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INHOMOGENEOUS BOUNDARY VALUE PROBLEMS IN A HALF SPACE

by J. BARROS NETO

We discuss in this paper the Dirichlet problem for elliptic operators with variable coefficients defined in a half space. General inhomogeneous boundary value problems have been extensively studied by Lions and Magenes in a series of papers [6]. In the papers, they consider elliptic operators with smooth coefficients defined on a bounded domain, except in the first of [6] where the case of an operator defined in R^n_+ is considered.

Our approach is however different. We use, rather than the Sobolev spaces H^m, the spaces $\mathfrak{D}^m$ obtained by completing the space of C^∞ functions with compact support, with respect to the Dirichlet norm [4]. These spaces which coincide with H^m when the domain is bounded are in the case of unbounded domains larger. In order to deal only with spaces of functions we have to impose a restriction on the dimension n of the Euclidean space, namely, we suppose always that $n > 2m$, what implies that $\mathfrak{D}^m$ is a subspace of $L^{q(m)}$, $\dfrac{1}{q(m)} = \dfrac{1}{2} - \dfrac{m}{n}$ (§ 1). The spaces $\mathfrak{D}^m$ are then normal spaces of distributions a fact very convenient in order to characterize the trace of elements of $\mathfrak{D}^m$ (§§ 4 and 5), to study the transposed problem (§ 8) and to apply the interpolation theory (§ 9). Our results apply to the Laplace operator in a three dimensional space.

The plan of the paper is the following. First we define the spaces $\mathfrak{D}^m_0(R^n_+)$, its dual $\mathfrak{D}^{-m}(R^n_+)$ as well as $\mathfrak{D}^m(R^n_+)$ and give some of their properties. An integro-differential operator $a(u, v)$, homogeneous of degree m, verifying an ellipticity condition (2.4), with smooth coefficients satisfying conditions i), ii) and iii) is considered and we prove that the corresponding partial differential operator A establishes an isomorphism from $\mathfrak{D}^m_0(R^n_+)$ onto $\mathfrak{D}^{-m}(R^n_+)$ (theorem 2.1). This solves the homogeneous Dirichlet problem for

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A. To study the inhomogeneous problem we characterize the trace on R^{n-1} of the elements belonging to $\mathfrak{D}^m(R^n_+)$. With the help of the trace theorem (theorem 5.1) and the isomorphism theorem (theorem 2.1) we prove theorem 6.1 solving the inhomogeneous Dirichlet problem for A.

The next step is the regularization of the solution. Theorem 7.1 proves regularization up to the boundary while lemma 8.1 proves the interior regularity.

After regularizing the solution we transpose our results (§ 8). Here, in order to carry through our argument we need another trace theorem (theorem 7.2) which extends theorem 5.1. We should point out that in § 8 we do not consider the transposition problem in its full generality (a question that presents many technicals difficulties) but rather study a particular case that leads us to the solution of the Dirichlet problem for a given function in $\mathfrak{D}^{-m}(R^n_+)$ and given boundary values assigned in Sobolev spaces $H^{\alpha}(R^{n-1})$ (theorem 8.3). Obviously, by choosing another elements in the dual of $\mathfrak{D}^{m,m}(R^n_+) \cap \mathfrak{D}^m_0(R^n_+)$ (see § 8) we can get isomorphism theorems of the same type as theorem 8.3 but involving only the spaces $\mathfrak{D}^m$ defined on R^n_+ and R^{n-1}.

Finally, we apply the interpolation theory and prove theorem 9.1. When interpolating between two given spaces one gets an abstract family of spaces that one likes to characterize. This is the aim of theorem 9.2 which follows as a particular case of theorem 9.3, a theorem about interpolation of spaces of integrable functions with change of measure. Our result is similar to a previous one of Stein and Weiss [11]; the methods are different.

For simplicity we have considered throughout this paper only the case of R^n_+ but it is clear that our results can be extended with slight modifications to more general unbounded domains. For instance they can be applied to the case of a complement of a ball in R^n.

The main results of this paper appeared without proof in [12]. We are indebted to J. L. Lions and E. Magenes for suggestions and criticisms.

1. Preliminaries.

Let R^n be the Euclidean space of dimension n; R^n_+ the half space $\{x = (x_1, \dots, x_n) \in R^n : x_n > 0\}$ and $\overline{R^n_+}$ the closed half space, i. e. the set of all elements $x \in R^n$ such that $x_n \geq 0$. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a n-tuple of integers $\alpha_i > 0$, we indicate by D^α the partial derivative $\dfrac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$ of order $|\alpha| = \alpha_1 + ... + \alpha_n$. Let $C_c^\infty(R^n)$ (resp. $C_c^\infty(R^n_+)$, resp. $C_c^\infty(\overline{R^n_+})$) the space of infinitely differentiable functions with compact support in R^n (resp. R^n_+, resp. $\overline{R^n_+}$). The dual of $C_c^\infty(R^n)$ (resp. $C_c^\infty(R^n_+)$) is the space of distributions in R^n (resp. R^n_+) that we denote by $\mathcal{D}'(R^n)$ (resp. $\mathcal{D}'(R^n_+)$).

DEFINITION 1.1. *We denote by* $\mathfrak{D}^m(R^n)$ *the completion of* $C_c^\infty(R^n)$ *with respect to the following norm:*

$$(1.1) \qquad \| \varphi \|_{\mathfrak{D}^m(R^n)} = \Big(\sum_{|\alpha| = m} \| D^\alpha \varphi \|^2_{L^2(R^n)} \Big)^{1/2}.$$

Clearly, $\mathfrak{D}^m(R^n)$ is a Hilbert space. However, if $n > 2m$, according to Sobolev's inequalities (10) we have:

$$(1.2) \qquad \| \varphi \|_{L^{q(m)}(R^n)} \leq C \cdot \| \varphi \|_{\mathfrak{D}^m(R^n)}, \text{ where } \frac{1}{q(m)} = \frac{1}{2} - \frac{m}{n}.$$

Then, $\mathfrak{D}^m(R^n)$ is a normal space of distributions [1] since from (1.2) it follows that $\mathfrak{D}^m(R^n) \subset L^{q(m)}(R^n)$ and the imbedding is continuous. The dual $\mathfrak{D}^{-m}(R^n)$ of $\mathfrak{D}^m(R^n)$ is a subspace of $\mathcal{D}'(R^n)$. We shall consider throughout this paper, only the case $n > 2m$.

Consider, now, the space V of functions u of $L^{q(m)}(R^n)$ such that $D^\alpha u \in L^{q(m-j)}(R^n)$, $|\alpha| = j$, $0 \leq j \leq m$, where

$$(1.3) \qquad \frac{1}{q(m-j)} = \frac{1}{2} - \frac{m-j}{n}, \quad 0 \leq j \leq m.$$

Equipped with the norm

$$(1.4) \qquad \| u \| = \sum_{|\alpha|=0}^{m} \| D^\alpha u \|_{L^{q(m-j)}(R^n)},$$

V is a reflexive Banach space.

We shall prove the

THEOREM 1.1. *The space* $\mathfrak{D}^m(R^n)$ *can be identified in the algebraic and topological senses to* V.

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[1] A topological vector space E is a normal space of distributions if $C_c^\infty(R^n) \subset E \subset \mathcal{D}'(R^n)$ the imbeddings being continuous and $C_c^\infty(R^n)$ being dense in E.

PROOF. If $1 \leq p < n$, we have, for all $\varphi \in C_c^\infty(R^n)$, the inequality due to Sobolev ([10])

$$(1.5) \qquad \| \varphi \|_{L^q(R^n)} \leq c \sum_{i=1}^{n} \| D_i \varphi \|_{L^p(R^n)}$$

where

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n},$$

the constant c depending only on p and n.

It is easy to see using (1.5) that the norms (1.1) and (1.4) coincide on $C_c^\infty(R^n)$. Thus, to prove the theorem it suffices to show that $C_c^\infty(R^n)$ is dense in V.

Let $\chi(x)$ be a function of $C_c^\infty(R^n)$ equal to 1 for $|x| \leq 1$ and to 0 for $|x| \geq 2$; let $\chi_R(x) = \chi\left(\frac{x}{R}\right)$ where R is a positive real number and write $u_R = \chi_R \cdot u$ for each $u \in V$. To prove that $C_c^\infty(R^n)$ is dense in V, it suffices to prove that the functions u_R belong to a bounded set of V. In fact, if it is so, then, since V is reflexive, there is a sequence (R_n) such that (u_{R_n}) converges weakly in V. It is easy to verify that (u_{R_n}) converges to u. Next, by regularization, we prove that there is a sequence of elements of $C_c^\infty(R^n)$ which converges weakly, in V, to u. The proof follows from the well known criterion of density in Banach spaces.

To prove that u_R belongs to a bounded set of V, it suffices to prove that $D^\alpha u_R$ belongs to a bounded set of $L^{q(m-|\alpha|)}(R^n)$, for all $0 \leq |\alpha| \leq m$. Write:

$$D^\alpha u_R = \chi_R \cdot D^\alpha u + \sum_{\substack{|\beta| + |\gamma| = |\alpha| \\ |\beta| > 0}} C_{\beta\gamma} D^\beta \chi_R \cdot D^\gamma u .$$

Since $\chi_R \cdot D^\alpha u$ converges to $D^\alpha u$ in $L^{q(m-|\alpha|)}(R^n)$ it suffices to verify that $D^\beta \chi_R \cdot D^\gamma u$ belongs to a bounded set in $L^{q(m-|\alpha|)}(R^n)$. Let $|\alpha| = j$, $|\gamma| = k$ and write

$$(1.6) \qquad \int_{R^n} | D^\beta \chi_R \cdot D^\gamma u |^{q(m-j)} \, dx \leq$$

$$\leq \frac{C_1}{R^{(j-k) \cdot q(m-j)}} \cdot \int_{|x| \leq 2R} | D^\gamma u |^{q(m-j)} \, dx .$$

(The integral at the right exists because, here, $q(m-k) > q(m-j)$). By

Holder's inequality we have:

$$(1.6') \qquad \int\limits_{|x| \leq 2R} |D^\gamma u|^{q(m-j)} \, dx \leq$$

$$\leq \left(\int\limits_{|x| \leq 2R} |D^\gamma u|^{q(m-j)\cdot\delta} \, dx \right)^{1/\delta} \cdot \left(\int\limits_{|x| \leq 2R} dx \right)^{1/\delta'},$$

where $\frac{1}{\delta} + \frac{1}{\delta'} = 1$. Now, we choose δ such that $q(m-j) \cdot \delta = q(m-k)$, that is,

$$\delta = \frac{n-2m+2j}{n-2m+2k} \text{ and } \delta' = \frac{n-2m+2j}{2(j-k)}.$$

From (1.6) and (1.6′) we get:

$$\int\limits_{R^n} |D^\beta \chi_R \cdot D^\gamma u|^{q(m-j)} \, dx \leq$$

$$\leq \frac{c_2 \cdot R^{\frac{2n(j-k)}{n-2m+2j}}}{R^{(j-k)\cdot q(m-j)}} \left(\int\limits_{R^n} |D^\gamma u|^{q(m-k)} \, dx \right)^{\frac{n-2m+2k}{n-2m+2j}} \leq c_3$$

because $q(m-j) = \dfrac{2n}{n-2m+2j}$ et $D^\gamma u \in L^{q(m-k)}(R^n)$, q.e.d.

Consider, now, the sesquilinear form:

$$(1.7) \qquad a(u, v) = (-1)^m \sum_{|p| = |q| = m} \int\limits_{R^n} a_{pq}(x) \, D^p u \cdot \overline{D^q v} \, dx,$$

defined on $\mathfrak{D}^m(R^n) \times \mathfrak{D}^m(R^n)$. We assume that the coefficients $a_{pq}(x)$ are functions belonging to $C^{2m}(R^n)$ and that (1.7) is continuous in $\mathfrak{D}^m(R^n) \times \mathfrak{D}^m(R^n)$. Furthermore, assume that *there is a constant* $c > 0$ *such that*

$$(1.8) \qquad |a(v, v)| \geq c \, \| v \|^2_{\mathfrak{D}^m(R^n)}, \text{ for all } v \in \mathfrak{D}^m(R^n).$$

Let us point out that the following set of conditions is sufficient in order that (1.8) holds:

i) *Suppose that the coefficients* $a_{pq}(x)$ *belong to* $C^{2m}(R^n)$ *and are uniformly bounded together with all their derivatives;*

ii) *There are constants* $\alpha > 0$ *and* $\beta \geq 0$, *such that* $Re\ a_{pp}(x) \geq \alpha > 0$ *and* $|a_{pq}(x)| \leq \beta$, *for all* $p \neq q$;

iii) *Finally, suppose that* $(\alpha - \beta\gamma^2) > 0$, *where* γ *is a positive constant such that*

$$\Big(\sum_{|\alpha|=m} \eta_\alpha^2\Big)^{1/2} \geq \gamma^{-1} \sum_{|\alpha|=m} \eta_\alpha ,$$

where η_α $(\alpha = (\alpha_1, \dots, \alpha_n))$ *are positive real numbers;*
then, it is a matter of verification that

$$\text{(1.9.)} \qquad \operatorname{Re}\ a(v, v) \geq (\alpha - \beta\gamma^2) \| v \|^2_{\mathfrak{D}^m(R^n)}, \text{ for all } v \in \mathfrak{D}^m(R^n).$$

Clearly, (1.9) implies (1.8) taking $c = \alpha - \beta\gamma^2$.

Next, suppose that f is a given element of $\mathfrak{D}^{-m}(R^n)$ and that we want to find an element $v \in \mathfrak{D}^m(R^n)$ such that

$$\text{(1.10)} \qquad a(u, v) = \langle f, \bar{v} \rangle \text{ for all } v \in \mathfrak{D}^m(R^n),$$

where $\langle\ ,\ \rangle$ represents the pairing between $\mathfrak{D}^m(R^n)$ and $\mathfrak{D}^{-m}(R^n)$. We observe that for a fixed element u of $\mathfrak{D}^m(R^n)$ the anti-linear form $v \longrightarrow a(u, v)$ is continuous in $\mathfrak{D}^m(R^n)$. There exists, then, a unique element $\mathcal{A} u \in \mathfrak{D}^{-m}(R^n)$ such that

$$\text{(1.11)} \qquad a(u, v) = \langle \mathcal{A} u, \bar{v} \rangle \text{ for all } v \in \mathfrak{D}^m(R^n).$$

It is easy to see that $\mathcal{A} \in \mathcal{L}(\mathfrak{D}^m, \mathfrak{D}^{-m})$, space of continuous linear operators form $\mathfrak{D}^m(R^n)$ into $\mathfrak{D}^{-m}(R^n)$. Relations (1.10) and (1.11) show that if u verifies (1.10) then $\mathcal{A} u = f$ and conversely. One can see, using (1.8), that the image $\mathcal{A}\,\mathfrak{D}^m$ is closed in $\mathfrak{D}^{-m}$ and that $\mathcal{A}$ is one-to-one. Also, one can see, easily, that $\mathcal{A}\,\mathfrak{D}^m$ is dense in $\mathfrak{D}^{-m}$. Consequently, $\mathcal{A}$ is an isomorphism from $\mathfrak{D}^m(R^n)$ onto $\mathfrak{D}^{-m}(R^n)$. We can summarize these results in the following.

THEOREM 1.2. *Given* $f \in \mathfrak{D}^{-m}(R^n)$ *there exists a unique element,* $u \in \mathfrak{D}^m(R^n)$ *such that* (1.10) *holds. We have* $Au = f$ *in the sense of distributions where:*

$$\text{(1.12)} \qquad Au = \sum_{|p|=|q|=m} D^q (a_{pq}(x) D^p u).$$

Furthermore, A *establishes an isomosphism of* $\mathfrak{D}^m(R^n)$ *onto* $\mathfrak{D}^{-m}(R^n)$.

2. The homogeneous Dirichlet problem in R^n_+.

The results discussed in section 1 suggest the following.

DEFINITION 2.1. *Denote by* $\mathfrak{D}^m(R^n_+)$ *the space*

$$\mathfrak{D}^m(R^n_+) = \{u \in L^{q(m)}(R^n) : D^\alpha u \in L^{q(m-j)}(R^n_+), |\alpha| = j\,;\ 0 \leq j \leq m\}$$

equipped with the norm

$$\| u \|_{\mathfrak{D}^m(R^n_+)} = \sum_{|\alpha|=j=0}^{m} \| D^\alpha u \|_{L^{q(m-j)}(R^n_+)} \tag{2.1}$$

where $q(m-j)$ is given by (1.4).

It is a reflexive Banach space. Denote by $\mathfrak{D}^m_0(R^n_+)$ the closure of $C^\infty_c(R^n_+)$ in $\mathfrak{D}^m(R^n_+)$. $\mathfrak{D}^m_0(R^n_+)$ is a normal space of distributions, its dual we denote by $\mathfrak{D}^{-m}(R^n_+)$. It can be, easily, seen that the elements of $\mathfrak{D}^{-m}(R^n_+)$ can be represented as $\sum_{|\alpha|=j=0}^{m} D^\alpha f_\alpha$ where $f_\alpha \in L^{q'(m-j)}(R^n_+)$; here $q'(m-j)$ denotes the conjugate exponent of $q(m-j)$.

In $\mathfrak{D}^m_0(R^n_+)$ consider the norm

$$\| u \|_{\mathfrak{D}^m_0(R^n_+)} = \Big(\sum_{|\alpha|=0} \| D^\alpha u \|^2_{L^2(R^n_+)} \Big)^{1/2}. \tag{2.2}$$

According to Sobolev's inequality (1.5) which holds also in R^n_+ it follows that in $\mathfrak{D}^m_0(R^n_+)$ the two norms (2.1) and (2.2) are equivalent.

Let

$$a(u, v) = (-1)^m \sum_{|p|=|q|=m} \int_{R^n_+} a_{pq}(x)\, D^p u\, \overline{D^q v}\, dx \tag{2.3}$$

be defined in $\mathfrak{D}^m(R^n_+) \times \mathfrak{D}^m(R^n_+)$, suppose that the coefficients $a_{pq}(x)$ are smooth functions defined in $\overline{R^n_+}$ and assume that

$$| a(v, v) | \geq c \cdot \| v \|^2_{\mathfrak{D}^m_0(R^n_+)}, \quad \text{for all } v \in \mathfrak{D}^m_0(R^n_+). \tag{2.4}$$

With the same argument used in section 1, we can prove the following.

Theorem 2.1. *The operator* $A = \sum_{|p|=|q|=m} D^q (a_{pq}(x) D^p)$ *establishes an isomorphism from* $\mathfrak{D}_0^m (R_+^n)$ *onto* $\mathfrak{D}^{-m} (R_+^n)$.

As we shall prove latter (section 5) the elements of $\mathfrak{D}_0^m (R_+^n)$ have zero Dirichlet data at the boundary R^{n-1} of R_+^n, i. e. $\gamma_j u = 0$, $0 \leq j \leq m-1$, for all $u \in \mathfrak{D}_0^m (R_+^n)$, where $\gamma_j u$ denotes the restriction (in a sense to be precised) to R^{n-1} of the normal derivative $\frac{\partial^j u}{\partial x_n^j}$, $u \in \mathfrak{D}^m (R_+^n)$. The theorem 2.1 states that there exists a unique solution of the homogeneous Dirichlet problem

$$(2.5) \qquad \begin{cases} Au = f \\ \gamma_j u = 0 \quad , \quad 0 \leq j \leq m-1, \end{cases}$$

for any $f \in \mathfrak{D}^{-m} (R_+^n)$.

In order to study the inhomogeneous Dirichlet problem

$$(2.6) \qquad \begin{cases} Au = f \\ \gamma_j u = g_j, \qquad 0 \leq j \leq m-1, \end{cases}$$

we need to define the restriction (or trace) of elements of $\mathfrak{D}^m (R_+^n)$ to the boundary R^{n-1}. Before doing this, we shall establish two properties of the space $\mathfrak{D}^m (R_+^n)$ and we shall introduce the spaces $\mathfrak{D}^\alpha (R^{n-1})$, α real.

3. Two properties of $\mathfrak{D}^m (R^n)$.

Theorem 3.1. *In* $\mathfrak{D}^m (R_+^n)$ *the subspace* $C_c^\infty (\overline{R_+^n})$ *of infinitely differentiable functions with compact support in* $\overline{R_+^n} = \{x \in R^n : x_n \geq 0\}$ *is dense.*

Proof. Let χ and χ_R be as in theorem 1.1 and consider their restrictions to $\overline{R_+^n}$ which we shall denote with the same notation. If $u \in \mathfrak{D}^m (R_+^n)$, let $u_R = \chi_R u$. As in theorem 1.1 we can prove that $u_R \to u$ weakly in $\mathfrak{D}^m (R_+^n)$ when $R \to +\infty$.

Let, then, u be a compact supported element of $D^m (R_+^n)$, define $v_\varepsilon (x) = u(x', x_n + \varepsilon)$, $\varepsilon > 0$ and denote by $u_\varepsilon (x)$ the restriction of $v_\varepsilon (x)$ to R_+^n. It can be seen that $v_\varepsilon (x) \in \mathfrak{D}^m (R_+^n)$ and $v_\varepsilon (x) \to u(x)$ in $\mathfrak{D}^m (R_+^n)$. We can then assume that u is the restriction to R_+^n of a compact supported element v_ε $\mathfrak{D}^m (R_{-\varepsilon}^n)$ where $R_{-\varepsilon}^n = \{x \in R^n : x_n > -\varepsilon\}$. Take, now, $\theta(x_n)$ a C^∞ function defined in R, equal to 1 when $x_n \geq 0$ and $= 0$ when $x_n \leq -\frac{\varepsilon}{2}$ and define $w(x', x_n) = \theta(x_n) \cdot v(x', x_n)$. We have: i) $w \in \mathfrak{D}^m(R^n_{-\varepsilon})$; ii) $w = 0$ in a neighborhood of $x_n = -\varepsilon$ and iii) the restriction of w to R^n_+ is u. Extend w to R^n defining it equal 0 for $x_n < -\varepsilon$ and denote by $\tilde{w}$ this extension. Then, $\tilde{w} \in \mathfrak{D}^m(R^n)$ and can be approached (thm. 1.1) by elements $\psi \in C_c^\infty(R^n)$ which can be assumed having support contained in $R^n_{-\varepsilon}$. It follows that u can be approached in $\mathfrak{D}^m(R^n_+)$ by the restriction φ of ψ to R^n_+, q. e. d.

Theorem 3.2. *There exists a continuous linear map P: $\mathfrak{D}^m(R^n_+) \rightarrow \mathfrak{D}^m(R^n)$ such that $Pu = u$ in R^n_+.*

Proof. Using theorem 3.1, it suffices to show that there exists a linear map P defined on $C_c^\infty(\overline{R^n_+})$ with values in $\mathfrak{D}^m(R^n_+)$ such that

$$\| P\varphi \|_{D^m(R^n_+)} \leq C \cdot \| \varphi \|_{D^m(R^n_+)}, \tag{3.1}$$

for all $\varphi \in C_c^\infty(\overline{R^n_+})$, and $P\varphi = \varphi$ in R^n_+.

Define P in the following way:

$$P\varphi = \begin{cases} \varphi(x', x_n) & \text{if} \quad x_n > 0 \\ \sum_{j=1}^{m} \lambda_j \varphi\left(x', -\frac{x_n}{j}\right) & \text{if} \quad x_n < 0 \end{cases} \tag{3.2}$$

the constants λ_j being given by

$$\sum_{j=1}^{m} \left(-\frac{1}{j}\right)^k \lambda_j = 1 \qquad 0 \leq k \leq m - 1. \tag{3.3}$$

One can check that P has the required properties, q. e. d.

4. Spaces $\mathfrak{D}^s(R^n)$.

In this section, $\xi = (\xi_1, \dots, \xi_n)$ and $x = (x_1, \dots, x_n)$ denote points of R^n. Let $\mathcal{S}(R^n)$ be the space of infinitely differentiable functions which are rapidly decreasing at ∞ and let $\mathcal{S}'(R^n)$ be the space of temperate distributions (dual of $\mathcal{S}(R^n)$). Let $\mathcal{F}$ be the isomorphism between $\mathcal{S}'$ and $\mathcal{S}'$ given by Fourier transform ([9]) and let $\mathcal{F}^{-1}$ be its inverse. If $\varphi \in \mathcal{S}(R^n)$ (resp. $T \in \mathcal{S}'(R^n)$) we shall denote its Fourier transform by $\widehat{\varphi}(\xi) = (\mathcal{F}\varphi)(\xi)$ (resp. $\widehat{T} = \mathcal{F}T$).

Suppose that s is a real number such that $-\frac{n}{2} < s < \frac{n}{2}$. Then $|\xi|^{2s}$ is locally integrale with respect to the Lebesgue measure in R^n and we can consider the measure μ_s of density $|\xi|^{2s}$ with respect to the Lebesgue measure in R^n. Let $L^2(\mu_s)$ be the space of square integrable functions with respect to μ_s. The dual of $L^2(\mu_s)$ can be identified to $L^2(\mu_{-s})$ with its natural pairing.

Furthermore, $L^2(\mu_s) \subset \mathcal{S}'(R^n)$. In fact, for each $\widehat{g} \in L^2(\mu_s)$, define

$$L_{\hat{g}}(\widehat{\varphi}) = \int_{R^n} (|\xi|^s \widehat{g}(\xi)) \cdot (|\xi|^{-s} \widehat{\varphi}(\xi))\, d, \tag{4.1}$$

for all $\widehat{\varphi} \in \mathcal{S}$. We want to verify that the linear functional $L_{\hat{g}}$ is continuous in the topology of $\mathcal{S}$. Firstly, we have:

$$|L_{\hat{g}}(\widehat{\varphi})| \leq C \cdot \left(\int_{R^n} |\xi|^{-2s} |\widehat{\varphi}(\xi)|^2 d\xi \right)^{\frac{1}{2}}. \tag{4.2}$$

Secondly, we observe that, since $-\frac{n}{2} < s < \frac{n}{2}$, then

$$\int_{|\xi| \leq 1} |\xi|^{-2s} d\xi \leq N_1 < +\infty. \tag{4.3}$$

On the other hand, let k be an integer, large enough such that:

$$\int_{|\xi| > 1} |\xi|^{-2s-2k} d\xi \leq M_1 < +\infty. \tag{4.4}$$

From (4.2), (4.3) and (4.4) we get:

$$L_{\hat{g}}(\widehat{\varphi}) \leq C \cdot N_1 \cdot (\sup_{\xi \in R^n} (|\widehat{\varphi}(\xi)|^2) + C \cdot M_1 \cdot \operatorname{Sup}_{\xi \in R^n} (|\xi|^{2k} \widehat{\varphi}(\xi)^2). \tag{4.5}$$

If, now, $\widehat{\varphi}$ converges to zero in $\mathcal{S}$, the two sup in the right-hand side of (4.5) can be made as small as we want, what proves that $L_{\widehat{g}}$ is continuous in $\mathcal{S}$. Finally, we observe that if $L_{\widehat{g}}$ is zero in $\mathcal{S}(R^n)$ then $\widehat{g} = 0$ a. e. in R^n. Consequently, the map $\widehat{g} \in L^2(\mu_s) \rightarrow L_{\widehat{g}} \in \mathcal{S}'(R^n)$ gives an imbedding of of $L^2(\mu_s)$ into $\mathcal{S}'(R^n)$.

DEFINITION 4.1. *Let* $-\frac{n}{2} < s < \frac{n}{2}$. *We define* $\mathfrak{D}^s(R^n)$ *as the completion of* $\mathcal{S}(R^n)$ *with respect to the norm* :

$$\|\varphi\|_s = \left(\int_{R^n} |\xi|^{2s} |\widehat{\varphi}(\xi)|^2 d\xi\right)^{\frac{1}{2}}. \tag{4.6}$$

When s is an integer we get the space introduced in the definition 1.1. Clearly, the space $\mathfrak{D}^s(R^n)$ is the inverse image by $\mathcal{F}$ of $L^2(\mu_s)$. Further properties of $L^2(\mu_s)$ are discussed in [2] and [4].

5. Trace of elements of $\mathfrak{D}^m(R^n_+)$.

To simplify our notations, we shall denote the variable x_n by t and the partial derivative D_n either by D_t or $\frac{\partial}{\partial t}$; $\mathcal{F}_{x'}$ will represent the partial Fourier transform with respect to the variables $x' = (x_1, \dots, x_{n-1})$; $\widehat{\varphi}(\xi', t)$ will be used to represent the partial Fourier transform $\mathcal{F}_{x'}\varphi(x', t)$ of a smooth function φ. If $\varphi(x', t) \in C_c^\infty(\overline{R^n_+})$, we define

$$\gamma_j \varphi(x') = \frac{\partial^j \varphi}{\partial t^j}(x_1, \dots, x_{n-1}, 0) \tag{5.1}$$

for all $0 \leq j \leq m-1$.

THEOREM 5.1 : *There exists a continuous linear map*

$$\gamma = (\gamma_0, \dots, \gamma_{m-1}) : \quad \mathfrak{D}^m(R^n_+) \rightarrow \prod_{j=0}^{m-1} \mathfrak{D}^{m-j-\frac{1}{2}}(R^{n-1})$$

with the following properties :

i) *for all* $\varphi \in C_c^\infty(\overline{R^n_+})$, (5.1) *holds;*
ii) γ *is onto;*
iii) $\gamma^{-1}(0) = \mathfrak{D}_0^m(R^n_+)$.

PROOF. 1. We shall prove, first, that

$$\gamma: \quad \varphi \to \gamma\varphi = (\gamma_0\varphi, \dots, \gamma_{m-1}\varphi)$$

from $C_c^\infty(\overline{R^n_+})$ equipped with the topology induced by $\mathfrak{D}^m(R^n_+)$ into $\prod_{j=0}^{m-1} C_c^\infty(R^{n-1})$ with the topology induced by $\prod_{j=0}^{m-1} \mathfrak{D}^{m-j-\frac{1}{2}}(R^{n-1})$ is a continuous linear map. Next we extend, by continuity, γ to $\mathfrak{D}^m(R^n_+)$ (thm. 3.1). Let, then, φ be an element of $C_c^\infty(\overline{R^n_+})$. Write:

$$(5.2) \qquad |\gamma_j\widehat{\varphi}|^2 = -\int_0^\infty \frac{\partial}{\partial t}\left(\frac{\partial^j\widehat{\varphi}}{\partial t^j}\cdot\overline{\frac{\partial^j\widehat{\varphi}}{\partial t^j}}\cdot\theta\right)t,$$

$0 \le j \le m-1$, where $\theta(\xi', t) = e^{-|\xi'|t}$. From here we derive:

$$(5.3) \qquad \int_{R^{n-1}} (|\xi'|^{m-j-\frac{1}{2}}|\gamma_j\widehat{\varphi}|)^2\, d\xi' \le$$

$$\le \int_0^\infty\int_{R^{n-1}} |\xi'|^{2\left(m-j-\frac{1}{2}\right)}\cdot\left|\frac{\partial}{\partial t}\left(\frac{\partial^j\widehat{\varphi}}{\partial t^j}\cdot\overline{\frac{\partial^j\widehat{\varphi}}{\partial t^j}}\,\theta\right)\right| d\xi'\, dt.$$

The integral on the right can be estimated by:

$$(5.4) \qquad \int_0^\infty\int_{R^{n-1}} |\xi'|^{2\left(m-j-\frac{1}{2}\right)}\left|\frac{\partial^j\widehat{\varphi}}{\partial t^j}\right|^2\left|\frac{\partial\theta}{\partial t}\right| dt +$$

$$+\int_0^\infty\!\!\int |\xi'|^{2\left(m-j-\frac{1}{2}\right)}\left|\frac{\partial^j\widehat{\varphi}}{\partial t^j}\right|\left|\frac{\partial^{j+1}\widehat{\varphi}}{\partial t^{j+1}}\right| |\theta|\, d\xi'\, dt.$$

Denote by I_1 (resp. I_2) the first (resp. the second) integral in (5.4). We get:

$$(5.5) \qquad I_1 = \int_0^\infty \int_{R^{n-1}} |\xi'|^{2\left(m-j-\frac{1}{2}\right)} \left| \frac{\partial^j \widehat{\varphi}}{\partial t^j} \right|^2 \frac{|\xi'|}{e^{|\xi'|t}} \, d\xi' \, dt \leq$$

$$\leq C_1 \int_0^\infty \int_{R^{n-1}} |\xi'|^{2(m-j)} \left| \frac{\partial^j \widehat{\varphi}}{\partial t^j} \right|^2 d\xi' \, dt \leq C_1' \, \| \varphi \|^2_{\mathfrak{D}^m(R^n_+)}$$

where the last estimate is obtained by inverse partial Fourier transform and taking in account the norm of $\mathfrak{D}^m(R^n_+)$. Next, we have:

$$I_2 = \int_0^\infty \iint |\xi'|^{(m-j)} \left| \frac{\partial^j \widehat{\varphi}}{\partial t^j} \right| \cdot |\xi'|^{(m-j-1)} \left| \frac{\partial^{j+1} \widehat{\varphi}}{\partial t^{j+1}} \right| \cdot \theta \, d\xi' \, dt \leq$$

$$\leq C_2 \int_0^\infty \iint |\xi'|^{2(m-j)} \left| \frac{\partial^j \widehat{\varphi}}{\partial t^j} \right|^2 d\xi' \, dt + C_2 \int_0^\infty \iint |\xi'|^{2(m-j-1)} \left| \frac{\partial^{j+1} \widehat{\varphi}}{\partial t^{j+1}} \right|^2 d\xi' \, dt,$$

and, as one can see, the two last integrals can be estimated by a constant times $\| \varphi \|^2_{\mathfrak{D}^m(R^n_+)}$. We get:

$$(5.6) \qquad I_2 \leq C_2 \, \| \varphi \|^2_{\mathfrak{D}^m(R^n_+)} .$$

Combining now (5.3), (5.5) and (5.6) we get:

$$(5.7) \qquad \| \gamma_j \varphi \|_{D^{m-j-\frac{1}{2}}(R^{n-1})} \leq C \cdot \| \varphi \|_{D^m(R^n_+)} , \quad \text{q. e. d.}$$

2. To prove that γ is onto we shall prove that given $f \in \prod_{j=0}^{m-1} \mathfrak{D}^{m-j-\frac{1}{2}}(R^{n-1})$ we can find an element $u \in \mathfrak{D}^m(R^n)$ such that its restriction to R^n_+, which is an element of $\mathfrak{D}^m(R^n_+)$ (thm. 3.2), has trace f (i. e., $\gamma u = f$) on R^{n-1}. For this it suffices to show that given $f_j \in \mathfrak{D}^{m-j-\frac{1}{2}}(R^{n-1})$ we can find $u \in$ $\in \mathfrak{D}^m(R^n)$ such that:

$$(5.8) \qquad \gamma_0 u = \dots = \gamma_{j-1} u = 0, \quad \gamma_j u = f_j, \quad \gamma_{j+1} u = \dots = \gamma_{m-1} u = 0.$$

Let:

$$(5.9) \qquad \widehat{u}(\xi', t) = (a_j t^j + \dots + a_{m-1} t^{m-1}) \, e^{|\xi'| t} \, q(|\xi'| t) \cdot \widehat{f}_j(\xi')$$

where $q(s) \in C_c^\infty(R)$, $\widehat{q}(0) = 1$, where $a_j, \dots, a_{m-1}$ are to be choosen in order that (5.8) holds. One can check, easily, that choosing

$$a_{j+k} = \frac{(-1)^k |\xi'|^k}{j!\, k!}, \qquad 0 \leq k \leq m-j-1 \tag{5.10}$$

then the relations (5.8) are verified.

Next, let us prove that *all the m-th order derivatives of u belong to* $L^2(R^n)$ or, equivalently, that $(\xi')^\alpha \frac{\partial^l \widehat{u}}{\partial t} \in L^2(R^n)$ where $|\alpha| + l = m$. Using (5.9) and (5.10), we can write:

$$\widehat{u}(\xi', t) = \sum_{k=0}^{m-j-1} \frac{(-1)^k |\xi'|^k}{j!\, k!} u_k(\xi', t) \cdot \widehat{f}_j(\xi') \tag{5.11}$$

where

$$u_k(\xi', t) = t^{j+k} e^{|\xi'| t} q(|\xi'|\, t).$$

To verify that $(\xi')^\alpha \frac{\partial^l u}{\partial t^l} \in L^2(R^n)$, $|\alpha| + l = m$, it suffices to verify that:

$$\frac{(-1)^k |\xi'|^{k+|\alpha|}}{j!\, k!} \frac{\partial^l u_k}{\partial t^l} \widehat{f}_j(\xi') \in L^2(R^n), \qquad 0 \leq k \leq m-j-1. \tag{5.12}$$

But we have:

$$\frac{\partial^l u_k}{\partial t^l} = \sum_{p+q+r=l} C_{pqr} \frac{\partial^p t^{j+k}}{\partial t^p} \cdot \frac{\partial^q e^{|\xi'| t}}{\partial t^q} \frac{\partial^r q(|\xi'|\, t)}{\partial t^r} = \tag{5.13}$$

$$= |\xi'|^{q+r} \sum_{p+q+r=l} C'_{pqr} t^{j+k-p} e^{|\xi'| t} q^{(r)}(|\xi'|\, t).$$

It suffices then, according to (5.12) an (5.13) to prove that:

$$|\xi'|^{|\alpha|+k+q+r} t^{j+k-p} e^{|\xi'| t} q^{(r)}(|\xi'|\, t) \widehat{f}_j(\xi') \in L^2(R^n)$$

where $|\alpha| + l = m$, $0 \leq k \leq m-j-1$. Let us compute:

$$I = \int_{R^n} |\xi'|^{2(|\alpha|+k+q+r)} t^{2(j+k-p)} e^{2|\xi'| t} |q^{(r)}(|\xi'|\, t)|^2 \cdot |\widehat{f}_j(\xi')|^2 \, d\xi'\, dt = \tag{5.14}$$

$$= \int_{R^{n-1}} |\xi'|^{2(|\alpha|+k+q+r)} |\widehat{f}_j(\xi')|^2 \left\{ \int_{-\infty}^{+\infty} t^{2(j+k-p)} e^{2|\xi'| t} |q^{(r)}(|\xi'|\, t)|^2 \, dt \, d\xi. \right.$$

But:

$$(5.15)\qquad \int_{-\infty}^{+\infty} t^{2(j+k-p)} e^{2|\xi'|t} |q^{(r)}(|\xi'|t)|^2 dt =$$

$$= \frac{1}{|\xi'|^{2(j+k-p)+1}} \int_{-\infty}^{+\infty} s^{2(j+k-p)} e^{2s} |q^{(r)}(s)|^2 ds = \frac{C}{|\xi'|^{2(j+k-p)+1}}.$$

Replacing (5.15) in (5.14) we get:

$$I = C \int_{R^{n-1}} |\xi'|^{2m-2j-1} |\widehat{f}_j(\xi')|^2 d\xi' < +\infty,$$

what proves that the m-th order derivatives of u belong to $L^2(R^n)$.

To complete the proof of 2, we use the following result ([9]), II pg. 40, remarques; also [3]).

LEMMA. *Suppose T is a distribution on R^n whose first order derivatives $D_i T$, $1 \leq i \leq n$, belong to $L^p(R^n)$, $1 < p < +\infty$, then*

$$T = S + C$$

where $S \in L^q(R^n)$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$.

Applying this lemma to our situation we conclude that each derivative of order $m-1$ of u is the sum of a function belonging to $L^{q(1)}(R^n)$, $\frac{1}{q(1)} = \frac{1}{2} - \frac{1}{n}$, plus a constant. But we can show that in our case, the constant must be zero. In fact, let v be a derivate of order $m-1$ of u; we can write, using partial Fourier transform in x', $\widehat{v} = (\xi')^\alpha \cdot \frac{\partial^l u}{\partial t^l}$ where $|\alpha| + l = m-1$. According to the lemma and to the fact that the m-th order derivatives of u (hence, all the first order derivatives of v) are in $L^2(R^n)$, we have:

$$v = w + C, \text{ where } w \in L^{q(1)}(R^n).$$

Take, now, $\varphi \in C_c^\infty(R^{n-1})$ such that $\int_{R^{n-1}} \varphi(x')\,dx' \neq 0$ and consider:

$$(5.16)\qquad \langle v, \varphi \rangle = \langle w, \varphi \rangle + C \int_{R^{n-1}} \varphi(x')\,dx'$$

(where $\langle\,,\,\rangle$ represents, here, the pairing between $C_c^\infty(R^{n-1})$ and $\mathcal{D}'(R^{n-1})$). We can write :

$$(5.17) \qquad \langle v, \varphi\rangle = \langle \widehat{v}, \widehat{\varphi}\rangle = \int_{R^{n-1}} (\xi')^\alpha \frac{\partial^l \widehat{u}}{\partial t^l} \cdot \widehat{\varphi}\, d\,\xi' .$$

An easy estimate shows that the integral in the right side of (5.17) equals $C(\varphi)\cdot(1/t^{\frac{n-2}{2}})$, hence goes to zero when $t \to +\infty$, because $n > 2m$. Also, w belonging to $L^{q(1)}(R^n)$, then :

$$(5.18) \qquad \langle w, \varphi\rangle = \int_{R^n} w(x', t)\,\varphi(x')\,dx'$$

as a function of t, goes to zero when $t \to +\infty$. Thus, from (5.16) and our hypothesis on φ it follows that $C = 0$, consequently all the derivatives of order $m-1$ of u belong to $L^{q(1)}(R^n)$. By induction, using the lemma and the same argument as above, we shall conclude that $u \in \mathfrak{D}^m(R^n)$.

3. Let us prove that $\gamma^{-1}(0) = \mathfrak{D}_0^m(R_+^n)$. First of all we remark that if $u \in \mathfrak{D}_0^m(R_+^n)$ then $\gamma u = 0$. Conversely, suppose that $u \in \mathfrak{D}^m(R_+^n)$ and $\gamma u = 0$. Consider the function

$$\alpha_k(t) = \begin{cases} 0 & \text{if} \quad 0 \le t \le \frac{1}{k} \\ k\cdot\left(t - \frac{1}{k}\right) & \text{if} \quad \frac{1}{k} \le t \le \frac{2}{k} \\ 1 & \text{if} \quad \frac{2}{k} \le t \end{cases}$$

(k is an integer > 0) and set $u_k(x', t) = \alpha_k(t)\cdot u(x', t)$. It suffices to verify that $u_k(x', t) \in \mathfrak{D}^m(R_+^n)$ and that $u_k(x', t) \to u(x'\, t)$ in $\mathfrak{D}^m(R_+^n)$ since, by regularizing the functions u_k, we get a sequence of functions of $C_c^\infty(R_+^n)$ which converges to u in $\mathfrak{D}^m(R_+^n)$, hence u will belong to $\mathfrak{D}_0^m(R_+^n)$. To prove that $u_k \to u$ in $\mathfrak{D}^m(R_+^n)$ it is enough to prove that $D^\alpha u_k \to D^\alpha u$ in $L^{q(m-|\alpha|)}(R_+^n)$ for all $0 \le |\alpha| \le m$. It is clear for $|\alpha| = 0$. We shall sketch the proof for the derivatives of order 1 ; the proof for derivatives of higher order is essentially the same.

One can verify that the only thing to be proven in that $\alpha_k'(t) \cdot u(x, t)$ converges to 0 in $L^{q(m-1)}(R^n_+)$, as $k \to +\infty$. Write

$$u = \int_0^t \frac{\partial u}{\partial \tau} \, d\tau \qquad 0 \leq t \leq \frac{2}{k}.$$

By Hölder's inequality we get:

$$| u |^{q(m-1)} \leq t^{q(m-1)-1} \int_0^{2/k} \left| \frac{\partial u}{\partial \tau} \right|^{q(m-1)} d\tau$$

Hence, it follows:

$$\int_0^\infty | \alpha_k'(t) \, u(x, t) |^{q(m-1)} \, dt = k^{q(m-1)} \int_0^{2/k} | u(x, t) |^{q(m-1)} \, dt \cdot$$

$$\leq k^{q(m-1)} \int_0^{2/k} t^{q(m-1)-1} \left\{ \int_0^{2/k} \left| \frac{\partial u}{\partial \tau} \right|^{q(m-1)} d\tau \right\} dt =$$

$$= C \cdot \int_0^{2/k} \left| \frac{\partial u}{\partial t} \right|^{q(m-1)} dt.$$

Finally

$$\int_{R^n_+} | \alpha_k'(t) \, u(x, t) |^{q(m-1)} \, dx' \, dt \leq C \int_0^{2/k} \int_{R^{n-1}} \left| \frac{\partial u}{\partial t} \right|^{q(m-1)} dt$$

and the right hand side $\to 0$ as $k \to +\infty$, q. e. d. The proof of theorem 6 is complete.

6. The inhomogeneous Dirichlet problem.

Let $a(u, v)$ be the sesquilinear form (2.4) and suppose that (2.5) holds. Let γ be the continuous linear map defined in theorem 5.1. Then:

THEOREM 6.1 *(A, γ) is an isomorphism from $\mathfrak{D}^m(R^n_+)$ onto $\mathfrak{D}^{-m}(R^n_+) \times \prod_{j=0}^{m-1} \mathfrak{D}^{m-j-\frac{1}{2}}(R^{n-1})$.*

PROOF. First we remark that by theorem 2.1 and theorem 5.1 (A, γ) is continuous and one to-one. Next, to see that it is onto, let $(f, g_0, \dots, g_{m-1})$ be an element of $\mathfrak{D}^{-m}(R^n_+) \times \prod_{j=0}^{m-1} \mathfrak{D}^{m-j-\frac{1}{2}}(R^{n-1})$. There is (theorem 5.1) an element $v \in \mathfrak{D}^m(R^n_+)$ such that $\gamma_j v = g_j$, $0 \leq j \leq m-1$. Since $Av \in \mathfrak{D}^{-m}(R^n_+)$, by theorem 2.1, there is a unique $u_0 \in \mathfrak{D}^m_0(R^n_+)$ such that:

$$Au_0 = f - Av.$$

Then $u = u_0 + v$ is the unique solution of (2.7). The fact that (A, γ) is an isomorphism follows from Banach's isomorphism theorem, q. e. d.

7. Regularization of solutions of the Dirichlet problem.

Suppose that the sesquilinear form (2.4) has smooth coefficients defined in $\overline{R^n_+}$ and suppose that conditions i), ii) and iii) of section 1 hold.

THEOREM 7.1. *Let f be an element of $\mathfrak{D}^{-(m-1)}(R^n_+) \cap \mathfrak{D}^{-m}(R^n_+)$ and let $u \in \mathfrak{D}^m_0(R^n_+)$ be the unique solution of the homogeneous Dirichlet problem. Then, all the partial derivatives $D_j u$, $1 \leq j \leq n$, belong to $\mathfrak{D}^m(R^n_+)$.*

PROOF 1. As in [8] (see also [1]), we start by regularizing the tangential derivatives. We have:

$$u(u, v) = \langle f, \bar{v} \rangle, \text{ for all } v \in \mathfrak{D}^m_0(R^n_+). \tag{7.1}$$

Let $h = (0, \dots, 0, h_j, 0, \dots, 0)$, $1 \leq j \leq n-1$, and denote by $\delta_h v$ the difference quotient

$$\frac{v(x_1, \dots, x_j + h_j, \dots, x_n) - v(x_1 \dots, x_n)}{h_j}$$

Since $\delta_h v \in \mathfrak{D}^m_0(R^n_+)$ we have also:

$$a(u, \delta_{-h} v) = \langle f, \overline{\delta_{-h} v} \rangle, \text{ for all } v \in \mathfrak{D}^m_0(R^n_+). \tag{7.2}$$

Write :

$$a(u, \delta_{-h} v) = (-1)^m \sum_{|p|=|q|=m} \int_{R^n_+} a_{pq}(x) D^p u \overline{(D^q (\delta_{-h} v)} dx =$$

$$= (-1)^m \sum_{|p|=|q|=m} \int_{R^n_+} a_{pq}(x+h) D^p (\delta_h u) \overline{D^q v} dx +$$

$$(-1)^m \sum_{|p|=|q|=m} \int_{R^n_+} \delta_h (a_{pq}(x)) \cdot D^p u \cdot \overline{D^q v} dx .$$

Replacing in (7.2) get the inequality :

$$\sum_{|p|=|q|=m} \int_{R^n_+} a_{pq}(x+h) D^p (\delta_h u) \overline{D^q} v dx \leq |\langle f, \overline{\delta_{-h} v} \rangle| \tag{7.3}$$

$$+ \sum_{|p|=|q|=m} \int_{R^n_+} |\delta_h (a_{pq}(x))| \cdot |D^p u| |D^p v| dx .$$

Since $f \in \mathfrak{D}^{-(m-1)} \cap \mathfrak{D}^{-m}$ we have :

$$|\langle f, \overline{\delta_{-h} v} \rangle| \leq C_2 \| f \|_{\mathfrak{D}^{-(m-1)}} \cdot \| Dv \|_{\mathfrak{D}_0^{m-1}} \leq C_2 \| f \|_{\mathfrak{D}^{-(m-1)}} \| v \|_{\mathfrak{D}_0^m} \tag{7.4}$$

provided that $|h|$ is small enough. On the other hand,

$$\int_{R^n_+} |\delta_h (a_{pq}(x))| \cdot |D^p u| |D^q v| dx \leq \tag{7.5}$$

$$C_3 \left(\int_{R^n_+} |D^p u|^2 dx \right)^{1/2} \cdot \left(\int_{R^n_+} |D^q v|^2 dx \right)^{1/2} ,$$

hence

$$\sum_{|p|=|q|=m} \int_{R^n_+} |\delta_h (a_{pq}(x))| \cdot |D^p u| |D^q v| dx \leq \tag{7.6}$$

$$C_3 (\sum_{|p|=m} \| D^p u \|_{L^2}) \cdot (\sum_{|q|=m} \| D^q v \|_{L^2}) \leq C_3 \gamma^2 \| u \|_{\mathfrak{D}_0^m} \cdot \| v \|_{\mathfrak{D}_0^m} ,$$

where γ is the constant which appears in condition iii) of section 1. Consider, now, the left hand side of (7.3). We have:

$$(7.7)\qquad \left| \sum_{|p|=|q|=m} \int_{R^n_+} a_{pq}(x+h)\, D^p(\delta_h u)\, \overline{D^q v}\, dx \right| \geq$$

$$\operatorname{Re} \sum_{|p|=m} \int_{R^n_+} a_{pp}(x+h)\, D^p(\delta_h u) \cdot \overline{D^p v}\, dx -$$

$$- \sum_{p \neq q} \int_{R^n_+} |a_{pq}(x+h)| \cdot |D^p(\delta_h u)|\, |D^q v|\, dx \geq$$

$$\operatorname{Re} \sum_{|p|=m} \int_{R^n_+} a_{pp}(x+h)\, D^p(\delta_h u) \cdot \overline{D^p v}\, dx - \beta\gamma^2 \, \| \delta_h u \|_{\mathfrak{D}^m_0} \cdot \| v \|_{\mathfrak{D}^m_0}.$$

From (7.3), (7.4), (7.6) and (7.7) it follows.

$$\operatorname{Re} \sum_{|p|=m} \int_{R^n_+} a_{pp}(x+h)\, D_p(\delta_h u) \cdot \overline{D^p v}\, dx - \beta\gamma^2 \, \| \delta_h u \|_{\mathfrak{D}^m_0} \cdot \| v \|_{\mathfrak{D}^m_0} \leq C_4 \| v \|_{\mathfrak{D}^m_0}.$$

Now, replacing v by $\delta_h u$ and noticing that $\operatorname{Re} a_{pp} > \alpha > 0$ (assumption ii of section 1) we get from (7.9) the inequality:

$$(7.9)\qquad (\alpha - \beta\gamma^2)\, \| \delta_h u \|^2_{\mathfrak{D}^m_0} \leq C_4 \| \delta_h u \|_{\mathfrak{D}^m_0}$$

and finally

$$(7.10)\qquad \| \delta_h u \|_{\mathfrak{D}^m_0} \leq C_5 .$$

By a standard argument we conclude that $D_i u \in \mathfrak{D}^m(R^n_+)$, $1 \leq i \leq n-1$.

2. Next we have to prove $D_n u \in \mathfrak{D}^m(R^n_+)$. Write

$$(7.11)\qquad Au = \sum_{|p|=|q|=m} D^q(a_{pq}(x)\, D^p u) =$$

$$= D_n^m \left(\sum_{|p|=m} a_p(x)\, D^p u \right) + D_n^{m-1} \left(\sum_{|p|=m} \sum_{i=1}^{n-1} D_i (a_{pq_i}(x)\, D^p u) \right) + \dots$$

$$\dots + \sum_{|p|=m} \sum_{|q|=m} D^q_\tau (a_{pq_\tau}(x)\, D^p u),$$

where D_τ^q denotes a tangential derivative of order $|q|$. Let

$$g = \sum_{|p|=m} a_{pq_0}(x)\, D^p u \qquad \text{(here } q_0 = (0, \dots, 0, m)). \tag{7.12}$$

Since $u \in \mathfrak{D}^m(R^n_+)$, it follows that $g \in L^2(R^n_+)$. On the other hand, $D_i u \in \mathfrak{D}^m(R^n_+)$, $1 \leq i \leq n-1$, hence, $D_i g \in L^2(R^n_+)$, $1 \leq i \leq n-1$. Finally, since $Au = f$, we get using (7.11):

$$D_n^m g = f - D_n^{m-1}\left(\sum_{|p|=m} \sum_{i=1}^{n-1} D_i\,(a_{pq_i}(x)\, D^p u)\right) + \dots \tag{7.13}$$

$$\dots + \sum_{|p|=m} \sum_{|q|=m} D_\tau^q (a_{pq_\tau}(x)\, D^p u)$$

and it is easy to verify that all the terms in the right hand side of (7.13) belong to $H^{-(m-1)}(R^n_+)$. Under these conditions, i. e.:

$$g,\ D_1 g, \dots, D_{n-1} g \in L^2(R^n_+) \quad \text{and} \quad D_n^m g \in H^{-(m-1)}(R^n_+). \tag{7.14}$$

We can conclude, using a result due to Lions ([6], lemma 11.2), that $g \in H^1(R^n_+)$.

Now, let us prove that, for all $|p| = m$, $D^p u \in H^1(R^n_+)$. In fact, we have, $D^p u \in L^2(R^n_+)$, for all $|p| = m$, because $u \in \mathfrak{D}^m(R^n_+)$. Also $D_j(D^p u) = D^p(D_j u) \in L^2(R^n_+)$, when $1 \leq j \leq n-1$. Thus we have to verify that $D_n(D^p u) \in L^2(R^n_+)$. It is trivial if D^p contains *at least* a tangential derivative. We, then, have to prove that $D_n(D_n^m u)$ belongs to $L^2(R^n_+)$. Rewrite g as follows:

$$g = a_{mm}(x)\, D_n^m u + \sum_{j=1}^{n-1} a_{p_j m}(x)\, D_n^{m-1} D_j u + \dots \tag{7.15}$$

(here to simplify our notations we replaced $q_0 = (0, \dots, 0, m)$ by m). By assumption $\operatorname{Re} a_{mm}(x) > 0$, thus, we get:

$$D_n^m u = \frac{1}{a_{mm}} \left\{ g - \sum_{j=1}^{n-1} a_{p_j m}(x)\, D_n^{m-1} D_j u - \dots \right\}. \tag{7.16}$$

It is a matter of verification that all the terms of the expression between brackets belong to $L^2(R^n_+)$. Taking the derivative D_n of both sides, we also get that the right hand side belongs to $L^2(R^n_+)$ (just use the fact that $g \in H^1$ and that tangential derivatives of order one of u belong to $\mathfrak{D}^m$). Thus $D_n^{m+1} u \in L^2(R^n_+)$.

Next, let us prove that, for all $|p| = m - 1$, $D^p u \in W^{1, q(1)}(R^n_+)$. In fact, let $|p| = m - 1$ and let $v_p = D^p u$. We have $v_p \in L^{q(1)}$ because $u \in \mathfrak{D}^m$. On the other hand, any first order derivative Dv_p is equal to a derivative of order m of u that, we proved above, belongs to $H^1(R^n_+)$, hence to $L^{q(1)}(R^n_+)$ according to Sobolev's imbedding theorem. By a recurrence argument, we can prove that for all $|p| = m - j$ $(0 \leq j \leq m)$, $D^p u \in W^{1, q(j)}(R^n_+)$ what proves, finally, that $D_i u \in \mathfrak{D}^m(R^n_+)$ for $j = 1, \ldots, n$, q.e.d.

For any open set $\Omega \subset R^n$ let $W^{k,q}(\Omega)$ be the Sobolev space of functions $u \in L^q(\Omega)$ with derivatives $D^p u$, in the sense of distributions, belonging to $L^q(\Omega)$ for $|p| \leq k$. When $q = 2$, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$.

The result of theorem 7.1 suggests the following.

DEFINITION 7.1. *Denote by* $\mathfrak{D}^{k,m}(R^n_+)$ *(k integer ≥ 0) the space of functions* $u \in W^{k\,q(m)}(R^n_+)$ *such that* $\mathfrak{D}^p u \in W^{k,q(m-j)}(R^n_+)$ *for* $0 \leq |p| = j \leq m$, *equipped with the norm:*

$$\| u \|_{\mathfrak{D}^{k,m}(R^n_+)} = \sum_{j=0}^{m} \| D^p u \|_{W^{k,q(m-j)}(R^n_+)} . \tag{7.17}$$

Clearly $\mathfrak{D}^{k,m}(R^n_+)$ is a Banach space. It is easy to verify that $u \in \mathfrak{D}^{k,m}(R^n_+)$ if and only if $D^p u \in \mathfrak{D}^m(R^n_+)$, $0 \leq |p| \leq k$. Then we can re-phrase the conclusion of theorem 7.1, by saying that $u \in \mathfrak{D}^{1,m}(R^n_+) \cap \mathfrak{D}_0^m(R^n_+)$.

We also point out that if we take f in $\bigcap_{j=0}^{h} \mathfrak{D}^{-(m-j)}(R^n_+)$ then the solution u of (2.6) belongs to $\mathfrak{D}^{k,m}(R^n_+) \cap \mathfrak{D}_0^m(R^n_+)$. The proof uses an induction argument and the same tecnique as those used in theorem 7.1.

We shall state, now, a trace theorem for elements of $\mathfrak{D}^{k,m}(R^n_+)$ that we shall use in next section.

THEOREM 7.2. *There is a continuous linear map*

$$\gamma = (\gamma_0, \ldots, \gamma_{m+k-1}) : \mathfrak{D}^{k,m}(R^n_+) \to \prod_{j=0}^{m-1} \left(\bigcap_{l=0}^{k} \mathfrak{D}^{m+k-(j+l)-\frac{1}{2}}(R^{n-1}) \right) \cdot \tag{7.18}$$

$$\cdot \prod_{j=m}^{m+k-1} H^{m+k-j-\frac{1}{2}}(R^{n-1})$$

such that :

i) *for each* $\varphi \in C_c^\infty(\overline{R_+^n})$, $\gamma_j \varphi = \dfrac{\partial^j \varphi}{\partial x_n^j}(x', 0)$ $0 \leq j \leq m + k - 1$ *;*

ii) γ *is onto ;*

iii) $\gamma^{-1}(0) = \mathfrak{D}_0^{k,m}(R_+^n)$.

Here, $D_0^{k,m}(R_+^n)$ denotes the closure in $\mathfrak{D}_0^{k,m}(R_+^n)$ of $C_c^\infty(R_+^n)$. On $\bigcap_{l=0}^{k} \mathfrak{D}^{m+k-(j+l)\frac{1}{2}}(R^{n-1})$ we consider the sup topology and $H^{m+k-j-\frac{1}{2}}(R^{n-1})$ is the Sobolev space that in our situation can be easily defined by means of Fourier transform. Clearly theorem 7.2 generalizes theorem 5.1. Its proof goes in the same way as in theorem 5.1. First, we have to prove that $C_c^\infty(\overline{R_+^n})$ is dense in $\mathfrak{D}^{k,m}(R_+^n)$ (see thm. 3.1) and that there is a continuous linear map $P: \mathfrak{D}^{k,m}(R_+^n) \rightarrow \mathfrak{D}^{k,m}(R^n)$ such that the restriction of Pu in R_+^n is u, for all $\mathfrak{D}^{k,m}(R^n)$. Next, for any $\varphi \in C_c^\infty(R_+^n)$ we represent $\gamma_j \varphi$ $\left(\text{restriction of the normal derivative } \dfrac{\partial^j \varphi}{\partial x_n^j} \text{ to } R^{n-1}\right)$ by means of (5.2). Going through a similar estimation as in thm. 5.1 we prove that (7.18) is continuous on elements of $C_c^\infty(\overline{R_+^n})$ and then, it can be extended continuously to $D^{k,m}(R_+^n)$. The proofs that γ is onto and that its kernel is $\mathfrak{D}_0^{k,m}(R_+^n)$ are similar to the ones given in theorem 5.1 and, for this reason, are left to the reader.

8. Transposition.

We assume that the sesquilinear form (2.4) verifies the same assumptions as in the previous section. Let $a^*(u, v) = \overline{a(v, u)}$ and let A^* be the formal adjoint of A. The following relation

$$(8.1) \qquad (u, A^* v) - (Au, v) = \sum_{j=0}^{m-1} \int_{R^{n-1}} S_{2m-j-1}\, u, \gamma_j \bar{v}\, dx' + \sum_{j=0}^{m-1} \int_{R^{n-1}} \gamma_j u \cdot T_{2m-j-1} \bar{v}\, dx',$$

where u and v are elements of $C_c^\infty(\overline{R_+^n})$, S_{2m-j-1} and T_{2m-j-1} are differential operators of order $2m - j - 1$ in R^{n-1}, is easily obtained by integration by parts ([6], II, pg. 144).

If we suppose that $a^*(u, v)$ verifies condition (2.5) then all the results in the previous sections hold for A^*. In particular, if follows from our remarks after definition 7.1 that

(8.2) *A^* is an isomorphism from $\mathfrak{D}^{m,m}(R_+^n) \cap D_0^m(R_+^n)$ onto $\bigcap_{j=0}^{m} \mathfrak{D}^{-j}(R_+^n)$.*

By transposing this result and applying it to a particular case, we shall be able to get another isomorphism theorem concerning the Dirichlet problem. Namely, each element $(f,(g_j)) \in D^{-m}(R_+^n) \times \prod_{j=0}^{m-1} H^{-j-\frac{1}{2}}(R^{n-1})$ define a continuous antilinear functional L on $\mathfrak{D}^{m,m}(R_+^n) \cap \mathfrak{D}_0^m(R_+^n)$ by setting

$$L(v) = \langle f, \bar{v} \rangle + \sum_{j=0}^{m-1} \langle g_j, T_{2m-j-1}\bar{v} \rangle \tag{8.3}$$

where the first pairing is between $\mathfrak{D}_0^m(R_+^n)$ and $\mathfrak{D}^{-m}(R_+^n)$, while the pairing in the summation is between $H^{j+\frac{1}{2}}(R^{n-1})$ and $H^{-j-\frac{1}{2}}(R^{n-1})$. As one can see, $\mathfrak{D}^{-m}(R_+^n) \cdot \prod_{j=0}^{m-1} H^{-j-\frac{1}{2}}(R^{n-1})$ can be identified to a subspace X of the dual of $\mathfrak{D}^{m,m} \cap \mathfrak{D}_0^m$. By (8.2), there is a unique u belonging to Z, dual of $\bigcap_{j=0}^{m} \mathfrak{D}^{-j}(R_+^n)$, such that

$$\langle u, A^* \bar{v} \rangle = \langle f, \bar{v} \rangle + \sum_{j=0}^{m-1} \langle g_j, T_{2m-j-1}\bar{v} \rangle, \tag{8.4}$$

for all $v \in \mathfrak{D}^{m,m} \cap \mathfrak{D}_0^m$. (The pairing in the left hand side of (8.4) is between $\bigcap_{j=0}^{m} \mathfrak{D}^{-j}$ and its dual Z). In particular, if $v = \varphi \in C_c^\infty(R_+^n)$, it follows from (8.4) that $Au = f$ in the sense of distributions.

DEFINITION 8.1. *Denote by H the space of functions $u \in Z$ such that $Au \in \mathfrak{D}^{-m}(R_+^n)$ equipped with the norm :*

$$\| u \|_H = (\| u \|_Z^2 + \| Au \|^2_{\mathfrak{D}^{-m}(R_+^n)})^{\frac{1}{2}} \tag{8.5}$$

It is a Banach space. Let, now, Y be the closure in H of subspace formed by those elements $u \in Z$ that verify (8.4) when $(f, (g_j))$ is arbitrarily given in X. We are going to see that $u \in Y$ if and only if $Au \in \mathfrak{D}^{-m}(R_+^n)$ and $\gamma_j u \in H^{-j-\frac{1}{2}}(R^{n-1})$, $0 \leq j \leq m-1$. The first assertion follows from the definition of Y. As for the second, we need to determine the trace of elements of Y.

Teorem 8.1. *$C_c^\infty(\overline{R_+^n})$ is dense in H.*

Proof. Each continuous anti-linear functional M on H can be represented in the following way:

$$M(w) = \langle \alpha, \overline{w} \rangle + \langle \beta, A\overline{w} \rangle \tag{8.6}$$

where $\alpha \in \bigcap_{j=0}^{m} \mathfrak{D}^{-j}(R_+^n)$ and $\beta \in \mathfrak{D}_0^m(R_+^n)$. On the other hand, an element $\xi \in Z$ dual of $\bigcap_{j=0}^{m} \mathfrak{D}^{-j}$, can be represented (not necessarily in a unique way) as a sum $\xi = \sum_{j=0}^{m} \xi_j$, where $\xi_j \in \mathfrak{D}_0^j(R_+^n)$ (here $\mathfrak{D}_0^0(R_+^n) = L^2(R_+^n)$). Hence, the pairing between and is given by:

$$\langle \alpha, \overline{\xi} \rangle = \sum_{j=0}^{m} \langle \alpha, \overline{\xi}_j \rangle \tag{8.7}$$

and (8.7) does not depend on the particular representation of ξ as a sum of elements $\xi_j \in \mathfrak{D}_0^j(R_+^n)$, $0 \leq j \leq m$. It follows that, if $\xi = \varphi \in C_c^\infty(\overline{R_+^n})$ then (8.7) reduces to

$$\langle \alpha, \overline{\varphi} \rangle = \int_{R_+^n} \alpha \cdot \overline{\varphi}\, dx. \tag{8.8}$$

Suppose, now, that $M(\varphi) = 0$ for all $\varphi \in C_c^\infty(\overline{R_+^n})$. Then we have:

$$M(\varphi) = \int_{R_+^n} \alpha\overline{\varphi}\, dx + \langle \beta, A\overline{\varphi} \rangle = 0, \text{ for all } \varphi \in C_c^\infty(\overline{R_+^n}). \tag{8.9}$$

Let $\tilde{\alpha}$ (resp. $\tilde{\beta}$) be equal to α (resp. β) in R_+^n and equal to 0 in R_-^n. We have, $\tilde{\alpha} \in L^2(R^n)$ and $\tilde{\beta} \in \mathfrak{D}^m(R^n)$. It follows from (8.9) that

$$(\tilde{\alpha}, \Phi)_{L^2} + \langle \tilde{\beta}, A\Phi \rangle = 0, \text{ for all } \Phi \in C_c^\infty(R^n). \tag{8.10}$$

We get

$$(8.11) \qquad A^* \widetilde{\beta} = - \widetilde{\alpha}, \text{ in the sense of distributions.}$$

Since $\widetilde{\beta} \in \mathfrak{D}^m(R^n)$ then $A^* \widetilde{\beta} \in \mathfrak{D}^{-m}(R^n)$, thus $\widetilde{\alpha} \in L^2(R^n) \cap \mathfrak{D}^{-m}(R^n)$. We can prove (lemma 8.1, below) that under our assumptions on A^* and $\widetilde{\alpha}$, the solution $\widetilde{\beta}$ of (8.11) belongs to $\mathfrak{D}^{m,n}(R^n)$. Consequently, β must belong to $\mathfrak{D}_0^{m,m}(R^n_+)$. Thus we have:

$$(8.12) \qquad \langle \beta, A\bar{w} \rangle = \langle A^*\beta, \bar{w} \rangle, \quad \text{for all } w \in H.$$

Replacing this relation in (8.6) we get:

$$(8.13) \qquad M(w) = \langle \alpha, \bar{w} \rangle + \langle \beta, A\bar{w} \rangle = \langle \alpha + A^*\beta, \bar{w} \rangle, \ w \in H.$$

Using (8.11), it follows that $M(w) = 0$ for all $w \in H$; consequently $C_c^\infty(\overline{R^n_+})$ is dense in H.

To complete the proof of theorem 8.1 we need to prove the

LEMMA 8.1. *Let $a(u, v)$ be the sesquilinear form* (1.7). *Assume that the coefficients are smooth and that conditions* i) ii) *and* iii) *hold. Let f be an element of $L^2(R^n) \cap \mathfrak{D}^{-m}(R^n)$. Then the unique element $u \in \mathfrak{D}^m(R^n)$, solution of $Au = f$, belongs to $\mathfrak{D}^{m,m}(R^n)$.*

PROOF. Consider iterated difference-quotients of order m, $\delta_h^p v$, where $p = (p_1, \dots, p_n)$ is a n-tuple of positive integers such that $p = m$ and $h = (h_1, \dots, h_{p_1}, \dots, h_{p_{n-1}+1}, \dots, h_{p_n})$ is a m-tuple of positive real numbers. The difference quotient $\delta_h^p v$ is easily defined by induction starting with difference-quotient of order 1 (see section 7). If $v \in \mathfrak{D}^m(R^n)$, it follows from our assumptions that:

$$(8.14) \qquad |\langle f, \delta_h^p \bar{v} \rangle| = |\langle \delta_h^p f, \bar{v} \rangle| \leq C \cdot \| v \|_{\mathfrak{D}^m(R^n)},$$

where C depends on f but not on v. With the same technique used in part 1 of the proof of theorem 7.1 we can conclude that $D^p u \in \mathfrak{D}^m(R^n)$ for all $|p| = m$. Denote by v_p the derivative $D^p u$. Since $u \in \mathfrak{D}^m(R^n)$, v_p belongs to $L^2(R^n)$. Also, since $v_p \in \mathfrak{D}^m(R^n)$, all derivatives of order m of v_p, belong to $L^2(R^n)$. We conclude, by Fourier transform, that $v_p \in H^m(R^n)$ for all $|p| = m$.

From this we shall derive that $u \in \mathfrak{D}^{m,m}(R^n)$. If fact, according to our remark just after definition 7.1, we have to prove that $v_\alpha = D^\alpha u \in \mathfrak{D}^m(R^n)$

for all $0 \leq |\alpha| \leq m$. It is trivial for $|\alpha| = 0$. Also, it is trivial for $|\alpha| = m$ because $H^m(R^n) \subset \mathfrak{D}^m(R^n)$. Let us prove that $v_\alpha \in D^m(R^n)$ for all $0 < |\alpha| < m$. By definition of $\mathfrak{D}^m(R^n)$ it is enough to prove that $D^p v_\alpha = D^p D^\alpha u \in L^{q(m-|p|)}(R^n)$ for all $0 \leq p \leq m$.

First, suppose that $m \geq |p| \geq m - |\alpha|$. Then,

$$D^p v_\alpha = D^p D^\alpha u \in H^{m-|p|}(R^n)$$

because $|p| + |\alpha| = m + r$ with $r \geq 0$ and $r \leq p$. But we know by Sobolev's imbedding theorem that $H^{m-|p|}(R^n) \subset L^{q(m-|p|)}(R^n)$.

Next, suppose that $|p| < m - |\alpha|$. We claim that it is enough to verify that:

$$D^p v_\alpha \in W^{m, q(m-|p|-|\alpha|)}(R^n) \quad \text{for all} \quad |p| < m - |\alpha|. \tag{8.15}$$

In fact, we have $W^{m, q(m-|p|-|\alpha|)}(R^n) \subset W^{|\alpha|, q(m-|p|-|\alpha|)}(R^n)$ and the last space is contained in $L^{q(m-|p|)}(R^n)$, again, by Sobolev's theorem, because $\dfrac{n}{q(m-p-|\alpha|)} > |\alpha|$ and

$$\frac{1}{q(m-|p|-|\alpha|)} - \frac{|\alpha|}{n} = \frac{1}{2} - \frac{m-|p|}{n} = \frac{1}{q(m-|p|)}$$

(see (1.4)). To prove (8.15) we proceed by induction on p. First let us verify that if $|p| = m - |\alpha| - 1$, then

$$D^p v_\alpha \in W^{m, q(1)}(R^n).$$

We have

$$D^p v_\alpha = D^p (D^\alpha u) \in L^{q(1)}(R^n), \quad \text{because} \quad u \in \mathfrak{D}^m(R^n) \quad \text{and}$$

$|p| + |\alpha| = m - 1$. Let γ be a n-tuple of positive integers such that $0 < |\gamma| \leq m$. It is easy to verify that

$$D^\gamma (D^p v_\alpha) \in H^1(R^n) \subset L^{q(1)}(R^n), \quad \text{then (8.15) holds for}$$

$|p| = m - |\alpha| - 1$. Suppose, now, that (8.14) is true for all p such that $|p| = m - |\alpha| - k$ and let us prove that it is true for $|p| = m - |\alpha| - (k+1)$. We have for such p

$$D^p v_\alpha = D^p (D^\alpha u) \in L^{q(k+1)}(R^n), \quad \text{because} \quad u \in \mathfrak{D}^m(R^n) \quad \text{and}$$

$|p| + |\alpha| = m - (k+1)$. Let $0 < |\gamma| \leq m$ and write

$$D^\gamma (D^p v_\alpha) = D^{\gamma'} (DD^p v_\alpha), \quad \text{where} \quad 0 \leq |\gamma'| \leq m-1.$$

Now, $DD^p v_\alpha$ is a derivative of order $m - |\alpha| - k$ of v_α, hence, by our induction hypothesis, belongs to $W^{m, q(k)}(R^n)$. It follows, then, that $D^\gamma (D^p v_\alpha) = D^{\gamma'} (DD^p v_\alpha)$ must belong to $W^{1, q(k)}(R^n)$, space which is contained in $L^{q(k+1)}(R^n)$, by Sobolev's theorem. Thus $D^p v_\alpha \in W^{m, q(k+1)}(R^n)$ for all $|p| = m - |\alpha| - (k+1)$, q. e. d.

THEOREM 8.2. *There is a continuous linear map* $\gamma' = (\gamma_0, \dots, \gamma_{m-1})$: $Y \longrightarrow \prod_{j=0}^{m-1} H^{-j-\frac{1}{2}}(R^{n-1})$ *such that, for all*

$$\varphi \in C_c^\infty(\overline{R^n_+}), \quad \gamma_j \varphi = \frac{\partial^j \varphi}{\partial x_n^j}(x', 0), \quad 0 \leq j \leq m-1.$$

PROOF. Let $h = (h_0, \dots, h_{m-1}) \in \prod_{j=0}^{m-1} H^{j+\frac{1}{2}}(R^{n-1})$. There is $w \in H^{2m}(R^n_+)$ such that

$$\gamma_0 w = \dots = \gamma_{m-1} w = 0, \quad T_m w = h_{m-1}, \dots, T_{2m-1} w = h_0$$

([6], II, pg. 145, lemma 1.1). Clearly $w \in \mathfrak{D}^{m,m}(R^n_+) \cap \mathfrak{D}_0^m(R^n_+)$ by Sobolev's imbedding theorem. Let u be an element of Y and consider the form:

$$L_h(u) = \langle u, A^* \overline{w} \rangle - \langle Au, \overline{w} \rangle \tag{8.16}$$

where the first pairing is between Z and $\bigcap_{j=0}^{m} \mathfrak{D}^{-j}$ $\left(\text{since } w \in H^{2m}, \text{ it is easy to see that } A^* w \in \bigcap_{j=0}^{m} \mathfrak{D}^{-j}\right)$ and the second pairing is between $\mathfrak{D}_0^m$ and $\mathfrak{D}^{-m}$.

One can verify that $L_h(u)$ does not depend on the choice of w but only on $h = (h_0, \dots, h_{m-1})$. Furthermore we have:

$$\begin{aligned} |L_h(u)| &\leq \|u\|_Z \cdot \|A^* w\|_{\bigcap_j \mathfrak{D}^{-j}} + \|Au\|_{\mathfrak{D}^{-m}} \cdot \|w\|_{\mathfrak{D}_0^m} \leq \\ &\leq C \cdot \{\|u\|_Z \cdot \|w\|_{\mathfrak{D}^{m,m} \cap \mathfrak{D}_0^m} + \|Au\|_{\mathfrak{D}^{-m}} \cdot \|w\|_{\mathfrak{D}^{m,m} \cap \mathfrak{D}_0^m}\} \\ &\leq C_1 \cdot (\|u\|_Z + \|Au\|_{\mathfrak{D}^{-m}}) \cdot \|w\|_{H^{2m}} \leq C_2 \cdot \|h\|_{\prod_j H^{j+\frac{1}{2}}}. \end{aligned} \tag{8.17}$$

Thus, we can write:

$$\langle u, A^* \overline{w} \rangle - \langle Au, \overline{w} \rangle = \sum_{j=0}^{m-1} \langle \sigma_j(u), \overline{h}_j \rangle \tag{8.18}$$

with $\sigma_j(u) \in H^{-j-\frac{1}{2}}(R^{n-1})$. Also we can verify that σ_j $(0 \leq j \leq m-1)$ is continuous on Y.

Finally, if we take $h_j \in C_c^\infty(R^{n-1})$, $0 \leq j \leq m-1$, it is known ([6], II) that w can be taken in $C_c^\infty(\overline{R^n_+})$. Also, if u is in $C_c^\infty(\overline{R^n_+})$, then taking in account (8.1) the right hand side of (8.18) can be written as

$$\sum_{j=0}^{m-1} \int_{R^{n-1}} \gamma_j(u) \cdot \overline{h}_j \, dx'.$$

It follows that for all $u \in C_c^\infty(\overline{R^n_+})$ we have $\sigma_j(u) = \gamma_j(u)$ because $C_c^\infty(R^{n-1})$ is dense in each $H^{-j-\frac{1}{2}}(R^{n-1})$. Thus $\sigma_j(u) = \gamma_j(u)$ for all $u \in Y$ because $C_c^\infty(\overline{R^n_+})$ is dense in Y. The proof is completed.

Summing up all the previous results, we can state the

THEOREM 8.3. $(A, \gamma): Y \longrightarrow \mathfrak{D}^{-m}(R^n_+) \times \prod_{j=0}^{m-1} H^{-j-\frac{1}{2}}(R^{n-1})$ *is an isomorphism onto.*

9. Interpolation.

In this section we apply Lions results on interpolation theory [5] to the two isomorphism theorems 6.1 and 8.3 and get, this way, another isomorphism theorems concerning the inhomogeneous Dirichlet problem. Next, we interpret the interpolated of the boundary value spaces; the theorem we prove contains also a result of Stein and Weiss about interpolation with change of measure.

We recall some definitions and properties of [5]. Let A and B be two Banach spaces contained in a topological vector space E. Let p and q be such that $1 < p$, $q < +\infty$ and suppose that α and β are real numbers such that $\theta = \frac{1}{p} + \alpha$ and $\theta_1 = \frac{1}{q} + \beta$ lie in the interval $(0, 1)$. Define $W(p, \alpha, A\,;\, q, \beta, B)$ as the space of vector valued functions $u(t)$ verifying the properties:

$$t^\alpha u(t) \in L^p(0, \infty\,;\, A) \tag{9.1}$$

$$t^\beta u'(t) \in L^q(0, \infty\,;\, B) \tag{9.2}$$

where $u'(t)$ denotes the derivative of $u(t)$ in the sense of distributions.

Equipped with the norm

$$\| u \|_W = \max \left\{ \left(\int_0^\infty t^{p\alpha} \| u(t) \|_A^p \, dt \right)^{\frac{1}{p}} ; \quad \left(\int_0^\infty t^{q\beta} \| u'(t) \|_B^q \, dt \right)^{\frac{1}{q}} \right\},$$

W is a Banach space. Lions has shown ([5]) that one can define the trace $u(0)$ of $u \in W$. The space of traces $T(p, \alpha, A; q, \beta, B)$ equipped with the norm

(9.4) $$\| f \|_T = \inf \{ \| u \|_w : u \in W \text{ and } u(0) = f \}$$

is a Banach space. It verifies the interpolation property, namely, if A_1 and B_1 are two other Banach space contained in a topological vector space E_1 and if Φ is a continuous linear map from A into A_1 and a continuous linear map from B into B_1 such that

$$\| \Phi a \|_{A_1} \leq C_1 \cdot \| a \|_A \quad \text{for all} \quad a \in A$$

$$\| \Phi b \|_{B_1} \leq C_2 \cdot \| b \|_B \quad \text{for all} \quad b \in B,$$

then Φ is a continuous linear map from $T(p, \alpha, A; q, \beta, B)$ into $T(p, \alpha, A_1; q, \beta, B_1)$ and its norm is $\leq C_1^{1-\gamma} \cdot C_2^{\gamma}$, where $\gamma = \dfrac{1}{m + \theta - \theta_1}$.

We apply these results to theorems 6.1 and 8.3, taking $p = q = 2$ and $\alpha = \beta$. After an obvious change of notation we can state

THEOREM 9.1. *(A, γ) is an isomorphism from $T(2, \alpha; \mathfrak{D}^m(R_+^n), Y)$ onto $\mathfrak{D}_{-m}(R_+^n) \times \prod_{j=0}^{m-1} T(2, \alpha; \mathfrak{D}^{m-j-\frac{1}{2}}(R^{n-1}), H^{-j-\frac{1}{2}}(R^{n-1}))$.*

The following theorem give us a characterization of the boundary value spaces which appear in theorem 9.1.

THEOREM 9.2. *The space $T(2, \alpha; \mathfrak{D}^{m-j-\frac{1}{2}}(R^{n-1}), H^{-j-\frac{1}{2}}(R^{n-1}))$ can be identified to the completion of $C_c^\infty(R^{n-1})$ with respect to the norm.*

(9.4) $$\int | \xi' |^{2\left(m-j-\frac{1}{2}\right)(1-\theta)} (1 + | \xi' |)^{-2\left(j+\frac{1}{2}\right)\theta} | \hat{\varphi}(\xi') |^2 \, d\xi'$$

where $\xi' = (\xi_1, \dots, \xi_{n-1})$ and $\theta = \dfrac{1}{2} + \alpha$.

PROOF. Using Fourier transform in R^{n-1}, we can see that the space $W(2, \alpha : \mathfrak{D}^{m-j-\frac{1}{2}}(R^{n-1}), H^{-j-\frac{1}{2}}(R^{n-1}))$ coincides with the space of functions $\widehat{u}(t, \xi')$ such that

$$(9.5) \qquad \int_0^\infty \int_{R^{n-1}} t^{2\alpha} |\xi'|^{2\left(m-j-\frac{1}{2}\right)} |\widehat{u}(t, \xi')|^2 \, dt \, d\xi' < +\infty$$

and

$$(9.6) \qquad \int_0^\infty \int_{R^{n-1}} t^{2\alpha} (1 + |\xi'|)^{-2\left(j+\frac{1}{2}\right)} \left| \frac{\partial \widehat{u}(t, \xi')}{\partial t} \right|^2 dt \, d\xi' < +\infty,$$

i. e., to the space $W(2, \alpha ; L^2(\mu_{m-j-\frac{1}{2}}), L^2(\nu_{-\left(j+\frac{1}{2}\right)}))$ where ν_t denotes the measure $(1 + |\xi'|)^{2t} \, d\xi'$. Theorem 9.2 will follows from the more general result

THEOREM 9.3. *Let X be a locally compact topological space, μ a Radon measure on X, $g_0(x)$ and $g_1(x)$ two functions ≥ 0, locally integrable with respect to μ and such that the set where $g_0 g_1$ is zero, has measure zero. Suppose that $1 < p$, $q < +\infty$, let $\theta = \frac{1}{p} + \alpha = \frac{1}{q} + \beta$ and let r be defined by $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$. Then:*

$$(9.7) \qquad T(p, \alpha, L^p(g_0 \mu) ; \quad q, \beta, L^q(g_1 \mu)) = L^r(\nu)),$$

where

$$\nu = g_0^{\frac{(1-\theta)r}{p}} \cdot g_1^{\frac{\theta r}{q}} \cdot \mu .$$

Furthermore,

$$(9.8) \qquad \|f\|_{L^r(\nu)} = C \cdot \|f\|_T$$

where the constant c depends only on p, α, q and β.

The proof of theorem 9.3 uses the following

LEMMA 9.1 ([5]). *Let v be a function defined a. e. in $(0, \infty)$ and such that*

$$(9.9) \qquad tv(t) \in L^p(0, \infty), \; tv'(t) \in L^q(0, \infty), \; \frac{1}{p} + \alpha = \frac{1}{q} + \beta = \theta \in (0, 1).$$

Then

$$|v(0)| \leq C \cdot \| t^{\alpha} v \|_{L^p}^{1-\theta} \cdot \| t^{\beta} v' \|_{L^q}^{\theta} . \tag{9.10}$$

where c depends only on p, α, q and β.

PROOF OF THEOREM 9.3. If $f \in T$, there exists $u \in W(p, \alpha, L^p(g_0 \mu); q, \beta, L^q(g_1 \mu))$ such that $u(x, 0) = f$. For x fixed in X, the function $t \longrightarrow u(x, t)$ verifies (9.9). Then we have:

$$|u(x,0)| \leq C \cdot \left(\int_0^{\infty} t^{p\alpha} |u(x,t)|^p \, dt \right)^{\frac{1-\theta}{p}} \cdot \left(\int_0^{\infty} t^{q\beta} \left| \frac{\partial u(x,t)}{\partial t} \right|^q dt \right)^{\frac{\theta}{q}}, \tag{9.11}$$

a. e. in x. From (9.11) we derive

$$\left(\int_X |u(x,0)|^r (g_0(x))^{\frac{(1-\theta)r}{p}} \cdot (g_1(x))^{\frac{\theta r}{q}} \, d\mu(x) \right)^{\frac{1}{r}} \leq \tag{9.12}$$

$$\leq C \cdot \left\{ \int_X \left[\left(\int_0^{\infty} t^{p\alpha} |u(x,t)|^p g_0(x) \, dt \right)^{\frac{(1-\theta)r}{p}} \cdot \left(\int_0^{\infty} t^{q\beta} \left| \frac{\partial u(x,t)}{\partial t} \right|^q g_1(x) \, dt \right)^{\frac{\theta r}{q}} \right] d\mu(x) \right\}^{\frac{1}{r}}.$$

Using Hölder's inequality we can estimate the right hand side of (9.12) by

$$\leq \left[\int_X \left(\int_0^{\infty} t^{p\alpha} |u(x,t)|^p g_0(x) \, dt \right)^{\frac{(1-\theta)r}{p}} \cdot d\mu(x) \right]^{\frac{1}{\gamma r}} \cdot$$

$$\cdot \left[\int_X \left(\int_0^{\infty} t^{q\beta} \left| \frac{\partial u(x,t)}{\partial t} \right|^q g_1(x) \, dt \right)^{\frac{\theta r}{q} \cdot \gamma'} \cdot d\mu(x) \right]^{\frac{1}{\gamma' r}}$$

where $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$. Now, choose γ such that $\frac{(1-\theta) r}{p} \gamma = 1$.

We have, $\gamma = \frac{p}{(1-\theta) r}$ and $\frac{1}{\gamma'} = \frac{r\theta}{q}$ $\left(\text{using } \frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}\right)$. We then get:

$$\left(\int_X |u(x,0)|^r (g_0(x))^{\frac{(1-\theta)r}{p}} \cdot (g_1(x))^{\frac{\theta r}{q}} \, d\mu(x) \right)^{\frac{1}{r}} \leq \tag{9.13}$$

$$\leq C \cdot \left(\int_X \int_0^\infty t^{p\alpha} \, | u(x,t) |^p \, g_0(x) \, d\mu \, dt \right)^{\frac{1-\theta}{p}} \cdot \left(\int_X \int_0^\infty t^{q\beta} \left| \frac{\partial u(x,t)}{\partial t} \right|^q g_1(x) \, d\mu \, dt \right)^{\frac{\theta}{q}} .$$

It follows that $f \in L^r(\nu)$, where $\nu = g_0^{\frac{(1-\theta)r}{p}} \cdot g_1^{\frac{\theta r}{q}} \mu$ and

$$(9.14) \qquad \| f \|_{L^r(\nu)} \leq C \cdot \left(\int_0^\infty t^p \, \| u(.,t) \|^p_{L^p(g_0 \mu)} \, dt \right)^{\frac{1-\theta}{p}} \cdot \left(\int_0^\infty t^{q\beta} \left| \frac{\partial u(.,t)}{\partial t} \right|^q_{L^q(g_1 \mu)} dt \right)^{\frac{\theta}{q}}$$

for each $u \in W$ such that $u(x, 0) = f$. According to lemma 2.1 of Lions [5] we have:

$$(9.15) \qquad \| f \|_T = \inf_{\substack{u \in W \\ u(x,0)=f}} \left(\int_0^\infty t^{p\alpha} \, \| u(.,t) \|^p_{L^p(g_0 \mu)} \, dt \right)^{\frac{1-\theta}{p}} \cdot$$

$$\cdot \left(\int_0^\infty t^{q\beta} \left\| \frac{\partial u(.,t)}{\partial t} \right\|^q_{L^q(g_1 \mu)} dt \right)^{\frac{\theta}{q}} .$$

Hence (9.14) and (9.16) yield

$$(9.16) \qquad \| f \|_{L^r(\nu)} \leq c \cdot \| f \|_T .$$

Conversely, suppose $f \in L^r(\nu)$, assume from now on that the constant in (9.10) is the best possible. Given $\varepsilon > 0$, let (v, t) be a function verifying the conditions of lemma 9.1 and such that

$$(9.17) \qquad | v(0) | \geq (c - \varepsilon) \, \| t^\alpha v \|^{1-\theta}_{L^p} \cdot \| t^\beta v' \|^\theta_{L^p} .$$

Furthermore, we can assume that $v(0) = 1$. Define (a. e. in X):

$$(9.18) \qquad u(x,t) = f(x) \cdot v \left(| f(x) |^\lambda \left(\frac{g_0(x)}{g_1(x)} \right)^{\frac{r}{pq}} t \right)$$

where λ is a real number to be fixed latter. Clearly, $u(x, 0) = f(x)$ a. e. We want to prove that $u \in W(p, \alpha, L^p(g_0 \mu);\, q, \beta, L^q(g, \mu))$ what amounts to prove that

$$(9.19) \qquad t^\alpha u(.,t) \in L^p(0, \infty;\ L^p(g_0 \mu)) ,$$

and

$$t^{\beta} \frac{\partial u(., t)}{\partial t} \in L^q(0, \infty ; L^q(g_1 \mu)) . \tag{9.20}$$

Consider the integral

$$I_1 = \int_0^\infty \int_X t^{p\alpha} | u(x, t) |^p g_0(x) \, d\mu \, dt . \tag{9.21}$$

Taking in account (9.18) we can see that

$$I_1 = \| t^{\alpha} v(t) \|_{L^p}^p \cdot \int_X | f(x) |^{p - \lambda p \theta} (g_0(x))^{1 - \frac{r\theta}{q}} \cdot g_1(x))^{\frac{r\theta}{q}} \, d\mu . \tag{92.2}$$

Now, if we choose such that $p - \lambda p \theta = r$, then the integral on the right hand side of (9.22) is finite because $f \in L^r(\nu)$ $\left(\text{observe that } 1 - \frac{r\theta}{q} = \frac{(1-\theta}{p}\right)$.

The real number λ being fixed, consider the integral

$$I_2 = \int_0^\infty \int_X t_{q\beta} \left| \frac{\partial u(x, t)}{\partial t} \right|^q g_1(x) \, d\mu \, dt . \tag{9.23}$$

A simple computation shows that

$$I_2 = \| t^{\beta} v' \|_{L^q}^q \cdot \int_X | f(x) |^{q(\lambda + 1 - \theta)} (g_0(x))^{\frac{(1-\theta)r}{p}} \cdot (g_1(x))^{1 - \frac{(1-\theta)r}{p}} \, d\mu . \tag{9.24}$$

Since $1 - \frac{(1-\theta)r}{p} = \frac{r\theta}{q}$, to prove that I_2 is finite we only have to check that $q(\lambda + 1) - \lambda q \theta = r$. This follows, easily, from our relations $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$ and $p - \lambda p \theta = r$. Thus, $f \in T(p, \alpha, L^p(g_0 \mu) ; q, \beta, L^q(g_1 \mu))$.

Finally, let us prove (9.8). From (9.15) it follows using (9.22) and (9.24):

$$\| f \|_T \leq I_1^{\frac{1-\theta}{p}} \cdot I_2^{\frac{\theta}{p}} = \| t^{\alpha} v \|_{L^p}^p \cdot \| t^{\beta} v' \|_{L^q}^q \cdot \| f \|_{L^r(\nu)} .$$

From (9.17) and $v(0) = 1$ we get:

$$\| f \|_T \leq (c - \varepsilon)^{-1} \| f \|_{L^r(\nu)} , \tag{9.25}$$

hence, making $\varepsilon \longrightarrow 0$:

$$\|f\|_T \leq C^{-1} \|f\|_{L^r(\nu)} . \tag{9.26}$$

Relation (9.8) follows from (9.16) and (9.26), q.e.d.

REMARKS. 1). Theorem. 9.3 contains proposition 4.2 of [5].

2). In [11] Stein and Weiss proved, using complex interpolation that $L^r(\nu)$ is an interpolation space between $L^p(g_0 \mu)$ and $L^q(g_1 \mu)$. This result is contained in theorem 9.3 where, in addition, we proved that $L^r(\nu)$ is a trace space in the sense of Lions ([5]).

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