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GENERAL MIXED PROBLEMS ON A HALF-SPACE FOR CERTAIN HIGHER ORDER PARABOLIC EQUATIONS IN ONE SPACE VARIABLE

by R. K. JUBERG (*)

In recent years much has been written on mixed problems for higher order parabolic equations. For works in the classical mode see, for example, Èidel'man [5], Èidel'man and Lipko [6], Isakova [12], Mihailov [19 *a, b*], Sobolev [23], Solonnikov [24], Zagorskii [25], Arima [2], Cattabriga [4*a*], Pini [21*a*], Juberg [13]; and for generalized solutions consult, for example, Agronovic and Visik [1], Mihailov [19*c*], Lions [17], Lions and Magenes [18], Browder [3], Lax and Milgram [16], Friedman [9], Hersh [11], Cattabriga [4*b*], Pini [21*b*]. The above list should in no sense be taken as being complete: for additional publications see the lists of references in those cited.

The existence theories, for both classical and generalized solutions, have been extended so as to encompass very general situations. This is true in regard to the equations that are considered as well as to the boundary conditions and the domains. However, on the question of uniqueness, very few general theorems have been published.

In this paper we consider general mixed problems in a half-space for certain simple equations in one space variable. The boundary operators are *unrestricted as to order and individual structure*. We prove general *uniqueness* results and also an existence theorem. The existence theorem is of interest because of the generality of the boundary operators. More significant, though, is the fact that in this result it is noted that the interrelationship of the boundary operators, beyond their independence, has an effect.

The method seems to be such that, if pursued, it would give rise to the best possible results; see the remark that follows the statements of the theorems in the next section. Moreover the method should provide a basis

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for extensions. For this reason we proceed in a direct manner and make use of specialized devices only when it is no longer reasonable to avoid them. Of central importance is a general representation theorem, Theorem 4. The principal results are derived primarily from it.

In Section 1 we state our principal results. Section 2 contains some auxiliary results preparatory to the proofs of the main theorems. The proofs of the uniqueness theorems are given in Section 3 and the existence is presented in Section 4. In Section 5 we give alternate formulas to those in Section 4 for the solution. The appendix contains the proofs of several lemmas stated in Section 2.

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1. Main Theorems.

Denote by R the topological product of the positive real line with the segment $(0, T)$, $\mathbb{R}^+ \times (0, T)$. Let the letter D with a variable subscript appended indicate the operation of differentiation with respect to this variable, for example, D_x, D_t , etc.

We consider the following problem: Find $u = u(x, t)$, locally bounded in $\mathbb{R}^+ \times [0, t)$, that solves the equation

$$(1.1) \quad D_x^{2m} u + (-1)^m D_t u = 0$$

for $(x, t) \in R$ and satisfies in addition the following conditions:

$$(1.2) \quad u(x, t) = 0 \quad (\exp Ax^{2m/2m-1}), x \rightarrow \infty,$$

uniformly in finite time intervals, and

$$(1.3) \quad u(x, 0+) = 0 \quad (0 < x < \infty),$$

$$(1.4) \quad B_j(D_x)u(0+, t) = f_j(t), \quad 0 < t < T, \quad (j = 1, \dots, m)$$

where the $B_j(\xi)$ are polynomials in the indeterminate ξ with constant coefficients.

The sense with which the boundary values are assumed is indicated in the statements of the theorems.

Let $\omega_k (k = 1, \dots, m)$ be the $2m$ -th roots of $(-1)^{m+1}$ that have negative real parts and form the $m \times m$ matrix

$$(1.5) \quad M(z) = (B_j(\omega_k z)).$$

The basic solvability criterion ([6], [24], [25]) is here expressed in terms of $M(z)$.

Basic Assumption.

$$(1.6) \quad \det M(z) \neq 0.$$

There are certain parameters that enter into the statements of the theorems, for which we make the following definition.

DEFINITION 1. Let \bar{r} be the maximum of the orders of the operators which appear in $B(D_x)$, the boundary operator with components $B_j(D_x)$ ($j = 1, \dots, m$). Denote by ϱ the largest integer such that $z^\varrho M^{-1}(z)$ is analytic at infinity: $M^{-1}(z)$ is the matrix inverse to $M(z)$.

DEFINITION 2. We shall say that $x \rightarrow u(x, \cdot)$ is continuous at $x = 0$ in the L^γ — sense locally, if for each $t \in (0, T)$

$$\lim_{x \rightarrow 0} \int_0^t |u(x, \tau) - u(0, \tau)|^\gamma d\tau = 0.$$

THEOREM 1. (*Uniqueness*) Suppose $u = u(x, t)$ satisfies (1.1) - (1.3). Further assume that $x \rightarrow u(x, \cdot)$ is continuous in the L^1 - sense locally at $x = 0$. If for $0 < t < T$

$$(1.7) \quad \lim_{x \rightarrow 0} \int_0^t (t - \tau)^p B(D_x) u(x, \tau) d\tau = 0,$$

where $p = \max((\bar{r} - 1)/2m, 0)$, then $u = 0$ in R .

COROLLARY. With the same assumptions as above, if either

$$\int_0^T |B(D_x) u(x, \tau)|^\gamma d\tau \rightarrow 0 \text{ as } x \rightarrow 0 \ (\gamma \geq 1)$$

or, more particularly, the expressions $B(D_x) u$ have trace zero on $[0, T]$ then $u = 0$ in R .

The corollary reduces immediately to the theorem, since, in either case,

$$\int_0^T |B(D_x) u(x, \tau)|^\gamma d\tau \rightarrow 0 \text{ as } x \rightarrow 0,$$

for some $\gamma \geq 1$. Which in turn implies (1.7).

The reader is referred to the fundamental paper on the theory of trace by Gagliardo [10]: for a survey on the theory see the exposition in Serrin [22].

THEOREM 2. (Uniqueness) Suppose $u = u(x, t)$ satisfies (1.1) - (1.3). Further assume that $x \rightarrow u(x, \cdot)$ is continuous in the $L^{2m+\varepsilon}$ -sense locally at $x = 0$, for some $\varepsilon > 0$. If for every $\Phi \in C^\infty$ with support compact in $[0, T]$

$$(1.8) \quad \lim_{x \rightarrow 0} \int_0^T \Phi(t) \int_0^t (t - \tau)^{p-1} B(D_x) u(x, \tau) d\tau dt = 0,$$

where $p = \max((\bar{r} - 1)/2m, 0)$ and the inner integral is to be replaced by $B(D_x) u(x, t)$ when $p = 0$, then $u = 0$ in R .

DEFINITION 3. For suitable functions $g = g(t)$ define

$$(1.9) \quad D^\alpha g(t) = \left(\frac{d}{dt}\right)^{1+[\alpha]} \frac{1}{\Gamma(1 - \alpha + [\alpha])} \int_0^t g(\tau) (t - \tau)^{-\alpha + [\alpha]} d\tau$$

where $[\alpha]$ denotes the largest integer in α , $\alpha \geq 0$.

DEFINITION 4. Let k be a non-negative integer and set $a = 1/2m$. Denote by $S^{ka, b, c}$, $0 < ka - [ka] + b \leq c < 1$, the set of functions, $g = g(t)$, $0 < t \leq T$, such that

(i) g is $[ka]$ -times continuously differentiable and $g^{(j)}(0) = 0$ for $j = 0, \dots, [ka] - 1$,

and

(ii)

$$(1.10) \quad \sup_{t, \tau} \sigma^c \frac{|D^{ka} g(t) - D^{ka} g(\tau)|}{|t - \tau|^b} + \sup_t t^{c-ka+[ka]} |D^{[ka]} g(t)| < \infty$$

where $\sigma = \min(t, \tau)$.

For simplicity of notation we shall suppress the dependence of $S^{ka, b, c}$ on the parameters a, b , and c and denote the space simply by S^k .

THEOREM 3. (Existence) Suppose $f \in S^{\bar{r}-e}$, where f is an m -vector. Then problem (1.1) - (1.4) has a unique solution $u = u(x, t)$ where $u \in C^{\bar{r}}(\bar{R} - \{(0, 0)\})$

and $\int_0^t |u(x, \tau)| d\tau = 0(1)$ as $x \rightarrow 0$ for each $t \in (0, T)$.

REMARK. It will become clear in the proof that a more precise theorem could be formulated by imposing the hypothesis component by component. From a more detailed study of the representations of the solution operators one would be lead to the appropriate conditions.

2. Auxiliary Results.

We denote the usual fundamental solution of equation (1.1) by $K = K(x, t)$ where

$$(2.1) \quad K(x, t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-ix\xi - \xi^{2m}t) d\xi,$$

and, further, we recall that

$$(2.2) \quad |D_x^j K(x, t)| \leq C_1(j) t^{-(1+j)/2m} \exp[-C_2(x^{2m}/t)^{1/2m-1}]$$

(see, for example Ladyženskaya [15], Rosenbloom [8] (chp. 4); or, also, the appendix in [13b] for the type of equation being considered).

The following theorem provides a quite general representation formula for solutions of equation (1.1) which satisfy (1.2) and (1.3). We shall see that the main results of this paper are rather direct consequences of it.

THEOREM 4. *Let $u = u(x, t)$ be locally bounded in $\mathbb{R}^+ \times [0, T)$, solve (1.1) in R , and satisfy (1.2) and (1.3). If $x \rightarrow u(x, \cdot)$ is continuous in the L^1 -sense locally at $x = 0$, then for $0 < t < T_0$, $T_0 = (C_2/A)^{2m-1}$, and $x > 0$*

$$(2.3) \quad u(x, t) = (-1)^m 2 \sum_{j=0}^{m-1} \int_0^t I^{2j} u(0, \tau) D_x^{2(m+j)-1} K(x, t - \tau) d\tau,$$

where

$$I^{2j} u(x, t) = \Gamma(2j)^{-1} \int_x^\infty u(\xi, t) (\xi - x)^{2j-1} d\xi; \quad I^0 u(x, t) = u(x, t).$$

In the proof of Theorem 4 we use the fact that here u is sufficiently often continuously differentiable in $\mathbb{R}^+ \times [0, T)$; in fact, $u \in C^\infty(\mathbb{R}^+ \times [0, T))$. This follows from (1.1), (1.3), and the local boundedness up to $t = 0$; for example, by an argument similar to that at the end of [16]. The proof of the theorem is then a straightforward deduction using Lemma 1 and Lemma 2, which follow.

LEMMA 1. Let E denote the set of functions, $v = v(x)$, in \mathbb{R}^+ which possess the following property: $|x^k D^j v(x)| \rightarrow 0$ as $x \rightarrow \infty$ for $0 \leq j, k < \infty$.

Then $(-D)^n$ and I^n commute on E , where $I^n v(x) = \Gamma(n)^{-1} \int_x^\infty v(\xi) (\xi - x)^{n-1} d\xi$.

PROOF. On repeatedly applying Leibnitz' rule for differentiating functions defined by integrals combined with integrations by parts we find simply that, for $j \leq n$,

$$\begin{aligned} (-D)^j I^n v(x) &= \Gamma(n)^{-1} (-D_x)^j \int_x^\infty v(\xi) (\xi - x)^{n-1} d\xi \\ &= \Gamma(n)^{-1} \left[(n-1) \dots (n-j+1) \delta_{jn} v(x) + \int_x^\infty v(\xi) (-D_x)^j (\xi - x)^{n-1} d\xi \right] \\ &= \Gamma(n)^{-1} \left[(n-1) \dots (n-j+1) \delta_{jn} v(x) + \int_x^\infty v(\xi) D_\xi^j (\xi - x)^{n-1} d\xi \right] \\ &= \Gamma(n)^{-1} \int_x^\infty (-D_\xi)^j v(\xi) \cdot (\xi - x)^{n-1} d\xi = I^n (-D)^j v(x), \end{aligned}$$

where δ_{jn} is the Kronecker delta function.

LEMMA 2. Let $u = u(x, t)$ solve (1.1) in R , $T = T_0$, and satisfy (1.2) and (1.3). If $u \in C^{2m-2}(\bar{R})$ ($\bar{R} \equiv$ closure of R), then

$$(2.4) \quad u(x, t) = (-1)^m 2 \sum_{j=0}^{m-1} \int_0^t D_x^{2(m-1-j)} u(0, \tau) D_x^{2j+1} K(x, t - \tau) d\tau.$$

The proof of Lemma 2 is presented in the appendix. Therefore we pass immediately to the proof of Theorem 4.

PROOF OF THEOREM 4. We shall use freely the fact that $u \in C^\infty(\mathbb{R}^+ \times \times [0, T])$. Then, from Lemma 2 applied to $(x, t) \rightarrow u(x + y, t)$ ($y > 0$), we have

$$(1) \quad u(x + y, t) = (-1)^m 2 \sum_{j=0}^{m-1} \int_0^t D_x^{2(m-1-j)} u(y, \tau) D_x^{2j+1} K(x, t - \tau) d\tau,$$

for any $y > 0$.

Now it follows simply from (2.2) and (1) that $u(\cdot, t) \in \mathcal{E}$, $t > 0$. Thus, using Lemma 1, we find on applying (1) to the function $I^{2m-2}u$, followed by the application of operator D_x^{2m-2} , that

$$(2) \quad u(x+y, t) = (-1)^m 2 \sum_{j=0}^{m-1} \int_0^t I^{2j} u(y, \tau) D_x^{2(m+j)-1} K(x, t-\tau) d\tau, \quad y > 0.$$

In addition, as one sees by an obvious change of integration variable,

$$I^{2j} u(y, \tau) = \Gamma(2j)^{-1} \int_0^\infty u(\xi+y, \tau) \xi^{2j-1} d\xi.$$

The conclusion then follows from (2) upon letting y tend to zero.

We next make note of certain properties of the spaces S^k , which we will use in the sequel. Observe, first, that on S^k the operational rules

$$(2.5) \quad D^{ja} = D^{ja-[ja]} D^{[ja]} \quad \text{and} \quad D^{ja} = (D^a)^j, \quad j \leq k,$$

are valid. The verification of these rules and also, in part, the proof of the following lemma (which is an expression of some additional essentially operational facts) involve manipulations of a similar nature. We shall give an indication of them by sketching the proof of the lemma.

LEMMA 3. (i) If $g \in S^k$, then $D^{ja} g \in S^{k-j}$.

(ii) If $g \in S^{k-j}$, then the function

$$t \rightarrow \Gamma(ja)^{-1} \int_0^t g(\tau) (t-\tau)^{ja-1} d\tau$$

is in S^k .

(iii) The operation $g \rightarrow D^{ja} g$, as a mapping of S^k into S^{k-j} , and the operation taking $g(t)$ into

$$\Gamma(ja)^{-1} \int_0^t g(\tau) (t-\tau)^{ja-1} d\tau,$$

as a mapping of S^{k-j} into S^k , are inverses of each other.

PROOF. Statement (i) follows directly from (2.5) and the definition of the space S^k .

Now consider statement (ii). We observe from definitions 3 and 4 that, for $g \in S^k$,

$$D^{ja} g(t) = \left(\frac{d}{dt}\right)^{1+[ja]-n} \Gamma(1-ja+[ja])^{-1} \int_0^t g^{(n)}(\tau) (t-\tau)^{-ja+[ja]} d\tau,$$

where $0 \leq n \leq 1 + [ja] \leq [ka] - 1$. This is, in essence, the first part of (2.5). Hence, without significantly compromising the statement, we suppose that $ka < 1$.

Set

$$(1) \quad h(t) = \Gamma(ja)^{-1} \int_0^t g(\tau) (t-\tau)^{ja-1} d\tau,$$

where $g \in S^{k-j}$. Then the application of the corresponding Riemann-Liouville operator of order $1-ka$ gives

$$(2) \quad \Gamma(1-ka)^{-1} \int_0^t h(\tau) (t-\tau)^{-ka} d\tau = \Gamma(1-(k-j)a)^{-1} \int_0^t g(\tau) (t-\tau)^{-(k-j)a} d\tau.$$

Hence, on differentiating (2),

$$(3) \quad D^{ka} h = D^{(k-j)a} g.$$

Further, since $\tau^{c-(k-j)a} g(\tau)$ is bounded on $(0, T)$, it follows from (1) that

$$(4) \quad t^{c-ka} |h(t)| \leq t^{c-ka} \Gamma(ja)^{-1} \int_0^t \tau^{c-(k-j)a} |g(\tau)| \tau^{-c+(k-j)a} (t-\tau)^{ja-1} d\tau$$

$$\leq (\text{Constant}) t^{c-ka} \int_0^t \tau^{-c+(k-j)a} (t-\tau)^{ja-1} d\tau = \text{Constant}.$$

Then, using (3) and (4), the conclusion follows directly from the definitions.

After observing that, for $g \in S^k$, $D^{ja} g$ is absolutely integrable on $(0, T)$, the proof of statement (iii) proceeds as for (3) above.

In the next lemma we give an alternate expression for the operator D^{ka} which will be found useful in obtaining certain limits in Lemma 5. The proof of this lemma as well as that of Lemma 5 will be deferred to the appendix.

LEMMA 4. If $g \in S^k$, $ka < 1$, then

$$(2.6) \quad D^{ka} g(t) = \Gamma(1 - ka)^{-1} \left\{ g(t) t^{-ka} + ka \int_0^t [g(t) - g(\tau)] (t - \tau)^{-ka-1} d\tau \right\}.$$

LEMMA 5. Suppose $g \in S^k$. Then for $j \leq k$

$$(2.7) \quad \lim_{y \rightarrow 0} \int_0^t g(\tau) D_y^{2m-1+j} K(y, t - \tau) d\tau = C(j, m) D^{ja} g(t),$$

where $C(j, m) = 0$ for j -even such that $j \not\equiv 0 \pmod{2m}$.

By continuing in the manner of the argument in Theorem 4 we are lead to the proof of the next theorem. This theorem provides the direct means for the proofs of our main results.

THEOREM 5. Suppose u satisfies the hypothesis in Theorem 4. Then in $\mathbb{R}^+ \times (0, T_0)$, $T_0 = (C_2/A)^{2m-1}$, we have for $x \in \mathbb{R}^+$,

$$(2.8) \quad B(D_x)u = D_t^{1+(\bar{r}-2)\alpha} VU$$

where V is an $m \times m$ matrix of linear combinations of Volterra type integral operators and U is a column vector with components $U_j = I^{2(j-1)} u(x, t)$, ($j = 1, \dots, m$).

PROOF. It follows from Theorem 4 that for $x, y > 0$ and $0 < t < T_0$

$$(1) \quad u(x + y, t) = (-1)^m 2 \sum_{j=0}^{m-1} \int_0^t I^{2j} u(x, \tau) D_y^{2(m+j)-1} K(y, t - \tau) d\tau.$$

This identity relates functions that are C^∞ with respect to all indicated variables in their ranges under consideration. Note also that the terms on the right side are convolutions with respect to t of functions vanishing of infinite order at $t = 0$.

From (1) we see that

$$\begin{aligned} D_x^r u(x + y, t) &= D_y^r u(x + y, t) \\ &= (-1)^m 2 \sum_{j=0}^{m-1} \int_0^t I^{2j} u(x, \tau) D_y^{2(m+j)-1+r} K(y, t - \tau) d\tau. \end{aligned}$$

Then, by Lemma 5, it follows that $D_x^r u(x, t)$ is a linear combination of terms of the form (constant) $D_t^{ka} I^{2j} u(x, t)$, where $k \leq 2(m-1) + r$ (see, Definition 1). Hence the conclusion follows from Lemma 3.

3. Proofs of Theorems 1 and 2.

The operator V appearing in (2.8) can be represented in a form which displays more clearly its structure and dependence on the boundary operator B . This could be done directly, but it would involve extensive and tedious calculations. Whereas, with considerable ease, it can be done through the use of the operational calculus in the field \mathcal{F} of *convolution quotients*, as developed by Mikusinski [20]; see also Erdelyi [7]. There is a *natural embedding* of continuous functions $f(t)$ into \mathcal{F} ; hence certainly for C^∞ functions. Furthermore, the operation of differentiation in functions vanishing at the origin has a natural unique correspondant in \mathcal{F} .

Let $U = U(t)$ be an m -vector with components $U_j \in C^\infty(\mathbb{R}^+)$ vanishing of arbitrary order at $t = 0$. Consider the expression

$$(3.1) \quad B(D_x) (-1)^m 2 \sum_{j=0}^{m-1} \int_0^t U_j(\tau) D_x^{2(m+j)-1} K(x, t - \tau) d\tau.$$

We shall embed this in the functions which are continuous from \mathbb{R}^+ into \mathcal{F} (see [20], Part 3, Chap. I). The operation of convolution of functions corresponds to multiplication in \mathcal{F} . Thus, setting

$$(3.2) \quad Q(x, t) = (-1)^m 2 K(x, t),$$

we have for the analogue of (3.1)

$$(3.3) \quad B(D_x) \sum_{j=0}^{m-1} U_j Q^{(2(m+j)-1)}(x),$$

where $Q(x)$ is the image of $Q(x, t)$ under the embedding, and U is the image of $U(t)$.

We adopt now some notation common to the operational calculus. In particular, the elements in \mathcal{F} that correspond to the operation of differentiation with respect to t and to the functions $\Gamma(\gamma)^{-1} t^{\gamma-1}$ will be denoted, respectively, by s and l^γ . Moreover, $s^\lambda = s l^{1-\lambda}$ for $0 \leq \lambda \leq 1$ and $s^{-\gamma} = l^\gamma$ for $\gamma \geq 0$. We note, also, that the operator D^γ acting on appropriate functions, say in $S^{ka, b, c}$ ($\gamma \leq k$), corresponds to s^γ .

Let $\omega_k (k=1, \dots, m)$ be the $2m$ -th roots of $(-1)^{m+1}$ which have negative real parts and set $a=1/2m$ (as in § 1). Using a contour integral representation valid for certain exponential functions from \mathbb{R}^+ into \mathcal{F} as given in [20], Appendix (Chp. VIII), one can show that

$$mQ^{(2m-1)}(x) = \sum_{k=1}^m \exp \omega_k s^a x.$$

Then, from the continuity properties of the exponentials and their derivatives, we find that

$$(3.4) \quad mQ^{(2(m+n)-1)}(x) = s^{2an} \sum_{k=1}^m \omega_k^{2n} \exp \omega_k s^a x.$$

Denote by \mathcal{C} the row vector with components $\exp \omega_k s^a x, (k=1, \dots, m)$ and let $\Omega^{(n)}$ be the column vector with components $\omega_k^{2(n-1)}, (k=1, \dots, m)$. Then (3.4) becomes $Q^{(2(m+n)-3)}(x) = \varepsilon \Omega^{(n)} m^{-1} s^{2a(n-1)}$ and (3.3) takes the form

$$(3.5) \quad B(D_x) \mathcal{C} \Omega m^{-1} \text{diag}(1, s^{2a}, \dots, s^{2(m-1)a}) U$$

where $\Omega = (\Omega^{(1)}, \dots, \Omega^{(m)}) = (\omega_k^{2(n-1)})$, an $m \times m$ matrix with k and n , respectively, as row and column indices. Moreover,

$$(3.6) \quad B(D_x) \mathcal{C} = (B_j(\omega_k s^a) \exp \omega_k s^a x).$$

Thus the image of (3.1) can be expressed as

$$(3.7) \quad (B_j(\omega_k s^a) \exp \omega_k s^a x) \Omega m^{-1} \text{diag}(1, s^{2a}, \dots, s^{2(m-1)a}) U.$$

Now it follows, as in the proof of Theorem 5, that (3.1) approaches $D_i^{1+(\bar{r}-2)a} VU$ as x tends to zero. Since (3.7) is continuous from \mathbb{R}^+ to \mathcal{F} , as x approaches zero it approaches $M(s^a) \Omega m^{-1} \text{diag}(1, s^{2a}, \dots, s^{2(m-1)a}) U$, (see (1.5)). To summarize, we have the correspondence

$$(3.8) \quad D_i^{1+(\bar{r}-2)a} VU \longleftrightarrow M(s^a) \Omega m^{-1} \text{diag}(1, s^{2a}, \dots, s^{2(m-1)a}) U;$$

and, hence, the added correspondences

$$(3.9) \quad D_i^{1+(\bar{r}-2)a} V \longleftrightarrow M(s^a) \Omega m^{-1} \text{diag}(1, s^{2a}, \dots, s^{2(m-1)a})$$

and

$$(3.10) \quad V \longleftrightarrow l^{1+(\bar{r}-2)a} M(s^a) \Omega m^{-1} \text{diag}(1, s^{2a}, \dots, s^{2(m-1)a}).$$

This latter correspondence, (3.10), prevails also when V acts on functions U which are merely locally integrable, that is, satisfy

$$\int_0^t |U_j(\tau)| d\tau < +\infty \quad \text{for each } t > 0.$$

PROOF OF THEOREM 1. We shall prove that $u = 0$ in $\mathbb{R}^+ \times (0, T_0)$, $T_0 = (C_2/A)^{2m-1}$. The general conclusion is then found by extending this result stepwise by increments of λT_0 , ($0 < \lambda < 1$), in the usual way.

We deduce, from Theorem 5, that

$$\begin{aligned} \frac{1}{\Gamma(p)} \int_0^x (\tau - \sigma)^{p-1} B(D_x) u(x, \sigma) d\sigma &= \\ &= \frac{\partial}{\partial \tau} \frac{1}{\Gamma(p + (2 - \bar{r})a)} \int_0^x (\tau - \sigma)^{p+(2-\bar{r})a-1} VU(x, \sigma) d\sigma, \end{aligned}$$

and on integrating from 0 to t we find that

$$\begin{aligned} (1) \quad \Gamma(p + 1)^{-1} \int_0^t (t - \tau)^p B(D_x) u(x, \tau) d\tau &= \\ &= \Gamma(p + (2 - \bar{r})a)^{-1} \int_0^t (t - \tau)^{p+(2-\bar{r})a-1} VU(x, \tau) d\tau \equiv h(x, t), \quad x \in \mathbb{R}^+. \end{aligned}$$

We deduce from (1) and the hypothesis (1.7) that $\lim_{x \rightarrow 0} h(x, t) = 0$. Since $x \rightarrow u(x, \cdot)$ is assumed to be continuous in the L^1 -sense locally at $x = 0$, the same is true for $x \rightarrow h(x, \cdot)$. Thus $h(0, t) = 0$ for almost all $t \in (0, T_0)$; or, rather,

$$(2) \quad \Gamma(p + (2 - \bar{r})a)^{-1} \int_0^t (t - \tau)^{p+(2-\bar{r})a-1} VU(0, \tau) d\tau = 0$$

almost everywhere in $(0, T_0)$.

Now (2) is the convolution product of a matrix $l^{p+(2-\bar{r})a} \mathcal{V}$ with the vector $U_0 = U(0, t)$, where \mathcal{V} is an $m \times m$ matrix, (see (3.10)). The convo-

lution determinant (that is, using convolution product) of this matrix corresponds in \mathcal{F} to the determinant of the matrix

$$l^{p+(2-\bar{r})a} l^{1+(\bar{r}-2)a} M(s^a) \Omega m^{-1} \text{diag}(1, s^{2a}, \dots, s^{2(m-1)a}),$$

which is simply $l^{mp+(m+1)/2} \det M(s^a)$. Recall that $p = \max((\bar{r} - 1)/2m, 0)$, \bar{r} is the maximum order of the components of $B(D_x)$, and $a = 1/2m$. Thus $l^{mp+(m+1)/2} \det M(s^a)$ is a polynomial in l^a . Therefore, clearly, the set of zeros of the convolution determinant of the matrix $l^{p+(2-\bar{r})a} \mathcal{C}$ is a discrete set, and a fortiori zero is in its support. Consequently, in accordance with the Titchmarsh theorem on the vanishing of convolution products, as expressed by Kalisch [14], we conclude that $U_0 = 0$ in $[0, T_0)$: that is, $I^{2(j-1)} u(0, t) = 0$ for $0 \leq t < T_0$, ($j = 1, \dots, m$). That $u = 0$ in $\mathbb{R}^+ \times (0, T_0)$ follows then from Theorem 4.

PROOF OF THEOREM 2. As in the above proof we will establish here only that $u = 0$ in $\mathbb{R}^+ \times (0, T_0)$.

As there we have for $x \in \mathbb{R}^+$,

$$\begin{aligned} (1) \quad I &\equiv \int_0^{T_0} \Phi(t) \left(\Gamma(p)^{-1} \int_0^t (t-\tau)^{p-1} B(D_x) u(x, \tau) d\tau \right) \\ &= \int_0^{T_0} \Phi(t) \frac{\partial}{\partial t} \left(\Gamma(p + (2-\bar{r})a)^{-1} \int_0^t (t-\tau)^{p+(2-\bar{r})a-1} VU(x, \tau) d\tau \right) dt \\ &= \int_0^{T_0} \Phi(t) \frac{\partial}{\partial t} h(x, t) dt \quad (\text{as above}). \end{aligned}$$

Thus, after an integration by parts and supposing the support of Φ to be in $[0, T_0)$,

$$(2) \quad I = - \int_0^{T_0} \Phi'(t) h(x, t) dt, \quad (\text{since } h(x, 0) = 0).$$

Then it follows from (1), (2) and the hypothesis (1.8) that

$$(3) \quad \lim_{x \rightarrow 0} \int_0^{T_0} \Phi'(t) h(x, t) dt = 0.$$

Now, since $x \rightarrow u(x, \cdot)$ is assumed to be continuous in the $L^{2m+\varepsilon}$ — sense locally at $x=0$, the same is true of $x \rightarrow h(x, \cdot)$. Therefore, from (3),

$$\int_0^{T_0} \Phi'(t) h(0, t) dt = 0 \quad \text{for every } \Phi \in C^\infty$$

with support in $[0, T_0)$. This implies that $h(0, t) = \text{constant}$ in $(0, T_0)$. Moreover, since $U_0 = U(0, t)$ is locally integrable of order $2m + \varepsilon$, we infer that $t \rightarrow h(0, t)$ is continuous in $[0, T_0)$ and $h(0, 0) = 0$. That is,

$$(4) \quad \Gamma(p + (2 - \bar{r})\alpha)^{-1} \int_0^t (t - \tau)^{p+(2-\bar{r})\alpha-1} VU(0, \tau) d\tau \equiv 0$$

for $0 \leq t < T_0$.

The proof is completed from (4) just as in the proof of Theorem 1.

4. Proof of Theorem 3.

We shall first of all derive formally a representation of the solution operator; formulas (4.8) and (4.9).

To this end consider the equation (see (3.9))

$$(4.1) \quad M(s^\alpha) \Omega m^{-1} \text{diag}(1, s^{2\alpha}, \dots, s^{2(m-1)\alpha}) H = f.$$

Thus

$$(4.2) \quad H = \text{diag}(1, s^{-2\alpha}, \dots, s^{-2(m-1)\alpha}) \Omega^* M^{-1}(s^\alpha) f$$

where Ω^* is the conjugate transpose of Ω . In addition we have

$$(4.3) \quad M^{-1}(z) = \frac{z^q}{\det M(z)} \sum_{k \geq 0} z^k M_k$$

where the M_k are constant $m \times m$ matrices. Then on combining (4.2) and (4.3) and setting

$$(4.4) \quad F^{(k)} = \Omega^* M_k f$$

we obtain the expression

$$(4.5) \quad H = \frac{\text{diag}(1, s^{-2\alpha}, \dots, s^{-2(m-1)\alpha})}{\det M(s^\alpha)} \sum_{k \geq 0} (s^\alpha)^{k+q} F^{(k)}.$$

Suppose

$$(4.6) \quad \det M(z) = C \prod_{j=1}^N (z - r_j).$$

Then, in (4.3), the index k ranges from 0 through $N - q - \varrho$ (see § 1, Definition 1).

Now for r constant we have the correspondence

$$(4.7) \quad \frac{1}{s^a - r} \longleftrightarrow \sum_{j=0}^{m-1} \frac{r^j}{\Gamma((1+j)a)} \left[t^{a(1+j)-1} + r^{2m} \int_0^t e^{r^{2m}\tau} (t - \tau)^{a(1+j)-1} d\tau \right] \equiv R(t; r)$$

(see [20], Part II, Chp. IV).

Then from the opening discussion in § 3 combined with (4.4)-(4.7), we obtain the formulas:

$$(4.8) \quad H_n = \frac{1}{C} \sum_{k \geq 0} D^{(k+q)a} R(\cdot; r_1) * \dots * R(\cdot; r_N) * l^{2(n-1)a} * F_n^{(k)}$$

($n = 1, \dots, m$), where symbol $*$ indicates a convolution product, and

$$(4.9) \quad u(x, t) = 2(-1)^m \sum_{n=1}^m \int_0^t H_n(\tau) D_x^{2(m+n)-3} K(x, t - \tau) d\tau.$$

Clearly, the operator $g \rightarrow \Phi * g$, $\Phi \in C^\infty(\mathbb{R}^+)$, maps S^k into itself. This together with Lemma 3, § 2, implies that the mapping $g \rightarrow R(\cdot; r) * g$ takes S^k into S^{k+1} . Therefore the operator defined by (4.8) carries $S^{\bar{r}-e}$ into $S^{\bar{r}+2(n-1)}$. That the function $u = u(x, t)$ defined by (4.8) and (4.9) provides the desired solution then can be verified directly from Definition 4, Lemma 5, Theorem 5, and the discussion in § 3. The uniqueness is a consequence of Theorem 1.

5. Formulas for Derivatives of Solutions.

We shall present here certain formulas for derivatives of solutions in which no differentiations occur on the data. These formulas obviously provide alternate expressions for the solutions also.

We assume now that $D_x^a u$ satisfies the hypotheses imposed on u in Theorem 4. Let $u(x)$ be the correspondent in \mathcal{F} of $u(x, t)$. Then from Theo-

rem 4, (4.5), and (4.9) together with § 3, in particular (3.3), we obtain the expression

$$(5.1) \quad D_x^\alpha u(x) = \frac{1}{CP(s^\alpha)} \sum_{\lambda \geq 0} (s^\alpha)^{q-p+\alpha} \sum_{\mu, \nu} (s^\alpha)^{2m\lambda+\mu-2\nu} Q^{(2(m+\nu)-1)}(x) G_{\nu+1}^{(2m\lambda+\mu)}$$

$0 \leq \mu \leq 2m - 1$, $0 \leq \nu \leq m - 1$, where

$$(5.2) \quad \det M(z) = Cz^p P(z), \quad P(0) \neq 0$$

and the $G_\nu^{(n)}$ ($\nu = 1, \dots, m$) are the components of the vector

$$(5.3) \quad G^{(n)} = \Omega^* \text{diag}(\omega_1^\alpha, \dots, \omega_m^\alpha) M_n f.$$

Since by (1.1) $s^k Q^{(L)}(x) = (-1)^{(m+1)k} D_x^{2mk} Q^{(L)}(x)$, (5.1) can be written in the form

$$(5.4) \quad D_x^\alpha u(x) = \frac{1}{CP(s^\alpha)} \Psi(x) \\ \equiv \frac{s^{q-p+\alpha}}{CP(s^\alpha)} \sum_{\lambda \geq 0} (-1)^{(m+1)\lambda} \left[\sum_{\mu \leq 2\nu} Q^{(2m(\lambda+1)+2\nu-1)}(x) (s^\alpha)^{\mu-2\nu} G_{\nu+1}^{(2m\lambda+\mu)} \right. \\ \left. + (-1)^{m+1} \sum_{\mu > 2\nu} Q^{(2m(\lambda+2)+2\nu-1)}(x) (s^\alpha)^{\mu-2\nu-2m} G_{\nu+1}^{(2m\lambda+\mu)} \right]$$

$0 \leq \mu \leq 2m - 1$, $0 \leq \nu \leq m - 1$.

Let Z denote the set of zeros of $P(z)$. Let $n(r)$ be the order of r as a zero of $P(z)$ and let $\pi(r)$ be the product of all functions $(r - \zeta)^{n(\zeta)}$, ζ complex, $\zeta \neq r$. Then we can express $1/P(z)$ as

$$(5.5) \quad \frac{1}{P(z)} = \sum_{r \in Z} \frac{1}{(n(r) - 1)!} D_r^{n(r)-1} \frac{1}{\pi(r)(z - r)}$$

where $\Sigma \equiv 1$ if Z is empty.

Recall the correspondence (4.7) and set

$$(5.6) \quad \Phi(x, \cdot; r) = R(\cdot; r) * \Psi(x, \cdot)$$

where $\Psi(x, \cdot) \longleftrightarrow \Psi(x)$ in (5.4). In terms of the foregoing we have the formulas for $D_x^\alpha u$, $\alpha \leq p - q$,

$$(5.7) \quad D_x^\alpha u(x, t) = \sum_{r \in Z} \frac{1}{(n(r) - 1)!} D_r^{n(r)-1} \frac{1}{\pi(r)} \Phi(x, t; r), \quad Z \text{ not empty,}$$

and

$$(5.8) \quad D_x^\alpha u(x, t) = \Psi(x, t), \quad Z \text{ empty.}$$

APPENDIX

PROOF OF LEMMA 2. Let H and H^* , respectively, be the operators defined by $v \rightarrow Hv = D_y^{2m} v + (-1)^m D_\tau v$ and $v \rightarrow H^*v = D_y^{2m} v - (-1)^m D_\tau v$. Let I_k denote the integral operator given by

$$(I_k v)(y, \tau) = \Gamma(k)^{-1} \int_0^y (y - \xi)^{k-1} v(\xi, \tau) d\xi.$$

Set

$$(1) \quad G(x, t; y, \tau) = K(x - y, t - \tau) - K(x + y, t - \tau).$$

Then, clearly,

$$(2) \quad D_y^k G(x, t; y, \tau) = \begin{cases} 0, & k\text{-even} \\ -2D_x^k K(x, t - \tau), & k\text{-odd.} \end{cases}$$

Set $v(y, \tau) = u(y + c, \tau)$, $c > 0$. Since $Hv = 0$, we find simply that

$$H(I_{2m-1} v) = \sum_{j=1}^{2m-1} \frac{1}{\Gamma(j)} y^{j-1} [D_y^j v]_{y=0}.$$

With this we form the identity

$$(3) \quad \begin{aligned} D_y^{2m-1} G \cdot H(I_{2m-1} v) - I_{2m-1} v \cdot H^*(D_y^{2m-1} G) \\ \equiv D_y^{2m-1} G \cdot \sum_{j=1}^{2m-1} \frac{1}{\Gamma(j)} y^{j-1} [D_y^j v]_{y=0}. \end{aligned}$$

Now rearrange (3) into the form

$$(4) \quad \begin{aligned} D_y^{2m-1} G \cdot D_y^{2m} (I_{2m-1} v) - I_{2m-1} v \cdot D_y^{2m} (D_y^{2m-1} G) - \\ - D_y^{2m-1} G \cdot \sum_{j=1}^{2m-1} \frac{1}{\Gamma(j)} y^{j-1} [D_y^j v]_{y=0} \equiv (-1)^{m+1} D_\tau (I_{2m-1} v \cdot D_y^{2m-1} G), \end{aligned}$$

and integrate with respect to y from o to Y . From (2) and the fact that $[I_k v]_{y=0} = 0$ for $k > 0$, we find that the left side takes the form

$$(5) \quad \sum_{j=0}^{m-2} (-1)^j [I_j v \cdot D_y^{2m-1+j} G - I_{2m-1-j} v \cdot D_y^{4m-2-j} G]_{y=Y} - \\ - \sum_{k=0}^{m-1} [D_y^{2k} v \cdot D_y^{2m-1-2k} G]_{y=0} - \sum_{j=1}^{2m-1} \sum_{i=1}^j \frac{(-1)^{i-1}}{\Gamma(j-i+1)} [D_y^j v]_{y=0} [y^{j-i} D_y^{2m-1-i} G]_{y=Y}.$$

We deduce simply from (1), (2), and (2.2) that for $t < T_0$ the first and last terms in (5) $\rightarrow 0$ as $Y \rightarrow \infty$. The right side of (4) when integrated with respect to y from 0 to Y is just

$$(6) \quad (-1)^{m+1} D_\tau \int_0^Y I_{2m-1} v \cdot D_y^{2m-1} G dy.$$

Next we integrate the identity ((5) \equiv (6)) with respect to τ from ε to $t - \varepsilon$ ($t < T_0$) and find that

$$(7) \quad -2(-1)^m \sum_{k=0}^{m-1} \int_\varepsilon^{t-\varepsilon} D_y^{2k} v(o, \tau) \cdot D_x^{2(m-k)-1} K(x, t-\tau) d\tau + o(1) \\ = \int_0^Y [I_{2m-1} v \cdot D_y^{2m-1} G]_{\tau=t-\varepsilon} dy - \int_0^Y [I_{2m-1} v \cdot D_y^{2m-1} G]_{\tau=\varepsilon} dy.$$

Letting $Y \rightarrow \infty$ in (7) gives us

$$(8) \quad -2(-1)^m \sum_{k=0}^{m-1} \int_\varepsilon^{t-\varepsilon} D_x^{2k} v(o, \tau) \cdot D_x^{2(m-k)-1} K(x, t-\tau) d\tau \\ = \int_0^\infty [I_{2m-1} v \cdot D_y^{2m-1} G]_{\tau=t-\varepsilon} dy - \int_0^\infty [I_{2m-1} v \cdot D_y^{2m-1} G]_{\tau=\varepsilon} dy.$$

Upon integrating by parts we find from (1), (2), and (2.2) that the right side in (8) becomes simply

$$(9) \quad - \int_0^\infty v(y, t-\varepsilon) G(x, t; y, t-\varepsilon) dy + \int_0^\infty v(y, \varepsilon) G(x, t; y, \varepsilon) dy.$$

Now let $\varepsilon \rightarrow 0$. Then, on replacing the right side in (8) by (9) and recalling that $v(y, \tau) = u(y + c, \tau)$ ($c > 0$), it follows from the hypotheses and the basic properties of the fundamental solution that

$$(10) \quad u(x + c, t) = 2(-1)^m \sum_{k=0}^{m-1} \int_0^t D_x^{2k} u(c, \tau) \cdot D_x^{2(m-k)-1} K(x, t - \tau) d\tau.$$

The result then follows from the hypotheses on letting $c \rightarrow 0$ in (10).

PROOF OF LEMMA 4. Set $h = D^{ka} g$. Then from Lemma 3

$$(1) \quad g(t) = \frac{1}{\Gamma(ka)} \int_0^t h(\tau) (t - \tau)^{ka-1} d\tau.$$

It was shown in [13b] for the case $m = 2$ (the general case being similar) that g is uniformly $ka + b$ Hölder continuous on any closed subset of $(0, T]$ and also that $t^{c-ka} g(t)$ is bounded in $(0, T]$. Thus we see that the right side in (2.6) is well-defined.

Now consider the function $F = F(z)$ defined by

$$(2) \quad F(z) = \frac{1}{\Gamma(1+z)} \left\{ g(t) t^z - z \int_0^t [g(t) - g(\tau)] (t - \tau)^{z-1} d\tau \right\}.$$

For fixed $t > 0$ the function F is analytic in $\text{Re } z > -ka - b$. By restricting z to $\text{Re } z > 0$ we can write the second term in (2) as a sum of two integrals; one of which can be evaluated. Hence we find that in $\text{Re } z > 0$

$$(3) \quad F(z) = \frac{1}{\Gamma(z)} \int_0^t g(\tau) (t - \tau)^{z-1} d\tau.$$

It follows from (1) and (3) that

$$(4) \quad F(z) = \frac{1}{\Gamma(ka+z)} \int_0^t h(\tau) (t - \tau)^{ka+z-1} d\tau.$$

The function on the right side of (4) is clearly analytic at least in $\text{Re } z > -ka$. Therefore relation (4) holds also in $\text{Re } z > -ka$. Since h is continuous in $(0, T)$, the conclusion follows upon letting z approach $-ka$.

PROOF OF LEMMA 5. Denote by $J(x, t)$ the function on which the limit acts in (2.7). Since

$$D_x^{2m-1+j} K(x, t - \tau) = ((-1)^m D_\tau)^{[ja]} D_x^{2m-1+j-2m[ja]} K(x, t - \tau),$$

we find upon substituting this into the expression for $J(x, t)$ and integrating by parts that

$$(1) \quad J(x, t) = (-1)^{(m+1)[ja]+m} 2 \int_0^t D_\tau^{[ja]} g(\tau) D_x^{2m-1+j-2m[ja]} K(x, t - \tau) d\tau.$$

Let $n = j - 2m[ja]$, $h = D^{[ja]} g$, and $C = 2(-1)^{(m+1)[ja]+m}$. Then rewrite (1) as

$$(2) \quad J(x, t) = C \left\{ h(t) \int_0^t D_x^{2m-1+n} K(x, t - \tau) d\tau - \int_0^t [h(t) - h(\tau)] D_x^{2m-1+n} K(x, t - \tau) d\tau \right\}.$$

After some simple calculations on (2.1) one finds that for $\gamma \leq 2m - 1$

$$(3) \quad D_x^\gamma K(0, t) = \begin{cases} 0, & \gamma\text{-odd} \\ (-i)^\gamma \csc \pi(\gamma + 1)a / (2m \Gamma(1 - (\gamma + 1)a)) t^{-(\gamma+1)a}, & \gamma\text{-even.} \end{cases}$$

Let us consider first the case $j \equiv 0 \pmod{2m}$: then $n = 0$. The second term in (2), by Definition 4, is clearly continuous to $x = 0$; and from (3), it vanishes there. As to the first term, it is simple to show that it approaches $(-1)^{m/2}$ as $x \rightarrow 0$.

For the remaining cases, when $j \not\equiv 0 \pmod{2m}$ and, hence, $n \neq 0$, we can carry out the integration in the first term in (2). Then it follows readily from Definition 4 that the right side in (2) is continuous to $x = 0$; and, hence, the assertion for these cases is obtained from (3) and Lemma 4.

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