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A STURM - LIOUVILLE THEOREM FOR NONLINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

By MELVYN S. BERGER

In this paper we prove the existence of a countably infinite number of distinct normalized eigenfunctions with associated eigenvalues $\lambda_n \longrightarrow \infty$ for the non-linear operator equation $Au = \lambda Bu$. Here A and B are certain non-linear operators acting in a reflexive Banach space X. If $X = \overset{\circ}{W}_{m,p}(G)$ we are able to prove a somewhat stronger result for non-linear elliptic eigenvalue problems of the form:

$$\sum_{|\alpha| \leq m} D^\alpha A_\alpha (x, u, \dots, D^m u) = \lambda \{ \sum_{|\alpha| \leq m-1} D^\alpha B_\alpha (x, u, \dots, D^{m-1} u)\}$$

$$D^\alpha u_{/\partial G} = 0 \qquad 0 \leq |\alpha| \leq m-1$$

where G is a bounded domain in R^N.

As in many non-linear problems, the method of solution is non-constructional and is based on the study of topological invariant appropriate to the problem. The invariant used here is the notion of category of a set due to Ljusternik and Schnirelmann. (cf. J. Schwartz [23]). In 1937-8, Ljusternik [17, 18] applied these methods to eigenvalue problems for second order ordinary differential equations.

The class of operators considered in our study is a non-linear generalization of a bounded self-adjoint operator, namely the class of abstract variational operators. The basic properties of these operators are taken up in PART I. Each abstract variational operator A gives rise to an infinite dimensional manifold ∂A_R. The relation between A and ∂A_R is taken up

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in I.2, for operators satisfying various monotonicity hypotheses. (cf. Leray-Lions [15]).

In PART II the non-linear elliptic partial differential operators of order $2m$, analogous to the abstract operators of PART I, are defined and investigated. The appropriate Sobolev space $\overset{\circ}{W}_{m,p}(G)$ is chosen by the order of growth of the non-linear operator A. Thus if A is linear, the appropriate Sobolev space in our study is the Hilbert space $\overset{\circ}{W}{}^{m}_{,2}(G)$. This part of our work should be read in conjunction with Vishik [25] where many interesting and difficult examples are considered. See also a paper of Meyers and Serrin [27].

PART III uses the previous results to construct the first eigenfunction and eigenvalue λ_1 directly, without use of any topological invariant. Higher eigenfunctions and eigenvalues pose quite a different problem as the notion of orthogonality has no immediate non-linear analogue. For second order ordinary differential equations higher order eigenfunctions can be studied, as in Nehari [20], by considering their zeros on the fundamental interval $[a, b]$.

The construction of higher order eigenfunctions is taken up in PART IV. The basic topological results on category are sketched and for the first time the assumption of oddness on the variational operators A and B plays a critical role. The asymptotic behavior of the eigenvalues $\{\lambda_n\}$ is also proved by topological arguments.

The present paper concludes with the example of PART V. Due to the lack of a principle of superposition we cannot expect non-linear eigenvalue problems to play the same role as in linear problems. Nonetheless non-linear eigenvalue problems arise in such diverse fields as the deformation of Riemannian structures in differential geometry, Reynolds number problems in steady-state viscous fluid flow, the Hartee-Fock approach to Schrodinger's equation for many particle systems, vibrations of heavy strings, rods and plates, non-linear programming and the utility theory of mathematical economics to mention only a few.

Fine surveys of the extensive previous studies in non-linear eigenvalue problems are to be found in the articles of L. Rall [21] and C. L. Dolph and G. J. Minty [8] and the bibliographies of the books of Vainberg [24], Krasnoselskii [11] and El'sgol'c [9].

Eingenvalue problems for non-linear elliptic partial differential equations have been studied by the author in [1], [2], [3], and [4], F. E. Browder [5] and [6] and N. Levinson [14]. The present work contains extensive generalizations of the research announcement [4], and Browder [6].

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PART I

Abstract non-linear operators arising from variational problems.

We shall study the class of non-linear operators that correspond to linear bounded self-adjoint operators with compact resolvents. It is within this framework that the classical Sturm-Liouville Theorem has a non-linear analogue.

A few comments on the study of non-linear operators via functional analysis are in order. First we shall study non-linear operator equations in reflexive Banach spaces. This enables us to carry over the direct method of the Calculus of Variations into an abstract setting and at the same time to study non-linearities within highly non-linear growth properties. Secondly it is important to specify the action of a non-linear operator on the weak topology of a reflexive Banach space X. This is superfluous in the linear case as the two possibilities of continuity from the strong or weak topology of X into the weak topology of X^* are automatically satisfied by any bounded linear operator of $X \longrightarrow X^*$.

Abstract non-linear operators can be classified independently of variational problems. This was carried out successfully by J. Schauder beginning in 1927 for completely continuous operators by introducing the topological methods of fixed point and mapping degree for this class of operators. Recently I. M. Vishik, G. J. Minty and F. E. Browder have studied various classes of monotone operators, which are also independent of variational arguments. The study of abstract non-linear operators arising from variational problems was carried out by Gateaux and Frechet among others. By introducing special topological methods for this class of operators both L. Ljusternik and M. Morse obtained many new and striking results. (For references to these works we refer to Elsgolc [9]). The present study combines elements of each approach mentioned above.

1.1 Abstract Variational Operators and Infinite Dimensional Manifolds.

Let X be a reflexive separable Banach space over the reals with conjugate space X^*. Suppose A is a mapping $X \longrightarrow X^*$, and denote by $\langle u, v \rangle$ the inner product of $u \in X$ and $v \in X^*$.

DEFINITION I.1. A functional $\Phi(u)$ has a Gateaux derivative $\Phi'(u, v)$ in the direction v if

$$\lim_{t \to 0} \left\{ \frac{\Phi(u + tv) - \Phi(u)}{t} \right\} = \Phi'(u, v) .$$

DEFINITION I.2. A is a variational operator if there is a functional $\Phi(u)$ defined on X such that the Gateaux derivative of $\Phi(u)$ in the direction v is $\langle v, Au \rangle$ for every $v \in X$.

LEMMA I.1.1 Let $A : X \to X^*$ be a mapping continuous from the strong topology of X into the weak topology of X^*. Then A is a variational operator if and only if for all $u, v \in X$

$$\text{(1)} \qquad \int_0^1 \langle u, A(su) \rangle \, ds - \int_0^1 \langle v, A(sv) \rangle \, ds = \int_0^1 \langle u - v, A(v + s(u - v)) \rangle \, ds .$$

Furthermore, the functional $\Phi(u)$ associated with $A(u)$ can be written

$$\text{(2)} \qquad \Phi(u) = \int_0^1 \langle u, A(su) \rangle \, ds .$$

PROOF :

Clearly if conditions (1) and (2) are valid,

$$\Phi(u + tv) - \Phi(u) = t \int_0^1 \langle v, A(u + stv) \rangle \, ds.$$

Hence the Gateaux derivative of $\Phi(u)$ in the direction v is $\langle v, Au \rangle$. On the other hand, if A is a variational operator, there is a functional $\Phi(u)$ such that

$$\frac{d}{dt} [\Phi(u + tv)] = \frac{d}{d\varepsilon} [\Phi(u + tv + \varepsilon v)]_{\varepsilon=0} = \langle v, A(u + tv) \rangle.$$

Integrating with respect to t between 0 and 1 we obtain

$$\Phi(u) - \Phi(v) = \int_0^1 \langle u - v, A(v + s(u - v)) \rangle \, ds.$$

Setting $v = 0$, $\Phi(0) = 0$ in this last formula we obtain

$$\Phi(u) = \int_0^1 \langle u, A(su) \rangle \, ds.$$

EXAMPLE

If A is linear, and X is a Hilbert space, formula (1) clearly is equivalent to the fact that A is self-adjoint. Thus the operators satisfying (1), can be regarded as non-linear generalizations of self-adjoint linear operators.

Using formula (2), we now define for each variational operator certain sets in X, that will be of interest throughout the present work.

DEFINITION I.3. Let R be a fixed positive number then

$$A_R \equiv \left\{ u/u \in X, \int_0^1 \langle u, A(su) \rangle \, ds \leq R \right\}$$

$$\partial A_R \equiv \left\{ u/u \in X, \int_0^1 \langle u, A(su) \rangle \, ds = R \right\}.$$

If A is a bounded linear self-adjoint operator, and X is Hilbert space ∂A_R represents a sphere in X with respect to the operator A. For non-linear operators A, ∂A_R is an infinite dimensional manifold and will serve as a non-linear normalization for elements $u \in X$. It will be of interest to determine the relationship between the properties of the operator A and the associated set ∂A_R. For the present we note that under the conditions of lemma I.1.1 ∂A_R is a closed set. This follows from the fact that ∂A_R is inverse image of the continuous function $\Phi(u)$ and the point R on the real axis.

1.2. Special Classes of Variational Operators.

First we consider the simplest class of variational operators arising in eigenvalue problems, monotone operators. These operators are analogous, on the one hand, to positive self-adjoint linear operators in a Hilbert space and on the other hand to operators arising from variational problems with convex integrands.

DEFINITION I.4.

Let $A : X \longrightarrow X^*$ be a variational operator. Then A is of class I if

(i) A is bounded

(ii) A is continuous from the strong topology of X to the weak topology of X^*

(iii) $A(-u) = -A(u)$

(iv) Coerciveness $\int_0^1 \langle u, A(su) \rangle \, ds \longrightarrow \infty$, as $\| u \| \longrightarrow \infty$

(v) Monotonicity, $\langle u - v, A(u) - A(v) \rangle \geq 0$, (for any $u, v \in X$).

LEMMA I.2.1.

Let A be as variational operators of class I, then ∂A_R is a closed, bounded set in X. Furthermore $\| u \| \geq k(R) > 0$ where $k(R)$ is a constant independent of $u \in \partial A_R$. A_R is a weakly closed, bounded convex set.

PROOF. The boundedness of ∂A_R and A_R follows from the coerciveness assumption (iv). Indeed suppose there is a sequence $\{u_n\} \in \partial A_R$ with $\| u_n \| \longrightarrow \infty$ then $\int_0^1 \langle u, A(su) \rangle \, ds \longrightarrow \infty$ by assumption, which is an obvious contradicition. To demonstrate the convexity of A_R, note for any t, $0 \leq t \leq 1$

$$\Phi(tu + (1-t)v) - \Phi(v) = t \int_0^1 \langle u - v, A(v + ts(u - v) \rangle \, ds \text{ by Lemma (1.1.1)}$$

$$\leq t \int_0^1 \langle u - v, A(v + s(u - v)) \rangle \, ds \ (by\ (v)) \leq t(\Phi(u) - \Phi(v)) .$$

Thus $\Phi(tu + (1-t)v) \leq R$ as required. The closure of A_R or ∂A_R can be demonstrated directly as follows. If $u_n \longrightarrow u$ strongly, $Au_n \longrightarrow Au$ weakly, and by lemma 1.1, the boundednes of ∂A_R and Schwarz's inequality

$$| \Phi(u) - \Phi(u_n) | \leq \int_0^1 \langle u - u_n, A(u_n + s(u - u_n)) \rangle \, ds \leq K \| u - u_n \|$$

where K is constant independent of n $\Phi(u) \longrightarrow \Phi(u_n)$ as $n \longrightarrow \infty$.

The weak closure of A_R is thus, a consequence of the theorem of Mazur as A_R is a bounded, closed, convex set. We now demonstrate that $\| u \|$ is uniformly bounded above 0 for $u \in \partial A_R$. Indeed, by monotonicity of A,

$$\langle u, Au \rangle \geq \int_0^1 \langle u, A(su) \rangle \, ds = R.$$

Hence by Schwarz's inequality and the boundedness of A

$$\| u \| \, \| Au \| \geq R \quad \text{or} \quad \| u \| \geq \frac{R}{\sup\limits_{u \in A_R} \| Au \|} = C(R) > 0.$$

The effect of the monotonicity assumption (v) in the direct method of the calculus of variations is to force the weak limit of a minimizing sequence to converge to the solution of the associated Euler-Lagrange equation. The following lemma is the abstract analogue of this fact.

LEMMA I.2.2.

Let A be a variational operator of class I. Suppose $u_n \longrightarrow u$ weakly in X, $Au_n \longrightarrow v$ weakly in X^*, and that $\langle u_n, Au_n \rangle \longrightarrow \langle u, v \rangle$, then $Au_n \longrightarrow Au$ weakly in X^*.

PROOF.

By monotonicity $\langle u_n - w, \; Au_n - Aw \rangle \geq 0$.

Letting $n \longrightarrow \infty$ $\langle u - w, \; v - Aw \rangle \geq 0$.

Setting $w = u - \lambda z$, $\lambda \geq 0$ $\langle \lambda z, \; v - A(u - \lambda z) \rangle \geq 0$.

Dividing by λ and letting $\lambda \longrightarrow 0$ $\langle z, v - Au \rangle \geq 0$.

As z is arbitrary, $v = Au$ and $Au_n \longrightarrow Au$ weakly.

We now extend the above results to a broader class of operators. These operators are termed « principally monotone » and are analogous to those operators arising from variational problems with integrand convex in the highest order derivatives.

DEFINITION I.5. (due to N. Meyers)

Let $A : X \longrightarrow X^*$ be a variational operator. Then A is of class II if A satisfies assumptions (i)-(iv) of Definition I.4 together with the extra coerciveness assumption $\langle u, Au \rangle \longrightarrow \infty$ as $\| u \| \longrightarrow \infty$ and in place of the monotonicity assumption (v) we have

(v′) $A(u, v) = P(u, v) + R(u, v)$ where $A, P, R : X \times X \to X^*$ and $A(u, u) = P(u) + R(u) = Au$ such that

(a) $\langle v - w, P(u, v) - P(u, w) \rangle \geq 0$
(b) If $u_n \to u$ weakly in X, and $\langle u_n - u, Pu_n - Pu \rangle \to 0$, then $Ru_n \to Ru$ weakly in X^*
(c) The form $\langle w, R(u) \rangle$ as a weakly continuous functional in both variables jointly.
(d) For fixed, v, $P(u, v)$ and $R(u, v)$ are continuous from the weak topology of X to the strong topology of X^*
(e) $P(u, v)$ and $R(u, v)$ are continuous from the strong topology of X to the weak topology of X^* in each variable uniformly, with respect to bounded sets in the alternate variable.

LEMMA I.2.3.

Let A be a variational opertion of class II. Then ∂A_R is a closed, bounded set with $\| u \|$ uniformly bounded above 0 for $u \in \partial A_R$. ∂A_R is homeomorphic to some sphere $\{\| u \| = K\}$ for sufficiently large R. Furthermore A_R is closed, bounded and weakly closed.

PROOF.

As in the previous lemma, the boundedness of ∂A_R and A_R is an immediate consequence of coerciveness.

Furthermore for $u \in \partial A_R$, by Schwarz's inequality

$$R = \int_0^1 \langle u, A(su) \rangle \, ds \leq \| u \| \sup_{0 \leq s \leq 1} \| A(su) \| \leq k(R) \| u \|$$

where $k(r)$ is a constant independent of $u \in \partial A_R$.

Thus $\| u \| \geq \dfrac{R}{k(R)} > 0$.

We now show that for R sufficienty large, the mapping $\sigma : u \to \left(\dfrac{k_R u}{\| u \|} \right)$ is a homeomorphism of ∂A_R and $\partial \Sigma_{k_R} = \{u / u \in X, \| u \| = k_R\}$ for sufficiently large k_R. In the inverse mapping σ^{-1} is defined by the dilation $v \to tv$. We choose k_R to a large number such that $\partial \Sigma_{k_R}$ lies entirely inside ∂A_R, i. e. $t \geq 1$. Thus to show σ^{-1} is well defined we show that if $t_1 v$ and $t_2 v \in \partial A_R$, for sufficiently for R, $t_1 = t_2$. This result follows from the coerciveness of

$\langle u, Au \rangle$ and $\int_0^1 \langle u, A(su) \rangle \, ds$ and lemma 1.1. Indeed, as $R \to \infty$ and $u \in \partial A_R$, $\| u \| \to \infty$. Otherwise there is a sequence $\{u_n\}$ such that $\| u_n \| \leq M$ $u_n \in \partial A_{Rn}$ with $R_n \to \infty$, hence $\| Au_n \| \leq k(M)$ and $\int_0^1 \langle u_n \, A(su_n) \rangle \, ds \leq k(M)$ where $\bar{k}(M)$ and $k(M)$ are constants independent of n. Thus $k_R \to \infty$ as $R \to \infty$. Now suppose $t_1, t_2 \geq 1$ and $\Phi(t, v) = \Phi(t_2 v)$ with $t_1 \neq t_2$, then, using lemma 1.1,

$$O = \int_0^1 \langle v, A(st_1 + (1-s) t_2) v \rangle \, ds .$$

As $st_1 + (1-s) t_2 \geq \min(t_1, t_2) \geq 1$ this equality is incompatible with the coercivity assumption that

$$\langle v, Av \rangle \to \infty \; as \; \| v \| \to \infty.$$

Finally we demonstrate the weak closure of A_R. This fact is equivalent to the weak lower-semicontinuity of the functional $\int_0^1 \langle u, A(su) \rangle \, ds$. Let $u_n \to u$ weakly, $u_n \in A_R$, then, setting $v_n(s) = u + s(u_n - u)$,

$$\underline{\lim}_{n \to \infty} \{\Phi(u_n) - \Phi(u)\} = \underline{\lim}_{n \to \infty} \int_0^1 \langle u_n - u, Av_n(s) \rangle \, ds$$

$$= \underline{\lim}_{n \to \infty} \int_0^1 \langle u_n - u, Pv_n(s) + Rv_n(s) \rangle \, ds =$$

$$= \underline{\lim}_{n \to \infty} \int_0^1 \langle u_n - u, Pv_n(s) \rangle \, ds \; (by) \; v'(c))$$

$$= \underline{\lim}_{u \to \infty} \int_0^1 \langle u_n - u, P(v_n(s), v_n(s)) - P(v_n(s), u) \rangle$$

$$+ \underline{\lim}_{n \to \infty} \int_0^1 \langle u_n - u, P(v_n(s), u) \rangle \geq 0 \qquad \text{by (v}'\text{(a), and v}'\text{(d))}$$

Thus $\underline{\lim}_{n \to \infty} \Phi(u_n) \geq \Phi(u)$ and $\Phi(u) \leq R$.

LEMMA I.2.4.

Let A be a variational operator of class II. Suppose $u_n \to u$ weakly in X, $Au_n \to v$ weakly in X^* and that $\langle u_n, Au_n \rangle \to \langle u, v \rangle$ then $Au_n \to Au$ weakly in X^*

PROOF.

Under the assumptions of the lemma, we note that

$$\langle u_n - u, \ Au_n - Au \rangle \to 0$$

and

$$\langle u_n - u, \ Ru_n - Ru \rangle \to 0.$$

Subtracting

$$\langle u_n - u, \ Pu_n - Pu \rangle \to 0 .$$

Hence by v′ (b) $Ru_n \to Ru$ weakly. Now let w be an arbitrary element of X, by v′(a) $\langle u_n - w, \ A(u_n, u_n) - A(u_n, w) \rangle \geq \langle u_n - w, R(u_n, u_n) - R(u_n, w) \rangle$ Letting $n \to \infty$ and using the fact that $Ru_n \to Ru$ weakly we obtain

$$\langle u - w, \ v - A(u, w) \rangle \geq \langle u - w, \ R(u, u) - R(u, w) \rangle.$$

Setting $w = u + \lambda z$, for $\lambda > 0$, and letting $\lambda \to 0$ as in lemma I.2.2. we obtain $Au = v$ and $Au_n \to Au$ weakly in X^*.

DEFINITION I.6.

Let $B : X \to X^*$ be a variational operator. Then B is of class III if

(i) B is continuous from the weak topology of X to the strong topology of X^*

(ii) $$B(-u) = -B(u)$$

(iii) $$\langle u, Bu \rangle > 0 \quad \text{for} \quad u \neq 0.$$

LEMMA I.2.5.

$\Phi(u) = \int_0^1 \langle u, B(su) \rangle \, ds$ is a weakly continuous functional. Thus ∂B_r is weakly closed, and on ∂B_r, $\| u \|$ is uniformly bounded above 0. Furthermore if A is a variational operator of class I or II, $\langle u, Bu \rangle$ is uniformly bounded away from 0 on ∂A_R, for sufficiently large R.

PROOF.

To prove $\Phi(u)$ is a weakly continuous functional, we let $u_n \longrightarrow u$ weakly in X. The using lemma 1.1

$$\Phi(u_n) - \Phi(u) = \int_0^1 (u_n - u, B[u + s(u_n - u)])\, ds.$$

As $B[u + s(u_n - u)]$ converges strongly to $B(u)$, $\Phi(u_n) \longrightarrow \Phi(u)$. Thus the weak closure of ∂B_r is immediate. If $\| u \| \longrightarrow 0$, for $u_n \in \partial B_r$, is weakly closed, $0 \in \partial B_r$; a fact contradicting (iii). A similar argument holds for the form $\langle u, Bu \rangle$ on ∂A_R.

LEMMA I.2.6.

Let X be a reflexive Banach space over the reals with a countable biorthogonal basis, and let B be a variational operator of class III defined on a bounded set S of X. Then for any $\varepsilon > 0$, there is a integer $N = N(\varepsilon)$ and a finite dimensional projection $P_N : X \longrightarrow R_N$ such that for any $u \in X$

$$| \Phi(P_N u) - \Phi(u) | < \varepsilon, \text{ where } \Phi(u) = \int_0^1 \langle u, B(su) \rangle\, ds\,.$$

PROOF.

This result is an immediate consequence of the fact that $\Phi(u)$ is a weakly continuous functional and Lemma 2 of Citlanadze [7]. We now make use of the fact that the variational operators A of class I, II, and III are odd functions

DEFINITION I.7.

Let R sufficiently large, so that ∂A_R is homeomorphic to a sphere, let $\partial \bar{A}_R$ be the set obtained by identifying u and $-u$ on ∂A_R.

LEMMA I.2.7.

$\partial \bar{A}_R$ is homeomorphic to P_∞, the infinite dimensional real projective space.

PROOF.

First we note that P_∞ can be obtained by identifying antipodal points of the sphere $\{\| x \| = k\}$. Thus we have the following diagram

$$\begin{array}{ccc} \partial A_R & \xleftrightarrow{\sigma} & \partial \Sigma_k \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ \partial \bar{A}_R & \xleftrightarrow[\bar{\sigma}]{} & P_\infty \end{array}$$

here $\bar{\sigma}$ denotes the induced mapping defined by σ and i.

1.3. Trajectories on Infinite Dimensional Manifolds ∂A_R.

Let A be a variational operator of class I or II. Then for fixed $R > 0$, a trajectory is a continuous function $f(u, t)$:

$\partial A_R \, x \, [-t_1, t_1] \rightarrow \partial A_R$, such that $f(u, 0) = u$. Then we can define trajectories on ∂A_R by means of the implicit function theorem. (For a finite dimensional spaces or Hilbert spaces, the methods of orthogonal trajectories have been long known.) We now study two additional ways of defining trajectories in spaces without a notion of orthogonality.

LEMMA I.3.1. (due to N. Meyers).

Let π be an arbitrary element of X a reflexive Banach space, then if $u \in \partial A_R f(u, t) = u + t\pi + a(t) u$ defines a trajectory on ∂A_R, for suitable (at) and t sufficiently small.

PROOF.

Let $\Phi(u) = \int_0^1 \langle u, A(su) \rangle \, ds$, then by lemma 1.1, setting $f(u, t) = u +$ $+ t\pi + a(t) u$, we must have

$$\frac{d}{dt} \Phi(f(u, t)) = 0 \text{ and thus we obtain}$$

$$a'(t) = \frac{\langle \pi, A(f(u, t)) \rangle}{\langle u, A(f(u, t)) \rangle}$$

$$a(0) = 0 .$$

By the existence theory of non-linear ordinary differential equations and the fact that $\langle u, Au \rangle$ is uniformly bounded above 0 we can conclude that $f(u, t)$ defines a trajectory on ∂A_R for sufficiently suall t. In order to study the dependence of trajectories $f(u, t)$ on the variable $u \in \partial A_R$, we prove the following:

LEMMA I.3.2.

Let $A : X \longrightarrow X^*$ be a variational operator of class I or II. Let $\mathfrak{g}$ be a compact set of ∂A_R, R sufffciently large, and suppose $C: \mathfrak{g} \longrightarrow \partial \Sigma$, is a continuous mapping. Then given $\varepsilon > 0$, there is a $t_\varepsilon > 0$, independent of $u \in \mathfrak{g}$, such that

$$f(u, t) = u + t\ Cu + t\theta(t, u)\, u$$

is a trajectory of ∂A_R for $|t| < t_\varepsilon$, where $\theta(t, u)$ is a continuous real-valued function with $\left| \theta(t, u) - \dfrac{\langle Cu, Au \rangle}{\langle u, Au \rangle} \right| \leq K_\varepsilon$, where K_ε is a constant depending only on R.

PROOF.

To show $f(u, t)$ defines a trajectory on ∂A_R we prove $\Phi(f(u, t)) - \Phi(f(u, 0)) = 0$ for suitable t. Using lemma 1.1 and setting $v(t, s, u) = u + ts(Cu + \theta(t, u)\, u)$ we obtain fron (1)

$$\int_0^1 \langle Cu + \theta(t, u)\, u, A(v(t, s, u)) \rangle\, ds = 0.$$

Let $G(\theta)$ be the right hand side of the equality, then $G(\theta) = \theta \langle u, Au \rangle + \langle Cu, Au \rangle + H(\theta)$.

$$\text{Where } H(\theta) = \int_0^1 \langle \theta u + Cu, Av(t, s, u) - Av(0, 0, u) \rangle\, ds\, .$$

We restrict θ for the time being to the interval $[-M, M]$, where M is a number to be determined independent of $u \in \mathfrak{g}$. Then the set $E_M = \{ v(t, s, u)$ $0 \leq s \leq 1$, $u \in \mathfrak{g}$, $|t| \leq B$, $|\theta| \leq M \}$ is compact in X, and thus on E_M, A is uniformly continuous from the strong topology of X to the strong topology of X^*. Thus give $\varepsilon > 0$, there is a $t_2 > 0$, such that for each t with $|t| \leq t_\varepsilon$, $|H(\theta)| < \varepsilon$.

Suppose now $\varepsilon > 0$ is given and t is chosen so small that $|H(\theta)| < \varepsilon$. Let $\theta' = \langle u, Au \rangle^{-1} \{\varepsilon - \langle Cu, Au \rangle\}$ then $G(\theta') = \varepsilon + H(\theta')$. Hence if $\theta' \in [-M, M]$, $|H(\theta')| < \varepsilon$ and $G(\theta') > 0$. Similarily letting $\theta'' = \langle u, Au \rangle \{-\varepsilon - \langle Cu, Au \rangle\}$ $G(\theta'') < 0$.

As $G(\theta)$ is a continuous real-valued function of θ, for some $\bar{\theta}$ between θ' and θ'', $G(\bar{\theta}) = 0$. Let $\bar{\bar{\theta}}$ be the largest such $\bar{\theta}$ where $G(\bar{\theta})$ changes sign. Then $\bar{\bar{\theta}}$ is a continuous function of u and t.

To complete the lemma we show that a number M can be chosen independent of $u \in \mathfrak{g}$, such that the set of real numbers $\theta'(u)$ and $\theta''(u) \in$ $\in [-M, M]$.

First as $u \in \partial A_R$, $(u, Au) \geq (R) > 0$ (for sufficiently R) and the set $\{\| Au \|\}$ is uniformly bounded, by M say.

Thus by Schwarz's inequality for θ'' we have

$$\theta \leq \langle u, Au \rangle^{-1} (\| Cu \| \, \| Au \| + \varepsilon) \leq \frac{1}{k(R)} (M + \varepsilon).$$

As $\overline{\overline{\theta}}$ lies between θ' and θ'' we have

$$\left| \theta + \frac{(Cu, Au)}{(u, Au)} \right| \leq \frac{2\varepsilon}{\langle u, Au \rangle} \leq \left(\frac{2}{k(R)} \right) \varepsilon = \overline{K}_\varepsilon$$

where $\overline{K}_\varepsilon$ is constant depending only on R.

PART II

Elliptic Non-linear Partial Differential Operators Arising from Variational Problems

In this section we apply the abstract principles of PART I to consider concrete non-linear operators arising in the theory of partial differential equations.

II.1. Notations and Preliminary Facts.

Let G be a bounded domain in R^N. We consider various classes of real-valued functions defined on G and their integrals with respect to N-dimensional Lebesgue measure. Derivatives (in the sense of L. Schwartz) are written

$$Dj = \frac{\partial}{\partial xj}, \; 1 \leq j \leq N, \; D^\alpha = \prod_j D_j^{\alpha_j} \text{ with } |\alpha| = \sum_{j=1}^{N} \alpha_j .$$

We consider the following Banach spaces

$$W_{m,p}(G)=\{u \mid D^\alpha u \in Lp(G) \mid \alpha \mid \leq m\} \qquad (1<p<\infty)$$

$$\overset{\circ}{W}_{m,p}(G)=\text{closure of } C_0^\infty(G) \text{ in } W_{m,p}(G).$$

If we choose the norm of $W_{m,p}(G)$ as

$$\| u \|_{m,p}=\{\sum_{|\alpha|\leq m} \| D^\alpha u \|_{0,p}^p\}^{\frac{1}{p}},$$

both $W_{m,p}(G)$ and $\overset{\circ}{W}_{m,p}(G)$ are separable reflexive Banach spaces. The space conjugate to $\overset{\circ}{W}_{m,p}(G)$ is denoted $W_{-m,q}(G)$, where $\dfrac{1}{p}+\dfrac{1}{q}=1$. An arbitrary element ψ of $W_{-m,q}(G)$ can be written $\psi=\sum_{|\alpha|\leq m} D^\alpha F_\alpha(x)$, where $F_\alpha(x) \in Lq(G)$. Furthermore $\overset{\circ}{W}_{m,p}(G)$ is a uniformly convex Banach space with biorthogonal basis [cf. Lions [16]].

A result of basic importance throughout the present work is Sobolev's Imbedding Theorem. Suppose X and X_1 are two topological spaces, then X is imbedded continuously in X_1, if the imbedding operator $i(y)=y$ is a continuous $(1-1)$ mapping from X to X_1 and we write $X \subset X_1$. If i is a compact mapping, we say the imbedding is compact.

THEOREM II.1.1 (Sobolev)

Let G be a bounded domain in R^N, then

(1) $\overset{\circ}{W}_{m,p}(G) \subset W_{s,t}(G)$ for $\dfrac{1}{p}-\dfrac{m-s}{N} \leq \dfrac{1}{t} \leq \dfrac{1}{p}$ the imbedding is compact if $\dfrac{1}{p}-\dfrac{m-s}{N}<\dfrac{1}{t}\leq\dfrac{1}{p}$

(2) $\overset{\circ}{W}_{m,p}(G) \subset C_{s,t}(G)$ for $\dfrac{1}{p}-\dfrac{m-s-t}{N}<0$ the imbedding is compact.

Defined on $\overset{\circ}{W}_{m,p}(G)$ we consider the operators

$$Au=\sum_{|\alpha|\leq m} D^\alpha A_\alpha(x, u, \dots, D^m u) \tag{1}$$

$$Bu=\sum_{|\alpha|\leq m-1} D^\alpha B_\alpha(x, u, \dots, D^{m-1} u) \tag{2}$$

and their associated non-linear Dirichlet forms

$$a(u, v) = \sum_{|\alpha| \leq m} \int_G A_\alpha (x, u, \dots, D^m u) D^\alpha v$$

$$b(u, v) = \sum_{|\alpha| \leq m-1} \int_G B_\alpha (x, u, \dots, D^{m-1} u) D^\alpha v.$$

We now extend the definition of each operator in the form (1) and (2) to the space $\overset{\circ}{W}_{m,p}(G)$. The operator $A: W_{m,p}(G) \longrightarrow W_{-m,q}(G)$ so defined will correspond to the abstract operators of PART I. Let $u, v \in \overset{\circ}{W}_{m,p}(G)$ and suppose $A_\alpha (k, u, \dots, D^m u) \in Lq$, $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$, for $|\alpha| \leq m$ then

$$a(u, v) = \sum_{|\alpha| \leq m} \int_G A_\alpha (x, u, \dots, D^m u) D^\alpha v$$

is a continuous linear functional on $\overset{\circ}{W}_{m,p}(G)$ in v. Thus we write $a(u, v) = \langle v, Au \rangle$, where $A: W_{m,p}(G) \longrightarrow W_{-m,q}(G)$. (A similar procedure holds for B).

We shall assume throughout this work that the functions $A_\alpha(x, u, \dots, D^m u)$ and $B_\alpha(x, u, \dots, D^{m-1}u)$ are obtained as Euler-Lagrange expressions from the functions $A(x, \dots, D^m u)$ and $B(x, \dots, D^{m-1}u)$ according to the formula

$$A_{p\alpha}(x, \dots, D^m u) = (-1)^{|\alpha|} A_\alpha (x, u, \dots, D^m u)$$

$$B_{p\alpha}(x, \dots, D^{m-1}u) = (-1)^{|\alpha|} B_\alpha (x, u, \dots, D^{m-1}u).$$

By lemma I.1.1 we note that $A(x, \dots, D^m u)$ is defined by the formula

$$\int_0^1 a(su, u)\, ds = \int_G A(x, u, \dots, D^m u).$$

II.2. Measure Theoretic Lemmas.

DEFINITION: II.1. A function $g(x, y_1, y_2, \dots, y_s)$ defined on $G \times R^s$ is continuous in the sense of Caratheodory if it is continuous with respect to $(y, \dots, y_s)$ for almost all $x \in G$ and measurable in G with respect to x for avery fixed $(y, \dots, y_s) \in R^s$.

LEMMA II.2.1 (Nemytski).

Let $g(x, y_1, \dots, y_0)$ be continuous in the sense of Caratheodory on $G \times R^n$. Then if $[u_i^{(n)}]$, $i = 1, \dots, s$, is a family of almost everywhere finite and measurable functions converging iu measure on G to a (finite almost everywhere) function u_i, $\{g(x, u_1^{(n)}, u_2^{(n)}, \dots, u_s)\}$ converges in measure to $g(x, u_1, u_2, \dots, u_s)$ on G. For the proof of this well-known result we refer to the book of Vainberg [24] (Theorem 18.6).

LEMMA II.2.2.

Let g_n, $g \in Lp(G) \, \| g_n \|_{0,p} \leq c \; (1 < p < \infty)$.
if $g_n \to g$ in measure on G, then
$g_n \to g$ weakly in $Lp(G)$.

PROOF.

(Lions-Leray [15]). Let $E(N) = \{x \mid x \in G, \; |g_n(x) - g(x)| \leq 1, \; n > N.\}$ Then meas $E(N) \to$ meas G as $N \to \infty$. Hence the set of functions θ_N, zero a. e. except in $E(N)$ are dense in $Lp(G)$ as $N \to \infty$. Thus

$$\int_G \theta_N(x)\,[g_n - g] \to 0$$

and the result follows immediately.

LEMMA II.2.3 (Serrin).

Let $u_n \to u$ weakly in $\overset{\circ}{W}_{mp}(G)$. Suppose on $G \times R^n \times R^t$, the functions $A_\alpha(x, y, z)$ are continuous in the sense of Caratheordory and satisfy the condition

$$(1) \qquad \sum_{|\alpha| = m} [A_\alpha(x, y, z') - A_\alpha(x, y, z)]\,[z'_\alpha - z_\alpha] > 0.$$

for $(z' \not\equiv z)$ a. e. in G where $z = (z_{\alpha_1}, \dots, z_{\alpha_t})$ and $|\alpha_i| = m$.

Then if

$$(2) \qquad \sum_{|\alpha| = m} \int_G [A_\alpha(x, u_n, \dots, D^\alpha u_n) - A_\alpha(x, u, \dots, D^\alpha u)]\,[D^\alpha u_n - D^\alpha u] \to 0$$

as $n \to \infty$, $D^\alpha u_n \to D^\alpha u$, $|\alpha| = m$, in measure on G.

For the proof of this result we refer to the forthcoming paper or Serrin and Meyers [27].

REMARK. For $u_n \longrightarrow u$ weakly in $\overset{\circ}{W}_{m,p}(G)$, we have a stronger result for the lower order derivatives of u_n. Indeed by the compactness of the imbedding $\overset{\circ}{W}_{m,p}(G) \longrightarrow \overset{\circ}{W}_{m-1,p}(G)$ we have $D^\alpha u_n \longrightarrow D^\alpha u$ strongly in $Lp(G)$.

LEMMA II.2.4 (Vainberg).

Let $g(x, y_1, \dots, y_s)$ be a function defined on $G \times R^s$, continuous in the sense of Caratheodory. Suppose the operator $g(x, u_1, \dots, u_s) = \tilde{g}(u_1, \dots, u_s)$ maps $Lp_1 \times Lp_2 \times Lp_3 \times \dots \times Lp_s$ into the space $Lp_1' \times Lp_2' \times \dots \times Lp_s'$. Then the opeaator $\tilde{g}$ is a continuous and bounded mapping if and only if $g(x, y_1, \dots, y_s)$ satisfies the growth condition

$$| g(x, y_1, \dots, y_s) | \leq k \left\{ 1 + \sum_{i=1}^{s} | y_i |^{p_i/p_i'} \right\} \quad \text{for some constants } k > 0.$$

For a proof of this result we refer to the book of Vainberg [24] (Theorem 19.2).

LEMMA II.2.5.

Suppose $u_n \longrightarrow u$ weakly in $\overset{\circ}{W}_{m,p}(G)$, and $v \in \overset{\circ}{W}_{m,p}(G)$ then if A defines a mapping $\overset{\circ}{W}_{m,p}(G) \longrightarrow W_{-m,p}(G)$ $A_\alpha(x, u_n, \dots, D^m v) \longrightarrow A_\alpha(x, u, \dots, D^m v)$ strongly in $Lq(G)$.

PROOF.

This result is an immediate consequence of Vainberg's theorem and Sobolev's Imbedding Theorem.

LEMMA II.2.6.

(Polynomial Growth Conditions) Let $A_\alpha(x, z_{\alpha 1}, \dots, z_{\alpha m})$ be a function defined on $G \times R \times \dots \times R^{n_m}$, continuous in the sense of Caratheodory and satisfying the growth condition:

$$| A_\alpha(x, z_{\alpha 1}, \dots, z_{\alpha m}) | \leq k_\alpha(x, z_{\alpha 1}, \dots, z_{\alpha \delta}) \left\{ 1 + \sum_{\alpha ; \beta = 1}^{m} | z_{\alpha\beta} |^{\sigma_{\alpha\beta}} \right\}$$

where $\sigma_{\alpha\beta} \leq \dfrac{N(p-1) + p(m - |\alpha|)}{N - p(m - |\beta|)}$ if $N > p(m - |\beta|)$ (with equality only for $|\alpha| = |\beta| = m$)

$< \infty$ if $N = p(n - |\beta|)$

$k_\alpha(x, z_{\alpha 1}, \dots, z_{\alpha\delta})$ is a continuous function of each variable defined on $G \times R^{n_1} \times \dots \times R^{n_\delta}$ for δ an integer and $\left(0 \leq \delta < m - \frac{N}{p}\right)$. Then the operator A is a continuous and bounded mapping $\overset{\circ}{W}_{m,p}(G) \to W_{-m,q}(G)$ with

$$\sum_{|\alpha| \leq m} \int_G A_\alpha(x, u, \dots, D^m u)\, D^\alpha v = \langle v, Au \rangle .$$

PROOF.

Using Vainberg's lemma and Sobolev's Imbedding Theorem, it is sufficient to show

$$A_\alpha(x, u, \dots, D^m u) \in Lq(\alpha) \quad \text{where} \quad D^\alpha u \in Lp(\alpha)$$

and

$$\frac{1}{p(\alpha)} = \frac{1}{p} - \frac{m - |\alpha|}{N}, \quad \text{with} \quad \frac{1}{p(\alpha)} + \frac{1}{q(\alpha)} = 1$$

i. e. the imbedding $\overset{\circ}{W}_{m,p}(G) \to \overset{\circ}{W}_{\alpha, p(\alpha)}(G)$ be continuous. By the polynomial growth assumption on A_α, we may consider each term $|z_{\alpha\beta}|^{\sigma_{\alpha\beta}}$ individually. Using Vainberg's Lemma and Sobolev's Imbedding Theorem again, $|D^\beta u|^{\sigma_{\alpha\beta}} \in Lq(\alpha)$ if

$$\sigma_{\alpha\beta} \leq \frac{p(\beta)}{q(\alpha)}, \quad \text{where} \quad \frac{1}{p(\beta)} = \frac{1}{p} - \frac{m - |\beta|}{N}.$$

Thus

$$\sigma_{\alpha\beta} \leq \frac{N(p-1) + p(m - |\alpha|)}{N - p(m - |\beta|)}.$$

In case $N \leq p(m - |\beta|)$ the results of the lemma follows immediately from part II of Sobolev's Imbedding Theorem.

REMARK: For the case $N = p(m - |\beta|)$, using the Imbedding Theorem of [4], we can obtain non-linearity of exponential growth for the functions A_α.

II.3. Special Classes of Non-linear Elliptic Operators.

Here we determine the hypotheses on $A_\alpha(x, z_{\alpha 1}, \dots, z_{\alpha m})$ necessary to define an operator $A: \overset{\circ}{W}_{m,p}(G) \to W_{-m,q}(G)$ belonging to one of the abstract classes I, II, and III.

LEMMA II.3.1.

Suppose the functions $A_\alpha(x, z_{1\alpha}, z_{2\alpha}, \ldots, z_{m\alpha})$ defined on $G \times R^{n_1} \times \ldots$ $\ldots R^{n_m}$ $|\alpha| \leq m$ are continuous in the sense of Caratheodory and form the Euler-Lagrange expression associated with the function $A(x, z_{1\alpha}, \ldots, z_{m\alpha})$.

Suppose (i) $a(u - v, u) - a(u - v, v) > 0 \quad u \neq v$

(ii) $a(u, u) \to \infty$ as $\| u \|_{m,p} \to \infty$

(iii) $A_\alpha(x, -z_{1\alpha}, \ldots, -z_{m\alpha}) = -A_\alpha(x, z_{1\alpha}, \ldots, z_{m\alpha})$

(iv) $A_\alpha(x, z_{1\alpha}, \ldots, z_{m\alpha})$ satisfy the growth conditions of lemma II.2.

then A is an operator of class I: $\overset{\circ}{W}_{m,p}(G) \to W_{-m,q}(G)$ where

$$\langle v, Au \rangle = \sum_{|\alpha| \leq m} \int_G A_\alpha(x, u, \ldots, D^m u)\, D^\alpha v.$$

The proof of this result is an immediate consequence of the definition I.4 and lemma II.2.6.

Before proceeding to studying operators of class II, the following notations will be important:

$$\langle v, Au \rangle = \sum_{|\alpha| = m} \int_G A_\alpha(x, u, \ldots, D^m u)\, D^\alpha w + \sum_{|\alpha| \leq m-1} \int_G A_\alpha(x, u, \ldots, D^m u)\, D^\alpha w$$

$$\langle w, A(u, v) \rangle = \sum_{|\alpha| = m} \int_G A_\alpha(x, u, \ldots, D^m v)\, D^\alpha w + $$

$$+ \sum_{|\alpha| \leq m-1} \int_G A_\alpha(x, u, \ldots, D^m v)\, D^\alpha w$$

where we substitute $D^\alpha v$ in place of $D^\alpha u$, if $|\alpha| = m$; and we write

$$\langle w, A(u, v) \rangle = \langle w, P(u, v) \rangle + \langle w, R(u, v) \rangle,$$

where

$$\langle w, P(u, v) \rangle = \sum_{|\alpha| = m} \int_G A_\alpha(s, u, \ldots, D^m v)\, D^\alpha w$$

$$\langle w, R(u, v) \rangle = \sum_{|\alpha| \leq m-1} \int_G A_\alpha(x, u, \ldots, D^m v)\, D^\alpha w.$$

LEMMA II.3.2.

Suppose the functions $A_\alpha(x, z_{1\alpha}, z_{2\alpha}, \dots, z_{m\alpha})$ defined on $G \times R^{n_1} \times \times R^{n_1} \times \dots \times R^{n_m}$, $|\alpha| \leq m$, are continuous in the sense of Caratheodory and form the Euler-Lagrange expression associated with the function $A(x, z_{1\alpha}, \dots, z_{m\alpha})$.

Sùppose (i) $a(u, u)$ and $\int_0^1 (a(su), u)\, ds \longrightarrow \infty$ as $\| u \|_{m, p} \longrightarrow \infty$

(ii) $A_\alpha(x, -z_{1\alpha}, \dots, -z_{m\alpha}) = -A_\alpha(x, z_{1\alpha}, \dots, z_{m\alpha})$

(iii) $A_\alpha(x, z_{1\alpha}, \dots, z_{m\alpha})$ satisfy the growth conditions of lemma II.2

(iv) $\langle v - w,\ P(u, v) - P(u, w) \rangle > 0 \quad v \neq w.$

Then A is an operator of class II : $\overset{\circ}{W}_{m, p}(G) \longrightarrow W_{-m, q}(G)$.

PROOF.

We define $\langle w, P(u, v) \rangle$, $\langle w, R(u, w) \rangle$ and $\langle w, A(u, v) \rangle$ as above. Then by virtue of lemma II.3.1, it is necessary to check only the assumptions (v′) of the definition. Hypothesis (v′ (a) and v′ (e) are automatically satisfied by virtue of lemma II.2.6. Furthermore hypothesis v′ (d) is an immediate consequence of lemma II.2.5. To check v′ (b) we let $u_n \longrightarrow u$ weakly in $\overset{\circ}{W}_{m, p}(G)$ and write

$$\langle u_n - u,\ P(u_n, u_n) - P(u, u) \rangle = \langle u_n - u,\ P(u_n, u_n) - P(u_n, u) \rangle$$
$$+ \langle u_n - u,\ P(u_n, u) - P(u, u) \rangle .$$

By virtue of hypothesis v′ (d), the latter term tends to 0 as $n \longrightarrow \infty$. Thus if $\langle u_n - u, Pu_n - Pu \rangle \longrightarrow 0$ we conclude $\langle u_n - u,\ P(u_n, u_n) - P(u_n, u) \rangle \longrightarrow 0$. Thus by Serrin's lemma $D^\alpha u_n \longrightarrow D^\alpha u \quad |\alpha| \leq m$, in measure on G. Thus by Nemytski's lemma $A_\alpha(x, u_n, \dots, D^\alpha u_n) \longrightarrow A_\alpha(x, u, \dots, D^\alpha u) \ |\alpha| \leq m$. Thus by the lemma of Leray-Lions

$$\lim_{n \to \infty} \langle w, R(u_n) \rangle = \lim_{n \to \infty} \int_G A_\alpha(x, u_n, \dots, D^\alpha u_n)\, D^\alpha w = \langle w, Ru \rangle .$$

Thus $R(u_n) \longrightarrow R(u)$, weakly.

Finally to check hypothesis $v'(c)$ we let $w_n \longrightarrow w$ weakly in $\overset{\circ}{W}_{m, p}(G)$ and $R(u_n) \longrightarrow v$ weakly in $W_{-m, q}(G)$. Thus by Sobolev's Imbedding Theorem $D^\alpha w_n \longrightarrow D^\alpha w$ strongly in $L_{p(\alpha)}$ for $|\alpha| \leq m - 1$.

Thus

$$\lim_{n\to\infty} \langle w_n , R(u_n) \rangle = \lim_{n\to\infty} \sum_{|\alpha|\leq m-1} \int_G A_\alpha (x, u_n , \dots , D^\alpha u_n) D^\alpha w_n$$

$$= \lim_{n\to\infty} \sum_{|\alpha|\leq m-1} \int_G A_\alpha (x, u, \dots , D^\alpha u) D^\alpha w = \langle w, Ru \rangle .$$

A special result applicable in this context, and not apparently in the abstract setting, pertains to the weak closure of ∂A_R.

THEOREM II.3.1.

Let A be a partial differential operator of class I or II. Suppose $u_n \longrightarrow u$ weakly in $\overset{\circ}{W}_{m,p}(G)$ and $Au_n \longrightarrow Au$ strongly in $W_{-m,q}(G)$. Then if $u_n \in \partial A_R$, $u \in \partial A_R$.

PROOF.

We consider the case of an operator in class II (the result for operators of class I follows by a similar argument). As $u_n \longrightarrow u$ weakly in $\overset{\circ}{W}_{m,p}(G)$ and $Au_n \longrightarrow Au$ strongly in $W_{-m,q}(G)$,

$$\langle u_n - u, \ P(u_n , u_n) - P(u_n , u) \rangle \longrightarrow 0.$$

Thus by Serrin's lemma, $D^\alpha u_n \longrightarrow D^\alpha u$ in measure in $G \mid \alpha \mid \leq m$. Thus by Nemytskii's lemma $A_\alpha (x, su_n D(su_n), \dots , D^m (su_n)) \longrightarrow A_\alpha (x, su, \dots , D^m (su))$ in measure on G.

Thus the integrals

$$\int_G A_\alpha (x, su_n , \dots , D^m su_n) D^\alpha u_n$$

due to the polynomial growth on A_α, are uniformly absolutely continuous and

$$\lim_{n\to\infty} \sum_{|\alpha|\leq m} \int_G A_\alpha (x, su_n , \dots , D^m (su_n)) D^\alpha u_n =$$

$$= \sum_{|\alpha|\leq m} \int_G A_\alpha (x, su, \dots , D^m (su)) D^\alpha u.$$

Hence if $u_n \in \partial A_R$, $u \in \partial A_R$.

Finally we consider operators of class III.

LEMMA II.3.3.

Suppose the function $B_\alpha(x, a\ z, \dots, z_{m-1,\alpha})$ defined on $G \times R^{n_1} \times \dots \times R^{n_{m-1}}$, $|\alpha| \leq m-1$, are continuous in the sense and form the Euler-Lagrange expression associated with the function $B(x, z_{1\alpha}, \dots, z_{m-1,\alpha})$.

Suppose, (i) $b(u, u) > 0 \quad u \neq 0$

(ii) $B_\alpha(x, -z, \dots, -z_{m-1,\alpha}) = -B_\alpha(x, z, \dots, z_{m-1,\alpha})$

(iii) $B_\alpha(x, z \cdot ,,, \cdot z_{m-1,\alpha})$ satisfy the growth conditions of lemma II.2

then B is an operator of class III : $\overset{\circ}{W}_{m,p}(G) \rightarrow W_{-m,q}(G)$ where

$$\langle v, Bu \rangle = \sum_{|\alpha| \leq m-1} \int_G B_\alpha(x, u, \dots, D^{m-1}u)\, D^\alpha v.$$

PROOF.

The only fact not immediate from the hypotheses of the lemma is that B is continuous from the weak topology of $\overset{\circ}{W}_{m,p}(G)$ to the strong topology of $W_{-m,q}(G)$. To demonstrate this fact, let $u_n \rightarrow u$ weakly in $\overset{\circ}{W}_{m,p}(G)$, then

$$\| Bu_n - Bu \|_{-m,q} = \sup_{\|v\|_{m,p} \leq 1} \langle v, Bu_n - Bu \rangle$$

$$= \sup_{\|v\|_{m,p} \leq 1} \sum_{|\alpha| \leq m-1} \int_G [B_\alpha(x, u_n, \dots, D^{m-1}u_n) - B_\alpha(x, u, \dots, D^{m-1}u)]\, D^\alpha v$$

$$\leq K \{ \sum_{|\alpha| \leq m-1} \| B_\alpha(x, u_n, \dots, D^{m-1}u_n) - B_\alpha(x, u, \dots, D^{m-1}u) \|_{0, q\alpha} \}$$

where K is a constant independent of n.

This last expression tends to 0 as $n \rightarrow \infty$ by virtue of lemma II.2.5.

PART III

The first eigen function

A variational argument, (independent of algebraic topology), carried out in this section is sufficient to demonstrate the existence of the first eigen function for the equation $Au = \lambda Bu$. In PART IV, essential use will be made of algebraic topology, to construct higher order eigen functions.

III.1. Solution of the Variational Problem.

Let X be a reflexive, separable Banach space over the reals with conjugate space X^*. We consider in X the variational problem :

$$(1) \qquad c_R = \sup_{\partial A_R} \int_0^1 \langle u, B(su) \rangle \, ds$$

where A is an operator of class I or II : $X \longrightarrow X^*$
B is an operator of class III : $X \longrightarrow X^*$
R is a fixed positive number, (to be specified later).

A solution of the variational problem (u) is an element $u \in \partial A_R$ such that

$$c_R = \int_0^1 \langle u, B(su) \rangle \, ds.$$

LEMMA III.1.1.

If R is chosen sufficiently large, the variational problem (1) has a solution.

PROOF :

By lemma I.2·5 $\Phi(u) = \int_0^1 \langle u, B(su) \rangle \, ds$ is a weakly continuous functional on X. Thus as ∂A_R is a bounded set, c_R is a finite number. Let $\{u_n\}$ be a sequence of elements of ∂A_R such that $\lim_{n \to \infty} \Phi(u_n) = c_R$. As ∂A_R is a bounded set in a reflexive Banach space, the sequence u_n can be refined to a weakly convergent subsequence with weak limit u. By reindexing this set, we can write $\lim_{n \to \infty} \Phi(u_n) = \Phi(u) = c_R$. However as only A_R, and not necessarily ∂A_R, is weakly closed, we conclude only that $u \in A_R$. We now show that u is a solution of the variational problem by proving $u \in \partial A_R$. To this end, we use the coerciveness assumption on the form $\int_0^1 \langle u, A(su) \rangle \, ds$. Choose R so large that if $u \in A_R$ and $su \in \partial A_R$, $s \geq 1$. Hence suppose $u \in A_R$ and not $\in \partial A_R$ then for some number $s > 1$ $su \in \partial A_R$.

Using lemma I.1.1 we write

$$\Phi(su) - \Phi(u) = (s-1)\int_0^1 \langle u, B[u + t(s-1)u]\rangle \, dt$$

As $s > 1$ and $u \neq 0$, $\Phi(su) - \Phi(u) > 0$, contradicting the maximality of $\Phi(u)$. Hence $u \in \partial A_R$.

III.2. **Solution of the Eigenvalue Problem.**

LEMMA III.2.1.

Let u be any solution of the variational problem (1) of III.1, then u is a non-trivial solution of the eigenvalue problem $Au = \lambda\, Bu$ where $\lambda = + \dfrac{\langle u, Au\rangle}{\langle u, Bu\rangle}$.

PROOF.

Suppose u is not a solution of the eigenvalue problem $Au = \lambda\, Bu$ for any λ, then $\| Au - \lambda\, Bu \| = a(\lambda) > 0$. Hence there is some $\pi(\lambda) \in X$ with $\| \pi(\lambda) \| = 1$ such that $\langle \pi, Au - \lambda\, Bu\rangle = a(\lambda)$. Now using lemma I.3.1 we construct a trajectory on ∂A_R, $f(u, t) = u + t\pi + a(t)\, u$ for $|t| \leq t_0$.

To obtain a contradiction, we move along the trajection just constructed until $\Phi(f(u, t)) > \Phi(u)$. We carry this out as follows: Using lemma 1.1, we write

$$(1) \qquad \Phi(u + t\pi + a(t)\, u) - \Phi(u) =$$

$$= t \int_0^1 \left\langle \pi + \frac{a(t)}{t} u,\ B\left[u + ts\left(\pi + \frac{a(t)}{t} u\right)\right]\right\rangle ds.$$

By the Mean-Value Theorem and lemma I.3.1 for $|z| < |t| < t_0$

$$a(t) = a'(z)\, t = \frac{\langle \pi, A\,(f(u, z))\rangle}{\langle u, A\,(f(u, z))\rangle}\, t \text{ where } f(u, z) = u + z\pi + a(z)\, u.$$

Rewriting (1), using the above result and the continuity properties of A and B along the trajectory $f(u, t)$

$$\Phi(u + t\pi + a(t)u) - \Phi(u) = \left(\frac{t}{\lambda}\right)\{\langle \pi, Au - \lambda Bu \rangle + \langle \pi, C(z) \rangle\}$$

$$\text{where } \|C(z)\| \longrightarrow 0 \quad \text{as} \quad z \longrightarrow 0 \quad \text{and} \quad \lambda = \frac{\langle u, Au \rangle}{\langle u, Bu \rangle}.$$

Thus by choosing $|t|$ sufficiently small and $\operatorname{sgn} t = \operatorname{sgn} \lambda$, we obtain $\Phi(u + t\pi + a(t)u) > \Phi(u)$, a contradiction.

We now formulate our results as follows:

THEOREM. III.1

Let A be an operator of class I or II $X \longrightarrow X^*$ where X is a reflexive Banach space over the reals. Let B an operator of class III: $X \longrightarrow X^*$. Then the eigenvalue problem $Au = \lambda Bu$ has at least one non-trivial solution (irrespective of the oddness assumptions on the operators A, B). This solution is normalized by the requirement that $u \in \partial A_R$ and characterized as a solution of the variational problem $\sup_{\partial A_R} \int_0^1 \langle u, B(su) \rangle \, ds$, for R sufficiently large. Furthermore

$$\lambda = + \frac{\langle u, Au \rangle}{\langle u, Bu \rangle}.$$

III.3. The Case of Elliptic Eigenvalue Problems.

By setting $X = \overset{\circ}{W}_{m,p}(G)$ and using the results of PART II we can immediately translate Theorem III. 1 into a result on non-linear elliptic eigenvalue problems. We note that the eigenfunction so obtain is to be understood in the weak or generalized sense.

DEFINITION.

A function $u \in \overset{\circ}{W}_{m,p}(G)$ is a generalized eigenfunction of the operator equation $Au = \lambda Bu$ if 1) for every $v \in \overset{\circ}{W}_{m,p}(G)$

$$\langle v, Au \rangle = \lambda \langle v, Bu \rangle \quad \text{for some } \lambda$$

and 2) $\|u\|_{m,p} \neq 0$

3) $u \in \partial A_R$; for sufficiently large R.

The assumption of oddness on the operators A and B has the following consequence in the case of second order partial differential operators:

LEMMA III.3.1.

Suppose $X = \overset{\circ}{W}_{1,p}(G)$ and the oddness assumptions on the operators A and B hold, then the eigenfunction constructed in Theorem III.2 can be considered as positive a. e. in G.

PROOF.

First we note that if $u \in \overset{\circ}{W}_{1,p}(G)$, $|u| \in \overset{\circ}{W}_{1,p}(G)$. Also

$$\begin{aligned}\int_0^1 \langle u, A(su)\rangle\, ds &= \sum_{|\alpha|\leq 1} \int_0^1 ds \int_G A_\alpha(x, su, \dots, D_j su \dots D_n su)\, D^\alpha u \\ &= \sum_{|\alpha|\leq 1} \int_0^1 ds \int_G A_\alpha(x, s\,|u|, D_j s\,|u| \dots)\, D^\alpha |u| \\ &= \int_0^1 \langle |u|, A(s\,|u|)\rangle\, ds.\end{aligned}$$

Hence if $u \in \partial A_R$ $|u| \in \partial A_R$. Also

$$\int_0^1 \langle u, B(su)\rangle\, ds = \int_0^1 \langle |u|, B(s\,|u|)\rangle\, ds.$$

Thus without loss of generality we may choose the minimizing sequence of the variational problem of III.1.1 from the positive a. e. functions of $\overset{\circ}{W}_{1,p}(G)$.

PART IV

Higher order eigen functions.

In this section a variational argument analogous to PART III is used to obtain an infinite sequence of distinct normalized eigenfunctions $\{u_n\}$ for the operator equation $Au = \lambda Bu$, with associated eigenvalues $\lambda_n \rightarrow \infty$. To achieve this result, we introduce additional constraints to the variational problem of PART III by using an invariant of algebraic topology, namely the Ljusternik-Schnirelmann category of sets.

IV.1. Summary of Topological Results.

DEFINITION.

Let X be a topological space and A a closed, compact subset of X. A has category 1 relative to X if it can be deformed on X to a point (i. e. A is homotopic on X to a point). A set B has category k relative to X if the least number of closed compact subsets of X with category 1 necessary to cover B is k, and we write $\text{cat}_X B = k$. The basic properties of category are listed below.

1) if $A \supseteq B$, $\text{cat}_X A \geq \text{cat}_X B$
2) $\text{cat}_X (A \cup B) \leq \text{cat}_X A + \text{cat}_X B$
3) if X is a separable metric space, $\dim_X A \geq \text{cat}_X A - 1$
4) If $\tau : A \longrightarrow X$ is homotopic to the identity $\text{cat}_X (\tau(A)) \geq \text{cat}_X A$
5) $\text{cat}_X A \geq \text{cat}_y A \;\; y \supseteq X$.

Let P^n be a n-dimensional real projective space. P^n can be regarded as obtained by identifying antipodal points of the sphere

$$s_n = \{x \mid x_0^2 + \dots + x_n^2 = 1, \quad x = (x_0 \dots x_n) \in R^{n+1}\}.$$

Schnirelmann [22] proved $\text{cat}_{pn} P^n = n + 1$. Furthermore $\text{cat}_{pn} P^m = m + 1$ (where $m < n$). This last result has been generalized by Citlanadze [7] as follows: if X and Y are projective spaces $X \subseteq y$ then $\text{cat}_X A = \text{cat}_y A$. Thus if we let P_∞ be the set obtained by identifying antipodal points of the unit sphere of a real, infinite dimensional, separable Banach space X $\text{cat}_{p\infty} P^m = \text{cat}_{pm} P^m = m + 1$. Thus P^∞ contains sets of every category $n = 1, 2, \dots$

We now partition $\partial \bar{A}_R$, defined in I.2, into a countably infinite family of classes. Let A be an operator of class I or II, and suppose R so large that ∂A_R is homeomorphic to some sphere $\partial \Sigma_k$ is homeomorphic to P_∞ and thus $\partial \bar{A}_R$ contains sets of every category $n = 1, 2, \dots$. Let $V \in \partial \bar{A}_R$, with $\text{cat}_{\partial \bar{A}_R} V \geq n$, and define

$$[V]_n = \{V \mid V \in \partial \bar{A}_R, \text{cat}_{\partial \bar{A}_R} V \geq n\}.$$

Thus

1) $[V]_1 \supset [V]_2 \supset \dots \supset [V]_n \dots \supset \dots$

and 2) $[V]_n$ is invariant under continuous deformations.

This procedure is a generalization of intersecting the unit sphere $\{\|x\| = 1\}$ with various spaces of finite dimension appropriate for linear eigenvalue problems.

IV.2. The Variational Problem.

Let X be a real reflexive, separable Banach space of infinite dimension with conjugate space X^*. We cousider in X a sequence of variational problems

$$c_n(R) = \sup_{[V]_n} \min_{V} \int_0^1 \langle u, B(su) \rangle \, ds \quad (n = 1, 3, \dots) \tag{1}$$

where A is an operator of class I or II: $X \longrightarrow X^*$

B is an operator of class III: $X \longrightarrow X^*$

V is a set such that $\mathrm{cat}_{\partial \bar{A}_R} V \geq n$

$$[V]_n = \{V \mid V \in \partial \bar{A}_R, \mathrm{cat}_{\partial \bar{A}_R} V \geq n\}$$

R is a positive number chosen so large that ∂A_R is homeomorphic to some sphere $\partial \Sigma_k$.

A solution of the variational problem (1) is an element $u \in \partial A_R$ such that

$$c_n(R) = \int_0^1 \langle u, B(su) \rangle \, ds.$$

In case $u \in A_R$ and not necessarily ∂A_R, we call u a «weak» solution.

Lemma IV.1.

If R is chosen sufficiently large, the variational problem (1) has a «weak» solution. The proof of this result is completely analogous to the first part of the proof of Lemma III.1.1. Clearly $\| u \| \neq 0$ for the weak solution.

We shall show in the next subsection that the « weak » solution u is actually a solution of the variational problem by showing that $u_n \longrightarrow u$ weakly and $Au_n \longrightarrow Au$ strongly for some sequence of $u_n \in \partial A_R$.

IV.3. Solution of the Eigenvalue Problem.

Using the notation of IV.2, we shall find a weak solution of the variational problem (1), which is also a solution of the eigenvalue problem $Au = \lambda Bu$ where $\lambda = + \dfrac{\langle u, Au \rangle}{\langle u, Bu \rangle}$.

LEMMA IV.3.1.

Suppose there is a sequence $u_n \in \partial A_R$, for sufficiently large R, with the properties

$$\text{(i)} \qquad \lim_{n \to \infty} \| Au_n + \lambda Bu_n \| = 0$$

$$\text{(ii)} \quad \lim_{u \to \infty} \int_0^1 \langle u_n , B(su_n) \rangle \, ds = \sup_{[V]_n} \min_{V} \int_0^1 \langle u, B(su) \rangle \, ds$$

then $\{u_n\}$ can be refined to a subsequence couverging weakly to a weak solution of the variational problem (1) and a non-trivial solution of the eigenvalue problem.

PROOF.

As ∂A_R is a bounded set, $\{u_n\}$ has a weakly convergent subsequence with weak limit u, and $\Phi(u) = \int_0^1 \langle u, B(su) \rangle \, ds$. Thus as $u \in A_R$, u is a weak solution of the variational problem (1). Furthermore as $\| Au_n + \lambda Bu_n \| \longrightarrow 0$ $\{Au_n\}$ converges strongly. By lemma 1.2.4. $Au_n \longrightarrow Au$. Thus $Au + \lambda Bu = 0$ and as $\| u \| \neq 0$, u is a non-trivial solution of the eigenvalue problem.

LEMMA IV.3.2.

Let X be a uniformly convex, separable Banach space over the reals. Suppose D is a continuous mappin of $\mathfrak{g} \longrightarrow X^*$ when $\mathfrak{g}$ is a compact subset of X. Define $C : \mathfrak{g} \longrightarrow \partial \Sigma_1$ by the formula $Cu = \pi$, where $\langle \pi, Du \rangle = \| Du \|$, then C is a well defined, continuous mapping.

PROOF.

As X is uniformly convex, $C(u) = \pi$ is a well-defined mapping. [see Wilansky [26]] To check continuity of C let $u_n \rightarrow u$ strongly in $\mathfrak{g}$ and suppose $Cu_n = \pi_n \; \| \pi_n \| = 1$.

As X is reflexive, π_n has a weakly convergent subsequence with weak limit V, $\| V \| \leq 1$ and we write $\pi_n \rightarrow V$ (after relabeling). Now $\| Du_n \| \rightarrow \| Du \| = \langle \pi, Du \rangle$. On the other hand $\| Du_n \| = \langle \pi_n , Du_n \rangle$ and passing to a subsequence we can write

$$\langle \pi_n , Du_n \rangle \rightarrow \langle V, Du \rangle .$$

Thus if $\| V \| = 1$, $\pi = V$ by the uniform convexity of X. On the other hand if $\| V \| < 1$, the equation $\langle V, Au \rangle = \| Au \|$ is impossible. Hence $\| \pi_n \| \rightarrow \rightarrow \| \pi \| = 1$.

Also the entire sequence π_n has the property that every weakly convergent subsequence tends to π.

Thus $\pi_n \rightarrow \pi$ weakly, (as X is reflexive) and, as $\| \pi_n \| \rightarrow \| \pi \|$ and X is uniformly convex

$$\pi_n \rightarrow \pi \text{ strongly.}$$

Thus $C(u) = \pi$ is a continuous mapping.

LEMMA IV.3.3.

Suppose there is no sequence $\{u_n\}$ satisfying the hypothesis of lemma IV.3.1. Let N be an integer $0 < N < \infty$, and $V \in [V]_N$ with $\sup\limits_{[V]_N} \min\limits_{V} \Phi(u) = c_R(N)$. Then there is a trajectory $f(u, t)$ mapping $V \rightarrow \overline{V}$ such that $\Phi(f(u, t)) \geq \Phi(u) + 2\delta$

where ever $| \Phi(u) - c_R(N) | \leq \delta$ and

$\delta > 0$ is independent of $u \in V$.

PROOF.

As there is no sequence $\{u_n\} \in \partial A_R$ such that $\lim\limits_{n \to \infty} \| Au_n + \lambda Bu_n \| = 0$ and $\lim\limits_{n \to \infty} \Phi(u_n) = c_R(N)$ there are numbers α and $\delta_1 > 0$ such that $\| Au + + \lambda Bu \| > \alpha$ whenever $| \Phi(u) - c_R(N) | < \delta_1$. Using lemma I.3.2. and IV.3.2, the trajectory $f(u, t) = \{u + t\pi + ta(t) u\} \in \partial A_R$ for $| t | < t_\varepsilon$, once

$\varepsilon > 0$ is given, Then using lemma 1.1 and the mean valne theorem,

$$\theta(u) = \Phi(u + t\pi + t\, a(t)u) - \Phi(u) = t \langle \pi + a(t)\, u, B[u + \sigma(\pi + a(t)u] \rangle$$
$$\text{where } 0 \leq \sigma \leq t.$$

Rewriting this last equation using lemma I.3

$$\theta(u) \geq t \left[\left(\frac{1}{\lambda}\right) < \pi, Au + \lambda Bu \rangle - | k\varepsilon \langle u, B(g(u)) \rangle + \langle \pi, Bg(u) - Bu \rangle | \right]$$
$$\text{where } g(u) = u + \sigma(\pi + a(t)\, u).$$

As in III.3 the left hand side of this inequality can be estimated using the uniformly continuity of B on the compact set $\widetilde{V}$

$$\text{where } \widetilde{V} = \{v \mid v = u + \sigma(\pi + a(t)\, u),\ 0 \leq \sigma \leq t,\ u \in V\}.$$

Thus

$$\theta(u) \geq t \left\{ \frac{1}{\lambda} \| Au + \lambda Bu \| - k_1 \varepsilon \right\}$$

$$\geq \frac{t}{\lambda} \{\alpha - k_1 \lambda \varepsilon\}.$$

We now choose $\varepsilon > 0$ so small that $k_1 \lambda \varepsilon < \frac{\alpha}{2}$. Thus we determine t_ε. Finally we choose $\delta > 0$ such that $\delta = \min\left(\delta_1, \frac{t_\varepsilon\, \alpha}{4\lambda}\right)$.

LEMMA IV.3.4.

There is a sequence $u_n \in \partial A_R$ for sufficiently large R, with the properties

(i) $$\lim_{n \to \infty} \| Au_n + \lambda Bu_n \| \to 0$$

(ii) $$\lim_{n \to \infty} \langle u_n, B(su_n) > ds = c_R(N).$$

PROOF.

Suppose there is no such sequence, then let $V \in [V]_N$ be chosen such that $\min_V \Phi(u) = c_R(N) - \delta$.

We obtain a centradiction by deforming V continuously into a set $\overline{V}$ such that

$$\min_{\overline{V}} \Phi(u) = c_R(N) + \delta.$$

To achieve this result, we use lemma IV.3.3. Suppose $u \in V$ and $|\Phi(u) - c_R(N)| \leq \delta$, then we move u along the trajectory $f(u, t)$ until $\Phi(f(u, t)) = c_R(N) + \delta$, (the preceding lemma guarantees that such a motion is always possible). On the other hand, if $\Phi(u) \geq c_R(N) + \delta$ we leave u fixed. As this deformation is continuous (by lemma IV.3.2), $\overline{V} \in [V]_N$ and $c_N(R) = \sup_{[V]_N} \min_{V} \Phi(u) \geq c_N(R) + \delta$ a contradiction.

Finally combining lemmas IV.3.1 and letting $N \to \infty$ we shall be able to obtain the following result.

Abstract Sturm-Liouville Theorem IV.3.1.

Let X be a uniformly convex separable Banach space over the reals with a countably infinite biorthogonal bases. Let A be an operator of class I or II: $X \to X^*$ and B be an operator of class III: $X \to X^*$. Then the equation $Au = \lambda Bu$ has a countably infinite number of distinct eigenfunctions u_N. If the operator A has the additional property (+) that whenever $v_n \to v$ weakly and $Av \to Av$ strongly, $v_n \to v$ strongly, the eigenfnnctions are normalized by the requirement that $u_N \in \partial A_R$ for sufficiently large R and characterized as solutions of the variational problem (1). Furthermore the associated eigenvalues $\lambda_N \to \infty$ as $N \to \infty$.

REMARK.

By virtue of Theorem II.3.1 the additional hypothesis mentioned in the above theorem is unnecessary for elliptic partial differential operators of class I or II.

Indeed for partial differential operators, we have the following result obtained by setting $X = \overset{\circ}{W}_{m,p}(G)$.

Sturm-Liouville Theorem for Non-linear Elliptic Operators IV.3.2.

Let A be an operator of clas I or II: $\overset{\circ}{W}_{m,p}(G) \to W_{-m,q}(G)$ and B be an operator of class III $\overset{\circ}{W}_{m,p}(G) \to W_{-m,q}(G)$. Then the operator equation $Au = \lambda Bu$ has a countably infinite number of distinct generalized eigenfunctions, u_N normalized by the requirement that $u_N \in \partial A_R$ for sufficiently large R and characterized as solution of the variational problem (1).

The only major result needing additional proof in the above theorems is the asymptotic behaviour of the eigenvalues λ_n. This fact will be established in the next subsection. In the abstract theorem, the additional property (+) is used as follows: by lemma IV.3.4, $u_n \to w$ weakly and $Au_n \to Au$

strongly with $u_n \in \partial A_R$. Hence by property $(+)$ $u_n \to u$ strongly and

$$\int_0^1 \langle u_n , A(su_n) \rangle \, ds \to \int_0^1 \langle u, A(su) \rangle \, ds = R.$$

IV.4. **Asymptotic Behaviour of Eigenvalues.**

LEMMA:

Let X be a reflexive Banach space with a countably infinite biortogonal bases. Then if B ia an operator of class III: $X \to X^*$ and $\Phi(u) = \int_0^1 \langle u, B(su) > ds$, for every $\varepsilon > 0$ then is an integer n such that $\sup\limits_{[V]_n} \min\limits_{V} \Phi(u) < \varepsilon$.

PROOF.

We show that every set V with $\mathrm{cat}_{\partial \bar{A}} V \geq n$ has a point u_0 such that $|\Phi(u_0)| < \varepsilon$. First we note $\Phi(u)$ is a weakly continuous functional defined on X with $\Phi(0) = 0$. By lemma I.2.6. given $\varepsilon < 0$ there is an integer n such that for every $u \in \partial \bar{A}_R$ $|\Phi(u) - \Phi(P^{(n)} n)| \leq \frac{\varepsilon}{2}$ (where $P^{(n)}$ is a projection operator $X \to R^n$. Secondly as $\Phi(0) = 0$, there is a $\delta >$ such that $|\Phi(u)| < \frac{\varepsilon}{2}$ if $\|x\| \leq \delta$. Now let $V \subset \partial \bar{A}_R$ is such that $\mathrm{cat}_{\partial \bar{A}_R} V > n$.

Case I. Then if V contains a point u such that $\|P^{(n)} u\| \leq \delta$, we obtain

$$|\Phi(u)| \leq |\Phi(u) - \Phi(P^{(n)}(u))| + |\Phi(P^{(n)} u)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and the lemma is proved.

Case II. On the other hand suppose $\|P^{(n)} u\| > \delta$. Then, for every $u \in \partial \bar{A}_R$, we set

$$H_n(u, t) = \frac{u(1-t) + tP^{(n)}(u)}{\|u(1-t) + tP^{(n)} u\|}$$

$H_n(u, t)$ is a continuous mapping as $(1-t)u + tP^{(n)} u = P^{(n)}u + (1-t)\widetilde{P}_n u$ where $P^{(n)} u + \widetilde{P}_n u = u$. Hence

$$\|(1-t)u + tP^{(n)} u\| = \|P^{(n)} u\| + (1-t)^2 \|P_n u\| \geq \|P^{(n)} u\|^2 \geq \delta^2.$$

Now $H(u, t)$ is a homotopy between V and P_{n-1} so $\operatorname{cat}_{\partial A_R} V \leq \operatorname{cat}_{\partial A_R} P_{n-1} \leq n$ Thus we have obtained a contradiction. Hence $\| P^{(n)} u \| \leq \delta$ and we return to Case I.

LEMMA IV.4.2.

For sufficiently large R, $c_R(N) \longrightarrow 0$. (This result is an immediate consequence of the preceding lemma and the definition of $c_R(N)$.)

LEMMA IV.4.3 : $\lambda_N \longrightarrow \infty$.

$$\lambda_N = \frac{\langle u_N , Au_N \rangle}{\langle u_N , Bu_N \rangle} \qquad \text{for } u_N \in \partial A_R.$$

By the preceding lemma $\int_0^1 \langle u_N , B(su_N) \rangle \, ds \longrightarrow 0.$

Thus $\{u_N\}$ has a weakly convergent subsequence with weak limit 0.
Thus $\langle u_N , Bu_N \rangle \longrightarrow 0$.
On the other hand $\langle u_N , Au_N \rangle \geq k(R) > 0$.
Thus $\lambda_N \longrightarrow \infty$.

PART V

Some examples

In this last section of the present work, we consider examples iilustrating the results proved in PARTS I-IV and giving direction for future studies.

V.1 The example of Ljusternik.

In [17], L. Ljusternik considered the equation

$$\text{(1)} \qquad \frac{1}{2k}\left(F_y - \frac{d}{dx} F_{y'}\right) = \lambda y^{2k-1}$$

$$y(a) = y(b) = 0$$

where $F(x, y, y')$ is a homogeneous function of degree $2k$ in the arguments y and y' and possess continuous derivatives up to and including order $2k+2$, for $k=1$, $F_{y'y'} > 0$

$$\text{and} \quad k > 1, \; F_{y'y'} \geq 0$$

with equality only for

$$F_{y'y'}(x, 0, 0) = 0.$$

By normalizing the solutions of (1) with the requirement that $\int_a^b y^{2k} = 1$, Ljusternik proved, (by considering the zeros of solutions) that (1) has a countably infinite number of distinct normalized eigenfunctions u_N with associated eigenvalues $\lambda_N \to \infty$. Furthermore on $[a, b]$, u_N has precisely $(N-1)$ zeros and this fact alone together with the normalized condition uniquely determines u_N apart from sign. In addition an infinite number of the solutions $\{u_N\}$ are linearly independent.

In a latter paper [18], Ljusternik also showed that the eigenfunctions u_N could be obtained by category argument as in PART IV. The essential idee being to calculate the families $\{[V]_N \, N = 1, 2 \ldots\}$ by means of the zeros of the normalized functions on $[a, b]$.

As nodal domains for even linear elliptic second order operators do not yield as precise results as the one-dimensional case, a more abstract approach for non-linear elliptic eigenvalue problems is needed.

V.2. **The example of Kolodner** [10].

(The heavy rotating string). In [10] I. Kolodner made a complete study of the two-point boundary value problem

$$u'' + \lambda (u^2 + s^2)^{-\frac{1}{2}} u = 0 \tag{1}$$

$$u(0) = u'(1) = 0 \qquad (0 \leq s \leq 1).$$

By considering the liuearized form of (1), namely

$$\begin{aligned} & y'' + \lambda \left(\frac{1}{s}\right) y = 0 \\ & y(0) = y'(1) = 0. \end{aligned} \tag{2}$$

Kolodner was able to give an elegant description of the spectrum of (1). The eigenvalues of (2), λ_n, form a discrete sequence of positive numbers increasing to ∞ with n. The eigenvalues of (1) are continuous for $\lambda_1 < \lambda < \infty$ and for λ such that $\lambda_n < \lambda \leq \lambda_{n+1}$, the multiplicity of λ is precisely n.

By converting (1) into a variational problem as in PART IV, we are able to guarantee the existence of a countable sequence $\lambda_n \longrightarrow \infty$ vith associated normalized eigenfunctions $\{u_N\}$ on ∂A_R. The continuity of the spectrum then arises by letting R vary from 0 to ∞.

The relation to the linear problem (2) is not studied in the present work and will be carried out in a general context in subsequent papers.

V.3. Quasi-linear Problems.

We consider following simple case of our work:

$$(-1)^{m+1} \Delta^m u + \lambda f(u) = 0 \text{ in } G$$

$$D^\alpha u / \partial G \qquad = 0 \qquad 0 \leq |\alpha| \leq m-1$$

where G is a bounded domain in R^N, Δ^m is the m-th iterated Leplacean, $f(x)$ is a continuous odd function with $x f(x) > 0 \qquad x \neq 0$.

Then by the results of PART IV, (1) has a countable number of eigenvalues $\lambda_n \longrightarrow \infty$ with associated generalized eigenfunctions $u_N \in \overset{\circ}{W}_{m,2}(G)$ satisfying the normalization $\left| \int_G u \, \Delta^m u \right| = R$ for each $R > 0$ so long as $f(u)$ satisfies the growth conditions:

(i) $\quad |f(t)| \leq k(1 + t^{\frac{N+2m}{N-2m}})$ if $N > 2m$

(ii) $\quad \lim\limits_{n \to \infty} \dfrac{t}{f(\log t)} = \infty$ if $N = 2m$.

If $N < 2m$ there is no growth restriction of $f(t)$. The eigenfunctions are smooth if the boundary of G, ∂G is smooth. See [2].

This example may be complicated by allowing $f(u)$ to depend on the derivatives of u up to order $2(m-1)$ and by adding to $\Delta^m u$. operator of class II: $\overset{\circ}{W}_{m,2}(G) \longrightarrow W_{-m,2}(G)$ as in Vishik [25].

A beautiful application to differential geometry of a quasi-linear elliptic eigenvalue problem can be found in Yamabe [30].

V.4. Non-linear Eigenvalue Problems of Non-Variational Type.

In the introduction of her book [13], O. Ladyzhenskaya asks the following question: Can the non-linear stationary Navier-Stokes equations for a viscous incompressible irrotational fluid flow in a bounded domain have more than one solution for large Reynolds number R?

This difficult problem can be regarded as quasi-linear elliptic eigenvalue problem of the form $Au = \lambda Bu$ where $u \in X$, the separable Hilbert space of solenoidal vectors of $W_{1,2}(G)$. Our results are not applicable in this case, as the non-linear operator B is not a variatioual operator. A simple example of a problem of non-variational type with no eigenvalues is given in the author's paper [4]. A problem of non-variational type arising in fluid mechanics is known as Taylor instability cf. Velte [29] and Cole [28].

V.5. Problems of the form $Au = \lambda Bu$, with non-linear A.

Finally we mention two important problems that can be regarded as non-linear eigenvalue problems of variational type:

1) Monge-Ampere equation $u_{xy}\, u_{yy} - u_{xy}^{2} = f(x, y, u)$

$$u/\partial G = 0$$

(cf. Krasnoselskii [12]).

2) Hartee-Fock Approximation for Schrodinger's equation with many particle systems (cf. Messiah [19] and Bethe [31]).

Both of these problems will be studied in subsequent papers.

BIBLIOGRAPHY

1. M. BERGER, *An eigenvalue problem for quasi-linear elliptic partial differential equations,* Bull. Amer. Math. Soc. 71, (1965) pp. 171-5.
2. M. BERGER, *An eigenvalue problem for non-linear elliptic partial equations,* Trans. Amer. Math. Soc. 120 (1965) pp. 145-184.
3. M. BERGER, *A Sturm-Liouville theorem for non-linear elliptic partial differential equations,* Proc. Nat. Acad. Sci. USA 53, (1965) pp. 1277-9.
4. M. BERGER, *Orlicz spaces and non-linear elliptic eigenvalue problems,* Bull. Amer. Math. Soc. 79 (1965) pp. 898-902.
5. F. BROWDER, *Variational methods for non-linear elliptic eigenvalue problems,* Bull. Amer. Math. Soc. 71 (1965) pp. 176-183.
6. F. BROWDER, *Ljusternik-Schnirelmann category and non-linear elliptic eigenvalue problems.* Bull. Amer Math. Soc. 71 (1965) pp. 644-8, Annals of Math. 72 (1965) pp. 459-477
7. E. S. CITLANADZE, *The method of orthogonal trajectories and non linear operators of va riational type in the space Lp.* Amer. Math. Soe. Translation Series 2, Vol. 5 (1957) (pp. 305-333).
8. C. DOLPH and G. MINTY, *On Non-linear Integral Equations of the Hammerstein type.* Nonlinear Integral Equations edited by P. Anselone, University of Wisconsin Press 1964, pp. 99-153.
9. L. EL'SGOLC, *Qualitative Mothods in Mathematical analysis,* Vol. 12, Trans. of Math. Monographs A. M. S. 1964.
10. I. KOLODNER, *Heavy Rotating String-A Non-linear Eigenvalue Problem.* Commun. Pure and Appl. Math. (1955) pp. 395-408.
11. M. KRASNOSELSKY, *Topological Methods in the theory of non-linear integral equations GITTL,* Moscow 1956, English trans. Macmillan, New York 1964.
12. M. KRASNOSELSKY, *Positive Solutions of Operator Equations,* Noordhoff, Groningen 1964.
13. O. LADYZHENSKAYA, *The mathematical theory of viscous incompressible flow,* Gordon and Breach, New York, 1963.
14. N. LEVINSON, *Positive eigenfunctions for* $\Delta u + \lambda f(u) = 0$, Arch. RAt. Mech. Anal. 11 (1962) pp. 258-72.
15. J. LERAY and J. LIONS, *Quelques resultats de Vishik.* Bull. Soc. Math. France 93, 1965 (97-107).
16. J. LIONS, *Problemes aux limit dans les equations aux derivees partielles,* Lecture Notes 1962. Universite de Montreal.
17. L. LJUSTERNIK, *Sur une class d'equations non-lineaires,* Math. Sbornik N. S. 2 (1937) pp. 1143-68.
18. L. LJUSTERNIK, *Quelques remarques supplementaires su les equations non-linaires de type de Sturm-Liouville,* Math. Sb. N. S. 4 (1938) pp. 227-232.
19. A. MESSIAH, *Quantum Mechanics* Vol. II, North Holland. Amsterdam 1962.
20. Z. NEHARI, *Characteristic values associated with a class of non-linear differential equations,* Acta Math. 105 (1961) pp. 141-75.

21. L. RALL, *Variational Methods for Non-linear Integral Equations, Non-linear Integral Equations,* ed. P. Anselone, University of Wisconsin Press, 1944 pp. 155-190.
22. L. SCHNIRELMANN, *Uber eine neue Kombinatoriske Invariante Monatsh.* Math. 37 1930 (131-4).
23. J. SCHWARTZ, *Seneralizing the Ljusternik-Schnirelmann theory of critical points,* Comm. Pure Appl. Math. 17 (1964) pp. 807-15.
24. M. VAINBERG, *Variational Methods for the investigation of non-linear operators GITTL,* Moscow 1959. Engl. trans. Holden Day San Francisco 1964.
25. M. VISHIK, *Quasilinear Strongly Elliptic Systems of Differential Equations in Divergence Form Trudy Mosk.* Math. Obsc. 12 1968 pp. 125-184.
26. A. WILANSKY, *Functional Analysis,* Blaisdell, New York 1964.
27. N. MEYERS and J. SERRIN, [to be published].
28. D. COLE, *Transition in circular Couette flow Journal of Fluid Mechanics* 21, 1965, pp. 385-425.
29. W. VELTE, *Stäbilitatsverhalten und Verzweigung stationer Lösungen der Navier-Stokesschen Gleichungen,* Arch. Rat. Mech. Analy. 17 pp. 97-125 (1964).
30. YAMABE, *On a Deformation af Riemannian Structures on Compact Manifolds,* Osaka Math. Jour. 12 (1960), 21-37.
31. H. BETHE, *Intermediate Quantum Mechanincs,* Benjamin 1964 New York.

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