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GROWTH AND DECAY PROPERTIES OF SOLUTIONS OF SECOND ORDER ELLIPTIC EQUATIONS (*)

PAUL C. FIFE

1. Introduction.

Let $\mathcal{D}$ be an unbounded domain in E^n, and L a linear elliptic partial differential operator of second order. The purpose of this paper is to investigate growth and decay properties of Phragmén Lindelöf type [18] as $|x| \to \infty$ for solutions $u(x)$ in $\mathcal{D}$ of $Lu \geq 0$ satisfying (for large $|x|$) $u = 0$ on the boundary $\partial\mathcal{D}$. Analogous results are given for solutions near singular points on the boundary. Emphasis will be placed on the effect of the domain on the growth estimates. Some of the results can be interpreted as providing classes of functions within which uniqueness holds for the Dirichlet problem when the domain is unbounded or when no boundary condition is prescribed at some point of the boundary.

The results which have been obtained in the past on this subject and which give growth conditions depending on $\mathcal{D}$ generally fall into three classes, depending on which properties of $\mathcal{D}$ enter into consideration. To describe them, we let $\Gamma_r = \mathcal{D} \cap \{x : |x| = r\}$, $\mathcal{S}_r = \{x : |x| < r\}$, $\mathcal{D}_r = \mathcal{D} \cap \mathcal{S}_r$, $\mathcal{D}'_r = \mathcal{S}_r - \mathcal{D}_r$, cap $(r) =$ capacity of $\mathcal{D}'_r - \mathcal{D}'_{r-1}$, and $\lambda(r) =$ the first eigenvalue of the Laplace-Beltrami operator acting on functions defined on Γ^r and vanishing on the boundary of Γ_r.

The three classes referred to above are those giving growth or convexity conditions in terms of (1) the ratio $m(r) = \text{meas}\,\mathcal{D}_r / \text{meas}\, S_r$; (2) cap (r); and (3) $\lambda(r)$. Type (3) has the advantage (see our Section 5) that it yields the best possible result in the case of harmonic functions in a cone; the second type on the other hand has the advantage that it yields necessary and sufficient conditions for regularity of the point at infinity (or a boun-

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dary point), considering only bounded solutions (see [5, 6]). The first two categories generally require less smoothness on the coefficients, but impose other conditions not necessary in the third class of results.

Results of the first type include those of Landis [16] and Novruzov [17]. Landis treats equations with only highest order terms, but with weak or no continuity requirements. He requires that $m(r)$ be smaller than a certain constant. Novruzov extends Landis' results to the case when certain lower order terms are present.

The second class is represented by Blohina [5]. Since the question of the possible growth at infinity is a certain kind of generalization of the problem of regularity of boundary points, it is natural to expect that results of the former kind could be framed in terms of Wiener-type sequences. Blohina announces such results, assuming the leading terms of the equation approach constants, and the lower order terms and derivatives of both approach zero at prescribed rates as $|x| \to \infty$. These conditions are relaxed when $\mathcal{D}$ is of « cylindrical type ».

The third class of results includes those of Dinghas [8, 9, 10] and Antohin [4] as well as of the present paper. Dinghas has given growth estimates and convexity conditions for $\mu(r) = \int_{\Gamma_r} u^2 \, d\omega$, where u is a harmonic function with boundary data ≤ 0. His method involves deriving an ordinary differential inequality for $\mu(r)$; this idea evidently goes back to Carleman [7]. Since the methods of the present paper are based on similar inequalities, I shall give Dinghas' argument in more detail for the case when $\mathcal{D}$ is a cone in E^n. Let Ω be an open set on the unit sphere, and $\mathcal{D}$ the cone consisting of rays through the origin and Ω. Then $\lambda(r) = \lambda(1) r^{-2}$. Clearly

$$\mu(r) = \int_{\Omega} u^2(r, \omega) \, d\omega,$$

$$\mu'(r) = 2 \int_{\Omega} u u_{rr} \, d\omega,$$

$$(\mu'(r))^2 \leq 4 \mu \int u_r^2 \, d\omega,$$

and

$$\mu''(r) = 2 \int u_r^2 \, d\omega + 2 \int u u_{rr} \, d\omega.$$

Since $u_{rr}+\frac{n-1}{r}u_r+\frac{1}{r^2}\Delta_1 u=0$, where Δ_1 is the Laplace-Beltrami operator on Ω, the last equation reads

$$\mu''(r)=2\int u_r^2\,d\omega+2\int u\left[-\frac{n-1}{r}u_r-\frac{1}{r^2}\Delta_1 u\right]$$

$$\geq\frac{(\mu')^2}{2\mu}-\frac{n-1}{r}\mu'-\frac{2}{r^2}\int u\Delta_1 u$$

$$\geq\frac{(\mu')^2}{2\mu}-\frac{n-1}{r}\mu'+\frac{2}{r^2}\lambda(1)\,\mu.$$

After a change of variable $\xi=\log r$, $q(\xi)=\mu(r)\,r^{n-2}$, this inequality becomes

$$2\frac{\ddot{q}}{q}-\left(\frac{\dot{q}}{q}\right)^2\geq 4\lambda(1)+(n-2)^2.$$

But

$$2\frac{\ddot{q}}{q}=2\frac{\ddot{q}}{\dot{q}}\frac{\dot{q}}{q}\leq\left(\frac{\ddot{q}}{\dot{q}}\right)^2+\left(\frac{\dot{q}}{q}\right)^2;$$

hence

$$\left(\frac{\ddot{q}}{\dot{q}}\right)^2\geq 4\lambda(1)+(n-2)^2.$$

From this one obtains the result that q is a convex function of $r^{\sqrt{4\lambda(1)+(n-2)^2}}$, from which fact growth or decay conditions can be derived.

Dinghas obtained in [9] the corresponding result when $\mathcal{D}$ is not necessarily a cone; he also obtained other related results in several other papers; example convexity conditions for $\mu_p(r)=\int_{\Gamma_r}u^p\,d\omega$ for subharmonic functions. In [10] he found convexity results for solutions of the equation

$$\Sigma_i\frac{\partial}{\partial x_i}\left(k(x)\frac{\partial u}{\partial x_i}\right)+c(x)\,u=0$$

in annular and semiannular domains. These results are given in terms of solutions of a certain associated linear second order differential equation.

Antohin has used this method to treat solutions of the general equation with C^1 leading coefficients in a domain contained in a half-space. He requires, however, that the coefficient $c(x)$ of the undifferentiated term be negative and large enough in absolute value, the magnitude depending on the other coefficients.

The present paper deals with the general operator with C^1 leading coefficients and locally bounded lower coefficients, in fairly general domains. Growth results near a finite singular boundary point are given in section 3, as well as results at infinity when the lower order terms and derivatives of the higher order terms vanish rapidly enough as $|x| \to \infty$. The theorem in section 4 generalizes these last results, replacing the stated conditions on the coefficients by a single considerably more general (and, unfortunately, complicated) « Assumption A. » Section 5 exhibits a case when these results are shown to be best possible, and section 6 exhibits a type of quasilinear elliptic equation to which the technique of the paper can be applied. I have obtained the results analogous to those of this paper for the case of L_p norms $(2 < p < \infty)$ and also growth results for $\int u^2\,dx$ where the integral is over surfaces $\varphi(x) = r$ and φ is some given function. However it is not yet clear whether these generalizations will lead to interesting results.

Regarding the general question of the asymptotic description of solutions of partial differential equations, many far reaching results have been obtained in cases when it is convenient to reduce the problem to an ordinary differential equation or inequality in a Banach space; for such results see Agmon and Nirenberg [1, 2, 3] and the references contained therein. The many results obtained include strong unique continuation theorems for general second order elliptic operators. Such theorems are related to, but more delicate than, the questions treated in the present paper. The same could be said for the interesting results on the reduced wave equation obtained by Kato [15] and Agmon [1]. Also related is the general problem of the nature of isolated singularities; see Serrin [19] and his bibliography. The methods of Agmon, Kato, Dinghas, and the present paper involve reducing the problem to an ordinary differential inequality. For simple domains Agmon's method could be applied to yield many of the results herein. I wish to thank him for some valuable discussions.

Finally, the work of Gilbarg, Hopf, Serrin, Herzog, and Huber [11-14, 20] on theorems of Phragmén-Lindelöf type should be mentioned.

2. Notation.

Let $\mathcal{D}$ be a domain in E^n (it may be the entire space), $\partial\mathcal{D}$ its boundary, $\mathcal{D}_{ab}$ (for $0 < a < b$) its intersection with the shell $\{x : a < |x| < b\}$, Γ_r its intersection with the sphere $\{x : |x| = r\}$, and Ω_r the projection of Γ_r

onto the unit sphere centered at the origin. $\mathcal{D}$ is to satisfy the two conditions:

D1. Γ_r varies continuously with r, in the sense that $\lim_{h \to 0}$ meas $\{\Omega_{r+h} - \Omega_r\} = 0$ for each $r > 0$.

D2. The boundary $\partial \mathcal{D}$ is regular enough so that the divergence theorem holds in each region $\mathcal{D}_{ab}$ for $0 < a < b$.

The only geometrical property of $\mathcal{D}$ that we shall be concerned with is the function $\lambda(r)$ defined by

$$\frac{\lambda(r)}{r^2} \leq \inf \int_{\Gamma_r} |\nabla_t \varphi|^2 , \tag{1}$$

where the infemum is taken over all smooth functions φ defined in a neighborhood of $\bar{\Gamma}_r$, vanishing on $\Gamma_r \cap \partial \mathcal{D}$, and satisfying $\int_{\Gamma_r} \varphi^2 = 1$. The notation $\nabla_t \varphi$ denotes the projection of the gradient of φ onto the plane tangent to Γ_r. Thus $\int_{\Gamma} | \nabla_t \varphi |^2$ is the Dirichlet integral of the restriction of φ to Γ_r, and the infemum is the first eigenvalue of the Laplace-Beltrami operator acting on functions Γ_r and vanishing on its boundary. If $\mathcal{D}$ is a cone, λ is constant.

Let L denote the elliptic operator

$$Lu \equiv (a_{ij}\, u_i)_j + b_i\, u_i + cu.$$

Here the summation convention is used, and subscripts on u denote differentiation: $u_i = \partial u / \partial x_i$.

We shall also have occasion to use the operator

$$L' \, u = (a_{ij}\, u_i)_j$$

and the function $A(x) = a_{ij}\, x_i\, x_j / | x |^2$, as well functions μ_i, l_i of the single variable r, satisfying

$$\mu_0(r) \leq a_{ij}(x)\, \xi_i\, \xi_j \leq \mu_1(r) \quad \text{for all} \quad x \in \Gamma_r \,, \; | \xi | = 1 \,;$$

$$a_{ij}(x)\, \xi_i\, \eta_j \leq \mu_2(r) \quad \text{for all} \quad x \in \Gamma_r \,, \; | \xi | = | \eta | = 1, \; \xi_i\, \eta_i = 0,$$

$$\frac{l_1(r)}{r} \leq \frac{L'(| x |)}{A(x)} \leq \frac{l_2(r)}{r} \quad \text{for all} \quad x \in \Gamma_r \,.$$

3. Cases when the lower order terms are inconsequential.

THEOREM 1. *Let $\mathcal{D}$ be bounded and let $0 \in \partial\mathcal{D}$. Suppose L is uniformly elliptic with coefficients continuous in $\overline{\mathcal{D}}$. Assume $a_{ij}(x)$ are differentiable in $\overline{\mathcal{D}} - \{0\}$, $c \leq 0$, and $a_{ij}(0) = \delta_{ij}$. Let $u(x)$ be continuous in $\overline{\mathcal{D}} - \{0\}$, a C^2 solution of $Lu \geq 0$ in $\mathcal{D}$; and let $u = 0$ on $\partial\mathcal{D} - \{0\}$ for small enough $|x|$. Then for each $\varepsilon > 0$ there is a constant C depending on u and ε such that for small r, either*

$$\int_{\Gamma_r} u^2 \geq C r^{1-\varepsilon} \left[1 - \int \exp\left(-\int \sqrt{4\lambda(r) + (n-2)^2}\right) dr / r\right] dr / r \tag{2}$$

or

$$\int_{\Gamma_r} u^2 \leq C^{-1} r^{1+\varepsilon} \int \exp\left[\int \sqrt{4\lambda(\bar{r}) + (n-2)^2}\, d\bar{r} / \bar{r}\right] d\bar{r} / \bar{r}, \tag{3}$$

with (2) holding if $u = 0$ on all of $\partial\mathcal{D} - \{0\}$, but $u \not\equiv 0$.

REMARK: It will be clear from the proof that the continuity of b_i at 0 may be relaxed provided $b_i(x) = 0\ (|x|^{-1})$ as $|x| \to 0$. Also, the requirement $a_{ij}(o) = \delta_{ij}$ may be always be effectuated by an affine transformation.

THEOREM 2. *Let $\mathcal{D}$ be unbounded and $\lambda(r) \geq \lambda_0$ for some $\lambda_0 > 0$. Suppose L is uniformly elliptic with coefficients Holder continuous in $\mathcal{D}$. Assume $a_{ij}(x)$ are differentiable in $\overline{\mathcal{D}}$,*

$$x_i\, \partial a_{ij} / \partial x_j = 0(1) \tag{4a}$$

$$b_i = o(|x|^{-1}) \tag{4b}$$

as $|x| \to \infty$, and $c \leq 0$. Let $u(x)$ be continuous in $\overline{\mathcal{D}}$, a C^2 solution of $Lu \geq 0$ in $\mathcal{D}$, and let $u = 0$ on $\partial\mathcal{D}$. Then there are constants $C, p, \beta_1 > 0$, and $\beta_2 > 0$, the latter three independent of u, such that either $u = 0$ or

$$\int_{\Gamma_r} u^2 \geq C r^p \int \left[\exp \int \sqrt{\beta_1 \lambda(r) + \beta_2}\, dr / r\right] dr / r. \tag{5}$$

Furthermore if $a_{ij}(x) \to \delta_{ij}^0$ and $x_i\, \partial a_{ij} / \partial x_j \to 0$ as $|x| \to \infty$, then $p = 1 - \varepsilon$, $\beta_1 = 4$, and $\beta_2 = (n-2)^2$, where $\varepsilon > 0$ is arbitrary. Then C depends on ε as well.

The proof of Theorem 1 will be deferred until later. We shall prove Theorem 2 under the assumption that $u > 0$ in $\mathcal{D}$. This involves no loss of generality because if u changes sign, one replaces $u(x)$ in the proof below by $u^+(x) = \max[0, u(x)]$. The arguments still hold, and the conclusion of Theorem 2 will be established with u replaced by u^+. But this implies a fortiori the conclusion as stated.

The proof employs the following three lemmas.

LEMMA 1. *Let $F(x)$ be a function continuous in $\bar{\mathcal{D}}$, with bounded piecewise continuous first derivatives in $\mathcal{D}$, and satisfying $F = 0$ on $\partial\mathcal{D}$. Then*

$$\int_{\Gamma_r} \partial F / \partial x_i = \frac{d}{dr} \int_{\Gamma_r} F(x)\, \nu_i(x), \tag{6}$$

where $\nu_i = x_i / r$, the x_i-component of the normal to Γ_r.

PROOF: For $r > r_0 > 0$, let

$$J(r) = \int_{\mathcal{D}_{r_0 r}} \partial F / \partial x_i .$$

Then for $h > 0$,

$$J(r+h) - J(r) = \int_{\mathcal{D}_{r,\, r+h}} \partial F / \partial x_i .$$

Now let $\Omega = \bigcap_{r \le \varrho \le r+h} \Omega_\varrho$, let T_0 be the cone with vertex at the origin and section Ω, $T_1 = T_0 \cap \mathcal{D}_{r,\, r+h}$, and $T_2 = \mathcal{D}_{r, r+h} - T_1$. Thus no ray from the origin passing through T_1 will intersect $\partial\mathcal{D}$ in the shell $r < |x| < r+h$, whereas every such ray passing through T_2 will intersect it. By condition $D\,1$ on $\mathcal{D}$, meas $T_2 = o(h)$ as $h \to 0$ and from the boundedness of $\partial F / \partial x_i$ we conclude that

$$\int_{T_2} \partial F / \partial x_i = o(h).$$

However we may write $\int_{T_1} = \int_r^{r+h} \varrho^{n-1}\, d\varrho \int_\Omega d\omega$,

so in all

$$J(r+h) - J(r) = \int_r^{r+h} \varrho^{n-1}\, d\varrho \int_\Omega (\partial F / \partial x_i)(\varrho\omega)\, d\omega + o(h).$$

Dividing by h and taking the limit, also using the fact that $\Omega \longrightarrow \Omega_r$ as $h \longrightarrow 0$, we find that

$$J'(r) = r^{n-1} \int_{\Omega_r} (\partial F / \partial x_i)\, d\omega = \int_{\Gamma_r} \partial F / \partial x_i \,.$$

On the other hand by the divergence theorem

$$J(r) = \int_{\partial \mathcal{D}_{r_0 r}} F(x)\, \nu_i(x),$$

where ν_i is the exterior normal to $\mathcal{D}_{r_0 r}$. Since $F = 0$ on $\partial \mathcal{D}$, $J(r) = \int_{\Gamma_r} F\nu_i - \int_{\Gamma_{r_0}} F\nu_i$. Differentiating and using the above expression for J', we obtain (6).

COROLLARY. *Let* $f(r) = \int_{\Gamma_r} A(x)\, u^2$. *Then*

$$\frac{1}{2} \frac{d}{dr} \left[\frac{df}{dr} - \int_{\Gamma_r} L'(|x|)\, u^2 \right] \geq \int_{\Gamma_r} a_{ij}\, u_i\, u_j - \frac{1}{2} \int b_i (u^2)_i - \int cu^2 \,. \tag{7}$$

PROOF: Since $Lu \geq 0$ and $u \geq 0$, we have (using the equation and (6))

$$0 \leq \int_{\Gamma_r} u\, Lu = u\, [(a_{ij}\, u_i)_j + b_i\, u_i + cu]$$

$$= \int \left\{ \frac{1}{2} (a_{ij}\, (u^2)_i)_j + \frac{1}{2} b_i\, (u^2)_i + cu^2 - a_{ij}\, u_i\, u_j \right\}$$

$$= \frac{1}{2} \frac{d}{dr} \int a_{ij}\, (u^2)_i\, \nu_j - \int a_{ij}\, u_i\, u_j + \frac{1}{2} \int b_i\, (u^2)_i + \int cu^2 \,.$$

But

$$\int_{\Gamma_r} a_{ij}\, (u^2)_i\, \nu_j = \int (a_{ij}\, \nu_j\, u^2)_i - \int (\partial\, (a_{ij}\, \nu_j) / \partial x_i)\, u^2$$

$$= \frac{d}{dr} \int (a_{ij}\, \nu_i\, \nu_j)\, u^2 - \int L'(|x|)\, u^2 = \frac{d}{dr} \int A(x)\, u^2 - \int L'(|x|)\, u^2 \tag{8}$$

(here we used the fact that $\nu_i = \partial |x| / \partial x_i$). Substituting this expression into the previous inequality yields (7), and the corollary is proved.

In the following we denote $\partial_\eta u(x) = a_{ij}(x)\nu_i(x)u_j(x)$. Also recall the definition of ∇_τ following (1).

LEMMA 2. *For any* χ_1, $0 < \chi_1 < 1$, *and any* $x \in \Gamma_r$,

$$|\nabla u(x)|^2 \geq (1-\chi_1)|\nabla_\tau u|^2 + \frac{\chi_1}{\chi_1 \mu_1^2 + \mu_2^2}(\partial_\eta u)^2 . \tag{9}$$

PROOF: Let $\tau = (\tau_1, \dots, \tau_n)$ be the unit vector in the direction of $\nabla_\tau u$, and $\nu = (\nu_1, \dots, \nu_n)$ as always the unit normal to Γ_r. Then $|\nabla_\tau u| = \partial_\tau u$, the latter denoting the directional derivative. We may express $\partial_\eta u$ as a linear combination

$$\partial_\eta u = \alpha_1 \partial_\tau u + \alpha_2 \partial_\nu u. \tag{10}$$

Expand all terms:

$$a_{ij}\nu_i u_j = \alpha_1 \tau_j u_j + \alpha_2 \nu_j u_j .$$

Hence

$$a_{ij}\nu_i = \alpha_1 \tau_j + \alpha_2 \nu_j ,$$

or

$$\alpha_2 = a_{ij}\nu_i \nu_j \leq \mu_1 ,$$

and

$$\alpha_1 = a_{ij}\nu_i \tau_j \leq \mu_2 .$$

Now we have

$$\nabla u|^2 = (\partial_\tau u)^2 + (\partial_\nu u)^2 = (1-\chi_1)(\partial_\tau u)^2 + \chi_1(\partial_\tau u)^2 + \frac{1}{\alpha_2^2}(\partial_\eta u - \alpha_1 \partial_\tau u)^2$$

$$= (1-\chi_1)(\partial_\tau u)^2 + \left(\chi_1 + \frac{\alpha_1^2}{\alpha_2^2}\right)(\partial_\tau u)^2 - \frac{2\alpha_1}{\alpha_2^2}\partial_\tau u \partial_\eta u + \alpha_2^{-2}(\partial_\eta u)^2$$

$$= (1-\chi_1)(\partial_\tau u)^2 + \left[\sqrt{\chi_1 + \frac{\alpha_1^2}{\alpha_2^2}}\,\partial_\tau u - \frac{\alpha_1/\alpha_2^2}{\sqrt{\chi_1 + \frac{\alpha_1^2}{\alpha_2^2}}}\,\partial_\eta u\right]^2 +$$

$$+ \frac{\chi_1}{\chi_1 \alpha_2^2 + \alpha_1^2}(\partial_\eta u)^2$$

$$\geq (1-\chi_1)(\partial_\tau u)^2 + \frac{\chi_1}{\chi_1 \alpha_2^2 + \alpha_1^2}(\partial_\eta n)^2 ,$$

which yields (9).

LEMMA 3. *Let* $f(r) = \int_{\Gamma_r} A(x)\, u^2$ *and* $H(r) = f'(r) - \int_{\Gamma_r} L'(|x|)\, u^2$.

Then

$$\int_{\Gamma_r} \frac{(\partial_\eta u)^2}{A(x)} \geq \frac{H^2}{4f}. \tag{11}$$

PROOF: We may rewrite (8) in the form

$$2\int u\partial_\eta u = \int \partial_\eta (u^2) = f'(r) - \int L'(|x|)\, u^2 = H(r). \tag{12}$$

From Schwarz' inequality,

$$H^2 \leq 4\int A(x)\, u^2 \int \frac{(\partial_\eta u)^2}{A} = 4f\int (\partial_\eta u)^2 / A,$$

which yields (11).

PROOF OF THEOREM 2: We proceed from (7), estimating the terms on the right separately. First, from (9),

$$a_{ij}\, u_i\, u_j \geq \mu_0 \int |\nabla u|^2$$

$$\geq \mu_0 (1 - \chi_1) \int (\partial_\tau u)^2 + \frac{\chi_1 \mu_0^2}{\chi_1 \mu_1^2 + \mu_2^2} \int \frac{(\partial_\eta u)^2}{A}.$$

By assumption (4b) on b_i, there is a function $\varepsilon_1(r) \to 0$ as $r \to \infty$ such that

$$\left| \int_{\Gamma_r} b_i\, u\, u_i \right| \leq \frac{1}{2} \operatorname*{Max}_{\Gamma_r} |\Sigma b_i| \left[\frac{1}{r} \int_{\Gamma_r} u^2 + r \int_{\Gamma_r} |\nabla u|^2 \right]$$

$$\leq \frac{\varepsilon_1(r)}{r} \left[\frac{f(r)}{r} + r \int |\nabla u|^2 \right].$$

Next, by Lemmas 2 and 3 and (1),

$$\int_{\Gamma_r} a_{ij}\, u_i\, u_j - \int b_i\, u u_i - \int c u^2 \geq$$

$$\geq \mu_0 \int_{\Gamma_r} |\nabla u|^2 - \varepsilon_1(r) \int |\nabla u|^2 - \varepsilon_1 r^{-2} f(r)$$

$$\geq (\mu_0 - \varepsilon_1)\left[(1-\chi_1)\int (\partial_\tau u)^2 + \frac{\chi_1 \mu_0}{\chi_1 \mu_1^2 + \mu_2^2}\int \frac{(\partial_\eta u)^2}{A}\right] -$$

$$- \varepsilon_1 r^{-2} f(r)$$

$$\geq \left[\frac{(\mu_0 - \varepsilon_1)(1-\chi_1)\lambda(r)}{\mu_0} - \varepsilon_1\right] r^{-2} f(r) +$$

$$+ \frac{\chi_1 \mu_0 (\mu_0 - \varepsilon_1)}{\chi_1 \mu_1^2 + \mu_2^2}(H^2/4f).$$

From this and (7) we now obtain

$$H'(r) \geq \beta_3 \lambda(r) r^{-2} f(r) + \beta_4 H^2/f \tag{13}$$

for large enough r; more specifically, for $r > r_0$, where r_0 is large enough that $\varepsilon_1(r)$ is small enough that β_3 and β_4 can be taken as positive constants (recall that we assume $\lambda(r) \geq \lambda_0 > 0$). Note that

$$0 < \beta_4 < 1/2. \tag{14}$$

Now let $\psi(r) = H(r)/f(r)$, so

$$\psi' = \frac{H'}{f} - \frac{Hf'}{f^2} = \frac{H'}{f} - \frac{H}{f}\left(H + \int_{\Gamma_r} L'(|x|)\, u^2\right)/f,$$

or

$$\frac{H'}{f} = \psi' + \psi^2 + \psi\left(\int L'(|x|)\, u^2\right)/f \leq \psi' + \psi^2 + \frac{l_2}{r}\psi.$$

Note that

$$L'(|x|) = \left(a_{ij}\frac{x_i}{r}\right)_j = \frac{1}{r}\left[a_{ij} - A(x) + x_i\, \partial a_{ij}/\partial x_j\right]. \tag{15}$$

Hence from assumption (4), uniform ellipticity, and the definition of $l_i(r)$, we see that $l_i(r)$ may be taken to be constants; this we shall do.

Dividing (13) by f, we find

$$\psi' + \psi^2 + \frac{l_2}{r}\psi \geq \beta_3 \lambda r^{-2} + \beta_4 \psi^2 ;$$

$$\psi' + (1-\beta_4)\psi^2 + \frac{l_2}{r}\psi \geq \beta_3 \lambda r^{-2} . \tag{16}$$

Now for some $r_1 \geq r_0$, set

$$F_i(r) = f(r_1) \exp\left\{ \int_{r_1}^{r} [\psi(\bar{r}) + l_i(\bar{r})/\bar{r}]\, d\bar{r} \right\},$$

$i = 1, 2$, so that $F_i(r_1) = f(r_1)$ and

$$\frac{F_1' - l_1 F_1 / r}{F_1} = \psi = \frac{f' - \int L'(|x|)\, u^2}{f} \leq \frac{f' - l_1 f / r}{f} .$$

Hence $(\log (e^{-\int l_1 dr/r} F_1))' \leq (\log (e^{-\int l_1 dr/r} f))'$, and so $f \geq F_1$ for $r \geq r_1$. In the same manner we obtain

$$F_1 \leq f \leq F_2 ,\, r \geq r_1 \tag{17a}$$

$$F_1 \geq f \geq F_2 ,\, r \leq r_1 . \tag{17b}$$

We now introduce the new variable $\xi = \log r$, denote $\dot{\psi} = d\psi / d\xi = r\psi'$, and rewrite (16) as

$$r\dot{\psi} + (1-\beta_4)(r\psi)^2 + l_2 (r\psi) \geq \beta_3 \lambda (r). \tag{18}$$

Next, for some $\sigma(\xi)$ we set

$$\begin{aligned} q(\xi) &= f(r_1) \exp \int_{\log r_1}^{\xi} [r\psi - \sigma(\xi)]\, d\xi = \\ &= F_i(r) \exp\left\{ - \int_{r_1}^{r} (l_i + \sigma)\, d\bar{r}/\bar{r} \right\}, \end{aligned} \tag{19}$$

$i = 1$ or 2, so that

$$\frac{\dot q}{q} = r\psi - \sigma,$$

$$r\dot\psi = \frac{\ddot q}{q} - \left(\frac{\dot q}{q}\right)^2 - r\psi + \dot\sigma =$$

$$= \frac{\ddot q}{q} - \left(\frac{\dot q}{q}\right)^2 - \frac{\dot q}{q} - \sigma + \dot\sigma,$$

and (18) reduces to

$$\frac{\ddot q}{q} - \beta_4\left(\frac{\dot q}{q}\right)^2 + \gamma(\xi)\frac{\dot q}{q} \geq \beta_3\lambda + \sigma - \dot\sigma - (1-\beta_4)\sigma^2 - l_2\sigma,$$

where

$$\gamma(\xi) = 2(1-\beta_4)\sigma - (1-l_2).$$

We now set σ to be the following constant:

$$\sigma = (1-l_2)/2(1-\beta_4)$$

(this is finite because of (14)), so that $\gamma = 0$ and

$$\frac{\ddot q}{q} - \beta_4\left(\frac{\dot q}{q}\right)^2 \geq \beta_3\lambda(r) + (1-l_2)^2/4(1-\beta_4). \tag{20}$$

Next we approximate

$$\frac{\ddot q}{q} = \frac{\ddot q}{\dot q}\,\frac{\dot q}{q} \leq \frac{1}{4\beta_4}\left(\frac{\ddot q}{\dot q}\right)^2 + \beta_4\left(\frac{\dot q}{q}\right)^2$$

to obtain

$$\left(\frac{\ddot q}{\dot q}\right)^2 \geq \beta_1\lambda(r) + \beta_2 > 0 \tag{21}$$

where

$$\beta_1 = 4\beta_3\beta_4\,;\ \beta_2 = (1-l_2)^2\beta_4/(1-\beta_4).$$

Inequality (20) implies that $\ddot q > 0$. Therefore two cases can occur:

Case 1: $\dot q(\xi) > 0$ for large enough ξ; or

Case 2: $\dot q(\xi) < 0$ for all $\xi > \log r_0$.

Setting $\pi^2 = \beta_1 \lambda + \beta_2$, we conclude that for large ξ,

$$\frac{\ddot{q}}{\dot{q}} \geq \pi \tag{22a}$$

in case 1, and

$$\frac{\ddot{q}}{\dot{q}} \leq -\pi \tag{22b}$$

in case 2.

In case 1, (22*a*) implies

$$\dot{q}(\xi) \geq \dot{q}(\xi_0)\, e^{\int_{\xi_0}^{\xi} \pi d\bar{\xi}} \quad ;$$

$$q(\xi) \geq q(\xi_0) + \dot{q}(\xi_0) \int_{\xi_0}^{\xi} e^{\int_{\xi_0}^{\bar{\xi}} \pi d\bar{\bar{\xi}}}\, d\bar{\xi}$$

for $\xi \geq \xi_0$, where ξ_0 is such that $\dot{q}(\xi_0) > 0$. Hence (changing back to the variable r),

$$q \geq C_1 \left(1 + \int e^{\int \pi dr/r}\, dr/r\right)$$

and from (17*a*) and (19),

$$f(r) \geq F_1(r) = q \exp\left\{\int_{r_0}^{r} (l_1 + \sigma)\, d\bar{r}/\bar{r}\right\} = C_2\, q r^{l_1 + \sigma},$$

which implies that (5) with $p = l_1 + \sigma$ holds in Case 1.

On the other hand in Case 2, $\dot{q} < 0$, so (22*b*) becomes

$$(\log(-\dot{q}))^{\cdot} \leq -\pi\,;$$

$$-\dot{q} \leq -\dot{q}(\xi_0)\, e^{-\int_{\xi_0}^{\xi} \pi d\bar{\xi}}$$

or

$$-\dot{q} \leq C\, e^{-\int \pi dr/r}$$

for some positive C. Hence in case 2,

$$q(\xi) = -\int_r^\infty \dot{q}\, d\bar{r}/\bar{r} \leq C \int_r^\infty e^{-\int \pi d\bar{r}/\bar{r}}\, d\bar{r}/\bar{r}.$$

[Note that $\lambda > \lambda_0 > 0$ guarantees that $\int^\infty e^{-\int \pi dr/r}\, dr/r < \infty$].

Observe that for $r = r_1$, $q = f(r_1)$ and

$$\dot{q} = (r\psi - \sigma)\, q =$$

$$= \left(\frac{r_1 H}{f} - \sigma\right) f(r_1) = r_1 H(r_1) - \sigma f(r_1) =$$

$$= r_1 \left[f'(r) - \int L'(|x|)\, u^2 - \frac{\sigma}{r} f\right]_{r=r_1}$$

$$\geq r_1 [f' - f(l_2 + \sigma)/r]_{r=r_1}$$

$$= r_1 [r^{l_2+\sigma} (r^{-l_2-\sigma} f)']_{r=r_1}.$$

We may assume the origin is located in the interior of the complement of $\mathcal{D}$. Then if $u \not\equiv 0$, the function $r^{-l_2-\sigma} f$ vanishes for some value of r, but is positive for some larger value. Hence its derivative is positive for some value of r, say $r = r_2$.

Let us suppose for the moment that the domain $\mathcal{D}$ lies outside but touches the sphere $|x| = r_0$ (thus $f(r)$ is defined only for $r > r_0$). Then we may choose r_1 equal to the value of r_2 found above, and the expression implies that $\dot{q}(r_1) > 0$. Thus case 2 is excluded and the conclusion (5) follows.

Now consider the other possibility that $\mathcal{D}$ contains points with $|x| < r_0$. We let $\mathcal{D}' = \mathcal{D} \cap \{x : |x| > r_0\}$, and let $v(x)$ be a bounded solution of $Lv = 0$ in $\mathcal{D}'$; $v = 0$ on $\partial\mathcal{D} \cap \partial\mathcal{D}'$; $v = u$ on Γ_{r_0}. More specifically, we construct v as follows. For every $\varrho > r_0$, let v_ϱ be the solution of the Dirichlet problem in $\mathcal{D}_{r_0\varrho}$ with zero Dirichlet data, except on Γ_{r_0}, where $v_\varrho = u$. The maximum principle shows (1) $v_\varrho(x) \geq 0$; (2) each $v_\varrho(x)$ attains its maximum on Γ_{r_0}; and (3) the functions v_ϱ form an increasing bounded sequence as $\varrho \to \infty$, hence they approach a limit $v(x) \geq 0$. The Schauder estimates (implied by the Hölder continuity assumption) tell us that the convergence process is uniform on the second derivatives as well, so v is the classical solution desired. Furthermore we know that v assumes a maximum on Γ_{r_0}

Let $w(x) = u(x) - v(x)$; it satisfies the hypotheses of Theorem 2 with $\mathcal{D}$ replaced by $\mathcal{D}'$; hence the above reasoning applies: either $w = 0$ or (5) holds for w. In the former case $v = u$ for $|x| \geq r_0$, which implies that u has a maximum at some point x_m with $|x_m| \leq r_0$. This is impossible unless $u = 0$. In the latter case we observe that $u = w + v \geq w$, so that u as well satisfies (5)- The proof of Theorem 2 is complete, except for the last assertion.

Assume that $a_{ij}(x) \to \delta^0_{ij}$ and $x_i\, \partial a_{ij} / \partial x_j \to 0$. Recalling (15) we observe now that $A(x) \to 1$ and $L'(|x|) \to (n-1)/r$ as $|x| \to \infty$. Therefore the functions $\mu_i(r)$ and $l_i(r)$ defined in section 2 could be taken so that $\mu_0, \mu_1 \to 1$; $\mu_2 \to 0$; $l_1, l_2 \to n-1$ as $r \to \infty$. Instead, for some given $\varepsilon_2 > 0$, we choose the r_0 in the foregoing proof (see the argument following (13)) large enough so that it is permissible to choose $\mu_0 \equiv 1 - \varepsilon_2$, $\mu_1 \equiv 1 + \varepsilon_2$, $\mu_2 \equiv \varepsilon_2$, $l_1 \equiv n - 1 - \varepsilon_2$, and $l_2 \equiv n - 1 + \varepsilon_2$ for $r \geq r_0$. We also assume r_0 is large enough that $\varepsilon_1 \leq \varepsilon_2$. Choose $\chi_1 = \varepsilon_2$. Then the coefficients in (13) may be expressed as

$$\beta_3 = 2 + \delta_1(\varepsilon_2); \; \beta_4 = \frac{1}{2} + \delta_2(\varepsilon_2),$$

where the δ_i are functions such that $\lim_{\varepsilon \to 0} \delta_i(\varepsilon) = 0$. In the same way we have $\sigma = 2 - n + \delta_3(\varepsilon_2)$ and $\pi^2 = 4\lambda(r) + (n-2)^2 + \delta_4(\varepsilon_2)$. In (5) we have seen that the exponent $p = \sigma + l_1$, so (5) can now be written as

$$\int_{\Gamma_r} u^2 \geq C r^{1-\varepsilon_2+\delta_3} \int \left[\exp \int \sqrt{4\lambda(r) + (n-2)^2 + \delta_4}\, dr/r\right] dr/r$$

for $r \geq r_0(\varepsilon_2)$. Therefore, given ε_1, we choose $\varepsilon_2, \delta_3, \delta_4$ so small (and r_0 correspondingly large) that the above right side dominates

$$r^{1-\varepsilon_1} \int [\exp \sqrt{2\lambda(r) + (n-2)^2}\, dr/r]\, dr/r$$

for $r \geq r_0$. We now need only enlarge the constant C sufficiently so that it holds for $r \leq r_0$ as well. This completes the proof of Theorem 2.

PROOF OF THEOREM 1. Again, we shall operate under the assumption that $u \geq 0$ in $\mathcal{D}$. If u changes sign one replaces u by $u^+ = \mathrm{Max}[0, u(x)]$; then by $u^- = \mathrm{Max}[0, -u(x)]$. In each case the arguments still hold and (2) and (3) are valid alternatives. Now if either u^+ or u^- satisfies (2), then u does all the more. But the only other case is when u^+ and u^- both satisfy (3); then the two corresponding inequalities may be added to yield (3) for u itself.

The proof of Theorem 2 is valid here up to Eq. (13), with the exception that now $\varepsilon_1(r) \to 0$ as $r \to 0$ (instead of $r \to \infty$). This follows from the fact that b_i are bounded (or more generally, $o(|x|)$). Thus (13) will hold not for large r, but for $r < r_0$, say, where r_0 is a small enough constant.

As in the case of the last assertion of Theorem 2, we find that $l_i(r) \to n - 1$. We continue as in the other proof up to (17), except that now we must choose $r_1 < r_0$. Continuing, we obtain (21) for small enough r.

Now let $\eta = -\xi$, $\bar{q}(\eta) = q(-\eta) = q(\xi)$, and $\bar{\pi}(\eta) = \pi(\xi)$. Then again $\bar{q}_{\eta\eta} > 0$, and either $\bar{q}_\eta > 0$ for large enough η, or $\bar{q}_\eta < 0$ for all η. In the first case

$$\frac{\bar{q}_{\eta\eta}}{\bar{q}_\eta} \geq \bar{\pi}(\eta) \tag{23a}$$

for large η, and in the second case

$$\frac{\bar{q}_{\eta\eta}}{\bar{q}_\eta} \leq -\bar{\pi}(\eta) \tag{23b}$$

for all $\eta > -\log r_0$.

In the first case for some η_0 and $\eta > \eta_0$,

$$\bar{q}(\eta) \geq \bar{q}(\eta_0) + \bar{q}_\eta(\eta_0) \int_{\eta_0}^{\eta} e^{\int_{\eta_0}^{\eta} \bar{\pi} d\bar{\eta}} \, d\bar{\eta}$$

$$\geq C\left(1 - \int e^{-\int \pi dr/r} dr/r\right)$$

for some $C > 0$ (here η_0 is chosen such that $\bar{q}_\eta(\eta_0) > 0$). The derivation of (2) is completed with the use of (17b), 19, and the ε-argument used in proving the last statement in Theorem 2.

In the second case $\bar{q}_\eta < 0$, so (23b) becomes

$$(\log(-\bar{q}_\eta))_\eta \leq -\bar{\pi}(\eta);$$

$$-\bar{q}_\eta \leq -\bar{q}_\eta(\eta_0)\, e^{\int_{\eta_0}^{\eta} \bar{\pi} d\bar{\eta}},$$

or

$$-\bar{q}_\eta \leq C^{-1} e^{\int \pi dr/r}$$

for some positive C. Hence

$$\bar{q} = -\int_0^r \bar{q}_\eta \, d\bar{r}/\bar{r} \leq C^{-1} \int_0^r e^{\int \pi d\bar{r}/\bar{r}} \, d\bar{r}/\bar{r} \, .$$

Again with (17*b*) and (19) ($i = 1$) this implies (3), and we have established the alternatives (2) or (3).

Suppose now that $u = 0$ on all of $\partial\mathcal{D} - \{0\}$, but $u \not\equiv 0$. As in the proof of Theorem 2, this implies there is a value $\eta_0 = -\log r_0$ for which $\bar{q}_\eta > 0$; hence the second case (23*b*) is excluded, and (2) holds.

4. The general case.

Here we relax the uniform ellipticity and other assumptions of Theorem 2, replacing them by a more general « Assumption A. ».

Let $B(x) = b_i x_i / A(x)$; $B_0(r)$ an arbitrary function of r;

$$B_1(r) \geq \sup_{\Gamma_r} | B(x) - B_0(r) |$$

[for best results one wants B_1 to be minimal, so B_0 should be some average of B over Γ_r];

$$B_2(r) \geq \sup A^{-1/2}(x) \left[b_i(x) y_i - B(x) a_{ij}(x) x_i y_j / r^2 \right],$$

where the supremum is over $x \in \Gamma_r$ and y_i such that $\Sigma y_i^2 = r^2$, $y_i x_i = 0$; and

$$C(r) = \sup_{\Gamma_r} r^2 c(x).$$

Let $\varkappa_1, \varkappa_2$ be arbitrarily chosen in the interval $0 < \varkappa_i < 1$. We further define

$$\beta(r) = \frac{\varkappa_1 \mu_0^2(r)}{\varkappa_1 \mu_1^2(r) + \mu_2^2(r)},$$

$$A_1(r) = \frac{2\mu_0 (1 - \varkappa_1) \lambda}{\mu_1} - 2B_2 (\lambda/\mu_1)^{1/2} - B_1^2/2 \, \varkappa_2 \, \beta - 2C \, ;$$

$$A_2(r) = \beta (1 - \varkappa_2)/2.$$

(If $\mu_1 = 0$, set $\varkappa_1 = 0$; and if $B_1 = 0$, set $\varkappa_2 = 0$). Let $\sigma(r)$ be an arbitrary differentiable function and

$$\gamma(r) = 2(1 - A_2)\sigma - (1 - l_2 - B_0),$$

$$\delta(r) = A_1 + (1 - l_2 - B_0)\sigma - (1 - A_2)\sigma^2 - r\sigma'(r),$$

and (if $\delta \geq 0$)

$$\pi(r) = 2\sqrt{\delta A_2} - \gamma.$$

We shall require that L and $\mathcal{D}$ satisfy the following assumption for large $|x|$.

Assumption A. There exists a function $\sigma(r)$ such that

$$4\delta(r) A_2(r) \geq \gamma^2(r).$$

In general Assumption A can always be guaranteed by making $\lambda(r)$ large enough or $C(x) < 0$ and $|c|$ large enough.

THEOREM 3. *Let $u(x)$ be continuous in $\bar{\mathcal{D}}$, a C^2 solution of $Lu \geq 0$ in $\mathcal{D}$, and let $u = 0$ on $\partial\mathcal{D}$ for large enough $|x|$. If, for some r_0, Assumption A holds for $r \geq r_0$, then there exists a constant C such that either*

$$\int_{\Gamma_r} u^2 \geq \frac{C}{\mu_1(r)} e^{\int (l_1(r)+\sigma(r))dr/r} \left(1 + \int \frac{1}{r} e^{\int \pi(r)dr/r} dr\right) \tag{24}$$

or

$$\int_{\Gamma_r} u^2 \leq \frac{C^{-1}}{\mu_0(r)} e^{\int (l_2(r)+\sigma(r))dr/r} \int_r^\infty e^{-\int \pi(\bar{r})d\bar{r}/\bar{r}} d\bar{r}/\bar{r}. \tag{25}$$

(Note that the constants implicit in the indefinite integrals may be absorbed in C; also note that if $\int^\infty e^{-\int \pi dr/r} dr/r = \infty$, condition (25) is vacuous).

COROLLARY 1. *If, in addition, the complement of $\mathcal{D}$ has a nonempty interior, Assumption A holds for all r where defined, and $u = 0$ on all of $\partial\mathcal{D}$, then either $u \equiv 0$ or (24) holds.*

The following result is an immediate consequence of Theorem 3.

COROLLARY 2. *Given any elliptic operator L with $a_{ij} \in C'$, b_i locally bounded, and c locally bounded from above, and any positive increasing function $g(r)$, there is a function $\lambda_0(r)$ with the following property. If $\mathcal{D}$ is a domain satisfying Hypotheses D 1 and D 2 and such that $\lambda(r) \geq \lambda_0(r)$, and $u(x)$ satisfies $Lu \geq 0$ in $\mathcal{D}$, $u = 0$ on $\partial\mathcal{D}$ for large $|x|$, then there is a constant C such that either*

$$\int_{\Gamma_r} u^2 \geq Cg(r)$$

or

$$\int_{\Gamma_r} u^2 \leq C^{-1} (g(r))^{-1} .$$

PROOF OF THEOREM 3 AND COROLLARY 1. The proof follows the general outline of the proof Theorem 2. We need an additional lemma.

LEMMA 4. *$b_i(x)\, u_i(x) = Q(x)\, \partial_\tau u(x) + \dfrac{B(x)}{r} \partial_\eta u$, where $\partial_\tau u$ is some directional derivative tangential to Γ_r, $\partial_\eta u = a_{ij} \nu_i u_j$, and*

$$|Q(x)| \leq A^{1/2}(x)\, B_2(r)/r$$

for

$$|x| = r.$$

PROOF: We may clearly express any such linear combination in the form

$$b_i u_i = Q_1\, \partial_\tau u + Q_2\, \partial_\nu u$$

for some $Q_i(x)$. However (10), in which $\alpha_2 = A(x) > 0$, shows that $\partial_\nu u$ can in turn be expressed in terms of $\partial_\tau u$ and $\partial_\eta u$. Thus

$$b_i u_i = Q_2\, \partial_\tau u + Q_3\, \partial_\eta u = Q_2 \tau_i u_i + Q_3\, a_{ij} \nu_j u_i ,$$

where τ_i are components of a unit vector tangential to Γ_r. Equating coefficients, we have

$$b_i = (Q_2 \tau_i + Q_3\, a_{ij} \nu_j),$$

hence

$$b_i \nu_i = Q_3\, a_{ij} \nu_i \nu_j = A(x)\, Q_3 ;$$

$$b_i \tau_i = Q_2 + Q_3\, a_{ij} \nu_i \tau_j .$$

Thus $Q_3 = b_i \nu_i / A = b_i x_i / rA = B(x)/r$, and (setting $y_i = r\, \tau_i$),

$$|Q_2| = |Q| = |b_i y_i - B(x)\, a_{ij} x_i y_j / r^2| / r \leq A^{1/2}(x)\, B_2(r)/r,$$

which establishes the lemma.

This yields the following expression for the second term on the right in (7):

$$\int b_i (u^2)_i = \int \left[Q(x)\, \partial_\tau (u^2) + \frac{B(x)}{r}\, \partial_\eta (u^2) \right].$$

Thus

$$\left| \int b_i (u^2)_i - \frac{B_0}{r} \int \partial_\eta (u^2) \right| \le \frac{B_2(r)}{r} \int A^{1/2} \, | \partial_\tau u^2 | + \frac{B_1(r)}{r} \int | \partial_\eta (u^2) |$$

$$= \frac{2B_2}{r} \int A^{1/2} u \, | \partial_\tau u | + \frac{2B_1}{r} \int u \, | \partial_\eta u | \le \frac{B_2}{r} \left[\varepsilon_1 \int | \partial_\tau u |^2 + \frac{1}{\varepsilon_1} \int A u^2 \right] +$$

$$+ \frac{B_1}{r} \left[\varepsilon_2 \int \frac{| \partial_\eta u |^2}{A} + \frac{1}{\varepsilon_2} \int A u^2 \right],$$

where ε_i are arbitrary positive numbers. Thus in our case (13) is replaced by

$$\frac{1}{2} H'(r) \ge \left\{ \frac{\lambda}{\mu_1 r^2} [\mu_0 (1 - \varkappa_1) - B_2 \, \varepsilon_1/2r] - \left[\frac{B_2}{2r\, \varepsilon} + \frac{B_1}{2r\, \varepsilon_2} + \frac{C}{r^2} \right] \right\} f +$$

$$+ [\beta - B_1 \, \varepsilon_2/2r] \, H^2/2f - B_0 H/2r,$$

which, with appropriate choices for ε_i, becomes

$$\left(H' + \frac{B_0}{r} H \right) \ge A_1(r) f/r^2 + A_2(r) \, H^2/f.$$

Setting $\psi = H/f$ and proceeding as before (σ is no longer necessarily con stant), we again obtain

$$\frac{\ddot{q}}{q} - A_2 \left(\frac{\dot{q}}{q} \right)^2 + \gamma \frac{\dot{q}}{q} \ge \delta$$

hence

$$\left(\frac{\ddot{q}}{\dot{q}} \right)^2 \ge \pi^2.$$

The rest of the argument is the same as before.

5. The case $Lu = \Delta u + \alpha r^{-2} x_i u_i = 0$.

This is one case in which an explicit solution is known, provided $\mathcal{D}$ is a cone; hence a check can be made on the accuracy of our estimates.

Let Ω be a domain on the unit sphere in E^n, and let $\mathcal{D}$ be the cone consisting of all rays through the origin and Ω. By definition (1),

$$\frac{\lambda(r)}{r^2} \leq \inf \int_{\Gamma_r} |\nabla_\tau \varphi|^2 = \frac{1}{r^2} \inf \int_{\Omega} |\nabla_\tau \varphi|^2 = \frac{\lambda_0}{r^2},$$

where λ_0 is the lowest eigenvalue of the Laplace-Beltrami operator Δ_1, acting on functions defined in Ω and vanishing on $\partial\Omega$. Thus we may (and shall) take $\lambda(r) \equiv \lambda_0$.

The explicit solution is constructed by separation of variables $u = R(r) K(\omega)$ where $r = |x|$ and (r, ω) are polar coordinates in E^n. The equation becomes

$$\frac{\frac{1}{r^{n-1}}(r^{n-1}R')'}{R} + \frac{1}{r^2}\frac{\Delta_l K}{K} + \frac{a R'}{rR} = 0.$$

Choosing K as the first eigenfunction: $\Delta_l K + \lambda_0 k = 0$, we obtain

$$\frac{1}{r^{n-1}}(r^{n-1} R')' + \frac{\alpha}{r} R' - \frac{\lambda_0}{r^2} R = 0,$$

which has solutions

$$R_\pm = r^{[-(n-2+\alpha)\pm\sqrt{4\lambda_0+(n-2+\alpha)^2}]/2}.$$

The corresponding solutions

$$u_\pm(x) = R_\pm(r) K(\omega)$$

satisfy

$$\int_{\Gamma_r} (u_\pm)^2 = C r^{n-1} (R_\pm)^2 = C r^{1-\alpha\pm\sqrt{4\lambda_0+(n-2+\alpha)^2}}. \tag{26}$$

We shall now see what results Theorem 3 yields for this particular case. Clearly $\mu_0 = \mu_1 = 1$, $\mu_2 = 0$, $l_1 = l_2 = n - 1$, $B(x) = \alpha = B_0$, $B_1 = B_2 = C = 0$, $\beta = 1$, $\Lambda_1 = 2\lambda_0$, $\Lambda_2 = 1/2$, $\gamma = \sigma - (2 - n - \alpha)$. We set $\sigma = 2 - n - \alpha$ so $\gamma = 0$, $\delta = 2\lambda_0 + \frac{1}{2}(n - 2 + \alpha)^2$ and $\pi = \sqrt{4\lambda_0 + (n - 2 + \alpha)^2}$.

Assumption A is clearly satisfied. In order to exclude the possible singularity at the origin we replace $\mathcal{D}$ now by the truncated cone $\mathcal{D}' = \mathcal{D} \cap \{x : |x| > 1\}$, not specifying the values of u on Γ_1. Then Theorem 3 yields the result that for $r \geq 1$, either

$$\int_{\Gamma_r} u^2 \geq C r^{1-\alpha}(1 + r^\pi) \geq 2\, C r^{1-\alpha+\sqrt{2\lambda_0+(n-2+\alpha)^2}}$$

or

$$\int_{\Gamma_r} u^2 \leq C^{-1} r^{1-\alpha-\sqrt{2\lambda_0+(n-2+\alpha)^2}}.$$

This is precisely the growth and/or decay behavior exhibited by the exact solution.

This and related arguments show that when $\mathcal{D}$ is conical near infinity or near the origin, and when the operator is that considered above, then Theorem 3 and its analog for growth near a finite boundary point give the best possible estimates.

6. A type of nonlinear equation.

In this section the preceding method will be employed to obtain growth estimates for quasilinear elliptic equations of the form

$$\partial^2 A_{ij}(u)/\partial x_i \partial x_j = 0 \tag{27}$$

(the summation convention is still used), where the functions $A_{ij}(u)$ are subject to certain restrictions outlined below.

Let $a_{ij}(u) = \frac{d}{du} A_{ij}(u)$. For each $u \geq 0$ we assume the form $a_{ij} \xi_i \xi_j$ to be positive definite:

$$0 < \mu_0(u) \leq a_{ij} \xi_i \xi_j \leq \mu_1(u) \tag{28}$$

for $|\xi|^2 = 1$. Furthermore we set

$$\mu_2(u) = \sup a_{ij} \xi_i \eta_j .$$

where the supremum is over vectors ξ_i and η_i such that $|\xi|^2 = |\eta|^2 = 1$, $\xi_i \eta_i = 0$. The main assumptions are that the following ratios are bounded independently of u for $u \geq 0$.

$$\alpha_1 = \sup_{u \geq 0} \frac{\mu_1}{\mu_0} < \infty, \tag{29a}$$

$$\alpha_2 = \sup_{u \geq 0} \frac{\mu_2}{\mu_0} < \infty ; \tag{29b}$$

furthermore we assume $\mu_0(u)$ is such that, for some positive constants α_0, α_3,

$$g(u) \equiv \int_0^u \xi \mu_0(\xi)\, d\xi \geq \alpha_0 u^2 \mu_0(u) \tag{30}$$

and

$$g(u) \leq \alpha_3 \left(\int_0^u \mu_0^{1/2}(\xi)\, d\xi \right)^2 \tag{31}$$

(in particular these are true if μ_0 is a nonnegative power of u).

Finally, we set

$$\varphi_{ij}(u) = \int_0^u \xi\, a_{ij}(\xi)\, d\xi, \tag{32}$$

and assume there exist constants l_1, l_2 such that

$$l_1 \leq \frac{\varphi_{ii}(u)}{\varphi_{ij}\,\xi_i\,\xi_j} - 1 \leq l_2 \tag{33}$$

for all $u \geq 0$, $|\xi|^2 = 1$.

THEOREM 4. *Let $\mathcal{D}$ be unbounded, and let $u \geq 0$ be a C^2 solution of (27) in $\mathcal{D}$ such that u is continuous in $\overline{\mathcal{D}}$ and $u = 0$ on $\partial\mathcal{D}$ for large enough $|x|$. Then there is a constant $C > 0$ such that either*

$$\int_{\Gamma_r} g(u(x)) \geq C r^{l_1+\sigma} \left[1 - \int \left[\exp \int - \sqrt{\beta_1\,\lambda(r) + \beta_2}\, dr/r \right] dr/r \right] \tag{34}$$

or

$$\int_{\Gamma_r} g(u(x)) \leq C^{-1}\, r^{l_2+\sigma} \int_r^\infty \left[\exp \int \sqrt{\beta_1\,\lambda(\bar{r}) + \beta_2}\, d\bar{r}/\bar{r} \right] d\bar{r}/\bar{r} \tag{35}$$

where

$$\sigma = (1 - l_2)/(2 - \tau/2), \tag{36}$$

$\varkappa_1$ *is arbitrary,* $0 < \varkappa_1 < 1$,

$$\tau = \varkappa_1\,\alpha_0/(\varkappa_1\,\alpha_1^2 + \alpha_2^2), \tag{37}$$

$$\beta_1 = \alpha_3\,\tau\,(1 - \varkappa_1), \tag{38}$$

$$\beta_2 = (1 - l_2)^2\,\tau/(4 - \tau). \tag{39}$$

Furthermore if $u = 0$ on all of $\partial\mathcal{D}$, then $u \equiv 0$ or (34) holds.

PROOF. In the following we shall often use commas to denote differentiations:

$$A_{ij,\,j} = \frac{\partial}{\partial x_j} A_{ij}(u(x)).$$

Also we retain the usage $u_j = \partial u/\partial x_j$ and the summation convention.

Equation (27) implies

$$\int_{\Gamma_r} u A_{ij,\,ij} = 0.$$

Lemma 1 may now be invoked to yield

$$0 = \int (uA_{ij,\,i})_{,\,j} - \int u_j A_{ij,\,i} = \frac{d}{dr}\int_{\Gamma_r} A_{ij,\,i}\, u\, \nu_j - \int a_{ij}\, u_i\, u_j . \tag{40}$$

We set $H(r) = \int_{\Gamma_r} A_{ij,\,i}\, u\, \nu_j$. On the one hand one sees

$$H(r) = \frac{1}{2}\int \partial_\eta (u^2),$$

where $\partial_\eta u = a_{ij}(u)\, \nu_i\, u_j(x)$; and on the other hand, by definition (32), $uA_{ij,\,i} = u a_{ij} u_i = \varphi_{ij,\,i}$, so

$$\begin{aligned} H(r) &= \int_{\Gamma_r} \varphi_{ij,\,i}(u)\, \nu_j = \int (\varphi_{ij}\nu_j)_{,\,i} - \int \varphi_{ij}(\nu_j)_{,\,i} \\ &= \frac{d}{dr}\int \varphi_{ij}\,\nu_i\,\nu_j - \frac{1}{r}\int [\varphi_{ii} - \varphi_{ij}\,\nu_i\,\nu_j]. \end{aligned} \tag{41}$$

On the basis of (40) we write

$$H'(r) = \int a_{ij}\, u_i\, u_j \geq \int \mu_0(u)\, |\nabla u|^2.$$

Lemma 2 holds as stated, with μ_1 and μ_2 in (9) functions of u. Furthermore defining

$$A(x) = u(x)^{-2}\, \varphi_{ij}(u(x))\, \nu_i\, \nu_j,$$

we note in passing, from (28), (32), (30), and (29a), that

$$\alpha_1\, g(u(x)) \geq A(x)\, u^2(x) \geq g(u(x)); \tag{42}$$

and proceed to define $f(r) = \int u^2 A(x)$. With this notation Lemma 3 is valid as written. Thus

$$H'(r) \geq \int \mu_0(u) \left[(1 - \varkappa_1) \, |\nabla_\tau u|^2 + \frac{\varkappa_1}{\varkappa_1 \mu_1^2 + \mu_2^2} (\partial_\eta u)^2 \right]. \tag{43}$$

From (28), (30) we obtain that $A(x) \geq \alpha_0 \mu_0(u(x))$, so by (37) the last term in this integrand may be replaced by

$$\frac{\varkappa_1 \alpha_0 \mu_0^2}{\varkappa_1 \mu_1^2 + \mu_2^2} \frac{(\partial_\eta u)^2}{A} \geq \tau \frac{(\partial_\eta u)^2}{A} .$$

Furthermore, setting $S(u) = \int\limits_0^u \mu_0^{1/2}(\xi)\, d\xi$, the first term on the right in (43) may be written

$$(1 - \varkappa_1) \int\limits_{\Gamma_r} |\nabla_\tau S|^2 \geq (1 - \varkappa_1) \lambda(r) r^{-2} \int\limits_{\Gamma_r} S^2 \geq (1 - \varkappa_1) \alpha_3 \lambda(r) r^{-2} \int g(u),$$

the last arising from assumption (31). Hence in all, (43) becomes

$$H'(r) \geq \beta_3 \lambda(r) r^{-2} f + \beta_4 H^2/f, \tag{44}$$

where $\beta_3 = (1 - \varkappa_1) \alpha_3$; $\beta_4 = \tau/4$.

Now from (41) and (33) we note that

$$f' - \frac{l_1}{r} f \geq H \geq f' - \frac{l_2}{r} f.$$

Thus defining $\psi(r) = H(r)/f(r)$ and $F_i(r)$ as in the proof of Theorem 2, we find that (16) and (17) hold. Again we define $\xi = \log r$ and $q(\xi)$ by (19), where now σ is the constant given by (36). This insures that $\gamma = 0$, so (20) and (21) hold, as well as the alternatives (22).

As before, the integration of (22) and application of (17*a*) and (19) implies alternatives which, with the aid of (42), can be put into the form (34), (35). The proof of the last statement of the theorem is also the same as before.

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