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# ON HOLOMORPHIC FUNCTIONS OF POLYNOMIAL GROWTH IN A BOUNDED DOMAIN

by RAGHAVAN NARASIMHAN (*)

Let $\Omega$ be a bounded open set in $\mathbb{C}^n$. For $z \in \Omega$, let

$$d(z) = d(z, \Omega)$$

denote the distance of the point $z$ from the boundary $\partial\Omega$ of $\Omega$.

DEFINITION. *A holomorphic function $f$ in $\Omega$ is said to be of polynomial growth if there exist constants $M, \varrho > 0$ such that, for $z \in \Omega$ we have*

$$|f(z)| \leq M d(z)^{-\varrho};$$

*$M$ and $\varrho$ may depend on $f$.*

The object of this note is to prove a theorem which is preliminary to certain questions on the existence of sections of coherent analytic sheaves on Stein spaces satisfying conditions of growth similar to the one occurring in the definition above. The result is the following.

THEOREM. *Let $g$ be holomorphic in $\overline{\Omega}$* (i. e. *in a neighborhood of $\overline{\Omega}$ and let $f$ be a function of polynomial growth in $\Omega$. If ( $g$ does not vanish identically on any non-empty open set in $\Omega$ and) $f/g$ is holomorphic in $\Omega$, then $f/g$ has polynomial growth.*

PROOF. Let $h = f/g$. It is sufficient to prove that for any $a \in \partial\Omega$ there is a neighborhood $U$ and constants $M, \varrho > 0$ with

$$|h(z)| \leq M d(z)^{-\varrho} \quad \text{for} \quad z \in \Omega \cap U.$$

We may suppose that $a = 0$. Further, it is sufficient to prove our inequality after a linear change of coordinates in $\mathbb{C}^n$. Thus, by the Weierstrass preparation theorem, we may make a linear change of coordinates such that

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$g$ assumes the form

$$g(z) = \omega(z) P(z); P(z) = z_n^p + \sum_{\nu=1}^{p} a_\nu(z_1, \ldots, z_{n-1}) z_n^{p-\nu}, a_\nu(0, \ldots, 0) = 0,$$

here $\omega$ and $P$ are holomorphic on the closure $\bar{U}$ of a neighborhood $U$ of 0 in $\mathbb{C}^n$ and $\omega$ has no zeros on $\bar{U}$. Further, we may suppose that $U = U' \times U''$, $U' \subset \mathbb{C}^{n-1}$, $U'' \subset \mathbb{C}$ and that, if $z' \in \bar{U}'$, $z' = (z_1, \ldots, z_{n-1})$, and $P(z'; t) = 0$, $t \in \mathbb{C}$, then $t \in U''$. [The notation $P(z'; t)$, $z' \in U'$, is self-explanatory].

We have only to obtain an estimate of the form

$$\left| \frac{f}{P}(z) \right| = |\omega(z) h(z)| \leq M d(z)^{-\varrho} \quad \text{for} \quad z \in \Omega \cap U.$$

Let $V$ be a neighborhood of $\bar{\Omega}$ in which $g$ is holomorphic; and let $A = \{z \in V \mid g(z) = 0\}$. We may clearly suppose that $0 \in A$ and that $U \subset V$. For $z = (z', z_n) \in U' \times U'' = U$, we define

$$\delta_A(z) = \min_{\nu} |z_n - z_n^{(\nu)}|$$

where $z_n^{(\nu)}$, $\nu = 1, \ldots, p$ are the zeros of the polynomial $P(z'; t)$ in $t$.

LEMMA 1. For $z \in U$, we have

$$|P(z)| \geq \delta_A(z)^p.$$

PROOF OF LEMMA 1. Given $z' \in U'$, let $\lambda_1, \ldots, \lambda_p$ be the zeros of $P(z'; t)$. Then

$$|P(z)| = |(z_n - \lambda_1) \ldots (z_n - \lambda_p)| \geq \delta_A(z)^p,$$

since, by definition, we have $|z_n - \lambda_j| \geq \delta_A(z)$ for each $j$.

Let $0 < \varepsilon \leq 1/2$ and let

$$\Delta_1 = \{z \in \Omega \cap U \mid \varepsilon\, d(z) \leq \delta_A(z)\},$$

$$\Delta_2 = \{z \in \Omega \cap U \mid \varepsilon\, d(z) > \delta_A(z)\}.$$

Clearly $\Delta_2$ is a neighborhood of $A \cap U \cap \Omega$.

Since $f$ has polynomial growth, there are constants $M, \varrho > 0$ with

$$|f(z)| \leq M d(z)^{-\varrho}, z \in \Omega.$$

Hence, for $z \in \Delta_1$, we have

$$\left| \frac{f}{P}(z) \right| \leq Md(z)^{-\varrho} \mid P(z) \mid^{-1} \leq Md(z)^{-\varrho}\, \delta_A(z)^{-p} \leq M\varepsilon^{-p}\, d(z)^{-\varrho-p}$$

by Lemma 1 and the definition of $\Delta_1$.

To deal with $\Delta_2$, we proceed as follows. For given $\zeta' = (\zeta_1, \dots, \zeta_{n-1}) \in U'$, let $L_{\zeta'}$ denote the complex line

$$z_1 = \zeta_1, \dots, z_{n-1} = \zeta_{n-1}.$$

Clearly $L_{\zeta'} \cap A$ contains at most $p$ points for any $\zeta' \in U'$.

Lemma 2. $\Delta_2 \cap L_{\zeta'}$ is relatively compact in $\Omega \cap L_{\zeta'}$ (considered as open sets in the plane $L_{\zeta'}$).

Proof of Lemma 2. Suppose that $\{z^{(k)}\}, (k = 1, 2, \dots)$ is a sequence of points of $\Delta_2 \cap L_{\zeta'}$ converging to a point $z^{(0)} \in \partial\Omega$. Since $\delta_A(z^{(0)}) = \lim_{k \to \infty} \delta_A(z^{(k)}) \leq \leq \lim_{k \to \infty} \varepsilon d(z^{(k)}) = 0$, $z^{(0)} \in A$. If then $W$ is a small neighborhood of $z^{(0)}$ not containing any other point of $L_{\zeta'} \cap A$, we see that for $z \in W \cap L_{\zeta'}$ we have

$$\delta_A(z) = \mid z_n - z_n^{(0)} \mid = \mid z - z^{(0)} \mid \geq d(z, \Omega) = d(z)$$

since $z^{(0)} \in \partial\Omega$. In particular, if $k$ is large enough,

$$d(z^{(k)}) \leq \delta_A(z^{(k)}) < \varepsilon d(z^{(k)})$$

by definition of $\Delta_2$, which is absurd. Hence no sequence of points of $\Delta_2 \cap L_{\zeta'}$ can converge to a point of $\partial\Omega$, which proves Lemma 2.

Lemma 3. If $\zeta' \in U'$, then, for any points $x_1, x_2$ lying in the same connected component of $\Delta_2 \cap L_{\zeta'}$, we have

$$\varepsilon^{2p}\, d(x_1) \leq d(x_2) \leq \varepsilon^{-2p}\, d(x_1).$$

Proof of Lemma 3. Let $z^{(1)}, \dots, z^{(m)}, m \leq p$, be the points in $\Delta_2 \cap L_{\zeta'}$ and let $D^{(j)} = \{z \in L_{\zeta'} \cap \Omega \mid \mid z_n - z_n^{(j)} \mid < \varepsilon d(z)\}$ $(j = 1, \dots, m)$. If $z \in D^{(j)}$, we have

$$d(z) \leq d(z^{(j)}) + \mid z - z^{(j)} \mid = d(z^{(j)}) + \mid z_n - z_n^{(j)} \mid < d(z^{(j)}) + \varepsilon d(z),$$

so that

$$d(z) < (1-\varepsilon)^{-1}\, d(z^{(j)}) < \varepsilon^{-1}\, d(z^{(j)}).$$

On the other hand

$$d(z) \geq d(z^{(j)}) - |z - z^{(j)}| > d(z^{(j)}) - \varepsilon\, d(z),$$

so that

$$d(z^{(j)}) < (1 + \varepsilon)\, d(z) < \varepsilon^{-1}\, d(z).$$

Hence, if $x, y$ are points of $D^{(j)}$, we have

$$\varepsilon^2\, d(x) < d(y) < \varepsilon^{-2}\, d(x).$$

Since, by Lemma 2, $\Delta_2 \cap L_{\zeta'} \subset D^{(1)} \cup \dots \cup D^{(m)}$, if $D$ is any connected component of $\Delta_2 \cap L_{\zeta'}$ and $x_1, x_2 \in D$, there exist indices $j_1, \dots, j_r$, $r \leq m \leq p$, and points $y_1, \dots, y_{r-1}$ with $y_\nu \in D^{(j_\nu)} \cap D^{(j_{\nu}+1)}$ such that

$$x_1, x_2 \in D^{(j_1)} \cup \dots \cup D^{(j_r)}.$$

If now $x_1, x_2 \in D$, we may suppose that $x_1 \in D^{(j_1)}$ and $x_2 \in D^{(j_s)}$, $s \leq r$. Then, by what we have seen above, if $s \geq 2$,

$$d(x_1) \leq \varepsilon^{-2}\, d(y_1) \leq \varepsilon^{-4}\, d(y_2) \leq \dots \leq \varepsilon^{-2(s-1)}\, d(y_{s-1}) \leq \varepsilon^{-2s}\, d(x_2),$$

so that $d(x_1) \leq \varepsilon^{-2p}\, d(x_2)$, and this is true also if $s = 1$. In the same way, $d(x_2) \leq \varepsilon^{-2p}\, d(x_1)$, and this proves Lemma 3.

We now proceed to the proof of our theorem.

If $\Delta$ is any open subset of $L_{\zeta'}$ for $\zeta' \in U'$, we denote by $\partial\Delta$ its boundary in $L_{\zeta'}$. We claim that if $\varepsilon$ is sufficiently small, then for all $\zeta'$ near $0 \in U'$, $\partial(\Delta_2 \cap L_{\zeta'}) \subset \Delta_1$. Because of Lemma 2, it is sufficient to show that $\partial(\Delta_2 \cap L_{\zeta'})$ does not meet $\zeta' \times \partial U''$. Now, if $\zeta'$ is restricted to a relatively compact subset $U_1'$ of $U'$, then $\zeta' \times \partial U''$ has a distance from $L_{\zeta'} \cap A$ which is bounded below. Hence $\delta_A(z)$ is bounded below on $(U_1' \times \partial U'') \cap \Omega$; since $d(z)$ is bounded above, if $\varepsilon$ is small enough, no point of $\zeta' \times \partial U''$, $\zeta' \in U_1'$, can have points of $\Delta_2 \cap L_{\zeta'}$ arbitrarily close to it. Hence, for $\zeta' \in U_1'$, if $\varepsilon$ is small, $\partial(\Delta_2 \cap L_{\zeta'}) \subset \Delta_1$.

If $z \in \Delta_2 \cap (U_1' \times U'')$, let $\zeta' = (z_1, \dots, z_{n-1})$ and let $D$ be the connected component of $\Delta_2 \cap L_{\zeta'}$ containing $z$. Since $\frac{f}{P}$ is holomorphic in $D$, we obtain, by the maximum principle,

$$\left| \frac{f}{P}(z) \right| \leq \sup_{x \in \partial D} \left| \frac{f}{P}(x) \right|,$$

Since $\partial D \subset \partial (\Delta_2 \cap L_{\zeta'}) \subset \Delta_1$, Lemma 1 gives, for $x \in \partial D$,

$$\left|\frac{f}{P}(x)\right| \leq M\, d(x)^{-\varrho}\, \delta_A(x)^{-p} \leq \frac{M}{\varepsilon^p}\, d(x)^{-\varrho-p} \quad (\text{since} \quad x \in \Delta_1)$$

$$\leq \frac{M}{\varepsilon^p}\, \varepsilon^{-2p(\varrho+p)}\, d(z)^{-\varrho-p} \quad (\text{by Lemma 3}).$$

Hence, for $z \in \Delta_2 \cap (U_1' \times U'')$,

$$\left|\frac{f}{P}(z)\right| \leq \sup_{x \in \partial D} \left|\frac{f}{P}(z)\right| \leq M_1\, d(z)^{-\varrho_1}, \quad M_1 = M\, \varepsilon^{-p-2p(\varrho+p)}, \quad \varrho_1 = \varrho + p.$$

Since for $z \in \Delta_1$, we have

$$\left|\frac{f}{P}(z)\right| \leq M\, \varepsilon^{-p}\, d(z)^{-\varrho-p} \leq M_1\, d(z)^{-\varrho_1}$$

we have

$$\left|\frac{f}{P}(z)\right| \leq M_1\, d(z)^{-\varrho_1} \quad \text{for} \quad z \in \Omega \cap (U_1' \times U'').$$

As already remarked, this proves the theorem.

Remark 1. It is easily seen that Theorem 1 when $f \equiv 1$ (case when $g$ has no zeros in $\Omega$ but may have zeros in $\bar{\Omega}$) is equivalent to the following statement:

Let $g$ be holomorphic in an open set $U \subset \mathbb{C}^n$. Let $A = \{z \in U \mid g(z) = 0\}$ and $d(z, A)$ be the distance of $z \in \Omega$ from $A$. Then, for any compact set $K \subset U$, there exist constants $c$, $\varrho > 0$ with

$$|g(z)| \geq c\, d(z, A)^{\varrho} \quad \text{for} \quad z \in K.$$

This is a very special case of a famous inequality of Lojasiewicz which asserts the validity of a similar inequality for all *real* analytic (and even *semi-analytic*) functions in open sets in $\mathbb{R}^n$; see e.g. S. Lojasiewicz: *Ensembles sémi-analytiques*; Inst. Hautes Etudes Sc. Paris (1965).

Remark 2. The assumption that $g$ be holomorphic in $\bar{\Omega}$ is essential. For example, if $g(z) = \exp(-1/z)$ in $\Omega = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0,\ |z| < 1\}$, we have $|g(z)| < 1$, $g(z) \neq 0$ in $\Omega$, but $1/g(z)$ tends to infinity like $\exp(1/d(z))$ when $z \to 0$ along the real axis.

If $\Omega = \left\{ z \in \mathbb{C} \mid |\operatorname{Im} z| < \frac{\pi}{2} |z|^2, \ |z| < 1 \right\}$ and $g(z) = \exp\left(-\exp\left(\frac{1}{z}\right)\right)$, again $|g(z)| < 1$ in $\Omega$. This time $1/g(z) \to \infty$ like $\exp\{\exp(c/d(z))\}$ for a suitable $c > 0$ when $z \to 0$ along the real axis.

It is possible to give examples where the growth is even faster.

REMARK 3. We have actually shown that there are constants $C, \alpha > 0$ depending only on $g$ such that if $|f(z)| \leq M\, d(z)^{-\varrho}$ and $f/g$ is holomorphic in $\Omega$, then $\left|\frac{f}{g}(z)\right| \leq MC^{\varrho+1}\, d(z)^{-\varrho-\alpha}$. Somewhat better inequalities can be obtained, but this requires more complicated reasoning.

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