ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 21, nº 2 (1967), p. 137-159

<http://www.numdam.org/item?id=ASNSP_1967_3_21_2_137_0>

GENERALIZED DISTRIBUTION SEMI-GROUPS OF BOUNDED LINEAR OPERATORS

by ERIC LARSSON

Introduction.

Let B be a Banach space and $\mathcal{L}(B)$ the algebra of all bounded linear operators from B to B. Set $R^+ = \{t \in R\,;\, t > 0\}$ and denote by $C_0(R^+)$ the set of all continuous functions with compact support in R^+. An ordinary semi-group of bounded linear operators from B to itself is a mapping L from R^+ to $\mathcal{L}(B)$ satisfying

(1) $$L(t+s) = L(t)\,L(s) \qquad \text{when} \qquad t, s \in R^+\,,$$

and a suitable continuity condition, usually

(2) $$L(t)\,a \longrightarrow L(t_0)\,a \qquad \text{when} \qquad t \longrightarrow t_0 \in R^+ \text{ and } a \in B.$$

To get a natural generalization of these semi-groups we consider the bounded linear operators defined by

$$L(\varphi)\,a = \int_0^\infty \varphi(t)\,L(t)\,a\,dt$$

when $\varphi \in C_0(R^+)$ and $a \in B$. L has the following properties :

$$L(\varphi + \psi) = L(\varphi) + L(\psi) \qquad \varphi, \psi \in C_0(R^+)$$

$$L(c\,\varphi) = c\,L(\varphi) \qquad \varphi \in C_0(R^+),\ c \in \mathbb{C}$$

$$L(\varphi * \psi) = L(\varphi)\,L(\psi) \qquad \varphi, \psi \in C_0(R^+)$$

Pervenuto alla Redazione l'8 Giugno 1966.

and the norm $\| L(\varphi) - L(\varphi_0) \|$ of $L(\varphi) - L(\varphi_0)$ in $\mathcal{L}(B)$ tends to 0 when $\varphi \xrightarrow{\text{uniformly}} \varphi_0$ with the support in a fixed compact subset of R^+. We observe that (1) and (2) correspond to the last two properties.

This leads to the following generalization. Let F be a topological convolution algebra of functions with support in R^+. By a F (distribution) semi-group of bounded linear operators from the Banach space B to itself we mean a mapping L from F to $\mathcal{L}(B)$ such that

$$L(\varphi + \psi) = L(\varphi) + L(\psi) \qquad \varphi, \psi \in F \tag{3}$$

$$L(c\varphi) = c\, L(\varphi) \qquad \varphi \in F,\ c \in \mathbb{C} \tag{4}$$

$$L(\varphi * \psi) = L(\varphi)\, L(\psi) \qquad \varphi, \psi \in F \tag{5}$$

$$\| L(\varphi) - L(\varphi_0) \| \to 0 \qquad \text{when} \qquad \varphi \to \varphi_0 \text{ in } F. \tag{6}$$

We also add the following auxiliary assumption. Let

$$\mathcal{R} = \{a\,;\, a = \sum_k L(\varphi_k)\, a_k, \varphi_k \in F,\ a_k \in B\}$$

and

$$\mathcal{N} = \{a\,;\, L(\varphi)\, a = 0 \quad \text{for every } \varphi \text{ in } F\}\,.$$

Then we assume that

$$\overline{\mathcal{R}} = B \text{ and } \mathcal{N} = \{0\}\,. \tag{7}$$

Distribution semi-groups were first introduced and studied by Lions [1]. His work has been continued in various directions by Foias [1], Peetre [1], [2], Yoshinaga [1], [2] and Da Prato-Mosco [1], [2]. All these authors consider the case $F = \mathcal{D}(R^+)$ — i. e. the space of all infinitely differentiable functions with compact support in R^+ topologized as in Schwartz [1] — and impose suitable growth conditions on the semi-groups at the origin and at infinity. In the present paper we extend some of their results to the case when F is a subspace of $\mathcal{D}(R^+)$ satisfying Gevrey conditions of a given exponent d. In the first section we give a brief presentation of the function spaces, refering to our paper [1]. Mainly following Lions [1] and Peetre [2], we then study different restrictions on the semi-groups at the origin and at infinity. In particular, we prove that a semi-group of ours is of a class σ_p if and only if the resolvent $R(\lambda) = (A - \lambda)^{-1}$ of the generator A of the semi-group satisfies

$$\| R(\lambda) \| \leq C\, |\operatorname{Re} \lambda|^{-1} (1 + |\operatorname{Re} \lambda|^{-1})^p \exp\,(|\lambda| / |\operatorname{Re} \lambda|)^{1/(d-1)}$$

for some constant C when $\operatorname{Re} \lambda > 0$. The paper ends with a section on normal semi-groups. Here we follow Foias [1]. It was professor Peetre who called my attention to this problem. I thank him for his kind interest.

The function spaces.

We give only the definitions and the basic facts. For the proofs and the details we refer to the section on generalized distribution spaces in Larsson [1].

Let $C^\infty(O)$ be the space of all infinitely differentiable functions on the open non-empty set $O \subset R$ and denote by $C_0^\infty(O)$ the sub-space of $C^\infty(O)$ containing all functions with compact support in O. For $d \geq 0$ we consider in $C^\infty(R)$ the quasi-norms

$$| \varphi, K |_{d, m} = \sup_{\substack{k \\ t \in K}} m^{-k} k^{-kd} |\varphi^{(k)}(t)|$$

where $m > 0$ and K is a compact set.

Definition 1. Let $G(d, O)$ be the space

$$\{\varphi ; |\varphi, K|_{d, m} < \infty \text{ for every } m > 0 \text{ and every compact } K \subset O\}$$

with the topology given by the quasi-norms $| \varphi, K |_{d, m}$. Put

$$G_0(d, O) = G(d, O) \cap C_0^\infty(O)$$

topologized as the inductive limit of all

$$G_0(d, K) = \{\varphi ; \varphi \in G(d, O), \text{ supp } \varphi \subset K\}$$

where K is compact in O and $G_0(d, K)$ is equiped with the topology defined by our quasi-norms $|\varphi, K|_{d, m}$. If $O = R^+ = \{t \in R ; t > 0\}$, we often omit R^+ and write $G(d)$ and $G_0(d)$, respectively.

$G(d, O)$ is a Fréchet space and $G_0(d, O)$ contains non-vanishing functions if and only if $d > 1$. In the following we restrict us to that case. The dual spaces of $G(d, O)$ and $G_0(d, O)$ are denoted by $G'(d, O)$ and $G_0'(d, O)$, respectively. We consider them under the strong and the weak topology. The convolution $T * S$ is defined in the natural way and is an element of $G_0'(d, R)$ with supp $T * S \subset \overline{\text{supp } T + \text{supp } S}$ when $T \in G_0'(d, R)$ and $S \in G'(d, R)$. In particular, it belongs to $G(d, R)$ when $S \in G_0(d, R)$. For the Laplace transform of a function in $G_0(d, R)$ we have the following important characterization. We use the notations

$$\widehat{\varphi}(\lambda) = \int \varphi(t)\, e^{\lambda t}\, dt \text{ (the Laplace transform of } \varphi)$$

and

$$|\varphi|_\mu = \int |\widehat{\varphi}(i\eta)| \exp(\mu |\eta|^{1/d})\, d\eta .$$

THEOREM 1. An entire analytic function Φ is the Laplace transform of an element $\varphi \in G_0(d, R)$ if and only if to every $\mu \in R$ there exists a constant C_μ such that

$$|\Phi(\lambda)| = |\Phi(\xi + i\eta)| \leq C_\mu \exp(S(\xi) - \mu |\eta|^{1/d})$$

where S is the support function of φ, defined by $S(\xi) = \sup \{x\,\xi\,;\, x \in \operatorname{supp} \varphi\}$. More precisely, to every compact set $K \subset R$ there is a constant C and to every $\mu \in R$ there exists $m > 0$ such that

$$|\widehat{\varphi}(\lambda)| \leq C\, |\varphi, K|_{d,m} \exp(S(\xi) - \mu |\eta|^{1/d})$$

and

$$|\varphi|_\mu \leq C\, |\varphi, K|_{d,m}$$

when $\varphi \in G_0(d, K)$. Further, to every given $m > 0$ we can find $\mu \in R$ such that

$$|\varphi, K|_{d,m} \leq C\, |\varphi|_\mu$$

when $\varphi \in G_0(d, K)$. Here the constant C is again only depending on K.

This shows that the quasi-norms $|\varphi, K|_{d,m}$ and $|\varphi|_\mu$ define the same topology on $G_0(d, K)$ and by that the same inductive limit on $G_0(d, O)$.

$G_0(d)$ semi-groups.

We define the $G_0(d)$ (distribution) semi-groups by the conditions (3)-(7) in the introduction. Write $\overline{R^+} = \{t \in R\,;\, t \geq 0\}$ and denote by $\overline{G'(d)}$ the space

$$\{T\,;\, T \in G'(d, R),\ \operatorname{supp} T \subset \overline{R^+}\}$$

considered under the strong topology. We observe that $\overline{G'(d)}$ is a convolution algebra and that $G_0(d) = G_0(d, R^+)$ is an ideal of $\overline{G'(d)}$. Further, we consider the sub-space $\overline{G_0(d)}$ of $\overline{G'(d)}$ containing all functions

$$\psi^+(t) = \begin{cases} \psi(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \qquad \text{where } \psi \in G_0(d, R).$$

Let L be a $G_0(d)$ semi-group. According to the definition, $L(\varphi)$ is a bounded operator on B when $\varphi \in G_0(d)$. We shall now generalize and define $L(T)$ for $T \in \overline{G'(d)}$. $L(T)$ will usually be an unbounded but pre-closed and densely defined operator. We follow the method of Peetre [2] and Yoshinaga [1].

DEFINITION 2. Let $T \in \overline{G'(d)}$ and $a \in \mathcal{R}$. Then we set

$$L(T)\,a = \sum_k L(T * \varphi_k)\,a_k$$

where $a = \sum_k L(\varphi_k)\,a_k$ with $\varphi_k \in G_0(d)$ and $a_k \in B$.

It is easily seen that the definition is consistent. $L(T)$ has the following properties.

THEOREM 2. Let $a \in \mathcal{R}$ and $T, S \in \overline{G'(d)}$. Then we have

(9) $$L(T)\,a \in \mathcal{R}$$

(10) $$L(T + S)\,a = L(T)\,a + L(S)\,a$$

(11) $$L(c\,T)\,a = c\,L(T)\,a \qquad c \in \mathbb{C}$$

(12) $$L(T * S)\,a = L(T)\,L(S)\,a$$

(13) $$L(T)\,a \to L(T_0)\,a \quad \text{when} \quad T \to T_0 \text{ in } \overline{G'(d)}.$$

(14) $L(T)$ is pre-closed.

We omit the elementary proof which follows directly from the definition. We only observe that we use $\mathcal{N} = \{0\}$ in the proof of (14).

In the following we let $\overline{L(T)}$ stand for the closure of $L(T)$ and write $L(\delta_t) = L(t)$ where δ_t is the unit mass at the point $t \in \overline{R^+}$. Then, essentially from Theorem 2, we have the following corollary.

COROLLARY. When $a \in \mathcal{R}$, we have

$$L(t)\,a \in \mathcal{R}$$

$$L(t+s)\,a = L(t)\,L(s)\,a$$

$$L(0)\,a = a$$

$$L(\varphi)\,a = \int_0^\infty \varphi(t)\,L(t)\,a\,dt \qquad \text{when} \qquad \varphi \in G_0(d)$$

and the mapping

$$\overline{R^+} \ni t \longrightarrow L(t)\,a \in B$$

is infinitely differentiable.

Regular $G_0(d)$ semi-groups.

Following Lions we now impose a restriction at the origin on our semi-groups. Put

$$G^{(a)}(d) = \{\varphi \in G(d, R);\ \text{supp}\ \varphi \subset (-\infty, a)\}$$

and let

$$G_+(d) = \bigcup_a G^{(a)}(d)$$

be the inductive limit space of all $G^{(a)}(d)$ when these subspaces are topologized by the quasi-norms $|\varphi, K|_{d,m}$.

DEFINITION 3. By a regular $G_0(d)$ semi-group of bounded linear operators from a Banach space B to itself we mean a mapping L from $G_+(d)$ to $\mathcal{L}(B)$ such that

(15) $\qquad L(\varphi + \psi) = L(\varphi) + L(\psi)$

(16) $\qquad L(c\,\varphi) = cL(\varphi) \qquad c \in \mathbb{C}$

(17) $\qquad L(\varphi) = 0 \qquad$ when $\text{supp}\ \varphi \subset R^- = \{t \in R;\ t < 0\}$

(18) $\qquad L(\varphi * \psi) = L(\varphi)\,L(\psi) \qquad$ when $\text{supp}\ \varphi,\ \text{supp}\ \psi \subset \overline{R^+}$

(19) $\qquad L(\varphi)\,a = \int_0^\infty \mu_a(t)\,\varphi(t)\,dt \quad$ for $\ a \in R\ $ where $\ \mu_a(0) = a$

and $\mu_a(t)$ is continuous for $t \geq 0$

(20) $\qquad \| L(\varphi) - L(\varphi_0) \| \longrightarrow 0 \quad$ when $\ \varphi \longrightarrow \varphi_0\ $ in $\ G_+(d)$.

Further, we add the auxiliary assumption:

(21) $$\overline{\mathcal{R}} = B \quad \text{and} \quad \mathcal{N} = \{0\}.$$

When $T \in \overline{G'(d)}$, we define $L(T)$ as in the preceeding section. Theorem 2 remains unchanged. As above, we set

$$\varphi^{+}(t) = \begin{cases} \varphi(t) & \text{when} \quad t \geq 0 \\ 0 & \text{when} \quad t < 0. \end{cases}$$

When $\varphi \in G_{+}(d)$, $\varphi^{+} \in \overline{G'(d)}$. Hence, $\overline{L(\varphi^{+})}$ is a densely defined and closed operator. A regular $G_0(d)$ semi-group can now be characterized in the following alternative way. The observation is new also for $\mathcal{D}(R^{+})$ semi-groups.

THEOREM 3. A $G_0(d)$ semi-group L can be continued to a regular $G_0(d)$ semi-group if and only if $\overline{L(\varphi^{+})}$ is bounded for every $\varphi \in G_{+}(d)$.

PROOF: Suppose that L is a regular semi-group. We prove that $\overline{L(\varphi^{+})} = L(\varphi)$ when $\varphi \in G_{+}(d)$. It is enough to show that $L(\varphi^{+})\,a = L(\varphi)\,a$ for every $a \in \mathcal{R}$. According to (19), there is to every $a \in \mathcal{R}$ a function μ, continuous on $\overline{R^{+}}$, such that

$$L(\varphi)\,a = \int_0^\infty \mu(t)\,\varphi(t)\,dt \quad \text{when} \quad \varphi \in G_{+}(d).$$

Take $\psi \in G_0(d)$ with $\int \psi(t)\,dt = 1$ and define $\psi_s(t) = \frac{1}{s}\,\psi\left(\frac{t}{s}\right)$. We have

$$L(\psi_s)\,L(\varphi^{+})\,a = L(\psi_s * \varphi^{+})\,a = \int_0^\infty \mu(t)\,(\varphi^{+} * \psi_s)(t)\,dt \quad \text{when} \quad a \in \mathcal{R}.$$

Since $\psi_s \to \delta$ when $s \to +0$, (13) of Theorem 2 implies that $L(\psi_s * \varphi^{+})\,a \to$ $\to L(\varphi^{+})\,a$. On the other hand

$$\int_0^\infty \mu(t)\,(\varphi^{+} * \psi_s)(t)\,dt \to \int_0^\infty \mu(t)\,\varphi^{+}(t)\,dt = \int_0^\infty \mu(t)\,\varphi(t)\,dt = L(\varphi)\,a.$$

Hence, $\overline{L(\varphi^{+})} = L(\varphi)$.

For the converse part of the proof assume that L is a $G_0(d)$ semi-group and that $\overline{L(\varphi^+)}$ is bounded for every $\varphi \in G_+(d)$. We set $L'(\varphi) = \overline{L(\varphi^+)}$ when $\varphi \in G_+(d)$. We have to prove that L' is regular. Because of Theorem 2 and its corollary, it only remains to show that L' is a continuous mapping from $G_+(d)$ to $\mathcal{L}(B)$. Let $\varphi_k \longrightarrow \varphi$ in $G^{(b)}(d)$ and $L'(\varphi_k) \longrightarrow D$ in $\mathcal{L}(B)$. Since $\varphi_k^+ \longrightarrow \varphi^+$ in $\overline{G'(d)}$ when $\varphi_k \longrightarrow \varphi$ in $G^{(b)}(d)$, $L'(\varphi_k)\,a = L(\varphi_k^+)\,a \longrightarrow$ $\longrightarrow L(\varphi^+)\,a = L'(\varphi)\,a$ when $a \in \mathcal{R}$. This gives $L'(\varphi)\,a = Da$ for every $a \in \mathcal{R}$. Consequently, $L'(\varphi) = D$ since $\mathcal{R}$ is dense in B. The closed graph theorem now proves that L' is continuous on the Fréchet spaces $G^{(b)}(d)$ and by that on $G_+(d)$.

T smooth $G_0(d)$ semi-groups.

As above, we define $\psi_s(t) = \dfrac{1}{s}\,\psi\left(\dfrac{t}{s}\right)$ when $s > 0$ and $\psi \in G_0(d)$. Let L be a $G_0(d)$ semi-group and $T \in \overline{G'(d)}$. We know that $L(T * \psi_s)\,a$ converges when $a \in \mathcal{R}$ and $s \longrightarrow +0$. We shall now characterize those L for which $\lim\limits_{s \to +0} L(T * \psi_s)\,b$ exists for every $\psi \in G_0(d)$ and every $b \in B$.

DEFINITION 4. A $G_0(d)$ semi-group L is called T smooth if

$$\overline{\lim_{s \to +0}}\, \| L(T * \psi_s)\,b \| < +\infty$$

for every $b \in B$ and every $\psi \in G(d)$.

We shall see that L is regular if L is T smooth for all $T \in \overline{G_0(d)}$. Since we always have $L(\delta)\,a = a$ when $a \in \mathcal{R}$, the δ smooth $G_0(d)$ semi-groups form an especially important class. This case has been studied by Peetre for $\mathcal{D}(R^+)$ semi-groups. We follow essentially him in the proofs.

THEOREM 4. If L is a T smooth $G_0(d)$, semi-group, then $\overline{L(T)}$ is bounded, $\overline{\lim\limits_{s \to +0}}\, \| L(T * \psi_s) \| < +\infty$ and $\lim\limits_{s \to +0} L(T * \psi_s)\,b$ exists for every $b \in B$ when $\psi \in G_0(d)$.

PROOF : $\overline{\lim\limits_{s \to +0}}\, \| L(T * \psi_s) \| < +\infty$ is a consequence of the Banach-Steinhaus theorem on uniform boundedness. $L(T * \psi_s)\,a$ converges when $a \in \mathcal{R}$ and $\psi \in G_0(d)$. Hence $\overline{\lim\limits_{s \to +0}}\, \| L(T * \psi_s) \| < +\infty$ implies that $L(T * \psi_s)\,b$ is a Cauchy filter for every $b \in B$ since $\mathcal{R}$ is dense in B. This gives the existence of $\lim\limits_{s \to +0} L(T * \psi_s)\,b$ for all $b \in B$ and the boundedness of $\overline{L(T)}$. The proof is complete.

THEOREM 5. L is a T smooth $G_0(d)$ semi-group if and only if to every $\tau > 0$ there exist constants C and $m > 0$ such that

$$\| L(T * \psi) \| \leq C \sup_k m^{-k} k^{-kd} \int_0^\tau t^k | \psi^{(k)}(t) | \, dt \tag{22}$$

when $\psi \in G_0(d)$ and $\operatorname{supp} \psi \subset (0, \tau)$.

PROOF: Assume that (22) is valid. Then

$$\| L(T * \psi_s) \| \leq C \sup_k m^{-k} k^{-kd} \int_0^{\tau s} t^k | \psi_s^{(k)}(t) | \, dt =$$

$$= C \sup_k m^{-k} k^{-kd} \int_0^\tau t^k | \psi^{(k)}(t) | \, dt$$

when $\psi \in G_0(d)$, $\operatorname{supp} \psi \subset (0, \tau)$ and $0 < s \leq 1$. Hence, L is a T smooth $G_0(d)$ semi-group.

Suppose on the other hand that L is such a semi-group. According to Theorem 4, we then have $\overline{\lim}_{s \to +0} \| L(T * \psi_s) \| < +\infty$. For simplicity's sake we restrict us to the case $\tau = 1$. Consider the Fréchet space $G_0(d, K)$ where $K = [2^{-1}, 2]$. The sets

$$M_n = \{ \psi \in G_0(d, K) ; \; \| L(T * \psi_s) \| \leq n, \; 0 < s \leq 1 \}$$

are closed and $G_0(d, K) = \bigcup_{n \in N} M_n$ since $\overline{\lim}_{s \to +0} \| L(T * \psi_s) \| < +\infty$. Using Baire's category theorem we get the existence of C and $m > 0$ such that

$$\| L(T * \psi_s) \| \leq C \sup_{t, k} m^{-k} k^{-kd} | \psi^{(k-1)}(t) | \leq C \sup_k m^{-k} k^{-kd} \int_{2^{-1}}^2 | \psi^{(k)}(t) | \, dt$$

when $\psi \in G_0(d)$, $\operatorname{supp} \psi \subset [2^{-1}, 2]$ and $0 < s \leq 1$.

If we apply this inequality to the function $2^{-\nu} \psi(2^{-\nu} t)$, we obtain

$$\| L(T * \psi_s) \| \leq C \sup_k m^{-k} k^{-kd} 2^{-\nu k} \int_{2^{-\nu-1}}^{2^{-\nu+1}} | \psi^{(k)}(t) | \, dt$$

when $\psi \in G_0(d)$, $\operatorname{supp} \psi \subset [2^{-\nu-1}, 2^{-\nu+1}]$ and $0 < s \leq 2^{\nu}$. It is easily proved that there is a sequence $\alpha_\nu \in G_0(d)$, $\nu = 0, \pm 1, \pm 2$, such that

$$\sum_{\nu=-\infty}^{+\infty} \alpha_\nu(t) = 1 \quad \text{when} \quad 0 < t < \infty$$

$$\operatorname{supp} \alpha_\nu = [2^{-\nu-1}, 2^{-\nu+1}]$$

and

$$|\alpha_\nu^{(k)}(t)| \leq C_k\, 2^{\nu k}$$

where $\sup_k l^{-k} k^{-kd} C_k < +\infty$ for every $l > 0$. When $\psi \in G_0(d)$ with $\operatorname{supp} \psi \subset [0, 1]$, we have

$$\|L(T * \psi)\| \leq \sum_{\nu=0}^{\infty} \|L(T * (\psi\, \alpha_\nu))\| \leq C \sum_{\nu=0}^{\infty} \sup_k m^{-k} k^{-kd}\, 2^{-\nu k} \int_{2^{-\nu-1}}^{2^{-\nu+1}} |(\psi\alpha_\nu)^{(k)}(t)|\, dt$$

$$\leq C \sum_{\nu=0}^{\infty} \sum_{k=0}^{\infty} m^{-k} k^{-kd} \sum_{j=0}^{k} \binom{k}{j} C_{k-j}\, 2^j \int_{2^{-\nu-1}}^{2^{-\nu+1}} t^j\, |\psi^{(j)}(t)|\, dt \leq$$

$$\leq C \sum_{k=0}^{\infty} 2^{-2k} \sum_{j=0}^{k} \binom{k}{j} (4m^{-1})^{k-j} (k-j)^{-(k-j)d}\, C_{k-j}\, (8m^{-1})^j j^{-jd}\, 2 \int_0^1 t^j\, |\psi^{(j)}(t)|\, dt.$$

Because of the properties of C_k, we obtain another constant C such that

$$\|L(T * \psi)\| \leq C \sup_j (8m^{-1})^j j^{-jd} \int_0^1 t^j\, |\psi^{(j)}(t)|\, dt.$$

The proof is complete.

We observe the following corollary which we need later for δ smooth $G_0(d)$ semi-groups and which proves that these semi-groups are regular.

COROLLARY. Let L be a T smooth $G_0(d)$ semi-group. Then to every $\tau > 0$ there are constants C and $m > 0$ such that

$$\|\overline{L(T * f)}\| \leq C \sup_k m^{-k} k^{-kd} \int_0^{\tau} t^k\, |f^{(k)}(t)|\, dt$$

when $f \in \overline{G_0(d)}$.

PROOF: Let $f \in \overline{G_0(d)}$ with $\operatorname{supp} f \subset [0, \tau)$. Take $\varrho \in G(d)$ such that

$$\varrho(t) = \begin{cases} 0 & \text{when} \quad t \leq 1 \\ 1 & \text{when} \quad t \geq 2. \end{cases}$$

Consider $\varphi(t) = \varrho\left(\frac{t}{s}\right) f(t)$ in $G_0(d)$. φ tends to f in $\overline{G'(d)}$ when $s \to +0$. Then, according to Theorem 2, $L(T * \varphi)\, a \to L(T * f)\, a$ for every $a \in \mathcal{R}$. We always have $\operatorname{supp} \varphi \subset (0, \tau)$. Hence, Theorem 5 gives constants C and $m > 0$ such that

$$\frac{\| L(T * \varphi)\, a \|}{\| a \|} \leq \| L(T * \varphi) \| \leq C \sup_k m^{-k} k^{-kd} \int_0^\tau t^k \,|\, \varphi^{(k)}(t) \,|\, dt \leq$$

$$\leq C \sup_k m^{-k} k^{-kd} \int_0^\tau t^k \,|\, f^{(k)}(t) \,|\, dt +$$

$$+ C \sup_k m^{-k} k^{-kd} \sum_{j=1}^{k} \binom{k}{j} \int_s^{2s} |\,(t/s)^j\,|\; t^{k-j} \,|\, f^{(k-j)}(t) \,|\; |\, \varrho^{(j)}(t/s) \,|\, dt \leq$$

$$\leq C \sup_k m^{-k} k^{-kd} \int_0^\tau t^k \,|\, f^{(k)}(t) \,|\, dt + D \sup_k (4m^{-})^k k^{-kd} \int_s^{2s} t^k \,|\, f^{(k)}(t) \,|\, dt.$$

Taking $s \to +0$ we get

$$\frac{\| L(T * f)\, a \|}{\| a \|} \leq C \sup m^{-k} k^{-kd} \int_0^\tau t^k \,|\, f^{(k)}(t) \,|\, dt$$

when $a \in \mathcal{R}$. Then $\overline{L(T * f)}$ must be bounded and satisfy the same inequality. The proof is complete.

$G_0(d)$ semi-groups of class σ_p.

We have considered $G_0(d)$ semi-groups restricted in different ways at the origin. In this section we also impose a restriction at infinity on our semi-groups. Following Peetre we make the following definition.

DEFINITION 5. Let $p \geq 0$. A $G_0(d)$ semi-group L is said to be of class σ_p if

$$\sup_{s>0} \| L(\varphi_s) \| (1+s)^{-p} < +\infty \tag{23}$$

when $\varphi \in G_0(d)$.

We observe that the semi-groups of class σ_p are δ smooth and can be characterized in the following way (cf. Theorem 5).

THEOREM 6. A $G_0(d)$ semi-group L is of class σ_p if and only if there are constants C and $m > 0$ such that

$$\| L(\varphi) \| \leq C \sup_k m^{-k} k^{-kd} \int_0^\infty (1+t)^p t^k \, | \varphi^{(k)}(t) | \, dt \tag{24}$$

when $\varphi \in G_0(d)$.

PROOF: (23) is a trivial consequence of (24). For the proof of the converse implication, let $(\alpha_\nu)_{-\infty}^{+\infty}$ be the partition of unity defined in the proof of Theorem 5. Using Baire's theorem as in that proof, we obtain constants C and $m > 0$ such that

$$\| L(\psi_s) \| \leq C(1+s)^p \sup_k m^{-k} k^{-kd} \int_{2^{-1}}^{2} | \psi^{(k)}(t) | \, dt$$

when $\psi \in G_0(d)$ and $\operatorname{supp} \psi \subset [2^{-1}, 2]$.

Again following Theorem 5, we get

$$\| L(\varphi) \| \leq \sum_{-\infty}^{+\infty} \| L(\varphi\alpha_\nu) \| \leq$$

$$\leq C \sum_{\nu=-\infty}^{+\infty} (1+2^{-\nu})^p \sup_k m^{-k} k^{-kd} 2^{-\nu k} \int_{2^{-\nu-1}}^{2^{-\nu+1}} | (\varphi\alpha_\nu)^{(k)}(t) | \, dt \leq$$

$$\leq C \sum_{k=0}^{\infty} 2^{-2k} \sum_{j=0}^{k} (4m^{-1})^{k-j} C_{k-j} (k-j)^{-(k-j)d} \binom{k}{j} (8m^{-1})^j j^{-jd} \, 2 \cdot$$

$$\cdot 3^p \int_0^\infty (1+t)^p t^j \, | \varphi^{(j)}(t) | \, dt \leq D \sup_j (8m^{-1})^j j^{-jd} \int_0^1 (1+t)^p t^j \, | \varphi^{(j)}(t) | \, dt.$$

The proof is complete.

In the same way as we proved the corollary of Theorem 5, we can get the following generalization of Theorem 6.

COROLLARY 1. Let L be of class σ_p. Then there are constants C and $m > 0$ such that $\overline{L(f)}$ is bounded and satisfies

$$\| \overline{L(f)} \| \leq C \sup_k m^{-k} k^{-kd} \int_0^\infty (1+t)^p t^k |f^{(k)}(t)| \, dt$$

for all $f \in \overline{G_0(d)}$.

If L is a $G_0(d)$ semi-group of class σ_p, we can define $\overline{L(f)}$ in some cases even when supp f is not compact. Take $\alpha \in \overline{G_0(d)}$ such that

$$\alpha(t) = \begin{cases} 1 & \text{when} \quad 0 \leq t \leq 1 \\ 0 & \text{when} \quad t \geq 2. \end{cases}$$

COROLLARY 2. Let $f \in G(d, \overline{R^+})$ and satisfy

$$\sup_k m^{-k} k^{-kd} \int_{X_1}^{X_2} (1+t)^p t^k |f^{(k)}(t)| \, dt \to 0$$

when $m > 0$ and $X_1, X_2 \to +\infty$.

Then $\overline{L(s\alpha_s f)}$ converges uniformly to a limit $\overline{L(f)}$ when $s \to +\infty$ and for some constants C and $m > 0$ we have

$$\| \overline{L(f)} \| \leq C \sup_k m^{-k} k^{-kd} \int_0^\infty (1+t)^p t^k |f^{(k)}(t)| \, dt$$

for all such functions f.

PROOF: According to Corollary 1 there are constants C and m such that

$$\| \overline{L(s\alpha_s f)} \| \leq C \sup_k m^{-k} k^{-kd} \int_0^\infty (1+t)^p t^k | s (\alpha_s f)^{(k)}(t) | \, dt \leq$$

$$\leq C \sup_k m^{-k} k^{-kd} \sum_{j=0}^k \binom{k}{j} \int_0^\infty (1+t)^p t^{k-j} |f^{(k-j)}(t)| \, |\alpha^{(j)}(t/s)| \, (t/s)^j \, dt \leq$$

$$\leq C \sup_k m^{-k} k^{-kd} \int_0^\infty (1+t)^p t^k |f^{(k)}(t)| \, dt +$$

$$+ D \, |\alpha, K|_{d, 3^{-1}m} \sup_k (3m^{-1})^k k^{-kd} \int_s^{2s} (1+t)^p t^k |f^{(k)}(t)| \, dt, \text{ where } K = \operatorname{supp} \alpha.$$

If the limit $\overline{L(f)}$ exists, this implies

$$\| \overline{L(f)} \| \leq C \sup_k m^{-k} k^{-kd} \int_0^\infty (1+t)^p t^k | f^{(k)}(t) | \, dt.$$

However, an inequality of the same kind gives that

$$\| L(s_1 \alpha_{s_1} f) - L(s_2 \alpha_{s_2} f) \| \to 0 \qquad \text{when} \qquad s_1, s_2 \to +\infty.$$

Hence, $\overline{L(f)}$ exists since $\mathcal{L}(B)$ is complete. We also observe that the limit $\overline{L(f)}$ is independent of α belonging to $\overline{G_0(d)}$ and satisfying $\alpha = 1$ in a neighborhood of the origin.

We can now give a characterization of our semi-groups L of class σ_p by their generators $\overline{L(-\delta')}$.

THEOREM 7. Let L be a $G_0(d)$ semi-group of class σ_p. Then $R(\lambda) = \overline{L(-e^{-\lambda t})}$ exists and there is a constant C such that

$$\| R(\lambda) \| \leq C | \operatorname{Re} \lambda |^{-1} (1 + 1/| \operatorname{Re} \lambda |)^p \exp (C | \lambda / \operatorname{Re} \lambda |^{1/(d-1)})$$

when $\operatorname{Re} \lambda > 0$. Further, $R(\lambda)$ is the resolvent of $\overline{L(-\delta')}$.

PROOF: When $\operatorname{Re} \lambda > 0$, $e^{-\lambda t}$ satisfies the condition of Corollary 2. Hence, $\overline{L(-e^{-\lambda t})}$ exists for $\operatorname{Re} \lambda > 0$ and

$$\| \overline{L(-e^{-\lambda t})} \| \leq C \sup_k m^{-k} k^{-kd} \int_0^\infty (1+t)^p t^k | \lambda |^k e^{-t \operatorname{Re} \lambda} \, dt \leq$$

$$\leq C | \operatorname{Re} \lambda |^{-1} (1 + 1/| \operatorname{Re} \lambda |)^p \exp C | \lambda / \operatorname{Re} \lambda |^{1/(d-1)} \int_0^\infty (1+t)^p e^{-t \varepsilon_0} \, dt$$

where C and $\varepsilon_0 > 0$. The inequality is established. Then it only remains to prove that $R(\lambda)$ is the resolvent of $\overline{L(-\delta')}$ when $\operatorname{Re} \lambda > 0$. We write $\overline{L(-\delta')} = A$. Take $\psi^+ \in \overline{G_0(d)}$ with $\psi \equiv 1$ in a neighborhood of the origin. For every $a \in D(A)$ there is a sequence $(a_\nu)_1^\infty$ such that $a_\nu \in \mathcal{R}$, $a_\nu \to a$ and

$Aa_\nu \to Aa$ when $\nu \to \infty$. If we set $\psi_{\lambda, s}(t) = -e^{-\lambda t}\psi(st)$, Theorem 2 implies

$$L(\psi^+_{\lambda, s}) L(-\delta') a_\nu = L(-\delta') L(\psi^+_{\lambda, s}) a_\nu =$$

$$= L(-\delta' * \psi^+_{\lambda, s}) a_\nu \quad = L(\delta) a_\nu + \lambda L(\psi^+_{\lambda, s}) a_\nu - sL(\psi'_{\lambda, s}) a_\nu$$

$$= a_\nu + \lambda L(\psi^+_{\lambda, s}) a_\nu - sL(\psi'_{\lambda, s}) a_\nu .$$

It is obvious that $\| sL(\psi'_{\lambda, s}) \| \to 0$ when $s \to +0$. Letting first $\nu \to \infty$ and then $s \to +0$, we get

$$R(\lambda) Aa = a + \lambda R(\lambda) a.$$

Hence, $R(\lambda)(A - \lambda) a = a$ when $a \in D(A)$.

For $a \in \mathcal{R}$ we just found

$$AL(\psi^+_{\lambda, s}) a = a + \lambda L(\psi^+_{\lambda, s}) a - sL(\psi'_{\lambda, s}) a.$$

Let b be an arbitrary element in B and let $(a_\nu)_1^\infty$ be a sequence in $\mathcal{R}$ converging to b. A is closed and

$$L(\psi^+_{\lambda, s}) a_\nu \to \overline{L(\psi^+_{\lambda, s})}\, b,$$

$$AL(\psi^+_{\lambda, s}) a_\nu \to b + \lambda \overline{L(\psi^+_{\lambda, s})}\, b - s\overline{L(\psi'_{\lambda, s})}\, b.$$

Hence, $\overline{L(\psi^+_{\lambda, s})}\, b \in D(A)$ and

$$A\overline{L(\psi^+_{\lambda, s})}\, b = b + \lambda \overline{L(\psi^+_{\lambda, s})}\, b - s\overline{L(\psi'_{\lambda, s})}\, b.$$

Letting $s \to +0$ and again using that A is closed, we get

$$AR(\lambda) b = b + \lambda R(\lambda) b.$$

Hence, $(A - \lambda) R(\lambda) b = b$ for every $b \in B$. The proof is complete.

We now turn to the converse theorem.

THEOREM 8. Let A be a closed and densely defined operator such that for some constant C the resolvent $(A - \lambda)^{-1} = R(\lambda)$ exists and satisfies

$$\| R(\lambda) \| \leq C |\operatorname{Re} \lambda|^{-1} (1 + 1/|\operatorname{Re} \lambda|)^p \exp C |\lambda/\operatorname{Re} \lambda|^{1/(d-1)}$$

when $\operatorname{Re} \lambda > 0$. Then there exists a $G_0(d)$ semi-group L of class σ_p such that $A = \overline{L(-\delta')}$.

Proof: When $\lambda = \xi + i\eta$ satisfies $\xi \geq D \, | \, \eta \, |^{1/d}$ for some fixed $D > 0$, there exists a constant C such that

$$\| R(\lambda) \| \leq C \, | \, \eta \, |^{-\frac{1}{d}} (1 + | \, \eta \, |^{-1/d})^p \exp C \, | \, \eta \, |^{1/d}.$$

Further, according to Theorem 1, we have a constant C with the property that to every $\mu > 0$ there is $m > 0$ satisfying

$$| \, \widehat{\varphi}(\xi + i\eta) \, | \leq C \, | \, \varphi, K \, |_{d, m} \exp (S(\xi) - \mu \, | \, \eta \, |^{1/d})$$

when $\varphi \in G_0(d, K)$. Let $l(r)$ be the curve $\xi = r \, | \, \eta \, |^{1/d}$ for $r > 0$. Then, the analyticity and our estimates of $R(\lambda)$ and $\varphi(\lambda)$ imply that

$$L(\varphi) = - \frac{1}{2\pi i} \int_{l(r)} \widehat{\varphi}(\lambda) \, R(\lambda) \, d\lambda$$

exists as a bounded operator, independent of $r > 0$, and that for some constants C and $m > 0$

$$\| L(\varphi) \| \leq C \, | \, \varphi, K \, |_{d, m}$$

when $\varphi \in G_0(d, K)$. When $\operatorname{supp} \varphi \subset R^-$, we obtain by shifting the integration path to $\xi = r \, | \, \eta \, |^{1/d} + q$ and letting $q \to + \infty$ that

$$L(\varphi) = 0.$$

Further,

$$\| L(\varphi_s) \| = \left\| (2\pi)^{-1} \int_{l(r)} \widehat{\varphi}(s\lambda) \, R(\lambda) \, d\lambda \right\| =$$

$$= \left\| (2\pi)^{-1} s^{-1} \int_{l(r)} \widehat{\varphi}(\lambda) \, R(\lambda / s) \, d\lambda \right\| \leq C(1 + s)^p .$$

In the proof that L is a $G_0(d)$ semi-group of class σ_p it still remains to be checked that $L(\varphi_1 * \varphi_2) = L(\varphi_1) \, L(\varphi_2)$ when $\varphi_1, \varphi_2 \in G_0(d)$, and that $\overline{\mathcal{R}} = B$ and $\mathcal{N} = \{0\}$.

Denote the curves $l(r)$ and $l(r+1)$ by l_1 and l_2, respectively.

$$L(\varphi_1)\,L(\varphi_2) = (2\pi i)^{-2}\int_{l_1}\widehat{\varphi}_1(\lambda)\,R(\lambda)\,d\lambda\int_{l_2}\widehat{\varphi}_2(\zeta)\,R(\zeta)\,d\zeta =$$

$$= (2\pi i)^{-2}\int_{l_1}\int_{l_2}\widehat{\varphi}_1(\lambda)\,\widehat{\varphi}_2(\zeta)\,(A-\lambda)^{-1}(A-\zeta)^{-1}\,d\lambda\,d\zeta.$$

Since $(A-\lambda)^{-1}(A-\zeta)^{-1} = (\lambda-\zeta)^{-1}(A-\lambda)^{-1} + (\zeta-\lambda)^{-1}(A-\zeta)^{-1}$, we obtain

$$L(\varphi_1)\,L(\varphi_2) = (2\pi i)^{-1}\int_{l_1}\left[(2\pi i)^{-1}\int_{l_2}(\lambda-\zeta)^{-1}\widehat{\varphi}_2(\zeta)\,d\zeta\right]\widehat{\varphi}_1(\lambda)\,R(\lambda)\,d\lambda +$$

$$+ (2\pi i)^{-1}\int_{l_2}\left[(2\pi i)^{-1}\int_{l_1}(\zeta-\lambda)^{-1}\widehat{\varphi}_1(\lambda)\,d\lambda\right]\widehat{\varphi}_2(\zeta)\,R(\zeta)\,d\zeta.$$

Because $\varphi_k \in G_0(d)$, there are constants C, $\varepsilon > 0$ and to every $\mu > 0$ a number $m > 0$ such that

$$|\widehat{\varphi}_k(\xi + i\eta)| \leq C\,|\varphi_k, K|_{d,m}\exp(\varepsilon\xi - \mu\,|\eta|^{1/d})$$

when $\xi \leq r\,|\eta|^{1/d}$. This implies that we can deform l_1 and l_2 to circles and get

$$(2\pi i)^{-1}\int_{l_2}(\lambda-\zeta)^{-1}\widehat{\varphi}_2(\zeta)\,d\zeta = -\widehat{\varphi}_2(\lambda) \quad \text{when} \quad \lambda \in l_1,$$

and

$$(2\pi i)^{-1}\int_{l_1}(\zeta-\lambda)^{-1}\widehat{\varphi}_1(\lambda)\,d\lambda = 0 \quad \text{when} \quad \zeta \in l_2.$$

Hence,

$$L(\varphi_1)\,L(\varphi_2) = -(2\pi i)^{-1}\int_{l_1}\widehat{\varphi}_1(\lambda)\,\widehat{\varphi}_2(\lambda)\,R(\lambda)\,d\lambda = L(\varphi_1 * \varphi_2).$$

Let now $\psi \in G_0(d, R)$ with $\psi \equiv 1$ in a neighborhood of the origin. When $\operatorname{Re}\lambda_0 > 0$ and $s > 0$, small enough, we have

$$L(\psi_{\lambda_0, s}) = (2\pi i)^{-1}\int_{l(r)}\widehat{\psi}(\zeta)\,R(\lambda_0 + s\zeta)\,d\zeta =$$

$$= R(\lambda_0) + (2\pi i)^{-1}\,s\int_{l(r)}\zeta\,\widehat{\psi}(\zeta)\,R(\lambda_0 + s\zeta)\,R(\lambda_0)\,d\zeta.$$

This gives that $L(\psi_{\lambda_0,s})$ tends uniformly to $R(\lambda_0)$ when $s \to +0$. Because $(A-\lambda_0)^{-1} B = D(A)$, we obtain that $\bigcup_{s>0} L(\psi_{\lambda_0,s}) B$ is dense in B. Since we also have $L(\varphi) = 0$ when $\varphi \in G_0(d, R^-)$ and $\| L(\varphi(t/s)) \| \to 0$ when $s \to +0$, we get $\overline{\mathcal{R}} = B$. The same argument gives that if $L(\varphi) a = 0$ for all $\varphi \in G_0(d)$, then $L(\psi_{\lambda_0,s}) a = 0$ when $s > 0$. This implies $R(\lambda_0) a = 0$. Hence, $a = 0$.

From Theorem 7 we have that $\lim_{s\to+0} L(\psi_{\lambda,s}) = L(-e^{-\lambda t})$ is the resolvent of $\overline{L(-\delta')}$ when $\operatorname{Re}\lambda > 0$. But we just found that $\lim_{s\to+0} L(\psi_{\lambda,s}) = (A-\lambda)^{-1}$. Hence, $A = \overline{L(-\delta')}$. The proof is complete.

Spectral representation of normal $G_0(d)$ semi-groups.

Here we specialize and consider $G_0(d)$ semi-groups N of normal bounded linear operators from a Hilbert space H to itself. In particular, this means that $N^*(\varphi) N(\varphi) = N(\varphi) N^*(\varphi)$ for all $\varphi \in G_0(d)$. We have $\overline{\mathcal{R}} \perp \mathcal{N}$ and $N(\varphi)\overline{\mathcal{R}} \subset \overline{\mathcal{R}}$ for every $\varphi \in G_0(d)$. Hence, if we restrict us to the Hilbert space $\overline{\mathcal{R}}$, the auxiliary assumption is automatically valid.

Following Foias [1] we shall give a spectral representation of our normal $G_0(d)$ semi-groups. For this we need two lemmas.

LEMMA 1. Let $T \neq 0$ in $G_0'(d)$ satisfy

(25) $$T(\varphi * \psi) = T(\varphi) T(\psi) \text{ when } \varphi, \psi \in G_0(d).$$

Then

$$T(\varphi) = \int_0^\infty e^{\lambda(T)t} \varphi(t)\, dt$$

where $\lambda(T)$ is a complex number. Further, $\lambda(T) \to \lambda(T_0)$ if $T \xrightarrow{\text{weakly}} T_0 \neq 0$.

PROOF : As above, we set $\frac{1}{s}\psi(t/s) = \psi_s(t)$ where $\psi \in G_0(d)$ and $\int \psi(t)\, dt = 1$. Take $\varphi \in G_0(d)$ such that $T(\varphi) \neq 0$. We have $T(-\varphi' * \psi_s) = -T(\varphi * (\psi_s)') = -T((\psi_s)')T(\varphi)$. Since $\varphi' * \psi_s \to \varphi'$ in $G_0(d)$, $T((-\psi_s)')$ converges to a complex number, say $\lambda(T)$, when $s \to +0$. We get $T(-\varphi') = T(\varphi)\lambda(T)$. Then, $T = Ce^{\lambda(T)t}$ for some constant C which is equal to 1 because of (25). At last $T'(\varphi) = \lambda(T) T(\varphi)$ gives directly that $\lambda(T) \to \lambda_0(T)$ when $T(\varphi) \to T_0(\varphi) \neq 0$. The proof is complete.

DEFINITION 6. Let Λ be a set of complex numbers. We define for $\eta \in R$

$$\xi_\Lambda(\eta) = \sup \{\xi ; \xi + i\eta \in \Lambda, \xi \in R\}.$$

We say that Λ is of class d if there exist constants A_Λ and B_Λ such that

$$\xi_\Lambda(\eta) \leq \max \{A_\Lambda, B_\Lambda |\eta|^{1/d}\}.$$

Let $C(\Lambda)$ be the space of all entire functions f satisfying

$$|f|_\Lambda = \sup_{\lambda \in \Lambda} |f(\lambda)| < +\infty.$$

We consider $C(\Lambda)$ under the norm $f \longrightarrow |f|$.

LEMMA 2. Λ is of class d if the mapping

$$G_0(d) \ni \varphi \longrightarrow \widehat{\varphi} \in C(\Lambda)$$

is continuous.

PROOF: The continuity of

$$G_0(d) \ni \varphi \longrightarrow \widehat{\varphi} \in C(\Lambda)$$

implies that to every compact set $K \subset R^+$ there are constants C and μ_0 such that

$$\sup_{\lambda \in \Lambda} |\widehat{\varphi}(\lambda)| \leq C |\varphi|_{\mu_0} \tag{26}$$

when $\varphi \in G_0(d, K)$. Take $\psi \in G_0(d)$ with $\int \psi(t)\, dx = 1$ and $\operatorname{supp} \psi \subset [1, 2]$. Let λ_0 be an arbitrary point in Λ and define

$$\psi_0(t) = \psi(t)\, e^{-\lambda_0 t}.$$

We have $\widehat{\psi_0}(\lambda) = \widehat{\psi}(\lambda - \lambda_0)$. Applying (26) to ψ_0 we get

$$1 \leq C |\psi_0|_{\mu_0} = C \int |\widehat{\psi}(i\eta - \lambda_0)| \exp(\mu_0 |\eta|^{1/d})\, d\eta \leq$$

$$\leq C \exp(\mu_0 |\eta_0|^{1/d}) \int |\widehat{\psi}(i\eta - \xi_0)| \exp(\mu_0 |\eta|^{1/d})\, d\eta$$

since $|\alpha+\beta|^{1/d} \leq |\alpha|^{1/d}+|\beta|^{1/d}$ when $d \geq 1$. ψ belongs to $G_0(d,[1,2])$. Then, according to Theorem 1, there is to every $\mu \in R$ a constant C_μ such that

$$|\widehat{\psi}(i\eta-\xi_0)| \leq C_\mu \exp(-\xi_0-\mu|\eta|^{1/d})$$

when $\xi_0>0$. Combining the last two inequalities, we obtain

$$1 \leq C \exp(-\xi_0+\mu_0|\eta_0|^{1/d})$$

where C is a constant. Hence, for another constant C,

$$\xi_0 \leq \mu_0|\eta_0|^{1/d}+C.$$

Consequently, Λ is of class d and the proof is complete.

We can now formulate a spectral representation theorem for our normal $G_0(d)$ semi-groups.

THEOREM 9. To every normal $G_0(d)$ semi-group N there exists a uniquely determined spectral measure E with the support of class d such that

$$N(\varphi)=\int \widehat{\varphi}(\lambda)\,dE(\lambda)$$

when $\varphi \in G_0(d)$.

PROOF: Apart from some obvious changes where we use the last two lemmas, the proof is identical with the proof of Theorem 1.1 in Foias [1] so we refer to that theorem.

To get a theorem in the opposite direction we prove the following lemma.

LEMMA 3. Let $\Lambda \subset \mathbb{C}$ be of class d. Then, to every $\tau>0$ there exist constants C and $m>0$ such that

$$\sup_{\lambda \in \Lambda}|\widehat{\varphi}(\lambda)| \leq C \sup_k m^{-k}k^{-kd}\int_0^\tau t^k|\varphi^{(k)}(t)|\,dt$$

when $\varphi \in G_0(d,R^+)$ and $\operatorname{supp}\varphi \subset (0,\tau)$.

PROOF: We consider the case $\tau=1$. Let $\varphi \in G_0(d)$ with $\operatorname{supp}\varphi \subset [2^{-1},2]$. According to Theorem 1, there is to every μ a number $m>0$ such that

$$|\widehat{\varphi}(\xi+i\eta)| \leq C \sup_k m^{-k}k^{-kd}|\varphi^{(k-1)}(t)|\exp(S(\xi)-\mu|\eta|^{1/d}) \leq$$

$$\leq C\exp(S(\xi)-\mu|\eta|^{1/d}) \sup_k m^{-k}k^{-kd}\int_{2^{-1}}^{2}|\varphi^{(k)}(t)|\,dt.$$

Here C is a constant, independent of μ, and

$$S(\xi) = \begin{cases} 2\xi & \text{when} \quad \xi \geq 0 \\ 2^{-1}\xi & \text{when} \quad \xi < 0. \end{cases}$$

If $\psi \in G_0(d)$ and $\operatorname{supp} \psi \subset [2^{-\nu-1}, 2^{-\nu+1}]$, we get, using the inequality on $\varphi(t) = 2^{-\nu}\psi(2^{-\nu}t)$, that

$$|\hat{\psi}(\lambda)| \leq C \exp\left(S(2^{-\nu}\xi) - \mu\, |2^{-\nu}\eta|^{1/d}\right) \sup_k m^{-k} k^{-kd} 2^{-\nu k} \int_{2^{-\nu-1}}^{2^{-\nu+1}} |\psi^{(k)}(t)|\, dt.$$

Consider now $\varphi \in G_0(d)$ with $\operatorname{supp} \varphi \subset (0, 1)$. Since Λ is of class d, there are constants μ and C such that $S(\xi) - \mu\, |\eta|^{1/d} \leq C$ when $\lambda = \xi + i\eta \in \Lambda$. Let $(\alpha_\nu)_{-\infty}^{+\infty}$ be the partition of unity used in the proof of Theorem 5. For some constant C we get

$$\sup_{\lambda \in \Lambda} |\hat{\varphi}(\lambda)| = \sup_{\Lambda} |\widehat{\varphi(\Sigma\alpha_\nu)}(\lambda)| \leq \sum_{\nu=0}^{\infty} \sup_{\Lambda} |(\widehat{\varphi\alpha_\nu})(\lambda)| \leq$$

$$\leq C \sum_{\nu=0}^{\infty} \sup_k m^{-k} k^{-kd} 2^{-\nu k} \int_{2^{-\nu-1}}^{2^{-\nu+1}} |\varphi\alpha_\nu)^{(k)}(t)|\, dt.$$

As in the proof of Theorem 5, this implies the existence of still another constant C such that

$$\sup_{\Lambda} |\hat{\varphi}(\lambda)| \leq C \sup_k (8m^{-1})^k k^{-kd} \int_0^1 t^k |\varphi^{(k)}(t)|\, dt.$$

The proof is complete.

We can now prove the converse of Theorem 9.

THEOREM 10. Let E be a spectral measure with $\Lambda = \operatorname{supp} E$ of class d. Then $N(\varphi) = \int \hat{\varphi}(\lambda)\, dE(\lambda)$ is a normal $G_0(d)$ semi-group which is δ smooth.

PROOF: $N(\varphi)$ exists since $\hat{\varphi}$ is bounded and continuous on $\operatorname{supp} E$. N is obviously linear. $N^*(\varphi) = \int \overline{\hat{\varphi}(\lambda)}\, dE(\lambda)$, $N(\varphi * \psi) = \int (\widehat{\varphi * \psi})(\lambda)\, dE(\lambda) = \int \hat{\varphi}(\lambda)\, dE(\lambda) \int \hat{\psi}(\lambda)\, dE(\lambda) = N(\varphi)\, N(\psi)$ and $N(\varphi)\, N^*(\varphi) = N^*(\varphi)\, N(\varphi)$. Set supp $E = \Lambda$. Then, according to Lemma 3,

$$\| N(\varphi) \| \leq \sup_{\Lambda} | \hat{\varphi}(\lambda) | \leq C \sup_{k} m^{-k} k^{-kd} \int_0^{\tau} t^k | \varphi^{(k)}(t) | \, dt$$

when supp $\varphi \subset (0, \tau)$ where C and $m > 0$ depend on τ. Since $\mathfrak{N} = \mathfrak{R}^1$, it now only remains to prove that $\mathfrak{N} = \{0\}$. Take $\varphi \in G_0(d)$ with $\int \varphi(t)\, dt = 1$. We have

$$N(\varphi_s)\, a = \int \hat{\varphi}(s\lambda)\, dE(\lambda)\, a = 0 \text{ for every } s > 0 \text{ when } a \in \mathfrak{N}.$$

$\hat{\varphi}(s\lambda)$ tends pointwise to 1 and is bounded in supp E. This implies that $N(\varphi_s)\, a \xrightarrow{\text{weakly}} a$. Hence, $a = 0$ and the proof is complete.

In particular, we have proved

THEOREM 11. Every normal $G_0(d)$ semi-group is δ smooth.

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