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TRIANGULATION OF SEMI-ANALYTIC SETS

by S. Lojasiewicz (Kraków)

The triangulability question for algebraic sets was first considered by van de Waerden [15] in 1929, and for analytic sets by Lefschetz [5], Koopman and Brown [4], and Lefschetz and Whitehead [6], in 1930-1933. In fact, [5] and [6] deal with triangulation of « analytical complex » (which is a finite disjoint collection with compact union of subsets of $\mathbb{R}^n$ each being an open and relatively compact subset of an analytic subset of an open set in $\mathbb{R}^n$). A lack of a convenient technique at that time was probably the reason for the proofs being rather sketched. Therefore, according to an opinion of many mathematiciens, it is of some interest to give a new detailed proof. This is just the purpose of the present paper, and actually we give a somewhat more general result: simultaneous triangulation of any locally finite collection of semi-analytic subsets of $\mathbb{R}^n$ or, owing to the Grauert imbedding theorem [3], of any countable real analytic manifold. Moreover, our formulation of the result is somewhat more precise; it follows e.g. that in any exemple of Milnor's type [9] the homeomorphism between the polyhedra has to have a singularity of non-algebraic type (it can not have the property (A) (see § 3)). However the frame of idea of the construction we give is that of Lefschetz ([5] and [6]).

The semi-analytic (resp. semi-algebraic) sets are those which can be locally given by analytic (resp. algebraic) inequalities (see [13], [14] and [8]). We may observe that the body of an « analytic complex » is exactly the same as a compact semi-analytic set. We will need certain basic properties of semi-analytic sets; these are only stated in § 1, and will be contained with proofs in papers to appear separately.

The method we use is that of normal decompositions of semi-analytic sets; it follows an old idea of Osgood [10] (see also [11]), and was applied

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by the author in [7] (1). The normal decompositions are certain special local stratifications in the sens of Thom [14]. For the semi algebraic case an elegant Whitney's stratification [17] can be also used.

The result was communicated on the Congress in Stockholm, 1962, and the construction was presented in details on a seminar at the Istituto Matematico Leonida Tonelli of the University in Pisa, spring, 1963.

Afther having written the manuscript the autor observed that had just appeared a thesis of B. Giesecke [2] concerning also the triangulation of semi-analytic sets; because of the differences which seem to exist in results and methods used, our article may be however of independent interest.

One should also mention a paper of K. Sato [19] on local triangulation of real analytic varieties.

§ 1. Preliminaries on semi-analytic sets.

In this § the results are only stated; the proofs will be contained in articles to be published soon.

I. *Definition of semi analytic set.* Let M be a real analytic manifold.

Let A, $G \subset M$ and let $f_1, \dots, f_r$ be real functions (each one defined on a subset of M); we say that A is described in G by $f_1, \dots, f_r$ iff $f_1, \dots, f_r$ are defined in G and $A \cap G$ is a finite union of finite intersections of sets of the form $\{x \in G : f_j(x) > 0\}$ or $\{x \in G : f_j(x) = 0\}$; we say that A is described at $c \in M$ by $f_1, \dots, f_r$ iff it is described by these functions in a neighborhood of c.

A subset A of M is said to be *semi analytic* iff it can be described at any $c \in M$ by a set (depending on c) of real analytic functions (at c). Equivalently, A is semi-analytic iff for each $x \in M$ the germ of A at x belongs to the smallest class S of germs at x (of subsets of M) satisfying $u, v \in S \Longrightarrow u \cup v,\ u \setminus v \in S$, and containing the germ at x of each set of the form $\{f > 0\}$ with f real analytic in a neighborhood of x.

The union of any locally finite family and the intersection of any finite family of semi-analytic sets is semi-analytic. The complement of any semi-analytic set is semi-analytic. A subset of any closed submanifold (2) M_1 of M is semi-analytic in M_1 if and only if it is semi analytic in M_1. The image of any semi-analytic set by an analytic isomorphisme is semi-analytic. A finite product of semi-analytic sets is semi-analytic.

(1) See also [8], where another application of this method is given.

(2) A submanifold of M is a subset of M which is locally the inverse image by a chart of a linear subvariety (of the same dimension for any point of this set), with the induced manifold structure.

II. *Normal decompositions.* A function $H(z_1, \dots, z_p; z)$ holomorphic at $(0, \dots, 0; 0)$ is called a distinguished polynomial in z iff it is a polynomial in z with coefficients vanishing at $(0, \dots, 0)$ except the leading one which $\equiv 1$; it is said to be real iff its values for real arguments are real.

Let M be a real analytic manifold of dimension n. Let $c \in M$. A *normal system at c* is a couple of an analytic chart $g: U \to g(U) \subset \mathbb{R}^n$ at c (3) such that $g(c) = 0$, and a system $\{H_l^k\}_{0 \le k < l \le n}$, where $H_l^k(z_1, \dots, z_k; z_l)$ is a real distinguished polynomial in z_l with discriminant $D_l^k(z_1, \dots, z_k) \not\equiv 0$, $(0 \le k < l \le n)$, such that in some neighborhood of $(0, \dots, 0)$

$$(\alpha) \qquad H_k^{k-1}(z_1, \dots, z_{k-1}; z_k) = H_l^k(z_1, \dots, z_k; z_l) = 0 \Longrightarrow H_l^{k-1}(z_1, \dots, z_{k-1}; z_l) = 0,$$

$$(\beta) \qquad D_l^k(z_1, \dots, z_k) = 0 \Longrightarrow H_k^{k-1}(z_1, \dots, z_{k-1}; z_k) = 0,$$

for $1 \le k < l \le n$.

A neighborhood $Q = g^{-1}(Q_0)$ of c, where $Q_0 = \{x: |x_i| < \delta_i\} \subset g(U)$, is said to be *normal* (with respect to the above normal system) iff H_l^k are holomorphic on $\{(z_1, \dots, z_n) \in \mathbb{C}^n: |z_i| \le \delta_i\}$, satisfy (α) and (β) in $\{(z_1, \dots, z_n) \in \mathbb{C}^n: |z_i| < \delta_i\}$, and

$$|z_i| < \delta_i, \; i = 1, \dots, k, \quad H_l^k(z_1, \dots, z_k; z_l) = 0 \Longrightarrow |z_l| < \delta_l$$

for each k, l with $0 \le k < l \le n$. Thus Q_0 is a normal neighborhood of 0 following the normal system $(e, \{H_l^k\})$, $e: \mathbb{R}^n \to \mathbb{R}^n$ being the identity map.

Every neighborhood of c contains a normal one.

The *normal decomposition* of Q (following the above normal system) is the decomposition

$$Q = \bigcup_{k=1}^{n} \bigcup_{\varkappa} \Gamma_\varkappa^k$$

where $\Gamma_\varkappa^k$ are the connected components of

$$V^k = g^{-1}(\{x \in Q_0: H_n^{n-1} = \dots = H_{k+1}^k = 0, \quad H_k^{k-1} \neq 0\}), \quad k = 0, \dots, n$$

(we put for convenience $H_0^{-1} = 1$). The sets $\Gamma_\varkappa^k$ are called *members* of the decomposition. Thus $\Gamma_\varkappa^k = g^{-1}(\Gamma_{0\varkappa}^k)$, where $\Gamma_{0\varkappa}^k$ are the members of the normal decomposition of Q_0 following the normal system $(e, \{H_l^k\})$ at 0.

A *normal decomposition at c* is the normal decomposition of a normal neighborhood following a normal system at c.

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(3) i. e. local analytic coordinates system at c.

1. Any normal decomposition is finite.

2. Let $Q = \bigcup_{k,\varkappa} \Gamma_\varkappa^k$ be a normal decomposition. Any $(\overline{\Gamma_\varkappa^k} \setminus \Gamma_\varkappa^k) \cap Q$ is a union of some Γ_ι^j with $j < k$; hence $V^k \cup \ldots \cup V^0$ are closed in Q, $k = 0, \ldots, n$.

3. Let $Q = \bigcup_{k,\varkappa} \Gamma_\varkappa^k$ be a normal decomposition at $0 \in \mathbb{R}^n$ following a normal system $(e, \{H_l^k\})$. Then any $\Gamma = \Gamma_\varkappa^k$ with $0 < k < n$ is of the form:

$$\Gamma = \{x : u = (x_1, \ldots, x_k) \in \Omega \quad \text{and} \quad x_j = \eta_j(u) \quad \text{for} \quad j = k+1, \ldots, n\}$$

with Ω open (in $\mathbb{R}^k$) and η_j analytic in Ω such that $0 \in \overline{\Omega}$, $H_j^k(u, \eta_j(u)) = 0$ in Ω and $\lim_{u \to 0} \eta_j(u) = 0$, $j = k+1, \ldots, n$.

4. Any member $\Gamma_\varkappa^k$ of any normal decomposition at c is a k-dimensional analytic submanifold (Γ_ν^n are open and $\Gamma_\iota^0 = \{0\}$) such that $c \in \overline{\Gamma_\varkappa^k}$.

5. Let $Q = \bigcup_{k,\varkappa} \Gamma_\varkappa^k$ be a normal decomposition as in 3. Let $0 < m < n$. Denote by π the projection $\mathbb{R}^n \ni (x_1, \ldots, x_n) \longrightarrow (x_1, \ldots, x_m) \in \mathbb{R}^m$ and by e_* the identity map $\mathbb{R}^m \longrightarrow \mathbb{R}^m$. The couple $(e_*, \{H_l^k\}_{0 \le k < l \le m})$ is a normal system at $0 \in \mathbb{R}^m$ and $Q_* = \pi(Q)$ is a normal neighborhood; let $Q_* = \bigcup_{j,\iota} \Gamma_{*\iota}^j$ be its normal decomposition. Then for any $\Gamma_\varkappa^k$ with $k \le m$ we have $\pi(\Gamma_\varkappa^k) = \Gamma_{*\varkappa'}^k$ (for some $\varkappa'$).

III. *Existence theorems.* A normal decomposition $Q = \bigcup_{k,\varkappa} \Gamma_\varkappa^k$ is said to be *compatible with a function* f defined in Q iff $f = 0$ or $f \neq 0$ on any member $\Gamma_\varkappa^k$; it is said to be *compatible with a subset* A of M iff any member is contained in A or in $M \setminus A$. If A is described at c by $f_1, \ldots, f_r$, then any normal decomposition of a sufficiently small neighborhood at c which is compatible with $f_1, \ldots, f_r$ is also compatible with A.

1. Let $c \in M$. There is a normal decomposition at c which is compatible with given functions $f_1, \ldots, f_r$ analytic at c, resp. with given semi-analytic sets $A_1, \ldots, A_s \subset M$; the normal neighborhood can be chosen arbitrarily small.

Let M be an affine space, let f be an analytic function at $c \in M$. A line λ though c is said to be *singular for f at c*, iff f vanish identically in a neighborhood of c in λ. Let A be a semi-analytic subset of M. A line λ through c is said to be *non singular for A at c*, iff A can be described at c by a set of functions for which λ is non-singular at c.

2. Let $f_1, \ldots, f_r, F$ be analytic functions at $c \in M$, such that $f_j = 0 \Longrightarrow F = 0$ in a neighborhood of c $(j = 1, \ldots, r)$, let λ be a non-singular line for

$f_1, \dots, f_r, F$ at c, and let χ be a hyperplane such that $\lambda \cap \chi = \{c\}$. There is a normal decomposition $Q = \bigcup_{k, \varkappa} \Gamma_\varkappa^k$ following a normal system $(g, \{H_l^k\})$ at c which is compatible with $f_1, \dots, f_r$ and such that g is affine, $g(\lambda)$ is the x_n-axis (i.e. $g_j = 0$ on λ for $j = 1, \dots, n-1$), $g(\chi)$ is the $(x_1, \dots, x_{n-1})$-hyperplane (i.e. $g_n = 0$ on χ), and $V^{n-1} = \{x \in Q : F(x) = 0\}$ (i.e. $H_n^{n-1} = 0 \Longleftrightarrow F = 0$ in Q). The normal neighborhood Q can be chosen arbitrarily small.

3. Let $A_1, \dots, A_s$ be semi-analytic subsets of M, let λ be a non-singular line for $A_1, \dots, A_s$ at $c \in M$ and let χ be a hyperplane such that $\lambda \cap \chi = \{c\}$. There is a normal decomposition following a normal system $(g, \{H_l^k\})$ at c which is compatible with $A_1, \dots, A_s$ and such that g is affine, $g(\lambda)$ is the x_n-axis and $g(\chi)$ is the $(x_1, \dots, x_{n-1})$-hyperplane. The normal neighborhood can be chosen arbitrarily small.

VI. *Some properties.* Let M be a real analytic manifold.

1. A subset A of M is semi-analytic if and only if for any $c \in M$ there is a normal decomposition at c which is compatible with A.

2. Any connected component of a semi analytic set is semi-analytic.

3. Any member of a normal decomposition is semi-analytic.

4. The closure, the interior and the boundary of any semi-analytic set are semi-analytic.

V. *Projection theorem.* Let M be a real analytic manifold, Λ an affine space. A subset A of $M \times \Lambda$ is said to be partially semi-algebraic (with respect to Λ) iff any $c \in M$ has a neighborhood U such that A can be described in $U \times \Lambda$ by a set of analytic functions $f_j(u, x)$ which are polynomials in x; thus any partially semi-algebraic set is semi-analytic. The intersection of any finite family of partially semi-algebraic sets is partially semi-algebraic. A union of a family of partially semi-algebraic sets is partially semi-algebraic, provided that each point of M has a neighborhood U such that $U \times \Lambda$ meet only finitely many sets of this family.

1. Any connected component of a partially semy-algebraic set is partially semi-algebraic.

2. SEIDENBERG THEOREM [4]. Let Λ_0 be another affine space and denote by π the projection $M \times \Lambda \times \Lambda_0 \ni (u, x, y) \rightarrow (u, x) \in M \times \Lambda$. If a subset A

(4) Formulated in [13] for semi-algebraic sets.

of $M \times \Lambda \times \Lambda_0$ is partially semi-algebraic with respect to $\Lambda \times \Lambda_0$, then $\pi(A)$ is partially semi-algebraic with respect to Λ.

VI. *Semi-algebraic sets.* A subset A of an affine space M is said to be *semi-algebraic* iff it can be described in M by a set of polynomials; it is said to be *locally semi-algebraic* iff it can be described at any $c \in M$ by a set (depending on c) of polynomials; each bounded locally semi-algebraic set is semi-algebraic. Let Ω be an open subset of M; an analytic function $f: \Omega \to \mathbb{R}$ is said to be analytic algebraic iff there is a polynomial $P(t, x) \not\equiv 0$ such that $P(x, f(x)) = 0$ in Ω. A subset A of M is locally semi-algebraic if and only if it can be described at any $c \in M$ by a set (depending on c) of analytic algebraic functions (at c).

All the facts stated in this § remain valid for M affine (and for affine charts g) if we replace the notions of semi analytic set and analytic function by those of locally semi-algebraic set and analytic-algebraic function.

Let P be the projective space derived from a finite dimensional vector space V (identified with the set of all lines through 0 in V); the canonical map (of P) is the map $\pi: V \setminus \{0\} \ni x \to \mathbb{R}x \in P$. For any projective hyperplane $P' \subset P$ (derived from a hyperplane V' of V) the set $P \setminus P'$ with its natural affine structure (such that for any $v \in V \setminus V'$ the restriction $\pi_{v+V}: v + V \to P \setminus P'$ is an affine isomorphism) is called an affine chart of P. Any affine space can be considered as an affine chart of a projected space.

Consider a multiprojective space i.e. a finite product of finite dimensional projective vector spaces $R = P_1 \times \ldots \times P_k$. The canonical map (of R) is the map $\pi = \pi_1 \times \ldots \times \pi_k: (V_1 \setminus \{0\}) \times \ldots \times (V_k \setminus \{0\}) \to R$, where π_i are the canonical maps of P_i, and V_i are the vector spaces from which P_i are derived; an affine chart of R is a product $\Lambda_1 \times \ldots \times \Lambda_k$ where Λ_i is an affine chart of P_i, $i = 1, \ldots, k$. A subset A of R is said to be semi-algebraic iff $\pi^{-1}(A)$ is semi algebraic. A subset of an affine chart Λ of R is semi algebraic in R iff it is semi-algebraic in Λ. A subset of R is semi-algebraic iff it can be described at any $c \in R$ by a set (depending on c) of polynomials in an affine chart of R. The union and the intersection of any finite family of semi-algebraic sets and the complement of any semi-algebraic set are semi-algebraic; a finite product of semi-algebraic sets is semi-algebraic.

Let R_1, R_2 be multiprojective spaces: a map of a subset of R_1 into R_2 is said to be semi-algebraic iff its graph is semi-algebraic. The composition of semi-algebraic maps is semi-algebraic. The image and the inverse image of any semi-algebraic set by any semi-algebraic map is semi-algebraic.

§ 2. Some lemmas.

For any C^1-manifold Λ denote by Λ_u its tangent space at $u \in \Lambda$; for any C^1-map $g : \Lambda \to \Lambda'$, where Λ' is another C^1-manifold, let $dg_u : \Lambda_u \to \Lambda'_{g(u)}$ be its differential at u; put $\operatorname{rank}_u g = \dim dg_u(\Lambda_u)$ and $\operatorname{rang} g = \sup\{\operatorname{rank}_u g : u \in \Lambda\}$. If $\operatorname{rank}_u g = \dim \Lambda'$, then $g(u)$ is an interior point of $g(\Lambda)$. Hence if $g(\Lambda)$ has no interior points, then $\operatorname{rank} g < \dim \Lambda'$.

LEMMA 1 (*Sard* (5)). Let Λ, Λ' be real analytic countable (6) manifolds and let $g : \Lambda \to \Lambda'$ be analytic. If $\operatorname{rank} g < \dim \Lambda'$, then $g(\Lambda)$ is meager (7).

PROOF. The lemma being trivial when $\dim \Lambda = 0$, assume it true if $\dim \Lambda < n$ and let $\dim \Lambda = n > 0$. Let $u \in \Lambda$; for some neighborhood U of u the set $Z = \{u \in U;\ \operatorname{rank}_u g < \operatorname{rank} g_U\}$ is semi-analytic and nowhere dense in U; let $Q = \bigcup_{k,\varkappa} \Gamma_\varkappa^k$ be a normal decomposition at u which is compatible with Z. It is sufficient to prove that each $g(\Gamma_\varkappa^k)$ is meager. If $k < n$, it is true by the induction hypothesis. Consider any Γ_ν^n. Since $\Gamma_\nu^n \cap Z = \varnothing$, $\operatorname{rank}_u g = p$ in Γ_ν^n where $p = \operatorname{rank} g_U < \dim \Lambda'$. Therefore any point of Γ_ν^n has a neighborhood whose image by g is contained in a p-dimensional submanifold of Λ' and hence $g(\Gamma_\nu^n)$ is meager.

LEMMA 2 (*Lefschetz-Whitehead*) (8). Let Λ be a C^1-manifold. Let M be an m-dimensional affine space, V its vector space, and S an $m-1$-dimensional C^1-submanifold of V such that $u \in S \Longrightarrow u \notin S_u$ (9). Let $g : \Lambda \to M$ and $h : \Lambda \to S$ be C^1-maps. If the map $\varphi : \Lambda \times \mathbb{R} \ni (u, \lambda) \to g(u) + \lambda h(u) \in M$ is of $\operatorname{rank} \leq m-1$, then $\operatorname{rank} h \leq m-2$.

PROOF. Let $u \in \Lambda$. By assumptions

$$d\varphi_{u,\lambda} :\ \Lambda_u \times \mathbb{R} \ni (v, \xi) \to dg_u(v) + \lambda\, dh_u(v) + \xi\, h(u) \in V$$

is of $\operatorname{rank} \leq m-1$ for any $\lambda \in \mathbb{R}$. When $\lambda \neq 0$, it follows the same for the map $(v, \xi) \to \frac{1}{\lambda} dg_u(v) + dh_u(v) + \xi\, h(u)$, and hence (letting $\lambda \to \infty$) for the

(5) This is a special case of Sard theorem [12].

(6) i. e. with countable basis.

(7) i. e. a counable union of nowhere dense sets.

(8) See [6], pp. 513-514.

(9) Any tangent space of a submanifold of M or V is identified with a vector subspace of V.

map $(v, \xi) \longrightarrow dh_u(v) + \xi h(u)$, i. e. $\dim (dh_u(\Lambda_u) + \mathbb{R} h(u)) \leq m - 1$. But since $dh_u(\Lambda_u) \subset S_{h(u)}$, we have $h(u) \notin dh_u(\Lambda_u)$, whence $\dim dh_u(\Lambda_u) \leq m - 2$.

REMARK. It is sufficient to assume that φ is of rank $\leq m - 1$ in an open set containing $\Lambda \times \{0\}$. For, u being fixed, the condition: rank $d\varphi_{u,\lambda} \leq \leq m - 1$ can be expressed by vanishing of some determinants which are polynomials in λ.

Let M be an affine space, V its vector space; denote by P the projective space derived from V (the set of directions in M).

Let Ω be an open subset of M and let $f\colon \Omega \longrightarrow \mathbb{R}$ be analytic. A direction $\sigma \in P$ is said to be *non-singular for* f iff for each $c \in \Omega$ the line $c + \sigma$ is non-singular for f at c.

LEMMA 3 (*Koopman and Brown* [10]). Let f be analytic and $\not\equiv 0$ in an open connected $\Omega \subset M$. Then the set of all singular directions (for f) is meager in P.

PROOF. Put $m = \dim M$. The ellipsoid $S = \{u \in V : |\psi(u)| = 1\}$, where $\psi\colon V \longrightarrow \mathbb{R}^m$ is a (linear) isomorphism, satisfies the assumption of the lemma 2. Let

$$\theta = \{(u, v) \in \Omega \times S : f(u + \xi v) \equiv 0 \quad \text{in a neighborhood of } \xi = 0\}$$

and let $\pi\colon \Omega \times V \ni (u, v) \longrightarrow v \in V$. Since the set in question is the image of $\pi(\theta)$ by the local homeomorphism $S \ni v \longrightarrow \mathbb{R} v \in P$, it is sufficient to prove that $\pi(\theta)$ is meager in S. We have

$$\theta = \left\{(u, v) \in \Omega \times V : |\psi(v)|^2 - 1 = 0,\ \frac{d^i}{d\xi^i} f(u + \xi v)_{\xi=0} = 0,\ i = 1, 2, \ldots\right\},$$

which implies that θ is an analytic subset of $\Omega \times V$ (for the ring of germs of real analytic functions at a point is noetherian). Let $c \in \Omega \times V$ and let $Q = \bigcup_{k,\varkappa} \Lambda^k_\varkappa$ be a normal decomposition at c which is compatible with θ; thus $Q \cap \theta$ is a union of some $\Lambda^k_\varkappa$ and it is sufficient to prove that $\pi(\Lambda)$ is meager in S for any $\Lambda = \Lambda^k_\varkappa \subset \theta$. Consider two analytic maps $g\colon \Lambda \ni (u, v) \longrightarrow u \in M$ and $h = \pi_\Lambda\colon \Lambda \ni (u, v) \longrightarrow v \in V$, and then the map $\varphi\colon \Lambda \times \mathbb{R} \ni (x, \lambda) \longrightarrow g(x) + \lambda h(x) \in M$. Since $\Lambda \subset \theta$, $h(\Lambda) \subset \pi(\theta) \subset S$. For any $x \in \Lambda$, $(g(x), h(x)) = x \in \theta$ and hence $f(g(x) + \lambda h(x)) \equiv 0$ in a neighborhood

[10] See [4], p. 242.

of $\lambda = 0$. Therefore $f(\varphi(x, \lambda)) = 0$ in the set

$$\Lambda^* = \{(x, \lambda) \in \Lambda \times \mathbb{R} : g(x) + \vartheta\lambda\, h(x) \in \Omega \quad \text{for} \quad 0 \leq \vartheta \leq 1\}$$

which is open in $\Lambda \times \mathbb{R}$ and contains $\Lambda \times \{0\}$. This implies that $\varphi(\Lambda^*)$ has no interior points (in M), whence the rank of φ in Λ^* is $\leq m-1$. By the lemma 2 and the remark, rank $h \leq m-2$ and hence, by the lemma 1, $\pi(\Lambda) = h(\Lambda)$ is meager in S, Q. E. D.

Let A be a semi-analytic subset of M. A direction $\sigma \in P$ is said to be *non-singular for A* iff for each $c \in M$ the line $c + \sigma$ is non-singular for A at c.

LEMMA 4. Set $\{B_\nu\}$ be a coutable collection of semi-analytic sets. The set of all directions which are simultaneously non-singular for each B_ν is dense in P.

In fact, consider any B_ν; there is a countable covering of M by open connected W_i such that B_ν can be described in W_i by a finite set of functions $\{f_{ij}\}$ analytic and $\not\equiv 0$ in W_i; since every direction which is simultaneously non-singular for all f_{ij} is also non-singular for B_ν, it follows from the lemma 3 that the set of all singular directions for B_ν is meager.

Consider now the affine space $M \times \mathbb{R}$ and put $n = \dim(M \times \mathbb{R})$, (i. e. $\dim M = n-1$).

Using the Weierstrass preparation theorem we derive easily the following lemma.

LEMMA 5. Any semi-analytic bounded subset of $M \times \mathbb{R}$ for which the direction $\{0\} \times \mathbb{R}$ (where 0 is the zero of V) is non-singular is partially semi algebraic (with respect to $\mathbb{R}$).

Denote by π the map $M \times \mathbb{R} \ni (u, t) \rightarrow u \in M$. An analytic submanifold $\psi \subset M \times \mathbb{R}$ is said to be *topographic* [11] iff $\pi(\psi)$ is an analytic submanifold of M and $\pi_\psi : \psi \rightarrow \pi(\psi)$ is an analytic isomorphism: then ψ is the graph of an analytic function $\pi(\psi) \rightarrow \mathbb{R}$ which we identify with ψ. The following lemma is trivial.

LEMMA 6. Let ψ be an analytic submanifold of $M \times \mathbb{R}$. Introduce a euclidean norm in M and let $A > 0$. If

$$(u, t),\ (u', t') \in \psi \Longrightarrow |\, t - t' \,| \leq A \,|\, u - u' \,|\,,$$

then ψ is topographic (and $|\, \psi(u') - \psi(u) \,| \leq A \,|\, u' - u \,|$ in $\pi(\psi)$). If

[11] This convenient terminology was proposed by A. ANDREOTTI.

$z \in V$ and $|z| < A$, then the image of ψ by the map

$$M \times \mathbb{R} \ni (u, t) \longrightarrow (u + tz, t) \in M \times \mathbb{R}$$

is also topographic.

We say that a subset Z of $M \times \mathbb{R}$ has property (P) iff the map $\pi_Z : Z \longrightarrow M$ is open. Any topographic submanifold (of $M \times \mathbb{R}$) of dimension $n - 1$ has property (P).

Lemma 7. For any function f analytic at $(c, 0) \in M \times \mathbb{R}$ and such that $f(c, t) \not\equiv 0$ there exist a neigborhood U and a function g such that f, g are analytic in U, $g(c, t) \not\equiv 0$ and the set $\{x \in U : f(x) g(x) = 0\}$ has property (P).

Proof. Let $H(u, t)$ be a distinguished polynomial in t at $(c, 0)$ ([12]) such that $H = 0 \Longleftrightarrow f = 0$ in a neighborhood of $(c, 0)$, its discriminant $D(u) \not\equiv 0$ ([13]). Let λ be a non-singular line for D at c; we can identify M with $M_1 \times \mathbb{R}$ (where M_1 is an affine space of dimension $m - 2$) in such a way that $c = (c_1, 0)$ and $\lambda = \{c_1\} \times \mathbb{R}$ for some $c_1 \in M_1$; thus $D(c_1, s) \not\equiv 0$. Let $H_1(v, s)$ be a distinguished polynomial in s at c such that $H_1 = 0 \Longleftrightarrow D = 0$ in a neighborhood of c. Consider the analytic function $F(v, t)$ defined in a neighborhood of $(c_1, 0)$ by the formula

$$F(v, t) = H(v, \xi_1, t) \dots H(v, \xi_q, t)$$

where ξ_j are the roots of H_1 at v ([14]); we have then $F(c_1, t) \not\equiv 0$ and $D(v, s) = H(v, s, t) = 0 \Longrightarrow F(v, t) = 0$ in a neighborhood of $(c_1, 0, 0)$. Assume $n = 2$, or $n > 2$ and the lemma true for $n - 1$. There exist an open neighborhood U_1 of $(c_1, 0)$ and a function g such that F, g are analytic in U_1, $g(c_1, t) \not\equiv 0$, $F = 0 \Longrightarrow g = 0$ in U_1, and the set $\{(v, t) \in U_1 : g(v, t) = 0\}$ has property (P), (in $M_1 \times \mathbb{R}$). In fact, if $n = 2$ we put $g = F$; in the second case we take U_1 and G for F according to the lemma (assumed true for $n - 1$) and we put $g = FG$. Now we choose an open neighborhood U of $(c_1, 0, 0)$ so that all the above relations hold and $(v, t) \in U_1$, if $(v, s, t) \in U$.

([12]) i. e. analytic at $(c, 0)$, polynomial in t with coefficients vanishing at c except the leading one which $\equiv 1$.

([13]) For the existence see e. g. [7], n⁰ 11.

([14]) See e. g. [7], n⁰ 13; we put $F \equiv 1$ when H_1 is of degree 0.

Then we have

$$\{(v, s, t) \in U: f(v, s, t)\, g(v, t) = 0\} = \{(v, s, t) \in U: H(v, s, t) = 0,\ D(v, s) \neq 0\} \cup$$
$$\cup \{(v, s, t) \in U: g(v, t) = 0\}.$$

Since both sets on the right side has property (P) (the first one is locally a topographic submanifold of dimension $n-1$), so has the set on the left. Therefore the lemma is proved by induction.

By lemma 7 and § 1, III, 2 we obtain

LEMMA 8. Let $B_1, \ldots, B_r$ be semi-analytic subsets of $M \times \mathbb{R}$ for which the line $\{c\} \times \mathbb{R}$ is non-singular at $(c, \gamma) \in M \times \mathbb{R}$. There exists a normal decomposition $Q = \bigcup_{k, \varkappa} \Gamma_\varkappa^k$ following a normal system $(g, \{H_l^k\})$ at (c, γ) which is compatible with $B_1, \ldots, B_r$ and such that g is affine, $g(\{c\} \times \mathbb{R})$ is the x_n-axis, $g(M \times \{\gamma\})$ is the $(x_1, \ldots, x_{n-1})$-hyperplane and $V^{n-1} \cup \ldots \cup V^0 = \bigcup_{k<n} \Gamma_\varkappa^k$ has property (P). The normal neighborhood can be chosen arbitrarily small.

From § 1, II, 3, 5 and V, 1 we get:

LEMMA 9. Let $Q = \bigcup_{k, \varkappa} \Gamma_\varkappa^k$ be a normal decomposition following a normal system $(g, \{H_l^k\})$ at (c, γ) with g affine and such that $g(\{c\} \times \mathbb{R})$ is the x_n-axis and $g(M \times \{\gamma\})$ is the $(x_1, \ldots, x_{n-1})$-hyperplane. Then all $\Gamma_\varkappa^k$ are partially semi-algebraic and those with $k < n$ are topographic. There is a normal decomposition $Q_* = \pi(Q) = \bigcup_{j, \iota} \Gamma_{*\iota}^j$ such that for any $\Gamma_\varkappa^k$ with $k < n$ we have $\pi(\Gamma_\varkappa^k) = \Gamma_{*\varkappa'}^k$ for some $\varkappa'$.

We call $Q_n = \bigcup_{j, \iota} \Gamma_{*\iota}^j$ the projected decomposition (of the previous one).

LEMMA 10. Let φ_ν be defined and analytic in (open) neighborhoods of $c \in M$, $\nu = \mp 1, \mp 2, \ldots$; assume that the sequence $\varphi_\nu(c)$ is strictly increasing, $\lim_{\nu \to \infty} \varphi_\nu(c) = \infty$, $\lim_{\nu \to -\infty} \varphi_\nu(c) = -\infty$, and that $\{\varphi_\nu\}$ is locally finite (as a family of graphs). Let G be a subset of M whose interior contains $\{c\} \times \mathbb{R}$. There exist restrictions φ_ν^* of φ_ν to open connected neighborhoods of c and an analytic isomorphism $g: M \times \mathbb{R} \to M \times \mathbb{R}$ such that:

1^0. $g((M \times \mathbb{R}) \setminus G) \subset M \times (-1, 1)$.

2^0. $\psi_\nu = g(\varphi_\nu^*)$ are disjoint, bounded, semi-analytic, topographic.

3^0. Any bounded subset of M is contained in some $\bigcap \{\Omega_\nu : |\nu| \geq N\}$, where $\Omega_\nu = \pi(\psi_\nu)$.

4^0. $\lim_{\nu \to \infty} \psi_\nu(u) = \infty$, $\lim_{\nu \to -\infty} \psi_\nu(u) = -\infty$ uniformly in any bounded subset of M.

5^0. $|\psi_\nu(u') - \psi_\nu(u)| \leq A\, |u' - u|$ if $u, u' \in \Omega_\nu$, with an A independent of ν, for a euclidean norm in V.

6^0. $\overline{\psi_\nu} \subset g(\varphi_\nu)$, $g(\varphi_\nu) \setminus \psi_\nu \subset M \times (-1, 1)$, (which implies that $\psi_\nu \cap \cap \{(u, t) : |t| \geq 1\}$ are closed).

Proof. Introduce a euclidean norm in V and identify M with V so that $c = 0$. We may assume (without loss in generality) that $\varphi_{-1}(0) < -1$ and $\varphi_1(0) > 1$. Choose $a_\nu > 0$ in such a way that φ_ν is defined in $\overline{U}_\nu$ where $U_\nu = \{u \in M : |u| < a_\nu\}$, and $\varphi_\nu(\overline{U}_\nu) \subset \Delta_\nu$, where Δ_ν are disjoint compact intervals, contained in $\{t : |t| > 1\}$. Put $\varphi_\nu^* = (\varphi_\nu)_{U_\nu}$. Take a function γ on $\mathbb{R}$ which is positive and constant on each Δ_ν as well on each component interval of $\mathbb{R} \setminus \cup \Delta_\nu$, and such that $\gamma(t) < a_\nu$ on Δ_ν and $\{(u, t) : |u| \leq \gamma(t)\} \subset \subset G \setminus \cup (\varphi_\nu \setminus \varphi_\nu^*)$. By the Whitney approximation theorem [15] we can find a function h positive and analytic on $\mathbb{R}$ and such that:

$$h(t) > \frac{1}{\gamma(t)} \sqrt{t^2 - 1} \text{ for } |t| \geq 1, \quad h(t) > 2k_\nu \quad \text{and} \quad |h'(t)| < \frac{1}{a_\nu} \quad \text{in} \quad \Delta_\nu$$

where k_ν is a constant satisfying $|\varphi_\nu(u') - \varphi_\nu(u)| \leq k_\nu\, |u' - u|$ for $u, u' \in U_\nu$. The inverse $\beta : \mathbb{R} \to \mathbb{R}$ of $t \to (e^t - e^{-t})/(e - e^{-t})$ is an increasing analytic isomorphism satisfying $\beta(\mp 1) = \mp 1$ and $|\beta'(t)| \leq \min(2, 1/t)$ in $\mathbb{R}$. We will prove that $g = g_2 \circ g_1$, where

$$g_1 : M \times \mathbb{R} \ni (u, t) \to (h(t)\, u, t) \in M \times \mathbb{R},$$

$$g_2 : M \times \mathbb{R} \ni (u, t) \to (u, \beta(t/\sqrt{1 + |u|^2})) \in M \times \mathbb{R},$$

is an analytic isomorphism which satisfies, together with φ_ν^*, the conditions 1^0-6^0.

First observe that $\{(u, t) : |t| \geq 1\} = g(Z)$, where

$$Z = \left\{(u, t) : |t| \geq 1,\ |u| \leq \frac{1}{h(t)} \sqrt{t^2 - 1}\right\} \subset \{(u, t) : |u| \leq \gamma(t)\} \subset G,$$

which yields 1^0; since $\overline{\varphi_\nu^*} \subset \varphi_\nu$ and $\varphi_\nu \setminus \varphi_\nu^* \subset \{(u, t) : |u| > \gamma(t)\} \subset (M \times \mathbb{R}) \setminus Z$,

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([15]) See [16].

we get 6^0. Now, we have

$$\Omega_\nu = \{\pi(g(u, \varphi_\nu(u))) : \ | u | < a_\nu\} = \{h(\varphi_\nu(u))\, u : \ | u | < a_\nu\} \supset$$
$$\supset \left\{u : \ | u | < \frac{1}{2}\sqrt{b_\nu^2 - 1}\right\}$$

where $b_\nu = \inf\{\, | t | : t \in \Delta_\nu\}$, since, because of $h(t) > \frac{1}{a_\nu}\sqrt{b_\nu^2 - 1}$ in Δ_ν, we have $| u | = \frac{a_\nu}{2} \Longrightarrow | h(\varphi_\nu(u))\, u | > \frac{1}{2}\sqrt{b_\nu^2 - 1}$. This gives 3^0 (for $b_\nu \to \infty$ as $| \nu | \to \infty$), and, after showing that ψ_ν are topographic, the property 4^0 will be a consequence of $\lim\limits_{\nu \to \mp\infty} \psi_\nu(0) = \mp\infty$ and the fact that the collection $\{\varphi_\nu^*\}$ and hence the collection $\{\psi_\nu\}$ is locally finite in $M \times \mathbb{R}$. Since φ_ν^* are analytic submanifolds which are disjoint, bounded and semi-analytic, so are ψ_ν. Thus, in view of the lemma 6, it remains to verify that

$$(u, t),\ (u', t') \in \psi_\nu \Longrightarrow | t - t' | \leq A\, | u' - u |$$

with an A independent of ν. Now, putting $\chi_\nu = g_1(\varphi_\nu^*)$, we have (v, t), $(v', t') \in \chi_\nu \Longrightarrow | t' - t | \leq | v' - v |$; in fact, $v = h(t)\, u$ and $v' = h(t')\, u'$ with u, u' such that $(u, t), (u', t') \in \varphi_\nu^*$, hence $| v' - v | \geq | h(t') |\, | u' - u | - | h(t') - h(t) |\, | u | \geq 2k_\nu \frac{1}{k_\nu} | t' - t | - \frac{1}{a_\nu} | t' - t |\, a_\nu = | t' - t |$. Let (v, s), $(v', s') \in \psi_\nu = g_2(\chi_\nu)$; then $s = \beta(t/\sqrt{1 + | v |^2})$ and $s' = \beta(t'/\sqrt{1 + | v' |^2})$ with t, t' such that $(v, t), (v', t') \in \chi_\nu$, whence $| t' - t | \leq | v' - v |$; assuming e.g. that $| v | \leq | v' |$ we have

$$\begin{aligned} | s' - s | \leq{} & | \beta(t'/\sqrt{1 + | v' |^2}) - \beta(t/\sqrt{1 + | v' |^2}) | + \\ & + | \beta(t/\sqrt{1 + | v' |^2}) - \beta(t/\sqrt{1 + | v |^2}) \\ \leq{} & 2 | t' - t | + | 1/\sqrt{1 + | v' |^2} - 1/\sqrt{1 + | v |^2} |\, \sqrt{1 + | v' |^2} \leq 3\, | v' - v |, \end{aligned}$$

Q. E. D.

For any $E \subset M$ and $\varphi, \varphi' : E \to \mathbb{R}$ such that $\varphi(u) \leq \varphi'(u)$ in E put

$$[\varphi, \varphi'] = \{(u, t) : \ u \in E,\ \varphi(u) \leq t \leq \varphi'(u)\};$$

thus in particular $[\varphi, \varphi] = \varphi$; if moreover $\varphi(u) < \varphi'(u)$ in E, we put

$$(\varphi, \varphi') = \{(u, t) : \ u \in E,\ \varphi(u) < t < \varphi'(u)\}.$$

The following lemma is trivial.

LEMMA 11. If $\varphi(u) < \varphi'(u)$ in $\bar{E}$ and the closures $\bar{\varphi}$, $\bar{\varphi}'$ are bounded functions (on $\bar{E}$), then $\overline{(\varphi, \varphi')} = [\bar{\varphi}, \bar{\varphi}']$.

Let F be a subset of another affine space N and let $\psi, \psi' : F \to \mathbb{R}$ satisfy $\psi(v) \le \psi'(v)$ in F. Assume that a map $g : E \to F$ satisfies the following condition

$$(s) \qquad u \in E,\ \varphi(u) = \varphi'(u) \Longrightarrow \psi(g(u)) = \psi'(g(u)).$$

By the map $h : [\varphi, \varphi'] \to [\psi, \psi']$ associated with g we will mean the map defined by the following formulas

$$h(u, t) = \begin{cases} (g(u), \psi(g(u)) + \dfrac{t - \varphi(u)}{\varphi'(u) - \varphi(u)}[\psi'(g(u)) - \psi(g(u))]) & \text{if } \varphi(u) \neq \varphi'(u), \\ (g(u), \psi(g(u))) & \text{if } \varphi(u) = \varphi'(u); \end{cases}$$

if $\varphi < \varphi'$ in E and $\psi < \psi'$ in F, by the map $h_0 : (\varphi, \varphi') \to (\psi, \psi')$ associated with g we will mean the map defined by the first formula. The following two lemmas are obvious.

LEMMA 12. We have $h_1, h_1' \subset h$ for the maps $h_1 : \varphi \to \psi$, $h_1' : \varphi' \to \psi'$ associated with g, and, if $\varphi < \varphi'$ in E, $\psi < \psi'$ in F, then $h_0 \subset h$ for the map $h_0 : (\varphi, \varphi') \to (\psi, \psi')$ associated with g. If $A \subset E$ and $g(A) \subset B \subset F$, then we have $h^* \subset h$ for the map $h^* : [\varphi_A, \varphi'_A] \to [\psi_B, \psi'_B]$ associated with $g_A : A \to B$.

LEMMA 13. If E, F are analytic submanifolds, φ, ψ analytic, and $g : E \to F$ is an analytic isomorphism, then the associated map $h_1 : \varphi \to \psi$ is also an analytic isomorphism. If moreover φ', ψ' are analytic, $\varphi < \varphi'$ on E, $\psi < \psi'$ on F, then (φ, φ'), (ψ, ψ') are analytic submanifolds and the associated map $h_0 : (\varphi, \varphi') \to (\psi, \psi')$ is also an analytic isomorphism.

We say that a map f (of a subset of an affine space M') into an affine space N' has property (A^{-1}), iff its graph is partially semi-algebraic with respect to N'.

LEMMA 14. Assume the condition (s) satisfied. If E is compact, $\varphi, \varphi', \psi, \psi'$ continuous, and g continuous, then the associated map $h : [\varphi, \varphi'] \to [\psi, \psi']$ is also continuous. If g has property (A^{-1}), φ, φ' are partially semi-algebraic with respect to $\mathbb{R}$, and ψ, ψ' semi-algebraic, then h has also property (A^{-1}).

PROOF. In view of the condition (s), the graph of h is the set

$$h = \{(u, t, v, s) : v = g(u),\ (s - \psi(v))(\varphi'(u) - \varphi(u)) =$$
$$= (t - \varphi(u))(\psi'(v) - \psi(v)),\ \varphi(u) \le t \le \varphi'(u),\ \psi(v) \le s \le \psi'(v)\}.$$

Under the assumptions of the first part of the lemma this set is compact and hence the conclusion follows. Under the assumptions of the second, this set is the projection by the map $(u, t, v, s, \eta, \eta', \zeta, \zeta') \to (u, t, v, s)$ of the intersection of the following eight subsets of the $(u, t, v, s, \eta, \eta', \zeta, \zeta')$-space:

$$\{\eta = \varphi(u)\},\ \{\eta' = \varphi'(u)\},\ \{\zeta = \psi(v)\},\ \{\zeta' = \psi'(v)\},$$

$$\{v = g(u)\},\ \{(s - \zeta)(\eta' - \eta) = (t - \eta)(\zeta' - \zeta)\},\ \{\eta \leq t \leq \eta'\},\ \{\zeta \leq s \leq \zeta'\},$$

each partially semi-algebraic with respect to the $(v, s, \eta, \eta', \zeta, \zeta')$-space; therefore, by § 1, V, h is partially semi-algebraic with respect to the (v, s)-space.

A locally finite simplicial complex (in M) is a locally finite collection K of disjoint open simplexes ([16]) such that each face of any simplex of K belongs to K. We put $|K| = \bigcup\{\sigma : \sigma \in K\}$. A locally finite cellular complex (in M) is a locally finite collection L of disjoint open cells ([17]) such that for any $\varrho \in L$, $\overline{\varrho}$ is a finite union of cells of L. Using the regular (barycentric) subdivision ([18]) we get

LEMMA 15. For any locally finite cellular complex L (in M) there is a locally finite simplicial complex K (in M) such that every cell of L is a (finite) union of simplexes of K.

§ 3. Triangulation theorem.

Let M be a real analytic manifold, E a subset of an affine space A. We say that a map $g : E \to M$ has property (A) iff its graph (in $A \times M$) is partially semi-algebraic with respect to A ([19]), then, by § 1, V, 2, the image of any semi-algebraic set is semi-analytic.

The following theorem will be proved in § 4.

THEOREM 1. Let $\{B_\nu\}$ be a locally finite collection of semi-analytic subsets of a finite dimensional affine space M. There exist a locally finite simplicial complex K with $|K| = M$ and a homeomorphism $\tau : M \to M$ (onto M) such that:

([16]) An open simplex in M is a subset of the form $c_0 \dots c_k = \{\sum_0^k t_i c_i : \sum_0^k t_i = 1, t_i > 0\}$ with $c_0, \dots, c_k$ independent; its faces are the simplexes $c_{\gamma_0} \dots c_{\gamma_s}$ with $\gamma_0 < \dots < \gamma_s$.

([17]) An open cell is a convex open bounded subset of an affine subspace of M.

([18]) See [18], p. 358, or [1], p. 131-132.

([19]) Thus a bijection $g : E \to F$: where E, F are subsets of affine spaces, has property (A) iff g^{-1} has property (A^{-1}).

(*a*) τ has property (A),

(*b*) for any $\sigma \in K$, $\tau(\sigma)$ is an analytic submanifold (of M) and $\tau_\sigma : \sigma \longrightarrow \tau(\sigma)$ is an analytic isomorphism,

(*c*) for any $\sigma \in K$ and any B_ν we have $\tau(\sigma) \subset B_\nu$ or $\tau(\sigma) \subset M \setminus B_\nu$.

By Grauert imbedding theorem [3] which ensures that every countable real analytic manifold is isomorphic with a closed analytic submanifold of a (real) affine space, it follows easily

Theorem 2. Let $\{B_\nu\}$ be a locally finite collection of semi-analytic subsets of a countable real analytic manifold M. There exist a locally finite simplicial complex K (in an affine space A) and a homeomorphism $\tau : |K| \longrightarrow M$ (onto M) such that the properties (*a*), (*b*), (*c*) hold.

In the case of a finite collection $\{B_\nu\}$ of bounded semi-algebraic sets the proof of the theorem does not require the use of lemma 10. Therefore we deal with only semi-algebraic sets (see § 1, VI), and we obtain.

Theorem 3. Let $B_1, \dots, B_k$ be bounded semi-algebraic subsets of an affine space M. There exist a finite simplicial complex K in M and a semi-algebraic homeomorphism $\tau : |K| \longrightarrow \bigcup_1^k \bar{B}_i$ such that the properties (*b*), (*c*) hold.

From this we deduce (see § 1, VI):

Theorem 4. Let $B_1, \dots, B_k$ be semi-algebraic subsets of a multiprojective space M. There exist a finite simplicial complex K (in an affine space) and a semi-algebraic homeomorphism $\tau : |K| \longrightarrow M$ (onto M) such that the properties (*b*), (*c*) hold.

In fact, it is sufficient to observe that the semi-algebraic map

$$P \ni \mathbb{R}x \longrightarrow \{x_i x_j\}_{i,j=1,\dots,n} \in \mathbb{R}^{n^2},$$

where $x = (x_0, \dots, x_n)$ and $\sum_0^n x_i^2 = 1$, is an analytic imbedding of P in $\mathbb{R}^{n^2}$ (i. e. yields an analytic isomorphism of P with an analytic submanifold of $\mathbb{R}^{n^2}$).

§ 4. Proof.

In this § we will prove the theorem 1. The affine space (of the theorem) will be denoted by M_1, its dimension by n. The proof will proceed by induction with respect to n, the theorem being trivial for $n = 1$. Thus we consider the case $n > 1$ and we assume that the theorem is true for the affine spaces of dimension $n - 1$.

We will say that a set E is compatible with a collection of sets $\{A_\nu\}$ iff, for each ν, $E \subset A_\nu$ or $E \cap A_\nu = \emptyset$. (Thus the condition (a) will mean that all $\tau(\sigma)$ should be compatible with $\{B_\mu\}$.

I. *A system of topographic manifolds.* There exists an analytic isomorphism $f: M_1 \to M \times \mathbb{R}$ where M is an affine space of dimension $n-1$, and a collection $\mathcal{C}$ of analytic submanifolds of $M \times \mathbb{R}$ such that the following properties hold:

(1) each $\Gamma \in \mathcal{C}$ is topographic, bounded and partially semi-algebraic (with respect to $\mathbb{R}$);

(2) $\mathcal{C}$ is locally finite;

(3) the sets $\Gamma \cap \{(u,t): |t| \geqq 2\}$, where $\Gamma \in \mathcal{C}$, are compact and mutually disjoint; they are empty when $\dim \Gamma < n$;

(4) the set $S = \bigcup \{\Gamma : \Gamma \in \mathcal{C}\}$ is closed and has property (P);

(5) for each $a \in M$ the sets $S \cap (\{a\} \times (0, \infty))$ and $S \cap (\{a\} \times (-\infty, 0))$ are unbounded;

(6) any connected subset of $M \times \mathbb{R}$ which is compatible with $\mathcal{C}$ is also compatible with $\{B'_\mu\}$, where $B'_\mu = f(B_\mu)$.

PROOF. By lemma 4, there is a direction in M_1 which is simultaneously non-singular for all B_μ. Therefore we can identify M_1 with $M \times \mathbb{R}$, where M is an affine space of dimension $n-1$, in such a way that every line $\{a\} \times \mathbb{R}$ is non-singular at each of its points for all B_μ. Using lemma 8 we can find a countable family $\mathcal{F}$ of normal decompositions $Q \times \Delta = V^n \cup \ldots \cup V^0$ which are compatible with all B_μ, such that the $Q \times \Delta$'s form a locally finite covering of $M \times \mathbb{R}$; consider the projected decompositions $Q = V_*^{n-1} \cup \ldots \cup V_*^0$ (lemma 9); since the union of all $Q \setminus V_*^{n-1}$ is meager, its complement (with respect to M) contains a point c. Now, we can choose, from the family $\mathcal{F}$, a sequence of normal decompositions $Q_\nu \times \Delta_\nu = V_\nu^{n-1} \cup \ldots$ $\ldots \cup V_\nu^0$, $\nu = 0, \pm 1, \pm 2, \ldots$, such that $c \in Q_\nu$, $\{c\} \times \mathbb{R} \subset \bigcup (Q_\nu \times \Delta_\nu)$, and such that the sequence of left ends of Δ_ν and that of right ones are strictly increasing. Then we can find a strictly increasing sequence γ_ν, $\nu = 0$, $\pm 1, \pm 2, \ldots$ such that $\gamma_\nu, \gamma_{\nu+1} \in \Delta_\nu$ (whence $\lim_{\nu \to \pm\infty} \gamma_\nu = \pm \infty$) and (c, γ_ν), $(c, \gamma_{\nu+1}) \in V_\nu^n$; since, by the choice of c, we have (lemma 9) $c \notin \pi(V_\nu^{n-2} \cup \ldots \cup V_\nu^0)$, where π is the map $M \times \mathbb{R} \ni (u,t) \to u \in M$, there is a neighborhood U_ν of c such that (see § 1, II, 2, 3):

$$U_\nu \times [\gamma_\nu, \gamma_{\nu+1}) \subset (Q_\nu \times \Delta_\nu) \setminus (V_\nu^{n-2} \cup \ldots \cup V_\nu^0),$$

$(U_\nu \times (\gamma_\nu, \gamma_{\nu+1})) \cap V_\nu^{n-1} = \varphi_{\nu 1} \cup \ldots \cup \varphi_{\nu k_\nu}$ where $\varphi_{\nu\sigma}$ are analytic and

$$\varphi_{\nu 1}(u) < \ldots < \varphi_{\nu k_\nu}(u) \text{ in } U_\nu,$$

$$U_\nu \times \{\gamma_\nu\} \subset V_\nu^n.$$

Put $G = \bigcup_{-\infty}^{\infty} U_\nu \times [\gamma_\nu, \gamma_{\nu+1})$ and $\varphi_{\nu 0} = \gamma_\nu$ in U_ν; it follows that any connected subset of G which is compatible with $\{\varphi_{\nu\sigma}\}$ is also compatible with $\{B_\mu\}$. In, fact, let E be such a subset; if $E \subset \varphi_{\nu\sigma}$ with $\sigma > 0$, then $E \subset V_\nu^{n-1}$; if $E \subset \varphi_{\nu 0}$, then $E \subset V_\nu^n$; finally if $E \subset G \setminus \bigcup \varphi_{\nu\sigma}$, then $E \subset \bigcup (U_\nu \times (\gamma_\nu, \gamma_{\nu+1}))$, whence $E \subset U_\nu \times (\gamma_\nu, \gamma_{\nu+1})$ for some ν, which implies $E \subset (Q_\nu \times \Delta_\nu) \setminus V_\nu^{n-1} \setminus (V_\nu^{n-2} \cup \ldots \cup V_\nu^0) = V_\nu^n$; therefore E is always contained in a component of some V_ν^{n-1} or V_ν^n.

By lemma 10, applied to G and $\varphi_{\nu\sigma}$ which can be ordered in a sequence φ_ν, $\nu = \pm 1, \pm 2, \ldots$ so that $\varphi_\nu(c)$ is strictly increasing, there exist restrictions φ_ν^* of φ_ν and an analytic isomorphism $g : M \times \mathbb{R} \to M \times \mathbb{R}$ such that the properties $1^0 - 6^0$ (of the lemma 10) hold; furthermore (by property 1^0) any connected subset of $\{(u, t) : |t| \geq 1\}$ which is compatible with $\{\psi_\nu\}$ is also compatible with $\{B_\nu''\}$, where $B_\nu'' = g(B_\nu)$, since, by property 6^0, it must be compatible with $\{g(\varphi_\nu)\}$.

Introduce a euclidean norm in the vector space V of M so that the property 5^0 holds. By lemma 4 we can find a $z \in V$ such that $|z| < A$ and the direction $\mathbb{R}(-z, 1)$ is non-singular for all B_μ'' and ψ_ν. Consider the map $l : (u, t) \to (u + tz, t)$ and put $f = l \circ g$. Then the direction $\{0\} \times \mathbb{R}$ is non singular for all $B_\mu' = l(B_\mu'') = f(B_\mu)$ and $\psi_\nu' = l(\psi_\nu)$; hence ψ_ν' are (by lemma 5) partially semi-algebraic and (by lemma 6) topographic. Furthermore $\{\psi_\nu'\}$ is locally finite (by property 4^0), ψ_ν' are bounded, disjoint (property 2^0), the sets $\psi_\nu' \cap \{(u, t) : |t| \geq 1\}$ are closed (by property 6^0), and any connected subset of $\{(u, t) : |t| \geq 1\}$ which is compatible with $\{\psi_\nu'\}$ is also compatible with $\{B_\mu'\}$. Finally, for any $a \in M$ the sets $(\{a\} \times (0, \infty)) \cap \bigcup \psi_\nu'$ and $(\{a\} \times (-\infty, 0)) \cap \bigcup \psi_\nu'$ are unbounded; in fact, it follows from the property 6^0 that, if $\psi_\nu(a - z) > 1$, ψ_ν meets the half-line $\{(a - tz, t) : t > 0\}$; hence, by properties 3^0 and 4^0, infinitely many of ψ_ν meet this half-line; similarly for any half-line $\{(a - tz, t) : t < 0\}$; thus the sets in question are infinite and the conclusion follows from the fact that $\{\psi_\nu'\}$ is locally finite.

Thus we see that the collection $\mathcal{C}_1$ consisting of all submanifolds $\psi_\nu' \cap \{(u, t) : |t| > 1\}$ and the hyperplanes $M \times \{1\}$ and $M \times \{-1\}$, satisfies the conditions (1) - (5) and the condition (6) for any connected subset of $\{(u, t) : |t| \geq 1\}$. Therefore it is sufficient to find a collection $\mathcal{C}_0$ of analytic submanifolds of $M \times (-2, 2)$ satisfying the conditions (1), (2), (4) and the condition (6) for any connected subset of $M \times (-1, 1)$, since then $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_0$ will satisfy (1) - (6).

Using lemmas 8 and 9 we can find a collection of truncated cones $T_\nu = \{(u, t) : |u - c_\nu| < r_\nu (1 - t/2), |t| < 1\}$ and normal decompositions $Q_{\nu\sigma} = \bigcup_{k, \varkappa} \Gamma^k_\varkappa (\nu, \sigma)$ each compatible with all B'_μ, such that $\Gamma^k_\varkappa (\nu, \sigma)$ with $k < n$ are partially semi-algebraic and topographic, $C_{\nu\sigma} = \bigcup_{k<n} \Gamma^k_\varkappa (\nu, \sigma)$ have property (P), $\{Q_{\nu\sigma}\}$ and $\{T_\nu\}$ are locally finite, $\overline{T}_\nu \subset \bigcup_\sigma Q_{\nu\sigma}$ and $M \times (-1, 1) = \bigcup T_\nu$. In fact, it is sufficient to take, for any $c \in M$, a finite set of normal decompositions $Q_i = \bigcup \Gamma^k_\varkappa (i)$ according to lemma 8, such that $\{c\} \times [-1, 1] \subset \subset \bigcup Q_i \subset \{(u, t) : |u - c|, |t| < 2\}$, and to choose $\{(u, t) : |u - c| \leq r (1 - - t/2), |t| \leq 1\} \subset \bigcup Q_i$ with $r > 0$. Now, let $\mathcal{C}_0$ consists of all $\Gamma^k_\varkappa (\nu, \sigma) \cap T_\nu$ with $k < n$, $\Sigma_\nu = \{(u, t) : |u - c_\nu| = r_\nu (1 - t/2), |t| < 1\}$ and $\pi_+ = M \times \{1\}$, $\pi_- = M \times \{-1\}$. Then the conditions (1) and (2) are obviously satisfied. To prove (4) observe that $S_0 = \bigcup \{\Gamma : \Gamma \in \mathcal{C}_0\} = \theta \cup \bigcup (C_{\nu\sigma} \cap T_\nu)$ where $\theta = \pi_+ \cup \cup \pi_- \cup \bigcup \Sigma_\nu$; since π_+, π_-, Σ_ν and $C_{\nu\sigma} \cap T_\nu$ have property (P), so has the set S_0; since θ and $\overline{T}_\nu \cap (\bigcup_\sigma C_{\nu\sigma})$ are closed and $\overline{T}_\nu \setminus T_\nu \subset \theta$, the set S_0 is closed. Finally, let E be a connected subset of $M \times (-1, 1)$, compatible with $\mathcal{C}_0$; then $E \cap T_\nu \neq \emptyset$ for some ν, which follows $E \subset T_\nu$ (since otherwise $E \subset \Sigma_\nu$); if $E \subset \Gamma^k_\varkappa (\nu, \sigma)$ for some $\sigma, \varkappa$ and $k < n$, then E is compatible with $\{B'_\mu\}$; in the oposite case we have $E \subset \bigcup \Gamma^n_\varkappa (\nu, \sigma)$, which follows that $E \subset \operatorname{int} B_\mu \cup \operatorname{int} ((M \times \mathbb{R}) \setminus B_\mu)$ (since $\Gamma^n_\varkappa (\nu, \sigma)$ are open), and this implies that E is compatible with $\{B'_\mu\}$. Q. E. D.

Remark. In the case of the theorem 3 we do not need to use the lemma 10 (we can assume $B_\mu \subset M \times (-1, 1)$ and take for $\mathcal{C}_1$ the set of hyperplanes $M \times \{k\}$, $k = \pm 1, \pm 2, \ldots, f$ being the identity; besides, we do not need then to triangulate the whole M).

To prove the theorem 1 it is now sufficient to satisfy its conditions for $\{B'_\nu\}$ (in $M \times \mathbb{R}$) instead of $\{B_\mu\}$ (in M_1).

II. *Triangulation of the projected system.* Let π be the map $M \times \mathbb{R} \ni (u, t) \longrightarrow u \in M$. There exist a locally finite simplicial complex K_1 with $|K_1| = M$ and a homeomorphism $\tau : M \longrightarrow M$ such that

(1) τ has property (A),

(2) any member of $\mathcal{K}_1 = \tau (K_1)$ is compatible with $\{\pi (\Gamma \cap \Gamma') : \Gamma, \Gamma' \in \mathcal{C}\}$, and for any $\varrho \in K_1$, $\tau (\varrho)$ is an analytic submanifold and $\tau_\varrho : \varrho \longrightarrow \tau (\varrho)$ is an analytic isomorphism.

In fact, by I (1), (2) and § 1, V, the sets $\pi (\Gamma \cap \Gamma' \cap (M \times (-2, 2)))$ with $\Gamma, \Gamma' \in \mathcal{C}$ are semi-analytic and form a locally finite family. For this family take a simplicial complex K_1 and a homeomorphism $\tau: M \to M$ according to our induction hypothesis; then we need only verify the property (2). Let $\beta \in \mathcal{K}_1$ and $\Gamma, \Gamma' \in \mathcal{C}$; we will show that $\beta \subset \pi(\Gamma \cap \Gamma')$ or $\beta \subset M \setminus \pi(\Gamma \cap \Gamma')$; if $\Gamma \neq \Gamma'$ or $\dim \Gamma < n$, then, by I (3), $\Gamma \cap \Gamma' \subset M \times (-2, 2)$, hence the alternative follows; in the opposite case we have $\Gamma = \Gamma'$ and $\dim \Gamma = n$; since β is contained in the set $\pi(\Gamma \cap (M \times (-2, 2)))$ or in its complement which is the union of two disjoint sets $M \setminus \pi(\Gamma)$, $\{u \in \pi(\Gamma) : |\Gamma(u)| \geq 2\}$, each compact because of I (1), (3), it implies that $\beta \subset \pi(\Gamma)$ or $\beta \subset M \setminus \pi(\Gamma)$.

Now, let K be a locally finite simplicial complex with $|K| = M$ such that

(3) each simplex of K is contained, together with one of its vertices, in a simplex of K_1.
One can take for K e. g. the regular (barycentric) subdivision of K_1 ([20]).
Then we have:

(4) any member of $\mathcal{K} = \tau(K)$ is compatible with $\{\pi(\Gamma_1 \cap \Gamma_2) : \Gamma_1, \Gamma_2 \in \mathcal{C}\}$;

(5) for any $\varrho \in \mathcal{K}$, $\tau(\varrho)$ is an analytic submanifold, and $\tau_\varrho : \varrho \to \tau(\varrho)$ is an analytic isomorphism.

III. *A prismatic stratification.* Let $\mathcal{L}$ denotes the collection of all non-empty sets of the form $(\beta \times \mathbb{R}) \cap \Gamma$ with $\beta \in \mathcal{K}$ and $\Gamma \in \mathcal{C}$. Then clearly $\bigcup\{\gamma : \gamma \in \mathcal{L}\} = S = \bigcup\{\Gamma : \Gamma \in \mathcal{C}\}$. For any $\beta \in \mathcal{K}$, denote by $s(\beta)$ the image by τ of the set of vertices of $\tau^{-1}(\beta)$; clearly, $s(\beta) \subset \overline{\beta}$. The following properties hold:

(1) each $\gamma \in \mathcal{L}$ is a topographic analytic submanifold of $M \times \mathbb{R}$ with $\pi(\gamma) \in \mathcal{K}$, bounded and partially semi-algebraic;

(2) $\mathcal{L}$ is a locally finite collection of disjoint sets;

(3) $\mathcal{K} = \pi(\mathcal{L})$ and for any $\gamma \in \mathcal{L}$ there are $\gamma', \gamma'' \in \mathcal{L}$ such that $\pi(\gamma') = \pi(\gamma'') = \pi(\gamma)$ and $\gamma' < \gamma < \gamma''$ on $\pi(\gamma)$;

(4) $S = \bigcup\{\gamma : \gamma \in \mathcal{L}\}$ is closed and has property (P);

(5) any connected subset of $M \times \mathbb{R}$ which is compatible with $\mathcal{L}$ is also compatible with $\{B'_\mu\}$;

(6) for any $\gamma \in \mathcal{L}$, $\overline{\gamma}$ is a continuous function on $\overline{\pi(\gamma)}$ and a union of members of $\mathcal{L}$: we have $\overline{\gamma}_\beta \in \mathcal{L}$ for any $\beta \in \mathcal{K}$ contained in $\overline{\pi(\gamma)}$;

(7) if $\gamma_1, \gamma_2 \in \mathcal{L}$ and $\pi(\gamma_1) = \pi(\gamma_2) = \beta$, then

$$\gamma_1 < \gamma_2 \text{ on } \beta \Longrightarrow \overline{\gamma}_1 \leq \overline{\gamma}_2 \text{ and } \overline{\gamma}_1 \not\equiv \overline{\gamma}_2 \text{ on } s(\beta).$$

([20]) See [17], p. 358, or [1]. pp. 131-132.

PROOF. For any $\gamma = (\beta \times \mathbb{R}) \cap \Gamma \in \mathcal{L}$, by II (4), we have $\beta \subset \pi(\Gamma)$ which, together with I (1) and II (1), (5), implies all the properties of (1). As to (2), the local finiteness of $\mathcal{L}$ is a consequence of that of $\mathcal{K}$ and $\mathcal{C}$ (see I (2)); if $\gamma_i = (\beta_i \times \mathbb{R}) \cap \Gamma_i \in \mathcal{L}$ and $\gamma_1 \cap \gamma_2 \neq \emptyset$, then $\beta_1 \cap \beta_2 \cap \pi(\Gamma_1 \cap \Gamma_2) \neq \emptyset$, hence, by II (4), $\beta_1 = \beta_2 \subset \pi(\Gamma_1 \cap \Gamma_2)$, which gives $\gamma_1 = \gamma_2$. The property (3) follows from I (5), since the members of $\mathcal{L}$ are continuous functions on connected sets; (4) coincide with I (4), and (5) is a consequence of I (6) in view of the fact that any $\Gamma \in \mathcal{C}$ is a union of some members of $\mathcal{L}$.

To prove (6), consider any $\gamma \in \mathcal{L}$ and put $\beta = \pi(\gamma)$. Since γ is bounded, we have $\pi(\overline{\gamma}) = \overline{\beta}$, and, by (4), $\overline{\gamma} \subset S$. For any $u \in \overline{\beta} \setminus \beta$, the set $(\{u\} \times \mathbb{R}) \cap \overline{\gamma}$ consists of one point, as a connected (21) subset of the set $(\{u\} \times \mathbb{R}) \cap S$ which is isolated (by I (1), (2)). Therefore $\overline{\gamma}$ is a continuous function on $\overline{\beta}$. Let $\beta_1 \in \mathcal{K}$, $\beta_1 \subset \overline{\beta}$; then $\overline{\gamma}_{\beta_1}$ is a continuous function on β_1, and is contained in the set $(\beta_1 \times \mathbb{R}) \cap S$; this set is a locally finite union of (disjoint) members of $\mathcal{L}$ each being a continuous function on β_1 (and β_1 is connected), hence $\overline{\gamma}_{\beta_1}$ must be one of them. This establish the second part of (6).

To prove (7), assume that $\gamma_1, \gamma_2 \in \mathcal{L}$ and $\pi(\gamma_1) = \pi(\gamma_2) = \beta$. Let $\gamma_1 < \gamma_2$ on β; then clearly $\overline{\gamma}_1 \leq \overline{\gamma}_2$ on $s(\beta)$; we have $\gamma_i \subset \Gamma_i$ with $\Gamma_i \in \mathcal{C}$ and $\beta = \tau(\varrho) \subset \widetilde{\beta} = \tau(\widetilde{\varrho})$, $\varrho \subset \widetilde{\varrho}$ with $\varrho \in K$ and $\widetilde{\varrho} \in K_1$ (see II (3)); now $\gamma_i \subset \widetilde{\gamma}_i = (\widetilde{\beta} \times \mathbb{R}) \cap \Gamma_i$, and, using II (2), by the same argument as for (1) and (2), we conclude that $\widetilde{\gamma}_2$ are continuous functions and $\widetilde{\gamma}_1 < \widetilde{\gamma}_2$ on $\widetilde{\beta}$; since, by II (3), $\widetilde{\varrho}$ contains a vertex of ϱ, we have $s(\beta) \cap \widetilde{\beta} \neq \emptyset$; but $\overline{\gamma}_i = \widetilde{\gamma}_i$ on $\overline{\beta} \cap \widetilde{\beta}$, which implies $\overline{\gamma}_1 < \overline{\gamma}_2$ on $s(\beta) \cap \widetilde{\beta}$. Q. E. D.

Let $\gamma_1, \gamma_2 \in \mathcal{L}$; we call γ_1, γ_2 a consecutive couple of $\mathcal{L}$ iff $\pi(\gamma_1) = \pi(\gamma_2)$, $\gamma_1 < \gamma_2$ (on $\pi(\gamma_1)$) and (γ_1, γ_2) does not contain any member of $\mathcal{L}$. Then the following property holds:

(8) If γ, γ' is consecutive, then for each $\beta \in \mathcal{K}$ contained in $\overline{\pi(\gamma)}$ we have $\overline{\gamma}_\beta = \overline{\gamma}'_\beta$ or $\overline{\gamma}_\beta$, $\overline{\gamma}'_\beta$ is consecutive.

In fact, $\overline{\gamma}_\beta, \overline{\gamma}'_\beta \in \mathcal{L}$ (by (6)), hence $\overline{\gamma}_\beta = \overline{\gamma}'_\beta$ or $\overline{\gamma}_\beta < \overline{\gamma}'_\beta$ (on β). Assume that $\gamma_1 \subset (\overline{\gamma}_\beta, \overline{\gamma}'_\beta)$ for some $\gamma_1 \in \mathcal{L}$; then, by (1), $\pi(\gamma_1) = \beta$. Take a point $x \in \gamma_1$.

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(21) This can be seen from $(\{u\} \times \mathbb{R}) \cap \overline{\gamma} = \bigcap_{\nu=1}^{\infty} \overline{(U_\nu \times \mathbb{R}) \cap \gamma}$, where $U_\nu = \tau(K_\nu)$ and $K_\nu = \{v : |v - \tau^{-1}(u)| < 1/\nu\}$, since $\{\overline{(U_\nu \times \mathbb{R}) \cap \gamma}\}$ is a decreasing sequence of compact connected sets (they are connected, as $U_\nu \cap \beta = \tau(K_\nu \cap \tau^{-1}(\beta))$ are).

Since $\mathcal{L}$ is locally finite, we have (by (4))

$$S \cap U \subset \bigcup \{\gamma'' : \gamma'' \in \mathcal{L},\ x \in \overline{\gamma''}\}$$

for some neighborhood U of x. Since $\pi(x) \in \overline{\pi(\gamma)}$, it follows, in view of (4), that $\pi(\gamma) \cap \pi(\gamma'') \neq \emptyset$ for some $\gamma'' \in \mathcal{L}$ such that $x \in \overline{\gamma''}$. Therefore $\pi(\gamma'') = \pi(\gamma)$ and (in view of (6) and (2)) $\gamma_1 \subset \overline{\gamma''}$. But this implies $\gamma'' \subset (\gamma, \gamma')$ (since otherwise $\gamma'' \leq \gamma$ or $\gamma' \leq \gamma''$, whence $\gamma_1 \leq \overline{\gamma}_\beta$ or $\overline{\gamma}'_\beta \leq \gamma_1$), which is a contradiction. Thus our alternative holds.

Denote by $\mathcal{L}^{\#}$ the collection of all sets (γ_1, γ_2) where γ_1, γ_2 is a consecutive couple of $\mathcal{L}$, and put $\mathcal{L}^* = \mathcal{L} \cup \mathcal{L}^{\#}$. Then we have:

(9) $\mathcal{L}^*$ is a locally finite collection of disjoint analytic submanifolds of $M \times \mathbb{R}$, and $\bigcup\{\gamma : \gamma \in \mathcal{L}^*\} = M \times \mathbb{R}$.

In fact, the first part of (9) follows from (1) and (2); to see the second, observe that $M = \bigcup\{\beta : \beta \in \mathcal{K}\}$, and that for each $\beta \in \mathcal{K}$ the set

$$\bigcup\{[\gamma, \gamma'] : \gamma, \gamma' \text{ cons. couple of } \mathcal{L} \text{ with } \pi(\gamma) = \beta\} =$$

$$= \bigcup\{(\gamma, \gamma'') : \gamma, \gamma' \text{ and } \gamma', \gamma'' \text{ cons. couples of } \mathcal{L} \text{ with } \pi(\gamma) = \beta\},$$

the equality following from (3), is a non-void (by (3)) closed-open ([22]) subset of $\beta \times \mathbb{R}$, and therefore coincides with the latter.

It follows, by (5):

(10) all the members of $\mathcal{L}^*$ are compatible with $\{B'_\mu\}$.

Finally we have:

(11) for any $\gamma \in \mathcal{L}^*$, $\overline{\gamma}$ is a union of members of $\mathcal{L}^*$; if $\gamma = (\gamma_1, \gamma_2) \in \mathcal{L}^{\#}$, any member of $\mathcal{L}^*$ contained in $\overline{\gamma}$ is of the form $(\overline{\gamma}_1)_\beta$ or $(\overline{\gamma}_2)_\beta$ or $((\overline{\gamma}_1)_\beta, (\overline{\gamma}_2)_\beta)$ with $\beta \in \mathcal{K}$, $\beta \subset \overline{\pi(\gamma)}$.

In fact, for any consecutive couple γ_1, γ_2 of $\mathcal{L}$ we have, by lemma 11,

$$\overline{(\gamma_1\, \gamma_2)} = [\overline{\gamma}_1, \overline{\gamma}_2] = \bigcup\{[(\overline{\gamma}_1)_\beta, (\overline{\gamma}_2)_\beta] : \beta \in \mathcal{K},\ \beta \subset \overline{\pi(\gamma_1)}\}$$

and $[(\overline{\gamma}_1)_\beta, (\overline{\gamma}_2)_\beta]$ is equal to $(\overline{\gamma}_1)_\beta$ or $(\overline{\gamma}_1)_\beta \cup ((\overline{\gamma}_1)_\beta, (\overline{\gamma}_2)_\beta) \cup (\overline{\gamma}_2)_\beta$, hence (11) follows in view of (6), (8) and (2).

([22]) This follows from the fact that $[\gamma, \gamma']$ or (γ, γ'') are closed resp. open in $\beta \times \mathbb{R}$, and that the first union is locally finite.

IV. *Corresponding rectilinear stratification and stratified map.* For any $\eta : r \longrightarrow \mathbb{R}$, r being the set of vertices of any open simplex ϱ (in M), denote by (η) the open simplex (in $M \times \mathbb{R}$) whose set of vertices is η; thus $(\eta): \varrho \longrightarrow \mathbb{R}$. We have obviously: $\eta_1 \leq \eta_2$ and $\eta_1 \not\equiv \eta_2$ on $r \Longrightarrow (\eta_1) < (\eta_2)$ on ϱ, and $\eta_1 \subset \eta_2 \Longrightarrow (\eta_1) \subset \overline{(\eta_2)}$.

Let $g = \tau^{-1}$. For any $\gamma \in \mathcal{L}$ put

$$\sigma(\gamma) = (\overline{\gamma}_{s(\beta)} \circ \tau) \quad \text{where} \quad \beta = \pi(\gamma).$$

Therefore, since $g(s(\beta))$ is the set of vertices of $g(\beta)$, $\sigma(\gamma)$ is a function on $g(\beta)$:

(1) $$\pi(\sigma(\gamma)) = g(\pi(\gamma)) \qquad (\text{for } \gamma \in \mathcal{L}).$$

By III (7),

(2) $$\gamma < \gamma' \quad \text{on} \quad \beta = \pi(\gamma) \Longrightarrow \sigma(\gamma) < \sigma(\gamma') \quad \text{on} \quad g(\beta)$$

$$(\text{for} \quad \gamma,\ \gamma' \in \mathcal{L} \quad \text{such that} \quad \pi(\gamma) = \pi(\gamma')).$$

Clearly,

(3) $$\gamma \geq a\ (\leq a) \quad \text{on} \quad \beta = \pi(\gamma) \Longrightarrow \sigma(\gamma) \geq a\ (\leq a) \quad \text{on} \quad g(\beta)$$

$$(\text{for} \quad \gamma \in \mathcal{L} \quad \text{and} \quad a \in \mathbb{R}).$$

Finally we have

(4) $$\gamma' \subset \overline{\gamma} \Longrightarrow \sigma(\gamma') \subset \overline{\sigma(\gamma)} \quad (\text{for} \quad \gamma', \gamma \in \mathcal{L}),$$

since $\beta' \subset \overline{\beta} \Longrightarrow s(\beta') \subset s(\beta)$ (for $\beta', \beta \in \mathcal{K}$).

Let $L^* = \{\sigma(\gamma) : \gamma \in \mathcal{L}\} \cup \{(\sigma(\gamma), \sigma(\gamma')) : \gamma, \gamma' \text{ cons. couple of } \mathcal{L}\}$. Then, by (1), (2) and (3) with III (2)

(5) L^* is a locally finite collection of disjoint open cells.

Furthermore,

(6) $$\bigcup \{\sigma : \sigma \in L^*\} = M \times \mathbb{R};$$

to see this, we observe (as before for III (9)) that for any $\varrho \in K$ the set

$$\bigcup \{[\sigma(\gamma), \sigma(\gamma')] : \gamma, \gamma' \text{ cons. couple of } \mathcal{L} \text{ with } \pi(\gamma) = \tau(\varrho)\} =$$

$$= \bigcup \{(\sigma(\gamma), \sigma(\gamma'')) : \gamma, \gamma' \text{ and } \gamma', \gamma'' \text{ cons. couples of } \mathcal{L} \text{ with } \pi(\gamma) = \tau(\varrho)\}$$

coincide with the set $\varrho \times \mathbb{R}$ (as a non-void closed-open subset of the latter).

Consider any $\gamma \in \mathcal{L}^*$ and put $\beta = \pi(\gamma)$. If $\gamma \in \mathcal{L}^{\#}$, we have $\gamma = (\gamma_1, \gamma_2)$; if $\gamma \in \mathcal{L}$ we put $\gamma_1 = \gamma_2 = \gamma$. In the first case the map

$$h^\gamma: \gamma = (\gamma_1, \gamma_2) \longrightarrow (\sigma(\gamma_1), \sigma(\gamma_2)),$$

and in the second the map

$$h^\gamma; \ \gamma \longrightarrow \sigma(\gamma)$$

associated with $g_\beta: \beta \longrightarrow g(\beta)$, is an analytic isomorphism, by the lemma 13, in view of II (5) and III (1).

Since, by II (1), the map $\tau_{g(\beta)}$ has property (A), its inverse g_β has property (A^{-1}); by (1), (4) and III (6) we obtain

$$g(\pi(\bar{\gamma}_1 \cap \bar{\gamma}_2)) \subset \pi(\overline{\sigma(\gamma_1)} \cap \overline{\sigma(\gamma_2)}),$$

i. e. the condition (s) of § 2, hence, by lemma 14, in view of III (6) and III (1), the map

$$\tilde{h}^\gamma: \bar{\gamma} = [\bar{\gamma}_1, \bar{\gamma}_2] \longrightarrow [\overline{\sigma(\gamma_1)}, \overline{\sigma(\gamma_2)}]$$

associated with $g_{\bar{\beta}}: \bar{\beta} \longrightarrow g(\bar{\beta})$ is continuous and has property (A^{-1}).

We have

$$(7) \qquad \gamma' \subset \bar{\gamma} \Longrightarrow h^{\gamma'} \subset \tilde{h}^\gamma \qquad (\text{for } \gamma', \gamma \in \mathcal{L}^*).$$

In fact, let $\beta' \subset \bar{\beta}$, $\beta' \in \mathcal{K}$; γ_1, γ_2 being as before, we have by (4), $\sigma((\bar{\gamma}_i)_{\beta'}) \subset \overline{\sigma(\gamma_i)}$, whence (see (1)) $\sigma((\bar{\gamma}_i)_{\beta'}) = (\overline{\sigma(\gamma_i)})_{g(\beta')}$; it follows, by lemma 12, that the map

$$[(\bar{\gamma}_1)_{\beta'}, (\bar{\gamma}_2)_{\beta'}] \longrightarrow [\sigma((\bar{\gamma}_1)_{\beta'}), \sigma((\bar{\gamma}_2)_{\beta'})]$$

associated with $g_{\beta'}$ is contained in $\tilde{h}^\gamma$; therefore, in view of III (11), by the lemma 12, the conclusion follows.

We put now $h = \bigcup \{h^\gamma: \gamma \in \mathcal{L}^*\}$. Then, by the definition of L^*, (5), (6) and III (9), $h: M \times \mathbb{R} \longrightarrow M \times \mathbb{R}$ is bijective and $h(\mathcal{L}^*) = L^*$. In view of (7) and III (11) we have [23] $h = \bigcup \{\tilde{h}^\gamma: \gamma \in \mathcal{L}^*\}$, which follows easily that h is a homeomorphism, since, by (5) and III (9), $\{\tilde{h}^\gamma: \gamma \in \mathcal{L}^*\}$ is a locally finite family of compact sets. By III (11), this yields:

(8) for any $\sigma \in L^*$, $\bar{\sigma}$ is a union of some members of L^*.

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[23] since any $\tilde{h}^\gamma$ (with $\gamma \in \mathcal{L}^*$) is the union of $h^{\gamma'}$ with $\gamma' \subset \bar{\gamma}$ (and $\gamma' \in \mathcal{L}^*$).

Finally, since (by local finiteness of $\mathcal{L}^*$) for any bounded $U \subset M \times \mathbb{R}$ the set $U \times (M \times \mathbb{R})$ meets only finitely many $\widetilde{h}^\gamma$, it follows (see § 1, V) that h has property (A^{-1}).

Thus $\tau^* = h^{-1} : M \times \mathbb{R} \rightarrow M \times \mathbb{R}$ is a homeomorphism satisfying the following conditions:

$$(9) \qquad \begin{cases} \tau^* \text{ has property } (A) \\ \tau^*(L^*) = \mathcal{L}^*, \\ \text{for any } \sigma \in L^*,\ \tau^*_\sigma : \sigma \rightarrow \tau^*(\sigma) \text{ is an analytic isomorphism.} \end{cases}$$

V. *Triangulation.* By (5) and (8), L^* is a locally finite cellular complex, hence, by lemma 15, there exists a locally finite simplicial complex K^* such that every cell of L^* is a finite union of simplexes of K^*; furthermore, by (6), $|K^*| = M \times \mathbb{R}$. Therefore, in view of (9), III (9) and III (10), τ^* with K^* satisfy the conditions of the theorem for $\{B'_\mu\}$.

This completes the proof of the theorem.

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