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SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

J. DUGUNDJI

Maps into nerves of closed coverings

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MAPS INTO NERVES OF CLOSED COVERINGS

by J. DUGUNDJI

Maps of spaces into nerves of open coverings, and their uses in homology, homotopy, and dimension theory, are well-known. In this paper, we will construct canonical maps of metric spaces into nerves of closed coverings. Such maps appear to be quite useful: not only do they have properties analogous to those of the maps into nerves of open coverings, but in addition, by using appropriate closed coverings these maps permit a fairly mechanical transfer of dimensional information from the nerve to the space, a matter that is not possible with maps into nerves of open coverings. This process will be illustrated in § 7, where we prove a contraction and a Borsuk-Eilenberg type of duality theorem ([1], [2], [10]) for metric spaces that need not be ANR, essentially by simply verifying the proposition on polytopes. Another application, to a homotopy-type theorem yielding a variant of Helly's theorem, will be given in § 5.

1. Intersection finite coverings.

Let Y be an arbitrary space and let $\mathcal{A} = \{A_\alpha \mid \alpha \in \mathfrak{a}\}$ be a covering of Y by arbitrary sets. We will denote the nerve of this covering by $N(\mathcal{A})$, and the vertex of $N(\mathcal{A})$ corresponding to the set A_α by α. Simplexes of $N(\mathcal{A})$ are denoted by $\sigma, \tau, \ldots$; for each simplex $\sigma = (\alpha_0, \ldots, \alpha_n) \in N(\mathcal{A})$, the family $\{A_{\alpha_0}, \ldots, A_{\alpha_n}\}$ of sets is denoted by $\widehat{\sigma}$ and we write $K(\widehat{\sigma}) = \bigcap_0^n A_{\alpha_i}$.

The covering $\mathcal{A}$ is called star-finite if each closed vertex-star $\overline{St\,\alpha}$ in

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$N(\mathscr{A})$ is a finite complex. We call $\mathscr{A} = \{A_\alpha \mid \alpha \in \mathfrak{g}\}$ intersection-finite (²) if each $\overline{St\,\alpha}$ is finite-dimensional (³); expressed directly in terms of the covering, $\mathscr{A}$ is intersection-finite if for each $\alpha \in \mathfrak{g}$ there exists an integer n_α such that no point of A_α belongs to more than n_α sets.

Let $\mathscr{A} = \{A_\alpha \mid \alpha \in \mathfrak{g}\}$ be an intersection-finite covering of Y. The deficiency $d(\widehat{\sigma})$ of $\widehat{\sigma}$ is $\dim \overline{St\,\sigma} - \dim \sigma$; it is clear that $0 \leq d(\widehat{\sigma}) < \infty$, and that $d(\widehat{\sigma})$ is the maximal number of sets that can be added to the family $\widehat{\sigma}$ and still have a non-empty intersection. The following simple lemma will be useful later:

1.1 LEMMA. Let $\mathscr{A} = \{A_\alpha \mid \alpha \in \mathfrak{g}\}$ be an intersection-finite covering, and let $\widehat{\sigma}$ be given. Let $\widehat{\tau} \neq \widehat{\sigma}$ be such that $K(\widehat{\sigma}) \cap K(\widehat{\tau}) \neq \emptyset$. If $d(\widehat{\sigma}) \geq d(\widehat{\tau})$, then $d(\widehat{\sigma \cup \tau}) < d(\widehat{\sigma})$ and $K(\widehat{\sigma \cup \tau}) = K(\widehat{\sigma}) \cap K(\widehat{\tau})$.

PROOF. Note that $\widehat{\tau}$ cannot be a proper subset of $\widehat{\sigma}$, since this would imply that $d(\widehat{\sigma}) < d(\widehat{\tau})$; because $\widehat{\sigma} \neq \widehat{\tau}$, we conclude that $\widehat{\sigma} \cup \widehat{\tau}$ properly contains $\widehat{\sigma}$ and consequently that $d(\widehat{\sigma \cup \tau}) < d(\widehat{\sigma})$. The rest is obvious.

A family $\mathfrak{B} = \{B_\beta \mid \beta \in \mathcal{B}\}$ of sets in a space Y is called-nbd-finite if each $y \in Y$ has a nbd U such that $\{\beta \mid U \cap B_\beta \neq \emptyset\}$ is finite; it is well-known [4 ; 82] that if $\mathfrak{B}$ is a nbd-finite family of closed sets in Y, then the union of any subfamily is also a closed set. This leads to

1.2 PROPOSITION. Let $\mathscr{F} = \{F_\alpha \mid \alpha \in \mathfrak{g}\}$ be a nbd-finite closed covering of a space Y. Then there exists a pairwise disjoint family $\{V_\alpha \mid \alpha \in \mathfrak{g}\}$ of open sets, indexed by the same $\mathfrak{g}$, such that

(1) $$V_\alpha \subset F_\alpha \text{ for each } \alpha \in \mathfrak{g}$$

(2) $$\{\overline{V}_\alpha \mid \alpha \in \mathfrak{g}\} \text{ is a nbd-finite closed covering of } Y.$$

PROOF. Well-order $\mathfrak{g}$ and use transfinite induction, replacing each F_α successively by $\overline{\operatorname{Int} F_\alpha} \subset F_\alpha$, to verify that $\{\overline{\operatorname{Int} F_\alpha} \mid \alpha \in \mathfrak{g}\}$ is also a nbd-finite covering. The required sets are $V_\alpha = \operatorname{Int} F_\alpha - \cup \{\overline{\operatorname{Int} F_\beta} \mid \beta < \alpha\}$.

(²) The usual terminology, «elementwise uniformly point-finite» is too long.

(³) $\overline{St\,\alpha}$ is not required to be a finite complex. It is clear that each star-finite covering is necessarily intersection-finite, but the converse need not be true, even for open coverings: cover E^1 by the open intervals $]n, n+1[$ and E^1.

A pairwise disjoint family of open sets in Y whose closures form a covering of Y is called a grating (4); thus, 1.2 says that any nbd-finite closed covering has a precise (5) refinement by a nbd-finite grating.

The notions of intersection-finite, star-finite, and nbd-finite coverings of a space Y are easily seen to be independent; although for *open* coverings star-finiteness implies nbd-finiteness, this need not be true for arbitrary (or even closed) coverings. Furthermore, although any covering of order (6) $n < \infty$ is necessarily intersection-finite, it need be neither star-finite nor nbd-finite (even though it be an open covering).

Dowker [3 ; 209] has shown that any nbd-finite open covering of a normal space has a nbd-finite intersection-finite open refinement. It follows from this that

1.3 PROPOSITION. Every open covering of a paracompact space has a refinement by a nbd-finite intersection-finite grating.

PROOF. Letting $\mathcal{V}$ be a nbd-finite intersection-finite refinement of the given $\mathcal{U}$, shrink [4 ; 152] $\mathcal{V}$ to a closed covering $\mathcal{F}$ and apply 1.2.

By Stone's theorem [4 ; 186] every metric space is paracompact, so every open covering of a metric space has a refinement by a nbd-finite intersection-finite grating.

2. Links and traverses.

Let P be a polytope; we consider all polytopes to be rectilinear and with the CW topology [4 ; 172]. We write $\sigma < \tau$ to indicate that σ is a *proper* face of τ, and we denote the barycenter of σ by $[\sigma]$. The first barycentric subdivision P' of P consists of all simplexes $([\sigma_1], \ldots, [\sigma_s])$ such that $\sigma_1 < \ldots < \sigma_s$. For each given $\sigma \in P$, the linked complex $Lk(\sigma)$ of σ is that subcomplex of P' consisting of all simplexes $([\sigma_1], \ldots, [\sigma_s])$ such that $\sigma < < \sigma_1 < \ldots < \sigma_s$, and $Tr(\sigma)$, the closed traverse of σ, is the join $([\sigma], Lk(\sigma))$ in P', which consists of all simplexes $([\sigma], [\sigma_1], \ldots, [\sigma_s])$ where $\sigma < \sigma_1 < \ldots < \sigma_s$, together with all faces of such simplexes. It is evident that

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(4) If $\mathcal{V} = \{V_\alpha \mid \alpha \in \mathfrak{a}\}$ is a grating, we denote the corresponding closed covering $\{\overline{V}_\alpha \mid \alpha \in \mathfrak{a}\}$ by $\overline{\mathcal{V}}$. We call $\mathcal{V}$ nbd-finite (resp. intersection-finite) according as the closed covering $\overline{\mathcal{V}}$ is nbd-finite (resp. intersection-finite). We say that $\mathcal{V}$ refines some covering $\mathcal{U}$ only whenever $\overline{\mathcal{V}}$ refines $\mathcal{U}$.

(5) For «precise», see [4 ; 161].

(6) A covering $\mathcal{A} = \{A_\alpha \mid \alpha \in \mathfrak{a}\}$ has order $\leq n$ if $\dim N(\mathcal{A}) \leq n$.

2.1 If $\sigma < \tau$, then $Tr(\tau) \subset Lk(\sigma) \subset Tr(\sigma) \subset St\,\sigma$.

2.2 $\dim\, Tr(\sigma) = \dim \overline{St\,\sigma} - \dim \sigma$.

Furthermore,

3.3 PROPOSITION. Let P be a polytope, $\dim P \leq n$, and let P^k denote the k-skeleton of P. Then

(1) The subcomplex $B_k = \cup\, \{Tr(\sigma) \mid \dim \sigma \geq k\}$ of P' has dimension $\leq n - k$.

(2) $P^{k-1} \cap B_k = \emptyset$

(3) P^{k-1} is a strong deformation retract of $P - B_k$.

PROOF. (1) and (2) are trivial. Ad (3): Removing the barycenter $[\sigma^n]$ of each n-simplex of P, and projecting from $[\sigma^n]$ onto $Fr(\sigma^n)$, we obtain a strong deformation retraction $r_n : P - B_n \to P^{n-1}$. Next, removing the barycenter of each σ^{n-1} and also the joins $([\sigma^{n-1}], [\sigma^n])$ whenever $\sigma^{n-1} < \sigma^n$ (that is, removing B_{n-1}) we have $r_n(P - B_{n-1}) = P^{n-1} - \cup\, \{[\sigma^{n-1}] \mid \sigma^{n-1} \in P\}$; following r_n by the projection from each $[\sigma^{n-1}]$ onto $Fr(\sigma^{n-1})$ we obtain a strong deformation retraction of $P - B_{n-1}$ onto P^{n-2}. The remaining details of the straightforward induction are omitted.

3. The main theorem.

Let $\mathcal{A} = \{A_\alpha \mid \alpha \in \mathfrak{g}\}$ be any covering of a space Y. We realize the nerve $N(\mathcal{A})$ as a rectilinear polytope in a real vector space with finite topology spanned by unit vectors in a fixed $1-1$ correspondence with the vertices of $N(\mathcal{A})$ and with each vertex placed at the unit point of the corresponding vector [4 ; 171].

3.1 THEOREM. Let Y be a metric space, and let $\mathcal{F} = \{F_\alpha \mid \alpha \in \mathfrak{g}\}$ be a nbd-finite intersection-finite closed covering of Y. Then there exists a continuous map $\lambda : \psi \to N(\mathcal{F})$ such that $\lambda^{-1}(Tr\,\sigma) = K(\widehat{\sigma})$ for each $\sigma \in N(\mathcal{F})$.

PROOF. Because $\mathcal{F}$ is nbd-finite, the family $\{K(\widehat{\sigma}) \mid \sigma \in N(\mathcal{F})\}$ is also a nbd-finite family of closed sets. For each integer $n \geq 0$, let

$$A_n = \cup\, \{K(\widehat{\sigma}) \mid d(\widehat{\sigma}) \leq n\}.$$

Then each A_n is closed in Y, and $A_n \subset A_{n+1}$ for each n. Moreover, because $\mathcal{F}$ is intersection-finite, we have $Y = \bigcup_0^\infty A_n$.

For each $n \geq 0$, we will construct a continuous $\lambda_n : A_n \to N(\mathcal{F})$ such that

(a) $\lambda_n^{-1}(Tr\ \sigma) = K(\widehat{\sigma})$ for each σ with $d(\widehat{\sigma}) \leq n$

(b) $\lambda_n \mid A_{n-1} = \lambda_{n-1}$.

We proceed by induction.

$n = 0$. The family of sets $\{K(\widehat{\sigma}) \mid d(\widehat{\sigma}) = 0\}$ is clearly pairwise disjoint. Furthermore, each $K(\widehat{\sigma})$ with $d(\widehat{\sigma}) = 0$ is both open and closed in A_0: it is evidently closed in A_0 and, because its complement in A_0 is a nbd-finite family of closed sets, its complement in A_0 is also closed. Therefore the map $\lambda_0 : A_0 \to N(\mathcal{F})$ which sends each $K(\widehat{\sigma})$ to $[\sigma]$ is continuous; and (a) is true where as yet (b) has no meaning.

$n = k + 1$. Assume that a continuous $\lambda_k : A_k \to N(\mathcal{F})$ satisfying (a) and (b) has been constructed. Before starting the construction of λ_{k+1} we establish

3.2 Lemma. (1). Let $d(\widehat{\sigma}) = k + 1$. If $\widehat{\tau} \neq \widehat{\sigma}$ and $d(\widehat{\tau}) \leq k + 1$, then $K(\widehat{\sigma}) \cap K(\widehat{\tau}) \subset A_k$. In particular, the family $\{K(\widehat{\sigma}) - A_k \mid d(\widehat{\sigma}) = k + 1\}$ is pairwise disjoint.

(2). Let $d(\widehat{\sigma}) = k + 1$. The $\lambda_k(K(\widehat{\sigma}) \cap A_k) \subset Lk\ \sigma$.

Proof of Lemma. (1) is immediate from 1.1 since, if $K(\widehat{\sigma}) \cap K(\widehat{\tau}) \neq \emptyset$, then $d(\widehat{\sigma} \cup \widehat{\tau}) < d(\widehat{\sigma})$ and $K(\widehat{\sigma}) \cap K(\widehat{\tau}) = K(\widehat{\sigma} \cup \widehat{\tau})$.

(2). Let $y \in K(\widehat{\sigma}) \cap A_k$; then $y \in K(\widehat{\sigma}) \cap K(\widehat{\tau})$ for some τ with $d(\widehat{\tau}) \leq k$. Let $\varrho = \sigma \cup \tau$; then according to 1.1 we have $d(\widehat{\varrho}) \leq k$; since $y \in K(\widehat{\varrho})$ and $\sigma < \varrho$ it follows from the property (a) of λ_k and from 2.1 that $\lambda_k(y) \in Tr(\varrho) \subset \subset Lk(\sigma)$. This proves the lemma.

We now construct λ_{k+1}. We shall first extend λ_k over each $K(\widehat{\sigma})$ for which $d(\widehat{\sigma}) = k + 1$. Let such a $\widehat{\sigma}$ be given; by 3.2 (2) we have $\lambda_k(K(\widehat{\sigma}) \cap A_k) \subset Lk(\sigma)$. Since $Lk(\sigma)$ is a CW complex, it is an ANE for metric spaces [12; 105] so, since $K(\widehat{\sigma}) \cap A_k$ is closed in $K(\widehat{\sigma})$, there exists a nbd U of $K(\widehat{\sigma}) \cap A_k$ in $K(\widehat{\sigma})$ and an extension $g : U \to Lk(\sigma)$ of $\lambda_k \mid K(\widehat{\sigma}) \cap A_k$. We can always choose $U \neq K(\widehat{\sigma})$ and an Urysohn function $c : K(\widehat{\sigma}) \to I$ such that $c^{-1}(1) = K(\widehat{\sigma}) - U$ and $c^{-1}(0) = K(\widehat{\sigma}) \cap A_k$.

Defining $h_\sigma : K(\widehat{\sigma}) \longrightarrow Tr(\sigma)$ by

$$h_\sigma(y) = \begin{cases} [\sigma] & y \in K(\widehat{\sigma}) - U \\ c(y)\cdot[\sigma] + (1 - c(y))\cdot g(y) & y \in U \end{cases}$$

we obtain a continuous extension h_σ of $\lambda_k \mid K(\widehat{\sigma}) \cap A_k$ over $K(\widehat{\sigma})$ which, because $c^{-1}(0) = K(\widehat{\sigma}) \cap A_k$ has the property $h_\sigma^{-1}(Tr\,\sigma - Lk\,\sigma) = K(\widehat{\sigma}) - A_k$.

We now define $\lambda_{k+1} : A_{k+1} \longrightarrow N(\mathcal{F})$ by

$$\lambda_{k+1}(y) = \begin{cases} \lambda_k(y) & y \in A_k \\ h_\sigma(y) & y \in K(\widehat{\sigma}),\ d(\widehat{\sigma}) = k+1. \end{cases}$$

This map is uniquely defined on A_{k+1} since, by 3.2 (1), the values of the h_σ all coincide on any intersection of sets $K(\widehat{\sigma})$. Moreover, because the family $\{K(\widehat{\sigma}) \mid d(\widehat{\sigma}) \leq n+1\}$ is nbd-finite and λ_{k+1} is continuous on each $K(\widehat{\sigma})$, we find [4 ; 83] that λ_{k+1} is continuous on A_{k+1}.

Finally, $\lambda_{k+1}^{-1}(Tr\,\sigma) = K(\widehat{\sigma})$ for each σ with $d(\widehat{\sigma}) \leq k+1$. This is true for all τ with $d(\widehat{\tau}) \leq k$: In fact, let τ be fixed; then for each σ with $d(\widehat{\sigma}) = k+1$, it is impossible that $Tr(\tau) \cap Tr(\sigma)$ contain $[\sigma]$, since then $\tau < \sigma$ and therefore $d(\sigma) \leq d(\tau) \leq k$. Consequently, for each σ with $d(\widehat{\sigma}) = k+1$, the intersection $Tr(\sigma) \cap Tr(\tau)$ is either empty or $\subset Lk(\sigma)$; and since no point of point $K(\sigma) - A_k$ is mapped by h_σ into $Lk(\sigma)$, it follows that $\lambda_{k+1}^{-1}(Tr\,\tau) = \lambda_k^{-1}(Tr\,\tau) = K(\widehat{\tau})$. In the remaining case, that $d(\widehat{\sigma}) = k+1$, we have $\lambda_{k+1}^{-1}(Tr\,\sigma) = h_\sigma^{-1}(Tr\,\sigma) = K(\widehat{\sigma})$ by the construction, and the inductive step is complete.

To complete the proof, let $\lambda : Y \longrightarrow N(\mathcal{F})$ be defined by

$$\lambda(y) = \lambda_n(y) \quad \text{if} \quad y \in A_n.$$

Because $\lambda_n \mid A_n = \lambda_{n+i} \mid A_n$ for all $i > 0$ and all n, λ is uniquely defined on $\bigcup_0^\infty A_n = Y$. It is continuous on Y: each $y \in Y$ has a nbd meeting at most finitely many $K(\widehat{\sigma})$, so it has a nbd lying on some A_n and $\lambda = \lambda_n$ is continuous on A_n. Furthermore, given any $\sigma \in N(\mathcal{F})$ we have $d(\widehat{\sigma}) = k$ for some k, and therefore by the property (a) of the λ_k, that $\lambda^{-1}(Tr\,\sigma) = \lambda_k^{-1}(Tr\,\sigma) = K(\widehat{\sigma})$. This completes the proof of the theorem.

Observe that

3.3 If $y \in F_{\alpha_1} \cap \dots \cap F_{\alpha_k}$ and only these sets, then

$$\lambda(y) \in Tr(\alpha_1, \dots, \alpha_k) - Lk(\alpha_1, \dots, \alpha_k).$$

For, let $\sigma = (\alpha_1, \dots, \alpha_k)$ and let $d(\widehat{\sigma}) = r$; since y does not belong to any more sets F_α, we must have $y \in K(\widehat{\sigma}) - A_{r-1}$ and therefore that

$$\lambda(y) = \lambda_r(y) = h_\sigma(y) \in Tr(\alpha_1, \dots, \alpha_k) - Lk(\alpha_1, \dots, \alpha_k).$$

If $\mathcal{F}$ is a nbd-finite intersection-finite closed covering of a metric space Y, we call a map $\lambda : Y \longrightarrow N(\mathcal{F})$ such that $\lambda^{-1}(Tr\,\sigma) = K(\widehat{\sigma})$ for each $\sigma \in N(\mathcal{F})$, a *canonical map.* Despite the arbitrariness in the above construction of λ, the homotopy class of a canonical map depends only on the covering:

3.4 THEOREM. Let Y be a metric space and $\mathcal{F}$ an intersection-finite nbd-finite closed covering. Then any two canonical maps $\lambda_0, \lambda_1 : Y \longrightarrow N(\mathcal{F})$ are homotopic.

PROOF. Let $y \in Y$ be given; then $y \in K(\widehat{\sigma})$ for some $\widehat{\sigma}$ and therefore

$$\lambda_0(y), \lambda_1(y) \in Tr(\sigma) \subset St\,\sigma \subset St\,p, \qquad p \text{ a vertex of } \sigma.$$

Thus, for each y the points $\lambda_0(y)$ and $\lambda_1(y)$ belong to a common open vertex-star in $N(\mathcal{F})$ and therefore [8; 214] $\lambda_0 \simeq \lambda_1$.

4. Relation to maps into polytopes.

Let P be a polytope and $f : Y \longrightarrow P$ continuous. If $\mathcal{A} = \{A_\alpha \mid \alpha \in \mathfrak{g}\}$ is any covering of Y that refines $\{f^{-1}(St\,p) \mid p \in P^0\}$, define a map $N(\mathcal{A})^0 \longrightarrow P^0$ by assigning to each vertex $\alpha \in N(\mathcal{A})^0$ a definite $p \in P^0$ such that $A_\alpha \subset f^{-1}(St\,p)$. This map is easily verified to send the vertices of a simplex of $N(\mathcal{A})$ to the vertices of a simplex in P; by extending linearly over each simplex of $N(\mathcal{A})$ we obtain a simplicial map $j : N(\mathcal{A}) \longrightarrow P$ called a projection. Any projection $j : N(\mathcal{A}) \longrightarrow P$ is continuous, because it is so on each closed simplex, and any two projections $N(\mathcal{A}) \longrightarrow P$ are homotopic [11; 235].

The Kuratowski maps (7) of a metric space Y into nerves $N(\mathcal{U})$ of nbd-finite open coverings have the following well-known property: for any polytope P, any continuous $f: Y \longrightarrow P$ homotopy factors through a suitable Kuratowski map $K: Y \longrightarrow N(\mathcal{U})$. The canonical maps of § 3 also have this property and, moreover, the closed coverings are very « orderly »:

4.1 THEOREM. Let Y be a metric space, let P be a polytope, and let $f: Y \longrightarrow P$ be continuous. Let $\mathcal{V} = \{V_\alpha \mid \alpha \in \mathfrak{g}\}$ be an intersection-finite nbd-finite grating that refines $\{f^{-1}(St\,p) \mid p \in P^0\}$, let $\lambda: Y \longrightarrow N(\overline{\mathcal{V}})$ be a canonical map, and let $j: N(\overline{\mathcal{V}}) \longrightarrow P$ be a projection. Then in the diagram

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "\lambda"'] & & P \\ & N(\overline{\mathcal{V}}) \arrow[ur, "j"'] & \end{tikzcd}$$

we have $f \simeq j \circ \lambda$.

PROOF. Let $y \in \overline{V}_{\alpha_1} \cap \ldots \cap \overline{V}_{\alpha_n}$ and only these sets; by 3.3 and 2.1 we find $\lambda(y) \in St\,\alpha_1$ and so $j \circ \lambda(y) \in St\,p$ for some vertex p such that $\overline{V}_{\alpha_1} \subset f^{-1}(St\,p)$ and therefore $f(y)$ also belongs to $St\,p$. Since $f(y)$ and $j \circ \lambda(y)$ belong to a common open vertex-star for each $y \in Y$, the maps f and $j \circ \lambda$ are homotopic. This completes the proof.

4.2 COROLLARY. If a metric space Y is dominated by a polytope, then it is dominated by the nerve $N(\overline{\mathcal{V}})$ of a grating $\mathcal{V}$ and, moreover, the canonical map $\lambda: Y \longrightarrow N(\overline{\mathcal{V}})$ has a left homotopy inverse.

PROOF. By 1.3 and 4.1 we have a grating $\overline{\mathcal{V}}$ and a diagram

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "\lambda"'] & & P \\ & N(\overline{\mathcal{V}}) \arrow[ur, "j"'] & \end{tikzcd}$$

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(7) If $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathfrak{g}\}$ is an open covering, we call Kuratowski maps $K: Y \to N(\mathcal{U})$ those continuous maps such that $K^{-1}(St\,\alpha) \subset U_\alpha$ for each $\alpha \in \mathfrak{g}$.

with $f \simeq j \circ \lambda$; by hypothesis there exists a $g: P \rightarrow Y$ such that $g \cdot f \simeq 1$ and therefore $(g \circ j) \circ \lambda \simeq 1$. This completes the proof.

Using 4.1, it is easy to see that each Kuratowski map $K: Y \rightarrow N(\mathcal{U})$ is homotopic to a map $\widehat{K}$ such that $\widehat{K}^{-1}(Tr\,\sigma) \subset K(\widehat{\sigma})$ for each $\sigma \in N(\mathcal{U})$.

5. Application to homotopy type and to Helly's theorem.

5.1 Theorem. Let Y be a metric space and $\mathcal{F} = \{F_\alpha \mid \alpha \in \mathfrak{g}\}$ a nbd-finite intersection-finite closed covering of Y. Assume that each $K(\widehat{\sigma})$ is $[d(\widehat{\sigma}) - 1]$-connected. Then Y dominates $N(\mathcal{F})$: in fact, each canonical map $\lambda: Y \rightarrow N(\mathcal{F})$ has a right homotopy inverse $g: N(\mathcal{F}) \rightarrow Y$ such that $g \circ \lambda$ is $\mathcal{F}$-close [8] to the identity map of Y.

Proof. Let $K^n = \cup \{Tr\,\sigma \mid d(\widehat{\sigma}) \leq n\}$; then each K^n is a closed subcomplex of $N(\mathcal{F})'$, $K^n \subset K^{n+1}$ for each n, and $\bigcup\limits_0^\infty K^n = N(\mathcal{F})$. We shall construct a continuous $g: N(\mathcal{F}) \rightarrow Y$ such that $g(Tr\,\sigma) \subset K(\widehat{\sigma})$ for each $\sigma \in N(\mathcal{F})$.

Define g^0 on the set of all vertices Q of $N(\mathcal{F})'$ by sending each $[\sigma]$ to any point of the corresponding $K(\widehat{\sigma})$. Clearly, $Q \supset K^0$ and we shall extend g^0 over $Q \cup K^n$ by induction on n.

Assume g^0 has been extended to a map g^n, $Q \cup K^n \rightarrow Y$ satisfying $g^n(Tr\,\sigma) \subset K(\widehat{\sigma})$ for each σ such that $d(\widehat{\sigma}) \leq n$. Let σ be any simplex with $d(\widehat{\upsilon}) = n + 1$; then g^n is defined on $Lk(\sigma)$: indeed, if $\tau \varepsilon Lk(\sigma)$, then τ is in the traverse of some σ_0 with $\sigma < \sigma_0$, consequently $d(\widehat{\sigma_0}) < d(\widehat{\sigma}) = n + 1$; furthermore, observing that g^n maps τ into $K(\widehat{\sigma_0}) \subset K(\widehat{\sigma})$, we find that $g^n\, Lk(\sigma) \subseteq K(\widehat{\sigma})$ as asserted. Now, according to 2.2, we have $\dim(Tr\,\sigma - Lk\,\sigma) = d(\widehat{\sigma})$ and according to hypothesis, $\pi_i(K(\widehat{\sigma})) = 0$ for $0 \leq i \leq d(\widehat{\sigma}) - 1$; therefore [9; 241] $g^n \mid Lk\,\sigma$ extends over $Tr(\sigma)$ with values in $K(\widehat{\sigma})$. Because each intersection $Tr(\sigma) \cap Tr(\sigma') \subset K^n$ whenever $d(\widehat{\sigma}) = d(\widehat{\sigma'}) = n + 1$, the piecewise extension of g^n over each $Tr(\sigma)$, $d(\widehat{\sigma}) = n + 1$, results in a continuous $g^{n+1}: Q \cup K^{n+1} \rightarrow Y$ such that $g^{n+1}(Tr\,\sigma) \subset K(\widehat{\sigma})$ whenever $d(\widehat{\sigma}) \leq n + 1$, and the inductive step is complete. The desired map $g: N(\mathcal{F}) \rightarrow Y$ is defined by setting $g(x) = g^n(x)$ whenever $x \in K^n$.

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(8) If $\mathcal{A} = \{A_\alpha \mid \alpha \in \mathfrak{g}\}$ is a covering of Y, two maps f, g of a space X into Y are called $\mathcal{A}$-close if for each $x \in X$ there is some A_α such that $f(x), g(x) \in A_\alpha$.

Now let $\lambda : Y \longrightarrow N(\mathcal{F})$ be a canonical map. We have

$$\lambda \circ g(Tr\,\sigma) \subset \lambda(K(\widehat{\sigma})) \subset Tr\,\sigma$$

for each σ, which shows that for each $x \in N(\mathcal{F})$, the points x and $\lambda \circ g(x)$ belong to a common open vertex-star of $N(\mathcal{F})$ and therefore that $\lambda \circ g \simeq 1$.

Finally, $g \circ \lambda(K(\widehat{\sigma})) \subset g(Tr\,\sigma) \subset K(\widehat{\sigma})$ for every $\sigma \in N(\mathcal{F})$, consequently $g \circ \lambda$ and 1_Y are $\mathcal{F}$-close. This completes the proof.

We apply 5.1 to obtain an extension of Helly's theorem :

5.2 Let $\mathcal{F} = \{F_1, \dots, F_k\}$ be k closed sets in a metric space Y such that each family of $(k-1)$ of these sets has a non-empty intersection. If for each $j \geq 0$ the intersection of every j of these sets is $(k-j-2)$-connected, and if [9] $H_{k-2}(\bigcup_1^k F_i)$ is a torsion group (or 0), then $\bigcap_1^k F_i \neq \varnothing$.

PROOF. Assume $F_1 \cap \dots \cap F_k = \varnothing$; then $N(\mathcal{F})$ is homeomorphic to $Fr\,(\sigma^{k-1})$ consequently $H_{k-2}(N) = Z$. Furthermore, for each $\sigma = (n_1, \dots$ $\dots, n_j) \in N(\mathcal{F})$ we have $d(\widehat{\sigma}) = k - 1 - j$ so it follows from 5.1 that $\bigcup_1^k F_i$ dominates $N(\mathcal{F})$. This implies that $H_{k-2}(\bigcup_1^k F_i)$ has a direct summand Z, which is a contradiction and the proof is complete.

In the case of E^2 and four closed sets $F_i \subset E^2$, this recovers the known result [14 ; 109] that if each F_i is 1-connected and each $F_i \cap F_j$ is 0-connected, then whenever every three of the F_i have non-empty intersection, so also will all four ; 5.2 indicates a generalization to $(n+2)$ closed sets in E^n, $n > 2$.

6. Finite-dimensional metric spaces.

In case Y is a finite-dimensional metric space, the use of suitable closed coverings (in fact, gratings) gives a simple relationship (6.4) between the dimension of certain subsets of and the dimension of their inverse images under a canonical map. We start with Morita's result [15 ; 36], [16 ; 68,78] that

6.1 Let C, X be metric spaces, and use the covering definition of dimension. Then

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(9) $H_n(X)$ denotes the singular integral n-homology group of X ; Z is the infinite cyclic group.

(a) If $p : C \to X$ is a continuous closed surjection, and if $\aleph(p^{-1}(x)) \leq n+1$ for each $x \in X$, then $\dim X \leq n + \dim C$.

(b) $\dim X \leq n$ if and only if there exists a 0-dimensional space C and a continuous closed surjection $p : C \to X$ such that $\aleph(p^{-1}(x)) \leq n+1$ for each $x \in X$.

The following lemma is known; a proof for separable metric spaces is given in [13; 59] and, for convenience, we give here a proof valid for arbitrary metric spaces.

6.2 Lemma. Let X be a metric space, $\dim X \leq n$, and let $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathfrak{g}\}$ be any nbd-finite open covering of X. Then there exists a closed covering $\{F_\alpha \mid \alpha \in \mathfrak{g}\}$ such that

(1) $F_\alpha \subset U_\alpha$ for each $\alpha \in \mathfrak{g}$

(2) $\{F_\alpha \mid \alpha \in \mathfrak{g}\}$ is nbd-finite and intersection-finite.

(3) $\dim (F_{\alpha_1} \cap \cdots \cap F_{\alpha_k}) \leq n - k + 1$ for every k sets and every $k > 0$.

Proof. Choose C and $p : C \to X$ satisfying 6.1 (b). Since $\{p^{-1}(U_\alpha') \mid \alpha \in \mathfrak{g}\}$ is an open covering of the 0-dimensional C, it has an open refinement $\mathcal{V} = \{V_\beta \mid \beta \in \mathcal{B}\}$ where $\mathcal{V}$ is a pairwise disjoint family of open sets covering C.

Define $\varphi : \mathcal{B} \to \mathfrak{g}$ by associating with each $\beta \in \mathcal{B}$ any $\alpha \in \mathfrak{g}$ such that $V_\beta \subset p^{-1}(U_\alpha)$, and for each $\alpha \in \mathfrak{g}$, let $W_\alpha = \cup \{V_\beta \mid \varphi(\beta) = \alpha\}$; then $\mathcal{W} = \{W_\alpha \mid \alpha \in \mathfrak{g}\}$ is also a pairwise disjoint open covering of C, and each $W_\alpha \subset p^{-1}(U_\alpha)$.

Because $\mathcal{W}$ is a pairwise disjoint open covering, each W_α is also closed in C and, since p is a closed map, $p(W_\alpha)$ is closed in X; the closed covering $p(\mathcal{W}) = \{p(W_\alpha) \mid \alpha \in \mathfrak{g}\}$ is a precise refinement of $\mathcal{U}$ and, since $\mathcal{U}$ is nbd-finite, so also is $p(\mathcal{W})$. We now show that $p(\mathcal{W})$ is the required covering.

Let $B = p(W_{\alpha_1}) \cap \cdots \cap p(W_{\alpha_k})$; since $W_{\alpha_1} \cap p^{-1}(B)$ is closed in C and $\widehat{p} = p \mid W_{\alpha_1} \cap p^{-1}(B) : W_{\alpha_1} \cap p^{-1}(B) \to B$ is surjective, we find that $\widehat{p}$ is a continuous closed surjection. Because p satisfies 6.1 (b), for each $b \in B$ there are at most $(n+1)$ points in C such that $p(c_1) = \ldots = p(c_{n+1}) = b$. Since $b \in p(W_{\alpha_s})$, $s = 1, \ldots, k$ there is at least one point in each W_{α_s} mapping into b; because the W_{α_s} are pairwise disjoint and there are already at least $(k-1)$ points from outside W_{α_1} mapping to b, there can be no more than $(n+1) - (k-1)$ points in W_{α_1} mapped by p (and therefore also $\widehat{p}$) to b. Thus, for each $b \in B$, we have $\aleph(\widehat{p}^{-1}(b)) \leq n - k + 2$ so, by 6.1 (a) we find $\dim B \leq n - k + 1$. Since the intersection-finiteness of $p(\mathcal{W})$ is a consequence of the property in (3), the proof is complete.

6.3 THEOREM. Let X be a metric space, $\dim X \leq n$, and let $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathfrak{g}\}$ be any open covering. Then there exists a grating $\mathcal{V} = \{V_\alpha \mid \alpha \in \mathfrak{g}\}$ such that

(1) $\overline{V}_\alpha \subset U_\alpha$ for each $\alpha \in \mathfrak{g}$

(2) $\overline{\mathcal{V}}$ is a nbd-finite closed cover of order $\leq n$ (and therefore also intersection-finite).

(3) $\dim (\overline{V}_{\alpha_1} \cap \cdots \cap \overline{V}_{\alpha_k}) \leq n - k + 1$ for all k sets and every $k > 0$.

PROOF. It is well-known [16 ; 12] that $\mathcal{U}$ has a precise nbd-finite open refinement $\mathcal{W} = \{W_\alpha \mid \alpha \in \mathfrak{g}\}$ of order $\leq n$. Shrink $\mathcal{W}$ to an open covering $\mathcal{L} = \{L_\alpha \mid \alpha \in \mathfrak{g}\}$ such that $\overline{L}_\alpha \subset W_\alpha$ for each α, and let $\mathcal{F}$ be a refinement of $\mathcal{L}$ satisfying 6.2. Finally, apply 1.2 to find a nbd-finite grating $\overline{\mathcal{V}}$ that refines $\mathcal{F}$. It is evident that $\mathcal{V}$ is the required grating.

If $\dim X \leq n$, we call a nbd-finite grating $\mathcal{V} = \{V_\alpha \mid \alpha \in \mathfrak{g}\}$ on X a standard grating of order $\leq n$ whenever $\dim (\overline{V}_{\alpha_1} \cap \cdots \cap \overline{V}_{\alpha_k}) \leq n - k + 1$ for every k sets and every $k > 0$; clearly, this implies that $\dim N(\overline{\mathcal{V}}) \leq n$. With this terminology, 6.3 says that every open covering of X has a standard grating of order $\leq n$ as refinement.

6.4 THEOREM. Let X be a metric space, $\dim X \leq n$, let $\mathcal{V} = \{V_\alpha \mid \alpha \in \mathfrak{g}\}$ be a standard grating of order $\leq n$, and let $\lambda : X \to N(\overline{\mathcal{V}})$ be a canonical map. Let $B_k = \bigcup \{Tr\,(\sigma) \mid \dim \sigma \geq k\}$. Then $\dim B_k \leq n - k$ and also $\dim \lambda^{-1}(B_k) \leq n - k$.

PROOF. We have already seen (2.3) that $\dim B_k \leq n - k$. Now

$$\lambda^{-1}(B_k) = \bigcup \{\lambda^{-1}(Tr\,\sigma) \mid \dim \sigma \geq k\} = \bigcup \{K(\widehat{\sigma}) \mid \dim \sigma \geq k\}$$

and each $K(\widehat{\sigma})$, being the intersection of at least $(k+1)$ sets, has dimension $\leq n - (k+1) + 1$; moreover, because $\overline{\mathcal{V}}$ is nbd-finite, so also is the family $\{K(\widehat{\sigma}) \mid \dim \sigma \geq k\}$ of closed sets, so by the Sum Theorem of dimension theory [16 ; 17] we conclude that

$$\dim \bigcup \{K(\widehat{\sigma}) \mid \dim \sigma \geq k\} \leq \max \dim K(\widehat{\sigma}) \leq n - k$$

and the proof is complete.

The fact that in 6.4 the dimension of the inverse image of B_k does not exceed that of B_k itself is important for the applications that follow, and is one reason for the usefulness of canonical maps.

7. Applications.

Let Y be a metric space dominated by a polytope, $\dim Y \leq n$. It is well-known (and trivial to prove) that if $\pi_i(Y) = 0$ for $0 \leq i \leq n$, then Y is contractible. The following is a generalization:

7.1 THEOREM. Let Y be a metric space dominated by a polytope, $\dim Y \leq n$.

Assume that $\pi_i(Y) = 0$ for $0 \leq i \leq k$. Then there exists a closed $E \subset Y$ with $\dim E \leq n - k - 1$, such that $Y - E$ is contractible in Y.

PROOF. By 6.3, 4.1 and 4.2, it follows that Y is dominated by the nerve of a standard grating $N(\overline{\mathcal{V}})$ of order $\leq n$ and that a canonical $\lambda : Y \rightarrow N(\overline{\mathcal{V}})$ has a left homotopy inverse $g : N(\overline{\mathcal{V}}) \rightarrow Y$. Let $B_{k+1} = \cup \{Tr(\sigma) \mid \dim \sigma \geq k+1\}$; by 2.3 there is a strong deformation retraction Δ_t of $N - B_{k+1}$ onto N^k, where $\Delta_0 = 1 \mid N - B_{k+1}$. Let $E = \lambda^{-1}(B_{k+1})$; then by 6.4 we have $\dim E \leq n - k - 1$ and also $\widehat{\lambda} = \lambda \mid Y - E$: $Y - E \rightarrow N - B_{k+1}$. Consider now the homotopy

$$h_t = g \circ \Delta_t \circ \widehat{\lambda} : (Y - E) \times I \rightarrow Y$$

For $t = 0$, we have $h_0 = g \circ \widehat{\lambda} = g \circ \lambda \mid Y - E$ which is homotopic to $1 \mid Y - E$. For $t = 1$, we have $h_1 = (g \mid N^k) \circ \Delta_1 \circ \widehat{\lambda}$; since $\pi_i(Y) = 0$ for $0 \leq i \leq k$, it follows that $g \mid N^k \simeq 0$ and consequently that $h_1 \simeq 0$. Thus, $0 \simeq 1 \mid Y - E$ and the theorem is proved.

Theorem 7.1. had previously been known only for separable metric spaces that are ANR; since every such ANR is dominated by a polytope [7 ; 243] our version contains the known results, and shows that it is the domination property, rather than the ANR property, that is involved in this matter.

The method used to prove 7.1 shows that by using standard gratings and canonical maps, many properties easily verifiable on polytopes can be readily transferred to spaces dominated by polytopes. As another application of this technique, we will obtain a Borsuk-Eilenberg type of duality theorem for maps into metric spaces that are dominated by polytopes, rather than for maps into ANR. To do this, we remark that, because of 6.3, the procedure in [5 ; 356] [12 ; 53] can be used to show.

7.2 Let X be a metric space, $A \subset X$ closed, and $\dim (X - A) \leq n$. Then there exists a space Z and a continuous map $\mu : X \to Z$ such that

(a) $\mu \mid A$ is a homeomorphism of A onto the closed subset $\mu(A) \subset Z$

(b) The open subspace $Z - \mu(A)$ is the nerve $N(\mathcal{F})$ of a standard grating of order $\leq n$ on $X - A$, and $\mu \mid X - A$ is a canonical map $\lambda : X - A \to N(\mathcal{F})$.

(c) Every nbd of a boundary point of $\mu(A)$ contains infinitely many closed simplexes of $N(\mathcal{F})$.

We identify A with $\mu(A)$ and denote the space Z by $A \cup N(\mathcal{F})$; as in [7 ; 232] and [6 ; 10] it follows that

7.3 Let f be a continuous map of $A \subset A \cup N(\mathcal{F})$ into a space Y. If either

(a) Y is a polytope, or a metric ANR, or

(b) Y is (10) LC^{n-1} and $\dim N(\mathcal{F}) \leq n$

then f can be extended over a nbd $U \supset A$ in $A \cup N(\mathcal{F})$; furthermore U can be taken such that if Q is the union of all closed simplexes of $N(\mathcal{F})$ contained in U, then no point of A is a limit point of $N(\mathcal{F}) - Q$.

We now establish the « duality » theorem :

7.4 Theorem. Let X be a metric space, $A \subset X$ a closed subset, and $\dim (X - A) \leq n$. Let Y be a metric space dominated by a polytope, and assume that $\pi_i(Y) = 0$ for $0 \leq i \leq k$. Then for each continuous $f : A \to Y$, there exists a closed $E \subset X - A$ with $\dim E \leq n - k - 2$, and a map $g \simeq f$ such that g is extendable over $X - E$.

Proof. Choose a polytope P and maps $K : Y \to P$, $\varphi : P \to Y$ such that $\varphi \circ K \simeq 1$. Consider the map $K \circ f \circ \mu^{-1}$ of $A \subset A \cup N(\mathcal{F})$ into P; by 7.3 (a), $K \circ f \circ \mu^{-1}$ has an extension $h : U \to P$ defined on some nbd $U \supset A$ in $A \cup N(\mathcal{F})$.

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(10) A metric space Y is LC^k if for each $y \in Y$ and nbd U containing y, there exists a nbd $V, y \in V \subset U$ such that for each $i \leq k$, every continuous $f : S^i \to V$ is nullhomotopic in U. Every ANR is LC^k for all $k \geq 0$ [7; 239].

Define $\widehat{h} = \varphi \circ h$; then $\widehat{h} \mid A \simeq f$. Let Q be the closed subpolytope described in 7.3; since $\pi_i(Y) = 0$ for $0 \leq i \leq k$, we can extend $\widehat{h} \mid Q$ over $Q \cup N(\mathcal{F})^{k+1}$ and, because of 2.3 (c), there is an extension of $\widehat{h} \mid Q$ to an $\widehat{H}: Q \cup [N(\mathcal{F}) - B_{k+2}] \longrightarrow Y$. The map $\widehat{H}$ together with $\widehat{h} \mid A$ determines a continuous map G of $A \cup N(\mathcal{F}) - (N(\mathcal{F}) - Q) \cap B_{k+2}$ into Y. Let $E = \lambda^{-1}[(N(\mathcal{F}) - Q) \cap B_{k+2}]$; then by 6.4 we have $\dim E \leq n - k - 2$; and $G \circ \mu \mid X - E$ is the desired extension. This completes the proof.

Note that, because of the manner in which E has been constructed, it follows that there exists a *single* fixed set $E_0 \subset X - A$, $\dim E_0 \leq n - k - 2$, which contains the exceptional set E for *every* map $f: A \longrightarrow Y$.

In case Y is an ANR, then according to the Borsuk-Dowker theorem ([11]), the given map f in 7.4 is itself extendable over $X - E$; thus, 7.4 contains the result of [1 ; 656] (which is proved in an entirely different way).

In case Y is only LC^{n-1}, then because of 7.3 (b), the proof shows that again f itself is extendable over $X - E$; thus, 7.4 contains a generalization of Borsuk's original result in [2 ; 162] for separable metric spaces. And if Y is LC^n then, again by the Borsuk-Dowker theorem, the exceptional set E depends only on the homotopy class of f.

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([11]) Let X be metric, $A \subset X$ closed and $f, g: A \rightarrow Y$ continuous. Assume that $g \simeq f$ and that g is extendable to a $G: X \rightarrow Y$. If either (1) Y is an ANR, or (2) Y is LC^n and $\dim(X - A) \leq n$, then f also has an extension $F: X \rightarrow Y$ which can be selected so that $F \simeq G$. For a proof, see [3; 212] or [12; 117].

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