ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

STANLEY KAPLAN

Abstract boundary value problems for linear parabolic equations

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 20, nº 2 (1966), p. 395-419
<http://www.numdam.org/item?id=ASNSP_1966_3_20_2_395_0>

ABSTRACT BOUNDARY VALUE PROBLEMS FOR LINEAR PARABOLIC EQUATIONS

STANLEY KAPLAN (1)

In a previous note [1] we sketched a new method for proving the existence of weak solutions to the first boundary value problem in the cylindrical domain

$$Q = \Omega \times (0, T) \qquad (\Omega \text{ a bounded open set in } R^n)$$

for the equation

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) = F \tag{1}$$

where the coefficients a_{ij} are real, bounded, and measurable, and the differential operator given by the sum in (1) is uniformly elliptic in Q. It is our purpose in the present work to describe the details of that method, along with some related results; we do this, however, for the more general « abstract parabolic boundary value problem » as it is set forth in [2], Chapter IV. To start, let us recall what is meant by a weak solution $u = u(x, t)$ of (1) which « assumes », for $t = 0$, given initial data $u_0 = u_0(x)$ and which « vanishes » on the lateral boundary $\partial\Omega \times (0, T)$. Let us put $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, and introduce the form

$$a(t; u, v) = \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial u}{\partial x_j} \overline{\frac{\partial v}{\partial x_i}}\, dx.$$

Pervenuto alla Redazione l' 8 Novembre 1965.

(1) Visiting Professor at the University of Pisa, 1964-65, under the auspices of the C. N. R.

We assume that $u_0 \in L^2[0, T; H]$ (i. e., the Hilbert space of those weakly measurable functions

$$F: (0, T) \longrightarrow H \quad \text{for which}$$

$$\| F \|^2_{L^2[0,T;H]} \equiv \int_0^T \| F(t) \|^2_H \, dt < \infty .)$$

Then, by a weak solution u, we mean a solution of the following:

PROBLEM: *Find u such that*

a) $u \in L^2[0, T; V]$ and

b) $$\int_0^T \{-(u(t), \varphi'(t))_H + a(t; u(t), \varphi(t))\} \, dt = (u_0, \varphi(0))_H + \int_0^T (F(t), \varphi(t))_H \, dt,$$

for all $\varphi \in L^2[0, T; V]$ such that $\varphi' \in L^2[0, T; H]$ and $\varphi(T) = 0$.

Here, *a*) expresses the « vanishing » of u on the lateral boundary of Q, and *b*) expresses both the fact that u satisfies (1), in the weak sense, and that $u(t)$ « assumes » the value u_0 for $t = 0$. In the present work, following [2], we start with two abstract separable Hilbert spaces H and V, with $V \subseteq H$ (the inclusion being continuous) and V dense in H, and a function $a(t; u, v)$ with these properties:

(i) *For almost every $t \in (0, T)$, $a(t; u, v)$ is a sesqui-linear form on $V \times V$* (i. e., it is linear in the first variable, complex-conjugate linear in the second.)

(ii) *For every $u, v \in V$, $a(t; u, v)$ is a measurable complex-valued function on $(0, T)$; moreover, for all $t \in (0, T)$ (except perhaps a set of measure zero which does not depend on u or v)*

$$| a(t; u, v) | \leq M \| u \|_V \| v \|_V$$

where M is a positive number independent of u and v.

(iii) *There exist constants $\lambda \geq 0$, $m > 0$, such that for all $t \in (0, T)$ (with the possible exception of a set of measure zero which does not depend on u) we have*

$$\text{Re } a(t; u, u) + \lambda \| u \|^2_H \geq m \| u \|^2_V .$$

THEOREM: ([2], Chapter IV, Theorem 1.1) *Given any $u_0 \in H$, $F \in L^2[0, T; H]$, there exists a unique solution u to the Problem above.*

The proof given in [2] holds even in the more general case where $F \in L^2[0, T; V']$: here, V' is the anti-dual of V, i. e., the space of continuous complex-conjugate linear functionals $F: V \longrightarrow \mathbb{C}\ (v \longrightarrow \langle F, v \rangle)$, given the usual norm (which makes it a Hilbert space, by the Riesz theorem.) Of course, in *b*) we now write $\langle F(t), \varphi(t) \rangle$ instead of $(F(t), \varphi(t))_H$. The theorem gives, with the appropriate choice of H, V, and $a(t; u, v)$, existence and uniqueness theorems for a large class of parabolic problems, including the special case already mentioned. Thus, in particular, one may consider (1) with lower order terms added, or the analogous parabolic equation of order $2m$ in the space variables, or even a parabolic system of equations, all with a great variety of boundary conditions (see [2], Chapter VI, for examples.)

To justify our seeking another method, let us reconsider the first boundary value problem for (1), but this time with inhomogeneous lateral boundary data. One usually assumes that the initial and boundary data are given as the trace of a function g defined in all of Q. Then, one looks for a solution u of (1) such that $u - g$ « vanishes » on $\Omega \times \{0\} \cup \partial\Omega \times (0, T)$. If we set $w = u - g$, then w must be a weak solution, in $L^2[0, T; V]$, of

$$\frac{\partial w}{\partial t} - \sum_{i, j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial w}{\partial x_j}\right) = F - \frac{\partial g}{\partial t} + \sum_{i, j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial g}{\partial x_j}\right).$$

For the above Theorem to apply, the right hand side of this equation must be in $L^2[0, T; V']$ $(= L^2[0, T; H^{-1}]$, in this case.) Thus, in general, we would need

A) $g \in L^2[0, T; H^1(\Omega)]$ and

B) $\dfrac{\partial g}{\partial t} \in L^2[0, T; V']$.

We would then be able to assert the existence of a unique solution to our problem, $u = w + g$, which would have these same two properties. Condition A) is natural enough; B), however, is perhaps a little strange. Thus, there is a reason for extending the above Theorem to cover equations with a larger class of inhomogeneous terms, which we do (Theorem 2, below,) under the assumption, however, that $u_0 = 0$; to some extent this loss can be recouped by making the initial data reappear as part of the right hand side, in the usual way (see Theorem 4, below.) Applying Theorem 2 to the first boundary value problem, we see that B) may be replaced by the two conditions

B′) $\displaystyle\int_0^T ds \int_0^T dt \, \frac{\| g(t) - g(s) \|_H^2}{| t - s |^2} < \infty$ and

C) $\displaystyle\int_0^T \frac{\| g(t) - g(0) \|_H^2}{t} \, dt < \infty.$

Thus, for any g satisfying A), B′) and C) there exists, by Theorem 2, a solution u of our problem which also satisfies A), B′), and C). By way of comparison, Theorem 4 gives a necessary and sufficient condition in order that the solution given in [2] satisfy B′) and C): it is, of course, a condition on u_0, i. e., that u_0 belong to a certain Hilbert space intermediate between V and H.

The heart of our method is Theorem 1, which is an existence theorem for weak solutions of an abstract parabolic equation on the whole real line; its proof uses the Fourier transform and the space $H^{\frac{1}{2}}$, thereby extending methods used in [2] (Chapter IV, Theorem 2.2) in the discussion of regularity of solutions. The easy lemmas at the beginning of Section 2 enable us to identify certain classes of functions on $(0, T)$, and bilinear forms defined on those function classes, with the restrictions of corresponding entities defined on the whole real line; then, Theorem 2 is seen to follow from Theorem 1, essentially by restriction.

Theorem 3 is a weak regularity theorem which asserts that under a slight further assumption on the inhomogeneous term, our solution u is continuous, as a function from $(0, T)$ to H, a fact which does not follow from the fact that u satisfies A), B′), and C) (see [3].) Theorem 3 is similar to Theorem 2.1 of Chapter IV of [2] in intent; its scope is somewhat different. Its proof is accomplished using methods introduced in [2] and [3].

Our dependence on ideas and methods introduced by J. L. Lions is obvious; in addition, we owe Prof. Lions a debt of gratitude for a helpful discussion of the present work, which influenced its final formulation. Prof. Lions was able to derive our Theorem 1 and Theorem 2 from the results of [2], using the theory of interpolation; this should appear in the forthcoming book on that subject by Lions and Peetre. We should also like to thank P. Grisvard for a useful discussion of Theorem 3, and, in particular, for pointing out to us an error in what we took to be the proof of a stronger result.

1. A parabolic equation on the whole real line.

We write $\mathcal{H}$ for $L^2[-\infty, \infty; H]$ and $\mathcal{V}$ for $L^2[-\infty, \infty; V]$; thus,

$$(u, v)_{\mathcal{H}} = \int_{-\infty}^{\infty} (u(t), v(t))_{\mathcal{H}}\, dt$$

and a similar expression applies for $(u, v)_{\mathcal{V}}$. We are given a continuous sesqui-linear form $A\langle u, v\rangle$ on $\mathcal{V} \times \mathcal{V}$:

$$\text{(2)} \qquad |A\langle u, v\rangle| \leq M \| u \|_{\mathcal{V}} \| v \|_{\mathcal{V}} \text{ for all } u, v \in \mathcal{V}$$

(M a positive constant) which, in addition, satisfies, for some $m > 0$,

$$\text{(3)} \qquad |\operatorname{Re} A\langle u, u\rangle| \geq m \| u \|_{\mathcal{V}}^2 \text{ for all } u \in \mathcal{V}.$$

We look for a solution of the following problem: find u which satisfies the equation

$$\left(\frac{\partial u}{\partial t}, v\right)_{\mathcal{H}} + A\langle u, v\rangle = \langle F, v\rangle$$

for every « test function » v. Our first step is to give this problem a precise meaning. To this end, we introduce the Fourier transform $u \longrightarrow \widehat{u} = \mathcal{F}(u)$ defined by

$$\widehat{u}(\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\tau} u(t)\, dt$$

for those u in $\mathcal{H}$ with compact support in $(-\infty, \infty)$. If u and v both have compact support and in addition assume only a finite number of values in H, then, by the Plancherel theorem, we have

$$\text{(4)} \qquad (\widehat{u}, \widehat{v})_{\mathcal{H}} = (u, v)_{\mathcal{H}}.$$

By a standard approximation argument, such functions are dense in $\mathcal{H}$; hence $\mathcal{F}$ extends, via (4), to a unitary transformation of $\mathcal{H}$ onto itself. Similarly $\mathcal{F}: \mathcal{V} \longrightarrow \mathcal{V}$ is unitary.

We consider the subspace $\mathcal{W}$ of $\mathcal{H}$ defined as follows: $u \in \mathcal{W}$ if and only if

$$\| u \|_{\mathcal{W}}^2 \equiv \int_{-\infty}^{\infty} (1 + |\tau|) \| \widehat{u}(\tau) \|_H^2 \, d\tau < \infty.$$

($\mathcal{W}$ is often called $H^{\frac{1}{2}}[-\infty, \infty; H]$; see [2], [3].) The significance of this space for us lies in the following: If $u \in \mathcal{H}$ has compact support, and is differentiable, i. e.,

$$\frac{\partial u}{\partial t} = \lim_{h \to 0} \frac{u(t+h) - u(t)}{h}$$

exists, the limit being taken in $\mathcal{H}$, then $i \tau \hat{u} \in \mathcal{H}$, where $i \tau \hat{u}$ denotes, of course, the function $\tau \longrightarrow i \tau \hat{u}(\tau)$, and, we have

$$\widehat{\frac{\partial u}{\partial t}} = i \tau \hat{u} \tag{5}$$

If both u and v are differentiable, with compact support, then applying (4), (5), and the Cauchy-Schwarz inequality, we conclude that

$$\left| \left(\frac{\partial u}{\partial t}, v \right)_{\mathcal{H}} \right| = \left| \int_{-\infty}^{\infty} i \tau (\hat{u}(\tau), \hat{v}(\tau))_H \, d\tau \right| \leq \| u \|_{\mathcal{W}} \| v \|_{\mathcal{W}}. \tag{6}$$

Since such functions are dense in $\mathcal{W}$, we may extend $\left(\frac{\partial u}{\partial t}, v \right)_{\mathcal{H}}$ in a unique manner to a continuous sesqui-linear form $B \langle u, v \rangle$ on $\mathcal{W} \times \mathcal{W}$; (6) tell us that

$$| B \langle u, v \rangle | \leq \| u \|_{\mathcal{W}} \| v \|_{\mathcal{W}} \text{ for all } u, v \in \mathcal{W}.$$

By a similar approximation argument, we conclude that

$$B \langle v, u \rangle = - \overline{B \langle u, v \rangle} \quad \text{for all } u, v \in \mathcal{W} \tag{7}$$

Finally, we introduce the Hilbert space $\mathcal{X} = \mathcal{V} \cap \mathcal{W}$, with the scalar product

$$(u, v)_{\mathcal{X}} = (u, v)_{\mathcal{V}} + \int_{-\infty}^{\infty} | \tau | (\hat{u}(t), \hat{v}(\tau))_H \, d\tau.$$

Since, by assumption, $\exists \, C > 0$ such that

$$\| h \|_H \leq C \| h \|_V \quad \text{for all } h \in V,$$

the scalar product in $\mathcal{X}$ is seen to be equivalent to the more natural one, i. e.,

$$(u, v)_{\mathcal{V}} + (u, v)_{\mathcal{W}}.$$

We introduce $\mathcal{X}$ because it is the natural domain of definition for the form

$$E\langle u, v\rangle = A\langle u, v\rangle + B\langle u, v\rangle;$$

more precisely, E is a continuous sesqui-linear form on $\mathcal{X} \times \mathcal{X}$. Thus E defines a continuous linear mapping L of $\mathcal{X}$ into $\mathcal{X}'$, the anti-dual of $\mathcal{X}$, by means of the relation

$$\langle Lu, v\rangle = E\langle u, v\rangle \quad \text{for all } v \in \mathcal{X}$$

where the brackets on the left denote the duality between $\mathcal{X}$ and $\mathcal{X}'$. Note that E is not coercive on $\mathcal{X}$; nonetheless, we have the following:

THEOREM 1: *L is an isomorphism of $\mathcal{X}$ onto $\mathcal{X}'$.*

PROOF: Since we already know that L: $\mathcal{X}$ into $\mathcal{X}'$ is continuous, it suffices to show that

(a) The range of L is dense in $\mathcal{X}'$, and

(b) $\exists\, N > 0$ such that

$$\| u \|_{\mathcal{X}} \leq N \| Lu \|_{\mathcal{X}'} .$$

((b) tells us that L^{-1} is continuous; (a) that it is densely defined.)

(a): Since $\mathcal{X}$ is a Hilbert space, it suffices to show that if $v \in \mathcal{X}$ is such that

$$\langle Lu, v\rangle = 0 \quad \text{for all } u \in \mathcal{X},$$

then $v = 0$. But, by definition, this means

$$E\langle u, v\rangle = 0 \quad \text{for all } u \in \mathcal{X};$$

in particular,

$$E\langle v, v\rangle = 0,$$

and, therefore, by (7),

$$0 = \operatorname{Re} E\langle v, v\rangle = \operatorname{Re} A\langle v, v\rangle,$$

from which it follows, by (3), that $v = 0$.

(b): We introduce the linear transformation $u \longrightarrow \tilde{u}$ defined as follows:

$$\tilde{u} = \mathcal{F}^{-1} \circ \mathcal{M} \circ \mathcal{F}(u)$$

where $\mathcal{M}$ is the multiplication operator defined by

$$\mathcal{M} u(\tau) = \frac{i\tau}{|\tau|} u(\tau).$$

It is obvious that

$$\|\tilde{u}\|_{\mathcal{V}} \leq \|u\|_{\mathcal{V}} \quad \text{for all } u \in \mathcal{V}$$

and

$$\|\tilde{u}\|_{\mathcal{X}} \leq \|u\|_{\mathcal{X}} \quad \text{for all } u \in \mathcal{X}.$$

Moreover, for $u \in \mathcal{W}$, it is clear that we have

$$B\langle u, \tilde{u}\rangle = \int_{-\infty}^{\infty} |\tau| \, \|\hat{u}(\tau)\|_H^2 \, d\tau .$$

If, for $u \in \mathcal{X}$ we write $F = Lu$, then, by definition,

$$E\langle u, v\rangle = \langle F, v\rangle \quad \text{for all } v \in \mathcal{X}.$$

Setting $v = u$ in the above, and taking first real parts and then absolute values of both sides, we obtain

$$\text{(8)} \qquad m\|u\|_{\mathcal{V}}^2 \leq |\operatorname{Re} A\langle u, u\rangle| = |\operatorname{Re} E\langle u, u\rangle| =$$
$$= |\operatorname{Re} \langle F, u\rangle| \leq |\langle F, u\rangle| \leq \|F\|_{\mathcal{X}'} \|u\|_{\mathcal{X}} .$$

Setting $v = \tilde{u}$ in the same equation, we obtain

$$B\langle u, \tilde{u}\rangle = \langle F, \tilde{u}\rangle - A\langle u, \tilde{u}\rangle$$

and hence

$$\int_{-\infty}^{\infty} |\tau| \, \|\hat{u}(\tau)\|_H^2 \, d\tau \leq \|F\|_{\mathcal{X}'} \|\tilde{u}\|_{\mathcal{X}} + M\|u\|_{\mathcal{V}} \|\tilde{u}\|_{\mathcal{V}}$$
$$\leq \|F\|_{\mathcal{X}'} \|u\|_{\mathcal{X}} + M\|u\|_{\mathcal{V}}^2$$
$$\leq \|F\|_{\mathcal{X}'} \|u\|_{\mathcal{X}} + \frac{M}{m}\|F\|_{\mathcal{X}'} \|u\|_{\mathcal{X}}$$

by (8). Adding this last inequality to (8), we have

$$\| u \|^2_{\mathcal{X}} \leq \left(1 + \frac{M+1}{m}\right) \| F \|_{\mathcal{X}'} \| u \|_{\mathcal{X}}$$

which gives (b), with

$$N = 1 + \frac{M+1}{m} .$$

2. Application to problems on a finite interval.

We here apply Theorem 1 to the study of the Problem of our introductory section. Thus, we are given $a(t; u, v)$ satisfying (i), (ii), and (iii) of that section. For the moment we assume that (iii) holds even with $\lambda = 0$; this further assumption will not enter in our final result. We may extend $a(t; u, v)$ in such a way that (i), (ii), and (iii) (with $\lambda = 0$) continue to hold on all of $(-\infty, \infty)$. Indeed, we may, for example, define

$$a(t; u, v) = \frac{M+m}{2} (u, v)_V \quad \text{for } t \notin (0, T).$$

With a so extended, we define

$$A \langle u, v \rangle = \int_{-\infty}^{\infty} a(t; u(t), v(t))\, dt \quad \text{for all } u, v \in \mathcal{V}.$$

A, so defined, is a sesqui-linear form on $\mathcal{V} \times \mathcal{V}$ which satisfies (2), and

$$\text{(9)} \qquad \operatorname{Re} A \langle u, u \rangle \geq m \| u \|^2_{\mathcal{V}} \quad \text{for all } u \in \mathcal{V},$$

an inequality which implies (3). Thus, with B, E, and L defined as in Section 1, Theorem 1 applies: given $F \in \mathcal{X}'$ there is a unique $u \, (= L^{-1}(F))$ in $\mathcal{X}$ such that $Lu = F$.

We shall say that

$$F = 0 \quad \text{for } t < 0$$

whenever $\langle F, v \rangle = 0$ for every $v \in \mathcal{X}$ such that

$$v = 0 \text{ (a. e.)} \quad \text{for } t > 0.$$

LEMMA 1: *If* $F = 0$ *for* $t < 0$, *then* $L^{-1}(F) = 0$ (a. e.) *for* $t < 0$.

PROOF: We write $u = L^{-1}(F)$; since $u \in \mathfrak{X}$, in particular the function $t \longrightarrow \| u(t) \|_H^2$ is in $L^1[-\infty, \infty; dt]$. Thus, by the well-known theorem of Lebesgue, the function

$$\varphi(t) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \| u(t) \|_H^2 \, dt$$

is a. e. defined, and we have $\varphi(t) = \| u(t) \|_H^2$ a. e. in $(-\infty, \infty)$. To accomplish our proof we shall in fact show that $\varphi(t)$ is defined and $= 0$ for all $t < 0$. Given $t < 0$, let $\varepsilon > 0$ be small enough so that $t + \varepsilon < 0$. We define $\varrho(\tau) = \varrho_{\varepsilon, t}(\tau)$ as follows: ϱ is continuous for all τ, $= 1$ for $\tau \leq t$, $= 0$ for $\tau \geq t + \varepsilon$, and linear in between. We shall first show that the equality

$$\text{Re } B\langle u, \varrho u \rangle = \frac{1}{2\varepsilon} \int_t^{t+\varepsilon} \| u(\tau) \|_H^2 \, d\tau \tag{10}$$

holds for all $u \in \mathcal{W}$. Since the mapping $u \longrightarrow \varrho u$ is continuous from $\mathcal{W}$ into $\mathcal{W}$ (in fact, it follows easily from (11) below that

$$\| \varrho u \|_{\mathcal{W}}^2 \leq \left(1 + \frac{1}{\varepsilon}\right) \| u \|_{\mathcal{W}}^2)$$

both sides of (10) are continuous functions of $u \in \mathcal{W}$. Thus, it is enough to prove (10) for u in $\mathcal{W}$ with compact support and such that $\dfrac{\partial u}{\partial \tau}$ exists (in $\mathcal{H}$.) In this case, $v = \varrho u$ is also differentiable in $\mathcal{H}$, and we have

$$\frac{\partial v}{\partial \tau} = \varrho \frac{\partial u}{\partial \tau} + \frac{\partial \varrho}{\partial \tau} u$$

where $\dfrac{\partial \varrho}{\partial \tau} = 0$ for $\tau < t$, $= -\dfrac{1}{\varepsilon}$ for $t < \tau < t + \varepsilon$, and $= 0$ for $t + \varepsilon < \tau$. Thus, we find that

$$B\langle u, v \rangle = \left(\frac{\partial u}{\partial \tau}, v\right)_{\mathcal{H}} = -\left(u, \frac{\partial v}{\partial \tau}\right)_{\mathcal{H}}$$

$$= -\left(u, \varrho \frac{\partial u}{\partial \tau}\right)_{\mathcal{H}} - \left(u, \frac{\partial \varrho}{\partial \tau} u\right)_{\mathcal{H}}$$

$$= -\overline{B\langle u, v \rangle} + \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \| u(\tau) \|_H^2 \, d\tau$$

from which (10) follows immediately.

From $u = L^{-1}(F)$, we obtain, in particular

$$\begin{aligned}\operatorname{Re} E\langle u, \varrho u\rangle &= \operatorname{Re} A\langle u, \varrho u\rangle + \operatorname{Re} B\langle u, \varrho u\rangle \\ &= \operatorname{Re}\langle F, \varrho u\rangle = 0\end{aligned}$$

since $\varrho u \in \mathcal{X}$, and vanishes for $\tau > 0$. Thus, by (10), we have

$$\int_{-\infty}^{\infty} \varrho_{\varepsilon, t}(\tau)\, a(\tau, u(\tau), u(\tau))\, d\tau + \frac{1}{2\varepsilon}\int_{t}^{t+\varepsilon} \| u(\tau)\|_H^2\, d\tau = 0 .$$

Letting $\varepsilon \to 0$, we obtain

$$\int_{-\infty}^{t} a(\tau; u(\tau), u(\tau))\, d\tau + \varphi(t) = 0 ;$$

since both terms on the left hand side of this equality are non-negative, our proof is complete.

Our next step is to characterize the restrictions to $(0, T)$ of the Hilbert spaces $\mathcal{V}$ and $\mathcal{W}$ (and the subspace of $\mathcal{W}$ consisting of those functions $u \in \mathcal{W}$ which vanish for $t < 0$.) The restriction of $\mathcal{V}$ to $(0, T)$ is of course just $L^2[0, T; V]$. Rather than treat simply $\mathcal{W}$, for later use it is convenient to consider $\mathcal{W}_\alpha$, for $1 > \alpha \geq 0$, where $\mathcal{W}_\alpha$ is the Hilbert space consisting of those $u \in \mathcal{H}$ for which

$$\| u \|^2_{\mathcal{W}_\alpha} = \int_{-\infty}^{\infty} (1 + |\tau|^{2\alpha}) \| \widehat{u}(\tau) \|_H^2\, d\tau < \infty.$$

In particular $\mathcal{W}_{\frac{1}{2}} = \mathcal{W}$. Using the Plancherel relation (4), an easy calculation gives the well-known identity

$$\| u \|^2_{\mathcal{W}_\alpha} = \| u \|^2_{\mathcal{H}} + \frac{1}{\sigma_\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ds\, dt\, \frac{\| u(t) - u(s)\|_H^2}{|t - s|^{1+2\alpha}} \tag{11}$$

for $0 < \alpha < 1$, where σ_α is given by

$$\sigma_\alpha = \int_{-\infty}^{\infty} \frac{|e^{i\theta} - 1|^2}{|\theta|^{1+2\alpha}}\, d\theta.$$

With this in mind, we define the spaces W_α, W_α^0, and W_α^T by means of the norms

$$\| u \|_{W_\alpha} = \left[\| u \|^2_{L^2[0,\, T;\, H]} + \frac{1}{\sigma_\alpha} \int_0^T \int_0^T ds\, dt \frac{\| u(t) - u(s) \|_H^2}{| t - s |^{1+2\alpha}} \right]^{\frac{1}{2}}$$

$$\| u \|_{W_\alpha^0} = \left[\| u \|^2_{W_\alpha} + \frac{1}{\sigma_\alpha} \int_0^T \frac{\| u(t) \|_H^2}{| t |^{2\alpha}}\, dt \right]^{\frac{1}{2}}$$

$$\| u \|_{W_\alpha^T} = \left[\| u \|^2_{W_\alpha} + \frac{1}{\sigma_\alpha} \int_0^T \frac{\| u(t) \|_H^2}{| T - t |^{2\alpha}} \right]^{\frac{1}{2}}.$$

We shall consider only α in the interval $\left[\frac{1}{2}, 1\right]$; thus for u to belong to W_α^0 (resp. W_α^T) it is necessary that u « vanish » at 0 (resp. T) sufficiently rapidly (a Dini condition in H.) Conforming to our previous notation, we write simply W^0 and W^T for $W^0_{\frac{1}{2}}$ and $W^T_{\frac{1}{2}}$.

LEMMA 2: (i) $u \in W_\alpha$ *if and only if there exists* $\tilde{u} \in \mathcal{W}_\alpha$ *such that*

$$u = \tilde{u} \text{ a. e. } in\ (0, T);$$

in that case $\tilde{u}$ *may be chosen so that*

$$\theta \| \tilde{u} \|_{\mathcal{W}_\alpha} \leq \| u \|_{W_\alpha} \leq \| \tilde{u} \|_{\mathcal{W}_\alpha} \tag{12}$$

where θ *is a positive constant independent of* u *and* $\tilde{u}$.

(ii) $u \in W_\alpha^0$ *(resp.* W_α^T*) if and only if there exists* $\tilde{u} \in \mathcal{W}_\alpha$ *with* $\tilde{u} = 0$ *for* $t < 0$ *(resp.* $t > T$*) such that*

$$u = \tilde{u} \text{ a. e. } in\ (0, T);$$

in that case, $\tilde{u}$ *may be chosen so that*

$$\eta \| \tilde{u} \|_{\mathcal{W}_\alpha} \leq \| u \|_{W_\alpha^0} \leq \| \tilde{u} \|_{\mathcal{W}_\alpha} \tag{13}$$

$$(resp.\ \eta \| \tilde{u} \|_{\mathcal{W}_\alpha} \leq \| u \|_{W_\alpha^T} \leq \| \tilde{u} \|_{\mathcal{W}_\alpha})$$

where η *is a positive constant independent of* u *and* $\tilde{u}$.

PROOF: (i): The «if» part is obvious, by (11). On the other hand, given $u \in W_\alpha$, we construct $\tilde{u}$ as follows: we extend u first to $(-T, T)$ by reflection about $t = 0$, then to $(-T, 3T)$ by reflection about $t = T$. To obtain $\tilde{u}$, we multiply the result by a fixed, real-valued, smooth function ϱ, defined on $(-T, 3T)$, which assumes the value 1 on $[0, T]$ and the value 0 outside of $\left(-\frac{T}{2}, \frac{3T}{2}\right)$; we put $\tilde{u} = 0$ outside of $(-T, 3T)$. To see that $\tilde{u}$ is in $\mathcal{W}_\alpha$ and that (12) holds, it suffices to consider the result of a reflection, about $t = 0$, for example. Thus, given $u \in W_\alpha$, we define u^* a. e. on $(-T, T)$ as follows:

$$u^*(t) = \begin{cases} u(t) & \text{for } 0 < t < T \\ u(-t) & \text{for } -T < t < 0 . \end{cases}$$

From considerations of symmetry, all we need estimate is the integral

$$I = \int_0^T dt \int_{-T}^0 ds \frac{\| u^*(t) - u^*(s) \|_H^2}{| s - t |^{1+2\alpha}};$$

we find that

$$I = \int_0^T dt \int_0^T d\sigma \frac{\| u(t) - u(\sigma) \|_H^2}{| \sigma + t |^{1+2\alpha}} \leq \int_0^T dt \int_0^T d\sigma \frac{\| u(t) - u(\sigma) \|_H^2}{| t - \sigma |^{1+2\alpha}}$$

from which our result follows.

(ii) First, suppose u is the restriction to $(0, T)$ of $\tilde{u} \in \mathcal{W}_\alpha$ with $\tilde{u} = 0$ for $t < 0$. Then, we have

$$\int_0^T \frac{\| u(t) \|^2}{| t |^{2\alpha}} H \, dt = 2\alpha \int_0^T \| u(t) \|_H^2 \left\{ \int_{-\infty}^0 \frac{ds}{| t - s |^{1+2}} \right\} dt$$

$$= 2\alpha \int_0^T dt \int_{-\infty}^0 ds \frac{\| \tilde{u}(t) - \tilde{u}(s) \|_H^2}{| t - s |^{1+2\alpha}}$$

$$\leq 2\alpha\sigma_\alpha \| \tilde{u} \|_{\mathcal{W}_\alpha}$$

and hence $u \in \overset{0}{W}_\alpha$. Conversely, given $u \in \overset{0}{W}_\alpha$, we construct its extension $\tilde{u}$ as follows: first we extend u to $(0, 2T)$ by reflection about $t = T$, and then to $(-\infty, \infty)$ by setting $\tilde{u} = 0$ outside of $(0, 2T)$. The calculation just above, along with that used in the proof of (i), yields our assertion. Similar reasoning applies in the case of W_a^T, and our proof is complete.

Now, given $u \in W^0$ (resp. W^T) and $v \in W^T$ (resp. W^0) we may define the sesquilinear form

$$\tilde{B}\langle u, v\rangle = B\langle \tilde{u}, \tilde{v}\rangle$$

where $\tilde{u}$ and $\tilde{v}$ are any extensions of u and v to $\mathcal{W}$, vanishing on the appropriate half-lines. For this definition to make sense, $B\langle \tilde{u}, \tilde{v}\rangle$ must not depend on which extension we choose.

LEMMA 3 : *If u and $v \in \mathcal{W}$, with $u = 0$ for $t < a$, and $v = 0$ for $t > a$, then $B\langle u, v\rangle = 0$.*

PROOF : Without loss of generality, we take $a = 0$. Let ϱ be a smooth real-valued function with support in $[-1, 1]$ such that

$$\int_{-1}^{1} \varrho(t)\, dt = 1.$$

We define $u_\varepsilon = \varrho_\varepsilon * u$, where $\varrho_\varepsilon(t) = \varepsilon^{-1}\varrho\left(\frac{t}{\varepsilon}\right)$; it is easy to see that $u_\varepsilon \longrightarrow u$ in $\mathcal{W}$ as $\varepsilon \longrightarrow 0$. Thus, it suffices to show that

$$B\langle u_\varepsilon, v\rangle \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0.$$

But, we have

$$B\langle u_\varepsilon, v\rangle = \int_{-\infty}^{\infty} dt \left(\frac{\partial u_\varepsilon}{\partial t}, v\right)_H$$

$$= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} ds\, \varrho_\varepsilon'(s)\,(u(t-s), v(t))_H$$

$$= \int_{-\varepsilon}^{0} dt \int_{-\varepsilon}^{t} ds\, \varrho_\varepsilon'(s)\,(u(t-s), v(t))_H .$$

Thus, taking $K = \sup | \varrho'(t) |$, we obtain

$$| B \langle u_\varepsilon , v \rangle | \leq \frac{K}{\varepsilon^2} \int_{-\varepsilon}^{0} dt \int_{-\varepsilon}^{t} ds \, \| u(t-s) \|_H \, \| v(t) \|_H$$

$$\leq \frac{K}{\varepsilon^2} \int_{-\varepsilon}^{0} dt \, \| v(t) \|_H \int_{0}^{\varepsilon} \| u(\sigma) \|_H \, d\sigma$$

$$\leq \frac{K}{2} \left(\int_{-\varepsilon}^{0} \frac{\| v(t) \|_H^2}{| t |} \right)^{\frac{1}{2}} \left(\int_{0}^{\varepsilon} \frac{\| u(\sigma) \|_H^2}{| \sigma |} \, d\sigma \right)^{\frac{1}{2}}.$$

But, both of these last integrals tend to zero as $\varepsilon \to 0$, by Lemma 2 (ii), and our proof is complete.

Thus, we have defined $\widetilde{B} \langle u, v \rangle$ as a continuous sesquilinear form on $W^0 \times W^T$ (or $W^T \times W^0$): continuity follows, in fact, from Lemma 2 (ii). We next extend this form to an even larger domain: we shall call W_*^0 the space of those $u \in W$ for which there exists an $h \in H$ such that $u - h \in W^0$. Such an h, if it exists, is uniquely determined by u, and, furthermore, if u is such that $\lim_{t \to 0+} u(t)$ exists (in H), then $h = \lim_{t \to 0+} u(t)$. This justifies our writing $h = u(0)$. We give W_*^0 the norm

$$\| u \|_{W_*^0} = \left(\| u \|_W^2 + \int_0^T \frac{\| u(t) - u(0) \|_H^2}{| t |} \, dt \right)^{\frac{1}{2}}.$$

It is clear that we have

$$W^0 \subset W_*^0 \subset W;$$

in fact, W^0 is the subspace of those $u \in W_*^0$ for which $u(0) = 0$. For u in W_*^0, v in W^T, we define

$$\widetilde{B} \langle u, v \rangle = \widetilde{B} \langle u - u(0), v \rangle.$$

REMARK: If f is differentiable in $L^2[0, T; H]$, then, as is easily seen, the mapping $f\colon [0, T] \to H$ is uniformly continuous (after modification on a set of measure zero), and, in fact, we have

$$\| f(t) - f(s) \|_H \leq \left\| \frac{\partial f}{\partial t} \right\|_{L^2[0, T; H]} | t - s |^{\frac{1}{2}} \text{ for } 0 \leq t, s \leq T.$$

From this it follows that f is in W_*^0; if, in addition, $f(T)=0$, then f is also in W^T. Using Lemma 2, one easily shows that the differentiable functions in $L^2[0,T;H]$ are dense in W_*^0, and those which vanish at $t=T$ are dense in W^T. If f is differentiable in $L^2[0,T;H]$, then

$$\widetilde{B}\langle f, v\rangle = \int_0^T \left(\frac{\partial f}{\partial t}(t), v(t)\right)_H dt \text{ for all } v \in W^T; \tag{14}$$

if, in addition, $f(T)=0$, then

$$\widetilde{B}\langle u, f\rangle = -\int_0^T \left(u(t)-u(0), \frac{\partial f}{\partial t}(t)\right)_H dt$$

$$= -\int_0^T \left(u(t), \frac{\partial f}{\partial t}(t)\right)_H dt - (u(0), f(0))_H$$

for all $u \in W_*^0$.

Thus, the following gives a solution of the Problem stated in our introductory section:

THEOREM 2: *Suppose* $a(t;u,v)$ *satisfies* (i), (ii), *and* (iii). *Then, given* $\varphi \in W_*^0$ *and* $\psi \in L^2[0,T;V']$, *there exists a unique* $u \in W^0 \cap L^2[0,T;V]$ *satisfying*

$$\int_0^T a(t;u(t),v(t))\,dt + \widetilde{B}\langle u, v\rangle \tag{15}$$

$$= \widetilde{B}\langle \varphi, v\rangle + \int_0^T \langle \psi(t), v(t)\rangle\,dt$$

for all $v \in W^T \cap L^2[0;T;V]$. (*The bracket in the integral on the right of* (15) *represents the duality between* V' *and* V, *of course*). *Furthermore, the mapping* $(\varphi,\psi) \longrightarrow u$ *is continuous from* $W_*^0 \times L^2[0,T;V']$ *into* $W_*^0 \cap L^2[0,T;V]$ (*given the natural topology of an intersection.*)

PROOF: *a*) First, assume that (iii) holds with $\lambda=0$. We write $\widetilde{\varphi}$ for the extension of $\varphi-\varphi(0)$ to the whole line, defined as in the proof of Lemma 2 (ii). In particular, $\widetilde{\varphi}=0$ for $t<0$. We define $\psi \in L^2[-\infty,\infty;V']$ as follows:

$$\widetilde{\psi}(t) = \begin{cases} \psi(t) & \text{for } t \in (0,T) \\ 0 & \text{for } t \notin (0,T). \end{cases}$$

Finally, we define $F \in \mathcal{X}'$ by setting, for $w \in \mathcal{X}$

$$\langle F, w \rangle = B \langle \tilde{\varphi}, w \rangle + \int_{-\infty}^{\infty} \langle \tilde{\psi}(t), w(t) \rangle \, dt.$$

Clearly, $F = 0$ for $t < 0$, by Lemma 3. Applying Theorem 1 (with $A \langle u, v \rangle$ defined as at the beginning of this section) $\tilde{u} = L^{-1}(F)$ exists in $\mathcal{X}$, and satisfies

$$A \langle \tilde{u}, w \rangle + B \langle \tilde{u}, w \rangle = \langle F, w \rangle \quad \text{for every } w \in \mathcal{X}.$$

By Lemma 1, $\tilde{u} = 0$ for $t < 0$. We take u to be the restriction of $\tilde{u}$ to $(0, T)$; by Lemma 2, $u \in W^0 \cap L^2[0, T; V]$. Given $v \in W^T \cap L^2[0, T; V]$, its extension $\tilde{v}$, as defined in Lemma 2 (ii) is in $\mathcal{X}$ and vanishes for $t > T$. Thus, putting $w = \tilde{v}$ in the last equation, we obtain (15). The uniqueness of u follows easily from Lemma 1; the continuity assertion follows from the fact that the mapping $(\varphi, \psi) \rightarrow u$ is the composition of $(\varphi, \psi) \rightarrow F$, $F \rightarrow \tilde{u}$, and $\tilde{u} \rightarrow u$, all of which are clearly continuous, in the appropriate topologies.

b) In the general case, we use the standard device of multiplication by an exponential. We observe that multiplication by $e^{\beta t}$, β a real constant, is an isomorphism of each of the spaces W^0, W^0_*, W^T, $L^2[0, T; V]$, and $L^2[0, T; V']$ onto itself. In addition, we have, for $w \in W^0_*$ and $v \in W^T$

$$\tilde{B} \langle e^{\beta t} w, v \rangle = \tilde{B} \langle w, e^{\beta t} v \rangle + \beta \int_0^T e^{\beta t} (w(t), v(t))_H \tag{16}$$

In fact, by the Remark above, it suffices to prove (16) for w a differentiable function in $L^2[0, T; H]$; but in this case, it is trivial, by (14). We next note that the form

$$a^*(t; u, v) = a(t; u, v) + \lambda (u, v)_H$$

satisfies (i), (ii), and the stronger version of (iii) used in *a*). Since $\lambda \geq 0$, and $\tilde{\varphi}$ and $\tilde{\psi}$ vanish for $t < 0$, we may define $F^* \in \mathcal{X}'$ by taking, for $w \in \mathcal{X}$

$$\langle F^*, w \rangle = B \langle e^{-\lambda t} \tilde{\varphi}, w \rangle + \lambda \int_{-\infty}^{\infty} e^{-\lambda t} (\tilde{\varphi}(t), w(t))_H \, dt$$

$$+ \int_{-\infty}^{\infty} e^{-\lambda t} \langle \tilde{\psi}(t), w(t) \rangle \, dt;$$

moreover, the mapping $(\varphi, \psi) \longrightarrow F^*$ is continuous, and $F^* = 0$ for $t < 0$. Thus, by the reasoning used in *a*), we may find $u^* \in W^0$ which satisfies

$$\int_0^T a^*(t\,;\, u^*(t), v(t))\, dt + \widetilde{B}\langle u^*, v\rangle = \langle F^*, v\rangle$$

for all $v \in W^T \cap L^2[0, T\,;\, V]$; moreover, the mapping $(\varphi, \psi) \longrightarrow u^*$ is continuous. Now, we set $u = e^{\lambda t} u^*$; the last equality becomes, using (16)

$$\begin{aligned}\int_0^T a(t\,;\, u(t), e^{-\lambda t} v(t))\, dt + \widetilde{B}\langle u, e^{-\lambda t} v\rangle &= \langle F^*, v\rangle \\ &= \langle F, e^{-\lambda t} v\rangle,\end{aligned}$$

which gives us (15), by our earlier observation about the isomorphism properties of multiplication by an exponential. Uniqueness and continuity are immediate consequences of the same observation.

3. Continuity of solutions.

Our problem here is the following: what further conditions on φ and ψ in Theorem 2 guarantee that our solution u is continuous, as a function from $[0, T]$ to H? We note first the following: if $\alpha > \frac{1}{2}$, then $W_\alpha \subset W_*^0$. In fact, if $\varphi \in W_\alpha$, then after modification on a set of measure zero, φ satisfies the Hölder-condition

$$\| \varphi(t) - \varphi(s) \|_H \leq K_\alpha \| \varphi \|_{W_\alpha} | t - s |^{2\alpha - 1} \tag{17}$$

for $0 \leq t, s \leq T$, where K_α is a positive number depending only on α. Let $\tilde{\varphi}$ be the extension of φ given by Lemma 2 (i); by a standard argument, it suffices to prove (17) in the special case in which $\tilde{\varphi}$ has compact support. Then, modifying φ on a set of measure zero, we have, for $0 \leq t \leq T$,

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\tau} \widehat{\tilde{\varphi}}(\tau)\, d\tau$$

and hence

$$\| \varphi(t) - \varphi(s) \|_H \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} | e^{i\tau(t-s)} - 1 | \, | \hat{\varphi}(\tau) | \, d\tau$$

$$\leq \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} \frac{| e^{i\tau(t-s)} - 1 |^2}{1 + | \tau |^{2\alpha}} \right]^{\frac{1}{2}} \cdot \| \tilde{\varphi} \|_{\mathcal{W}_\alpha}$$

from which (17) follows, using Lemma 2 (i).

THEOREM 3: *Suppose* $\varphi \in W_\alpha$, *for some* $\alpha > \frac{1}{2}$, *and* $\psi \in L^2[0, T; V']$. *Then the solution* u *corresponding, by Theorem 2, to* (φ, ψ) *may be redefined on a set of measure zero so as to give a continuous mapping from* $[0, T]$ *into* H, *with* $u(0) = 0$.

PROOF: For the sake of simplicity we give the proof only in the special case considered in the introductory section: $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, where Ω is a bounded open set in R^n. Then, below, we indicate what must be done in the general case.

First, we consider the unbounded operator on H, A, defined by means of the relation

$$(Au, v)_H = (u, v)_V \text{ for all } v \in V;$$

the domain of A consists of those u in V for which the mapping $v \longrightarrow (u, v)_V$ is a continuous anti-linear functional on the set V, given its topology as a subset of H. In general, A is self-adjoint and $\geq \lambda_0 > 0$ ([2], Chapter II). In our special case, it is well-known that A (which is just the negative Laplacian, defined on functions which vanish at $\partial\Omega$) has a compact inverse. The problem

$$A\sigma = \lambda\sigma \quad (\lambda \text{ a complex number})$$

thus has an infinity of positive eigenvalues $0 < \lambda_0 = \lambda_1 \leq \lambda_2 \leq \ldots$ and corresponding eigenfunctions $\{\sigma_k\}$, elements of V, which in fact form a complete orthonormal basis in H. For u in H, we write

$$u_k = (u, \sigma_k)_H \qquad k = 1, 2, \ldots$$

Then, $\| u \|_H^2 = \sum_{k=1}^{\infty} | u_k |^2$; moreover, $u \in V$ if and only if $\{\lambda_k^{\frac{1}{2}} u_k\}$ is square summable ; in that case, we have

$$\| u \|_V^2 = \sum_{k=1}^{\infty} \lambda_k \mid u_k \mid^2.$$

Finally, for $u \in V'$ we define

$$u_k = \langle u, \sigma_k \rangle \qquad k = 1, 2, \dots ;$$

it is then easy to see that $\{\lambda_k^{-\frac{1}{2}} u_k\}$ is square-summable, and

$$\| u \|_{V'}^2 = \sum_{k=1}^{\infty} \lambda_k^{-1} \mid u_k \mid^2.$$

We shall, in fact, prove the following, which by Lemma 2, implies our Theorem : If $u \in \mathcal{X}$ and satisfies

$$B\langle u, v \rangle = B \langle \varphi, v\rangle + \int_{-\infty}^{\infty} \langle \psi (t), v (t) \rangle \, dt \text{ for all } v \in \mathcal{X} \tag{18}$$

where $\varphi \in \mathcal{W}_\alpha \left(\alpha > \frac{1}{2}\right)$ and $\psi \in L^2 [-\infty, \infty ; V']$, then, after modification on a set of measure zero, the mapping $u : (-\infty, \infty) \to H$ is bounded and uniformly continuous. Our proof consists of two steps :

a) Let $\varrho (\tau, \lambda)$ be a non-negative function with the property

$$\sup_{k=1,2,3,\dots,} \int_{-\infty}^{\infty} \frac{d\tau}{\varrho (\tau, \lambda_k)} = M < \infty.$$

Let $u \in \mathcal{H}$ be such that

$$||| \widehat{u} |||_\varrho^2 \equiv \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \varrho (\tau, \lambda_k) \mid \widehat{u}_k (\tau) \mid^2 d\tau < \infty.$$

Then, after modification on a set of measure zero, $u : (-\infty, \infty) \to H$ is uniformly continuous, and we have

$$\sup_{-\infty<t<\infty} \| u (t) \|_H^2 \leq \frac{M}{2\pi} ||| \widehat{u} |||_\varrho^2 \tag{19}$$

PROOF: If $\widehat{u}$ has compact support, then, redefining u on a set of measure zero, we have

$$u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\tau} \widehat{u}(\tau)\, d\tau\,;$$

thus, u is uniformly continuous, and we have

$$| u_k(t) |^2 = \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\tau} \widehat{u}_k(\tau)\, d\tau \right|^2$$

$$\leq \frac{1}{2\pi} \left(\int_{-\infty} \frac{d\tau}{\varrho(t, \lambda_k)} \right) \left(\int_{-\infty}^{\infty} \varrho(\tau, \lambda_k) \, | \widehat{u}_k(\tau) |^2 \, d\tau. \right.$$

Hence,

$$\| u(t) \|_H^2 = \sum_{k=1}^{\infty} | u_k(t) |^2 \leq \frac{M}{2\pi} \||\, \widehat{u} \,\||_\varrho^2$$

which gives (19). For general $u \in \mathcal{H}$, we approximate $\widehat{u}$ simultaneously in $\mathcal{H}$ and in the $\||\cdot\||_\varrho$ norm by a sequence $\{v_n\}$ of functions in $\mathcal{H}$, with compact support. The sequence $\{\check{v}_n\}$ of inverse Fourier transforms approaches u in $\mathcal{H}$; it is also a Cauchy sequence in the uniform norm, by (19), and hence has as limit a uniformly continuous H-valued function, which must agree almost everywhere with u.

b) We now show that under the assumptions made on u above, and with the choice

$$\varrho(\tau, \lambda) = \frac{\lambda^2 + |\tau|^2}{\lambda + |\tau|^{2-2\alpha}}\,,$$

the hypotheses of *a*) are satisfied. Since $\alpha > \frac{1}{2}$, an easy calculation verifies the first. For the second, we first note that

(20) $$i\tau \widehat{u}_k(\tau) = i\tau \widehat{\varphi}^k(\tau) + \widehat{\psi}_k(\tau) \text{ a. e. in } (-\infty, \infty)\,;$$

in fact, taking $v(t) = \varrho(t)\, \sigma_k$, where ϱ is an arbitrary test function, (18), combined with (6), gives

$$\int_{-\infty}^{\infty} i\tau\, \widehat{u}_k(\tau) \overline{\widehat{\varrho}(\tau)}\, d\tau = \int_{-\infty}^{\infty} i\tau\, \widehat{\varphi}_k(\tau) \overline{\widehat{\varrho}(\tau)}\, d\tau + \int_{-\infty}^{\infty} i\tau\, \widehat{\psi}_k(\tau) \overline{\widehat{\varrho}(\tau)}\, d\tau$$

from which (20) follows at once. Thus, we may conclude that

$$|\widehat{u}_k(\tau)|^2 \leq 2\left[|\widehat{\varphi}_k(\tau)|^2 + \frac{|\widehat{\psi}_k(\tau)|^2}{|\tau|^2}\right] \text{ a. e.}$$

and therefore

$$\varrho(\tau, \lambda_k)\,|\widehat{u}_k(\tau)|^2 = \frac{\lambda_k^2}{\lambda_k + |\tau|^{2-2\alpha}}\,|\widehat{u}_k(\tau)|^2 + \frac{|\tau|^2}{\lambda_k + |\tau|^{2-2\alpha}}\,|\widehat{u}_k(\tau)|^2$$

$$\leq \lambda_k\,|\widehat{u}_k(\tau)|^2 + 2\,|\tau|^{2\alpha}\,|\widehat{\varphi}_k(\tau)|^2 + \frac{2}{\lambda_k}\,|\widehat{\psi}_k(\tau)|^2$$

Hence, we have, applying Plancherel

$$|||\,u\,|||_{\varrho}^2 \leq \|\,u\,\|_V^2 + 2\,\|\,\varphi\,\|_{\mathcal{W}_\alpha}^2 + 2\,\|\,\psi\,\|_{L^2[-\infty,\,\infty;\,V']}^2$$

and our proof is complete.

For the general case, the eigenfunction expansion is, of course, not available. But this expansion is just a special case of the von Neumann diagonalization theorem used by Lions in a similar context ([2], Chapter IV, Theorem 2.1, and [3]). Using this theorem, the proof goes through, formally without change (an integral over a half-line replacing the sum over discrete eigenvalues).

4. Comparison with the solution in [2].

In [2], Chapter IV, an existence theorem (Theorem 1.1) is given which provides a solution $u \in L^2[0, T; V]$ corresponding to given initial data $u_0 \in H$. To compare the scope of this theorem with our Theorem 2 it will suffice to answer the following question: For which $u_0 \in H$ does there exist $\varphi \in W_*^0 \cap L^2[0, T; V]$ such that $\varphi(0) = u_0$? In the formulation of our answer, and its proof, it is again more convenient to treat only the same special case considered in Section 3; the remarks made there about the extension to the general case apply again here.

THEOREM 4: *Given $h \in H$, there exists $\varphi \in W_*^0 \cap L^2[0, T; V]$ such that $\varphi(0) = h$ if and only if*

$$\sum_{k=1}^{\infty} (\log \lambda_k)\,|\,h_k\,|^2 < \infty \tag{20}$$

PROOF: Given $h \in H$, we define

$$\varphi^*(t) = \sum_{k=1}^{\infty} h_k e^{-\lambda_k t} \sigma_k .$$

It is easily seen that $\varphi^* \in W \cap L^2[0, T; V]$; in fact

$$\| \varphi^* \|^2_{L^2[0, T; V]} = \int_0^T \left\{ \sum_{k=1}^{\infty} \lambda_k | h_k |^2 e^{-2\lambda_k t} \right\} dt \leq \frac{1}{2} \| h \|^2_H$$

and

$$\| \varphi^* \|^2_W = \int_0^T \int_0^T \left\{ \sum_{k=1}^{\infty} | h_k |^2 \frac{| e^{-\lambda_k t} - e^{-\lambda_k s} |^2}{| t - s |^2} \right\} dt\, ds$$

$$\leq K \| h \|^2_H$$

where $K = \int_0^\infty \int_0^\infty \frac{| e^{-t} - e^{-s} |^2}{| t - s |^2} dt\, ds < \infty.$

Moreover, we have

$$\int_0^T \frac{\| \varphi^*(t) - h \|^2_H}{| t |} dt = \int_0^T \left\{ \sum_{k=1}^{\infty} | h_k |^2 \frac{| e^{-\lambda_k t} - 1 |^2}{| t |} \right\} dt. \tag{21}$$

But since

$$\int_0^T \frac{| e^{-\lambda t} - 1 |^2}{| t |} dt \sim \log \lambda \text{ as } \lambda \to +\infty,$$

the « if » part of our assertion follows from (21) by Fubini's theorem. Conversely, given $\varphi \in W^0_* \cap L^2[0, T; V]$ with $\varphi(0) = h$, we apply Theorem 2 to assert the existence of a $u \in W^0 \cap L^2[0, T; V]$ which satisfies

$$\tilde{B} \langle u, v \rangle + \int_0^T (u(t), v(t))_V \, dt = - \tilde{B} \langle \varphi, v \rangle - \int_0^T (\varphi(t), v(t))_V \, dt \tag{22}$$

for all $v \in W^T \cap L^2[0, T; V]$. We shall show that $u + \varphi = \varphi^*$ a. e. in $[0,T]$; hence φ^* will belong to W^0_* (with $\varphi^*(0) = h$) and the « only if » part will follow from (21), using Fubini's theorem again. To this end, we write

$$\psi_k(t) = (u + \varphi)_k(t)\, e^{\lambda_k t};$$

in (22), we take $v(t) = e^{\lambda_k t} \varrho(t) \sigma_k$, where ϱ is an arbitrary smooth real-valued function with compact support in $(0, T)$. (22) then gives us (see the Remark of Section 2)

$$-\int_0^T \psi_k(t)\, \varrho'(t)\, dt = \lambda_k \int_0^T \psi_k(t)\, \varrho(t)\, dt - \int_0^T e^{\lambda_k t} \varrho(t)\, (u(t) + \varphi(t), \sigma_k)_V\, dt$$

$$= \lambda_k \int_0^T \psi_k(t)\, \varrho(t)\, dt - \int_0^T e^{\lambda_k t} \varrho(t)\, (u(t) + \varphi(t), \Lambda\sigma_k)_H\, dt$$

$$= 0,$$

from which it follows at once that, after modification on a set of measure zero, $\psi_k(t)$ is constant on $[0, T]$. But since

$$\int_0^T \frac{\| \varphi_k(t) - h_k \|_H^2}{|t|}\, dt < \infty$$

we must have $\psi_k(t) = h_k$ a. e. in $[0, T]$, and hence, for every $k = 1, 2, \ldots$

$$(u + \varphi)_k(t) = \varphi_k^*(t) \text{ a. e. in } [0, T],$$

from which our assertion follows.

COROLLARY: *Suppose $a(t; u, v)$ is as Theorem 2. Given $h \in H$, there exists a $u \in W_*^0 \cap L^2[0, T; V]$ with $u(0) = h$, satisfying*

$$\widetilde{B}\langle u, v \rangle + \int_0^T a(t; u(t), v(t))\, dt = 0$$

for all $v \in W^T \cap L^2[0, T; V]$ if and only if h satisfies (20).

PROOF: If such a u exists, then h satisfies (20), by Theorem 4. Conversely, if h satisfies (20), then, by Theorem 4, there is a $\varphi \in W_*^0 \cap L^2[0, T; V]$

with $\varphi(0) = h$. Then, by Theorem 2, there exists $w \in W^0 \cap L^2[0, T; V]$ satisfying

$$\widetilde{B}\langle w, v\rangle + \int_0^T a(t; w(t), v(t))\, dt = -\widetilde{B}\langle \varphi, v\rangle - \int_0^T a(t; \varphi(t), v(t))\, dt$$

for all $v \in W^T \cap L^2[0, T; V]$; $u = w + \varphi$ clearly has all of the desired properties.

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