEMMA 14. *For any $z \in H_g$ the matrix of the system L*

$$\mathcal{L} = \begin{pmatrix} \theta_2[\mu](0, z) \\ \dfrac{\partial^2 \theta_2[\mu]}{\partial u_\alpha \, \partial u_\beta}(0, z) \end{pmatrix}$$

is of maximal rank $= 1 + \dfrac{1}{2} g(g+1)$.

PROOF. Suppose that at a point $z_0 \in H_g$ the rank of $\mathcal{L}$ is $< 1 + \dfrac{1}{2} g(g+1)$. We must have therefore a linear relation

$$a_0 \, \theta_2[\mu](0, z_0) = \sum_{i \le k} a_{ik} \frac{\partial^2 \theta[\mu]}{\partial u_i \, \partial u_k}(0, z_0)$$

valid for every μ with a_0 and a_{ik} constants not all zero.

Multiplying these relations by $\theta_2[\mu](c, z_0)$ and summing over μ we then get

$$a_0 \{\theta(u + c, z_0)\, \theta(u - c, z_0)\}_{u=0} = \sum_{i \le k} a_{ik} \frac{\partial^2}{\partial u_i \, \partial u_k} \{\theta(u + c, z_0)\, \theta(u - c, z_0)\}_{u=0} .$$

Taking into account the fact that $\theta(u, z) = \theta(-u, z)$, we obtain

$$a_0 \{\theta(c, z_0)\}^2 = \sum_{i \le k} a_{ik} \frac{\partial^2}{\partial c_i \, \partial c_k} \{\theta(c, z_0)\}^2 .$$

For the function $\psi(c) = \{\theta(c, z_0)\}^2$ there are the periodicity conditions:

$$\psi(c + z_0 m) = e^{-2\pi i({}^t m z_0 m + 2\, {}^t m c)} \psi(c)$$

for any $m \in \mathbf{Z}^g$. Therefore from the above linear relations we obtain

$$\sum_{i \leq k} a_{ik} \left(m_i \frac{\partial \theta}{\partial c_k} (c, z_0) + m_k \frac{\partial \theta}{\partial c_i} (c, z_0) \right) = 2\pi i \sum_{i \leq k} a_{ik}\, m_i\, m_k\, \theta (c, z_0)\,.$$

Take all m's equal zero except $m_i = m_k = 1$. Identifying the coefficients in the Fourier expansion on both sides we get for any $p \in \mathbf{Z}^g$

$$a_{ik} (p_i + p_k) = a_{ik}\,.$$

Therefore $a_{ik} = 0$. Since $\theta (c, z_0)$ is not identically zero we must also have $a_0 = 0$ and the lemma is proved.

b) From the theory of transformation of theta functions we borrow the following fact:

There is a subgroup G of finite index in Γ such that for any $\gamma \in G$:

$$\gamma : z \longrightarrow (Az + B)(Cz + D)^{-1}$$

each one of the functions $g(u, z) = \theta_2 [\mu] (u, z)$ satisfies the functional equation:

$$(1) \quad g({}^t(Cz + D)^{-1} u, (Az + B)(Cz + D)^{-1}) = \varrho \det (Cz + D)^{\frac{1}{2}}\, e^{2\pi i\, {}^t u^t (Cz+D)^{-1}\, Cu}\, g(u, z)$$

where $\det\,(Cz + D)^{\frac{1}{2}}$ is a determined branch of this square root and where ϱ is an eighth root of unity.

This theorem can be found in C. L. Siegel [28], pg. 395; with the notations of that paper if one takes $T = 2I$ one has $\vartheta (x, 0, 2z, 2u) = \theta_2 [x] (u, z)$ and the group G is the « Theta gruppe der Stufe T ».

Going back to the system L by virtue of the lemma 14 and Cramer's rule on can replace the system L by a system having the same set of solutions and of the form

$$\left\{ \sum_{1 \leq a \leq s} \lambda_\mu\, \mathcal{D}_{\mu a} (z) = 0 \right.$$

where $\mathcal{D}_{\mu a} (z)$ are the determinants of the minors of maximal rank extracted from the matrix $\mathcal{L}$.

Now we remark that the functions $\theta_2 [\mu] (u, z)$ are even functions of u so that their first derivatives vanish at $u = 0$. $\mathcal{D}_{\mu a}$ thus appear as Wronskian determinants extracted from the Wronskian matrix for the functions $\theta_2 [\mu] (u, z)$. From the functional equation (1) and a known property of Wronskians of automorphic forms we deduce that the functions $\mathcal{D}_{\dot{a}\mu} (z)$ satisfy a transformation formula of the form :

$$\mathcal{D}_{\mu\alpha}((Az+B)(Cz+D)^{-1}) = \varrho_0 \det(Cz+D)^{k/2}\,\mathcal{D}_{\mu\alpha}(z)$$

for any $\gamma \in G$, where ϱ_0 is an eighth root of unity depending on γ and k is an integer Both ϱ_0 and k are independent of μ and α.

The system L has thus the same solutions as the system L'

$$L' \equiv \{(\sum_{1\leq\alpha\leq s} \lambda_\mu\,\mathcal{D}_{\mu\alpha}(z))^8 = 0.$$

In this the coefficients are now modular forms with respect to G.

c) For the system K representing the equations of the Kummer variety a system of equations with G-modular forms as coefficients valid on a certain Zariski open set U of H_g was given by Wirtinger [31]. An outline of his method will be given in the last section. From this we deduce that the ramification sets N_s on U can be defined by a set of equations obtained by equating to zero a set of G-modular forms.

d) The method of C. L. Siegel based on the normal equation of $K_g(z)$ has however the advantage of giving the equation on a very explicit set $\mathcal{E} \subset H_g$ and not on an unspecified Zariski open set U of H_g. Moreover some geometrical facts can be proved along this line. As an instance we derive here from this method the following

PROPOSITION 12. *For any s with $0 \leq s \leq g-2$ and for any $z_0 \in \mathcal{E}$ we can find a Zariski neighborhood $V(z_0)$ of z_0 in $\mathcal{E}$ and a finite set of meromorphic and G-automorphic functions $\{m_\alpha(z)\}_{1\leq\alpha\leq k}$ regular on $V(z_0)$ such that*

$$N_s \cap V(z_0) = \{z \in V(z_0) \mid m_\alpha(z) = 0, \text{ for } 1 \leq \alpha \leq k\}.$$

PROOF. Let $f(\eta, s, z, x) = 0$ be the normal equation of $K_g(z)$, $z \in \mathcal{E}$. We remark that all coefficients $\alpha_i(\eta, s, z)$ (η and s being considered as indeterminates) are non zero. Moreover the functions

$$\beta_j(s, z) = \frac{\alpha_j(\eta, s, z)}{\alpha_0(\eta, s, z)}$$

for $1 \leq j \leq \varrho$ do not depend on the parameters η and are meromorphic functions of s and z.

Because of the transformation formula (1) for $\gamma \in G$ the Kummer varieties $K_g(z)$ and $K_g(\gamma z)$ coincide. Therefore on has

$$\beta_j(s, \gamma z) = \beta_j(s, z) \qquad 1 \leq j \leq \varrho.$$

Let $z_0 \in \mathcal{E}$ and select $s = s_0$ integral such that $\alpha_0(\eta, s_0, z_0) \not\equiv 0$. On a Zariski neighborhood $U(z_0)$ of z_0 in $\mathcal{E}$ we then have $\alpha_0(\eta, s_0, z) \not\equiv 0$. Let $f = \sum_0^{\varrho} \alpha_i \, \omega_i(x)$ where $\omega_i(x)$ are the different monomials in the x's of the proper degree. Consider the substitution $x = S\lambda$ and set

$$\omega_i(S\lambda) = \Sigma \, c_{ij} \, \omega_j(\lambda) \qquad c_{ij} \in \mathbf{Z}[s]$$

where $\omega_j(\lambda)$ are the different monomials in the λ's of the proper degree.

We then have

$$f(\eta, x, z, S\lambda) = \alpha_0(\eta, s, z) \sum_j \Big(\sum_i \beta_i(s, z) \, c_{ij}\Big) \, \omega_j(\lambda).$$

Develop α_0, β_i and c_{ij} in power series in $s - s_0$:

$$\alpha_0(\eta, s, z) = \sum_\nu \alpha_{0\nu}(\eta, s_0, z)(s - s_0)^\nu \; ; \; \beta_i(s, z) = \Sigma \, \beta_{i\mu}(s_0, z)(s - s_0)^\mu$$

$$c_{ij} = \sum_\varrho c_{ij\varrho}(s - s_0)^\varrho$$

so that one obtains

$$f(\eta, s, z, S\lambda) = \sum_j \Big(\sum_{\mu+\nu+\varrho=\sigma} \alpha_{0\nu}(\eta, s_0, z) \sum_i \beta_{i\mu}(s_0, z) \, c_{ij\varrho}(s - s_0)^\sigma \Big) \, \omega_j(\lambda).$$

Since f is a polynomial in $s - s_0$, we deduce that there is an integer N independent of s_0 and z such that the set of equations

$$K' = \begin{cases} \{g'_\tau(z, \lambda) \equiv \displaystyle\sum_{\mu+\varrho=\tau} \sum_j \sum_i \beta_{i\mu}(s_0, z) \, c_{ij\varrho} \, \omega_j(\lambda) = 0 \\ |\tau| \leq N \end{cases}$$

has the same solutions as the set of equations K for any $z \in U(z_0)$.

Let $g(z)$ be any G-modular form of weight $4kh$ such that $g(z_0) \neq 0$. This is certainly the case if h is large enough. Replace the system L' with the system

$$L'' = \Big\{ g(z)^{-1} \{\Sigma \lambda_\mu \mathcal{D}_{\mu\alpha}\}^{8h} = 0.$$

In a Zariski neighborhood $V(z_0)$ of $z_0 \in \mathcal{E}$ then, the equations K' and L'' have G-automorphic, meromorphic coefficients, regular on $V(z_0)$. By the elimination procedure described above we then get on $V(z_0)$ the conclusion we wanted.

It is known (and it can be derived from the pseudoconcavity of the modular group cf. [5]) that any meromorphic and G-automorphic function is the quotient of two G-modular forms. Moreover with a finite set of

G-modular forms one can obtain a one to one holomorphic imbedding τ of H_g/G into a projective space. By the use of compactification of the modular space H_g/Γ this fact was first proved by W. Baily [7] and for more general conditions by H. Cartan [11]. It can however be proved directly as is shown in C. L. Siegel [28]. The image of $\tau(H_g/G)$ is contained in an algebraic variety of the same dimension. In particular it follows from the previous proposition that

COROLLARY : *For any s $0 \leq s \leq g-2$ each irreducible component of $N_s \cap \mathcal{E}$ has an image by τ which is an algebraic set. In particular the whole Jacobi space in $\tau(H_g/G)$ lies in an algebraic set of dimension $3g-3$.*

15. *The Wirtinger method.* We give here an outline of Wirtinger's method of writing the equations of the Kummer variety.

a) First we introduce theta functions with characteristic. For $x, y \in \mathbf{Z}^g$ we set

$$\theta(x, y; u, z) = \Sigma\, e^{\pi i\, {}^t\left(m+\frac{x}{2}\right) z \left(m+\frac{x}{2}\right) + 2\, {}^t\left(m+\frac{x}{2}\right)\left(u+\frac{y}{2}\right)}$$

$$= e^{\pi i \left\{\frac{1}{4}\, {}^t xzx + {}^t x \left(u+\frac{y}{2}\right)\right\}}\, \theta\left(u + \frac{y}{2} + z\frac{x}{2};\, z\right).$$

Since $\theta(x+2a, y+2b; u, z) = e^{\pi i\, {}^t xb}\, \theta(x, y; u, z)$, up to constant factors $\neq 0$ one obtains all of these functions when x, y run over a set of representatives of $\mathbf{Z}^g/2\mathbf{Z}^g$. One obtains thus 2^{2g} linearly independent functions over $\mathbf{C}$. Among them those for which ${}^t xy$ is even are even functions of u. Their number is $2^{g-1}(2^g+1)$. Those for which ${}^t xy$ is odd are odd functions of u and their number is $2^{g-1}(2^g-1)$. Their periodicity properties are as follows :

$$\theta(x, y; u+n+zm, z) = e^{-\pi i \{{}^t mzm + 2\,{}^t mu - {}^t nx + {}^t my\}}\, \theta(x, y; u, z),$$

for $n, m \in \mathbf{Z}^g$. With the notations of n. 11 one has

$$\theta_2[\mu](u, z) = \theta(\mu, 0; 2u, 2z).$$

Let

$$\gamma : z \longrightarrow \hat{z} = (Az+B)(Cz+D)^{-1}$$

be a transformation of the modular group.

We let

$$\widehat{u} = {}^t(Cz + D)^{-1} u$$

$$\widehat{x} = {}^tAx + {}^tCy + \{{}^tAC\}$$

$$\widehat{y} = {}^tBx + {}^tDy + \{{}^tBD\}$$

where for a square matrix M, $\{M\}$ denotes the vector whose components are the diagonal elements of M.

One has the following transformation formula (cf. C. L. Siegel [28], theorem 8)

$$(1) \qquad \theta(x, y; \widehat{u}, \widehat{z}) = \varrho \det (Cz + D)^{\frac{1}{2}} e^{-\pi i \varphi(x, y)} e^{\pi i \psi(u)} \theta(\widehat{x}, \widehat{y}; u, z)$$

where

ϱ is a constant of absolute value 1 depending only on γ

$$\varphi(x, y) = \frac{1}{4}({}^txA + {}^tyC)({}^tBx + {}^tDy + 2\{{}^tBD\}) - \frac{1}{4}{}^txy$$

$$\psi(u) = {}^tu(Cz + D)^{-1} Cu .$$

We see that by this transformation even functions are changed into even functions. Moreover, if (x_1, y_1), (x_2, y_2) are two systems of characteristics with $x_1 + x_2 \equiv \sigma \pmod 2$, $y_1 + y_2 \equiv \tau \pmod 2$ then $\widehat{x}_1 + \widehat{x}_2 \equiv \widetilde{\sigma} \pmod 2$ $\widehat{y}_1 + \widehat{y}_2 \equiv \widetilde{\tau} \pmod 2$ with $\widetilde{\sigma}$ and $\widetilde{\tau}$ depending only upon σ, τ and γ.

We will need the addition theorem given by the formulae:

$$(2) \quad \theta(x, y; u + v, z)\, \theta(x, y; u - v, z) = \sum_{\mu} (-1)^{{}^t(x+\mu)y}\, \theta_2[\mu](v,z)\, \theta_2[x + \mu](u,z)$$

$$(3) \quad 2^g\, \theta_2[\mu](v, z)\, \theta_2[\mu + x](u, z) = \sum_{y} (-1)^{{}^t(\mu+x)y}\, \theta(x,y; u+v,z)\, \theta(x, y; u - v, z).$$

A special case of (1) is formula (1) of n. 14 and a special case of (2) is formula (2) of n. 11.

b) We consider for given $z \in H_g$ the ideal $\mathcal{J}(K_g(z))$ of all homogeneous polynomials in the ring $\mathbf{C}[\dots, \lambda_\mu, \dots]$ vanishing on $K_g(z)$. It is defined by the conditions

$$\mathcal{J}(K_g(z)) = \{p \in \mathbf{C}[\dots, \lambda_\mu, \dots,] \mid p(\dots, \theta[\mu](u), \dots) \equiv 0\}.$$

We want to writhe a basis for polynomials of degree 4 in that ideal.

LEMMA 15. *The* $2^{g-1}(2^g+1)$ *products* $\theta_2[\sigma](u)\,\theta_2[\mu](u)$ *are linearly independent at any point* $z \in H_g$ *such that*

$$(\alpha) \qquad \theta(x, y\,;\,0, z) \neq 0 \text{ for all } x, y \text{ with } {}^t xy \equiv 0 \pmod 2.$$

PROOF. Setting $u = v$ in (2) we get the formula

$$(2)_0 \qquad \theta(x, y\,;\,2u, z)\,\theta(x, y\,;\,0, z) = \Sigma(-1)^{{}^t(x+\mu)y}\,\theta_2[\mu](u, z)\,\theta_2[x+\mu](u, z)$$

Since $\theta(x, y\,;\,0, z) = 0$ if ${}^t xy \equiv 1 \pmod 2$ and by the assumption (α) on the left hand side we get $2^{g-1}(2^g+1)$ linearly independent functions. These being linear combinations of the prodncts $\theta_2[\sigma](u)\,\theta_2[\mu](u)$, we get the conclusion.

Our problem, under assumption (α), is thus reduced to write a basis for all quadratic relations among the products $\theta_2[\sigma](u)\,\theta_2[\mu](u)$. From (3) setting $u = v$ we get

$$(3)_0 \qquad 2^g\,\theta_2[\mu](u, z)\,\theta_2[\mu+x](u, z) = \sum_{{}^t xy \text{ even}} (-1)^{{}^t(\mu+x)y}\,\theta(x, y;\,0, z)\,\theta(x, y;\,2u, z).$$

It follows that under assumption (α) any quadratic relation among the products $\theta_2[\sigma](u)\,\theta_2[\mu](u)$ gives a quadratic relation for the even theta functions $\theta(x, y, 2u, z)$ and conversely.

c) Let

$$\Sigma\, C(\alpha, \beta\,;\,x, y)\,\theta(\alpha, \beta\,;\,u, z)\,\theta(x, y\,;\,u, z) = 0 \;\forall\, u$$

be, for given $z \in H_g$, a quadratic relation among the even thetafunctions with characteristic. Making use of the periodicity conditions it follows that any relation of that type is a linear combination of relations of the form

$$\sum_{\substack{\alpha+x\equiv\sigma \pmod 2 \\ \beta+y\equiv\tau \pmod 2}} C(\alpha, \beta, x, y)\,\theta(\alpha, \beta\,;\,u, z)\,\theta(x, y\,;\,u, z) = 0 \;\forall\, u.$$

d) For $\sigma = \tau = 0$ we are reduced to find all linear relations among the functions $\{\theta(x, y\,; u, z)\}^2$ where ${}^t xy \equiv 0 \pmod 2$. Setting $v = 0$ in (2) we get:

$$(2)_1 \qquad \{\theta(x, y\,;\,u, z)\}^2 = \sum_\mu (-1)^{{}^t(x+\mu)y}\,\theta_2[\mu](0, z)\,\theta_2[x+\mu](u, z).$$

Let A be the matrix indexed by (x, y) with ${}^t xy \equiv 0 \pmod 2$ and μ, given by

$$A = \left((-1)^{{}^t(x+\mu)y}\,\theta_2[\mu](0, z)\right).$$

Then all linear relations among the functions $\{\theta(x, y; u, z)\}^2$, ${}^t xy \equiv 0 \pmod 2$, are given by the conditions:

$$\operatorname{rank}\left(\{\theta(x, y; u, z)\}^2, A\right) = \operatorname{rank} A$$

Note that setting $v = 0$ in (3) we get for $x = 0$,

$$(3)_1 \qquad 2^g\, \theta_2[\mu](0, z)\, \theta_2[\mu](u, z) = \Sigma(-1)^{{}^t\mu y} \{\theta(0, y; u, z)\}^2$$

so that if

$$(\beta) \qquad \theta_2[\mu](0, z) \neq 0 \quad \text{for all } \mu$$

then among the functions $\{\theta(x, y; u, z)\}^2$, ${}^t xy \equiv 0 \pmod 2$, there are exactly 2^g linearly independent. In this case rank $A = 2^g$.

e) For $\sigma = 0$ $\tau \neq 0$ we are reduced to find all linear relations among the products $\theta(x, y; u, z)\, \theta(x, y + \tau; u, z)$ where ${}^t xy$, ${}^t x(y + \tau)$ and therefore also ${}^t x\tau$, are even.

Changing in (2) u into $u + \dfrac{\tau}{4}$ and v in $v + \dfrac{\tau}{4}$ and setting $v = 0$ we get

$$(2)_2 \quad \theta(x, y + \tau; u, z)\, \theta(x, y; u, z) = \sum_{{}^t\mu\tau \equiv 0 \pmod 2} (-1)^{{}^t(x+\mu)y}\, \theta(\mu, \tau; 0, 2z)\, \theta(x + \mu, \tau; 2u, 2z).$$

Since $\tau \neq 0$ among the functions $\theta(x + \mu, \tau, 2u, 2z)$ for which ${}^t(x + \mu)\tau$ is even there are 2^{g-1} linearly independent ones.

Let B be matrix indexed by (x, y), ${}^t xy \equiv 0 \bmod 2$, and μ

$$B = \left((-1)^{{}^t(x+\mu)y}\, \theta(\mu, \tau; 0, 2z)\right).$$

Then all linear relations of the type under consideration are given by the conditions

$$\operatorname{rank}\left(\theta(x, y + \tau; u, z)\, \theta(x, y; u, z), B\right) = \operatorname{rank} B.$$

Note that from (3) changing u to $u + \dfrac{\tau}{4}$, v to $v + \dfrac{\tau}{4}$, and setting $v = 0$ and $x = 0$ we get

$$(3)_2 \quad 2^g\, \theta(\mu, \tau; 0, 2z)\, \theta(\mu, \tau; 2u, 2z) = \Sigma(-1)^{{}^t\mu y}\, \theta(0, y + \tau; u, z)\, \theta(0, y; u, z)$$

so that if

$$(\gamma) \qquad \theta(\mu, \tau; 0, 2z) \neq 0 \text{ for every } \mu \text{ with } {}^t\mu\tau \equiv 0 \pmod 2$$

then the rank of B equals 2^{g-1}.

f) To get all other relations Wirtinger proceeds as follows. For every modular transformation γ one writes the relations given in *e*) in the variables $\widehat{u}$ and $\widehat{z}$ and notices that by (1) the rank of B is not affected.

Let

$$\Sigma\, C(x, y)\, \theta(x, y + \tau\,; \widehat{u}, \widehat{z})\, \theta(x, y\,; \widehat{u}, \widehat{z}) = 0 \;\forall\, u$$

be one of these relations. From the transformation formula (1) we then obtain a relation

$$\Sigma\, C(x, y)\, \varepsilon(x, y, \tau, \gamma)\, \theta(\widehat{x}, \widehat{y+\tau}\,; u, z)\, \theta(\widehat{x}, \widehat{y}\,; u, z) = 0 \;\forall\, u$$

where ε denotes an eighth root of unity. Conversely from a relation of this second type we derive one of the type given in *e*) at the point $\widehat{z}$.

Now the functions $\theta(\widehat{x}, \widehat{y+\tau}\,; u, z)$ and $\theta(\widehat{x}, \widehat{y}\,; u, z)$ are both even as are their transforms. For any choice of $(\sigma, \varrho) \neq 0$ there exists a trasformation γ such that the sum of the characteristics in the product $\theta(\widehat{x}, \widehat{y+\tau}\,; u, z)\, \theta(\widehat{x}, \widehat{y}\,; u, z)$ is congruent to (σ, ϱ) (mod 2). In this way one obtains all remaining relations.

g) These considerations can be summarized as follows:

Let

$$W = \{z \in H_g \mid \theta(x, y\,; 0, z) \neq 0 \text{ for } {}^t xy \equiv 0 \pmod 2\}$$

Let

$$X_{ab} = \{z \in H_g \mid \text{rank } A = a, \text{rank } B = b\}.$$

There is a finite number of polynomials of degree 4 *in* $\mathbf{C}[\ldots, \lambda_\mu \ldots]$ *whose coefficients are polynomials over* $\mathbf{Q}\left(e^{\frac{\pi i}{4}}\right)$ *in the thetanulls* $\theta(\mu, \tau\,; 0, 2z)$ *for* ${}^t\mu\tau \equiv 0$ (mod 2) *such that for any point* $z \in W \cap X_{ab}$ *they form a basis for the homogeneous polynomials of degree* 4 *in the ideal* $J(K_g(z))$ *of the Kummer variety. The closure of the set* $X_{2g,\, 2g-1}$ *is* H_g. Now Wirtinger proves that for a certain Zariski open set U of H_g containing the diagonal matrices these equations of 4*th* order are defining equations for the Kummer variety on $U \cap W$. The Jacobi space J intersects U (cf. [18]) and moreover $J \cap W \neq \emptyset$ by a theorem of Farkas [14]. Combining these results one can write by equating to zero G-modular forms the equations of an analytic set having $\overline{J}$ as an irreducible component.

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