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A THEOREM ON GENERALIZED ABSOLUTE RIESZ SUMMABILITY

by S. M. MAZHAR

1.1. Let Σa_n be a given infinite series and $\{\lambda_n\}$ be an increasing sequence of positive numbers tending to infinity with n. We write

$$A_\lambda(\omega) = A_\lambda^0(\omega) = \sum_{\lambda_n < \omega} a_n , \qquad \omega > \lambda_1 ,$$
$$= 0 \qquad \omega \leq \lambda_1 ,$$

$$A_\lambda^\alpha(\omega) = \sum_{\lambda_n < \omega} (\omega - \lambda_n)^\alpha a_n , \qquad \alpha > 0,$$

$$= \int_{\lambda_1}^{\omega} (\omega - t)^\alpha \, dA_\lambda(t)$$

and

$$C_\lambda^\alpha(\omega) = A_\lambda^\alpha(\omega) / \omega^\alpha .$$

A series Σa_n is said to be summable by Riesz means of « type » λ and « order » α or, simply, summable (R, λ, α), $\alpha \geq 0$ to the sum s if

$$\lim_{\omega \to \infty} C_\lambda^\alpha(\omega) = s,$$

where s is any finite number [8].

The series Σa_n is said to be summable $| R, \lambda, \alpha |$, $\alpha \geq 0$, if the function $C_\lambda^\alpha(\omega) \in BV(h, \infty)$, that is to say, if

$$\int_h^\infty \left| \frac{d}{d\omega} C_\lambda^\alpha(\omega) \right| d\omega < \infty ,$$

where h is a finite positive number [6, 7].

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Similarly the series Σa_n is said to be summable $|R, \lambda, \alpha|_k$, $\alpha > 0$, $k \geq 1$, $\alpha k' > 1$, $\frac{1}{k} + \frac{1}{k'} = 1$, if the integral

$$\int_h^\infty \omega^{k-1} \left| \frac{d}{d\omega} C_\lambda^\alpha(\omega) \right|^k d\omega$$

is convergent [4]. (1)

1.2. In 1915 Hardy and Riesz [2] proved the following interesting theorem concerning the Riesz summability of an infinite series.

THEOREM A. If $\lambda_1 > 0$ and Σa_n is summable (R, λ, α), then the series $\Sigma a_n \lambda_n^{-\alpha}$ is summable (R, l, α), where $l_n = e^{\lambda_n}$.

Analogous problem was considered by Tatchell [9] for absolute Riesz summability. He proved the following theorem:

THEOREM B. If $\alpha \geq 0$ and Σa_n is summable $|R, \lambda, \alpha|$, then $\Sigma a_n \lambda_n^{-\alpha}$ is summable $|R, l, \alpha|$, where $l_n = e^{\lambda_n}$ (2).

The object of the present note is to establish the corresponding result for the generalized absolute Riesz summability, namely summability $|R, \lambda, \alpha|_k$ for integral values of α. In a subsequent note it is proposed to discuss the non-integral case.

2.1. In what follows we shall prove the following theorem:

THEOREM. *If α is a positive integer and Σa_n is summable $|R, \lambda, \alpha|_k$, then $\Sigma a_n \lambda_n^{-\alpha+1/k'}$ is summable $|R, l, \alpha|_k$, where $l_n = e^{\lambda_n}$, $k \geq 1$, $\lambda_1 > 0$ and $\frac{1}{k} + \frac{1}{k'} = 1$.*

It is evident that for $k = 1$ our theorem includes the above theorem of Tatchell for integral values of α.

2.2. We require the following lemmas for the proof of this theorem:

LEMMA 1 [3]. *If $\alpha > 0$ and $B_\alpha(\omega)$ is the Rieszian sum of type λ and order α of the series $\Sigma a_n \lambda_n$*, then

$$\omega^{\alpha+1} \frac{d}{d\omega} C_\lambda^\alpha(\omega) = \alpha B_{\alpha-1}(\omega) = \frac{d}{d\omega} B_\alpha(\omega).$$

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(1) See also Borwein [1] who defined the summability $|R, n, \alpha|_k$.

(2) This theorem for the case $\alpha = 1$ is due to Mohanty [5].

LEMMA 2 [2]. *If l is a positive integer, then*

$$A_\lambda(t) = \frac{1}{l\,!}\left(\frac{d}{dt}\right)^l A_\lambda^l(t).$$

3.1. PROOF OF THE THEOREM. Under the hypothesis of the theorem we have by Lemma 1

$$\int_{\lambda_1}^{\infty} \omega^{-(1+\alpha k)} \mid B_{\alpha-1}(\omega) \mid^k d\omega < \infty \tag{3.1.1}$$

and we have to establish the convergence of the integral

$$\int_{l_1}^{\infty} \omega^{-(1+\alpha k)} \mid E_{\alpha-1}(\omega) \mid^k d\omega, \tag{3.1.2}$$

where $E_{\alpha-1}(\omega)$ is the Rieszian sum of order $(\alpha - 1)$ and of type l, of the series $\Sigma a_n \lambda_n^{-\alpha+1/k'} e^{\lambda_n}$.

By writing $\omega = e^x$ in the above integral (3.1.2) we find that the required condition can also be written in the form

$$\int_{\lambda_1}^{\infty} e^{-\alpha x k} \mid E_{\alpha-1}(e^x) \mid^k dx < \infty . \tag{3.1.3}$$

We have

$$\begin{aligned} E_{\alpha-1}(e^x) &= \int_{l_1}^{e^x} (e^x - u)^{\alpha-1}\, dE(u) \\ &= \int_{\lambda_1}^{x} (e^x - e^t)^{\alpha-1}\, dE(e^t) \\ &= \int_{\lambda_1}^{x} (e^x - e^t)^{\alpha-1}\, e^t\, t^{-\alpha-1/k}\, dB(t) \\ &= \left[(e^x - e^t)^{\alpha-1}\, e^t\, t^{-\alpha-1/k}\, B(t)\right]_{\lambda_1}^{x} \\ &\quad - \int_{\lambda_1}^{x} B(t)\, \frac{d}{dt}\left\{(e^x - e^t)^{\alpha-1}\, e^t\, t^{-\alpha-1/k}\right\} dt. \end{aligned}$$

Applying Lemma 2 and integrating $(\alpha - 1)\cdot$ times we have

$$E_{\alpha-1}(e^x) = [(e^x - e^t)^{\alpha-1} e^t \cdot t^{-\alpha-1/k} B(t)]_{\lambda_1}^x +$$

$$+ C\,[3] \left[\sum_{i=1}^{\alpha-1} (-1)^i \left(\frac{d}{dt}\right)^{\alpha-i-1} B_{\alpha-1}(t) \left(\frac{d}{dt}\right)^i \{(e^x - e^t)^{\alpha-1} e^t\, t^{-\alpha-1/k}\}\right]_{\lambda_1}^x$$

$$+ C \int_{\lambda_1}^x B_{\alpha-1}(t) \left(\frac{d}{dt}\right)^{\alpha} \{(e^x - e^t)^{\alpha-1} e^t\, t^{-\alpha-1/k}\}\, dt$$

$$= C B_{\alpha-1}(x)\, e^{\alpha x}\, x^{-\alpha-1/k} + C \int_{\lambda_1}^x B_{\alpha-1}(t) \left(\frac{d}{dt}\right)^{\alpha} \{(e^x - e^t)^{\alpha-1} e^t\, t^{-\alpha-1/k}\}\, dt$$

$= L_1 + L_2$, say.

Since

$$\int_{\lambda_1}^{\infty} e^{-x\alpha k} \,|\, L_1 \,|^k\, dx \leq C \int_{\lambda_1}^{\infty} x^{-(1+\alpha k)} \,|\, B_{\alpha-1}(x) \,|^k\, dx < \infty,$$

it is, therefore, by virtue of Minkowski's inequality sufficient to prove that

$$\int_{\lambda_1}^{\infty} e^{-x\alpha k} \,|\, L_2 \,|^k\, dx < \infty\, .$$

Now

$$L_2 = O\left\{\int_{\lambda_1}^x |\, B_{\alpha-1}(t) \,|\, t^{-\alpha-1/k}\, e^{\alpha t}\, dt\right\} + O\left\{\int_{\lambda_1}^x |\, B_{\alpha-1}(t) \,|\, \sum_{i=1}^{\alpha-1} e^{ix}\, e^{(\alpha-i)t}\, t^{-\alpha-1/k}\, dt\right\}$$

$= L_{21} + L_{22}$.

Applying Hölder's inequality, we observe that

$$\int_{\lambda_1}^{\infty} e^{-x\alpha k} \,|\, L_{21} \,|^k\, dx = O\left\{\int_{\lambda_1}^{\infty} e^{-x\alpha k} \int_{\lambda_1}^x |\, B_{\alpha-1}(t) \,|^k\, t^{-(1+\alpha k)}\, e^{\alpha t}\, e^{\alpha x(k-1)}\, dt\, dx\right\}$$

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[3] Where C denotes a constant not necessarily the same at each occurrence.

$$= O\left\{\int_{\lambda_1}^{\infty} t^{-(1+\alpha k)} \mid B_{\alpha-1}(t) \mid^k dt\right\}$$

$$= O(1).$$

Also, in order to show that

$$\int_{\lambda_1}^{\infty} e^{-x\alpha k} \mid L_{22} \mid^k dx < \infty$$

it is sufficient to prove the convergence of the integral

$$\int_{\lambda_1}^{\infty} e^{-x\alpha k}\left\{\int_{\lambda_1}^{x} \mid B_{\alpha-1}(t) \mid e^{ix} e^{(\alpha-i)t} t^{-\alpha-1/k} dt\right\}^k dx$$

for $1 \leq i \leq \alpha - 1$.

Using Hölder's inequality we find that the above integral is

$$< \int_{\lambda_1}^{\infty} e^{-x\alpha k+ixk} \int_{\lambda_1}^{x} \mid B_{\alpha-1}(t) \mid^k t^{-(1+\alpha k)} e^{(\alpha-i)t} dt\, e^{(\alpha-i)x(k-1)} dx$$

$$= C \int_{\lambda_1}^{\infty} t^{-(1+\alpha k)} \mid B_{\alpha-1}(t) \mid^k e^{(\alpha-i)t} \int_{t}^{\infty} e^{x(i-\alpha)} dx\, dt$$

$$= C \int_{\lambda_1}^{\infty} t^{-(1+\alpha k)} \mid B_{\alpha-1}(t) \mid^k dt < \infty,$$

by hypothesis.

This completes the proof of the theorem.

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