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Calculus of variations for integrals depending on a convolution product

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CALCULUS OF VARIATIONS FOR INTEGRALS DEPENDING ON A CONVOLUTION PRODUCT

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1. Introduction.

This paper considers existence theorems in problems of the calculus of variations for integrals in an infinite interval, whose integrands depend on an unknown function of one real variable, its derivative, and a convolution integral. The direct methods of the calculus of variations are applied in connection with properties of lower semicontinuity of regular integrals.

Specifically consider $F(x, y, d, p)$ a real valued continuous function defined on E^4.

DEFINITION: The class K is the collection of all functions $y = y(x)$, $-\infty < x < \infty$, satisfying

1. $y(x)$ is absolutely continuous in every finite interval,
2. $y'(x)$ is L-integrable on $(-\infty, \infty)$, or briefly $y'(x) \in L^1(-\infty, \infty)$,
3. $F[x, y(x), y'(x), p(x)]$ is L-integrable on $(-\infty, \infty)$, where $p(x)$ is one of the following convolution integrals:

$$a.\ p(x) = y' * y' = \int_{-\infty}^{\infty} y'(x-t)\, y'(t)\, dt,$$

$$b.\ p(x) = |y'| * |y'| = \int_{-\infty}^{\infty} |y'(x-t)|\, |y'(t)|\, dt,$$

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$$c.\ p(x) = y' * g = \int_{-\infty}^{\infty} y'(x-t)\, g(t)\, dt, \text{ where } g(x),\ -\infty < x < \infty, \text{ is}$$

given and in $L^1(-\infty, \infty)$, or

$$d.\ p(x) = |y'| * |g| = \int_{-0}^{\infty} |y'(x-t)|\, |g(t)|\, dt \text{ with } g(x) \text{ given as above}$$

It will be assumed K is non-empty.

Known properties of seminormal functions are used to prove sufficient conditions for

$$I[y] = \int_{-\infty}^{\infty} F[x, y(x), y'(x), p(x)]\, dx$$

to be lower semicontinuous in K with respect to uniform convergence on $(-\infty, \infty)$. An example is given showing that, under the hypotheses given, $I[y]$ need not be lower semicontinuous in K with respect to uniform convergence on every compact set in $(-\infty, \infty)$.

Let P be a given compact set contained in E^2.

DEFINITION: The class $\bar{K}$ is any non-empty subclass of K satisfying

1. The graph of every $y(x) \in \bar{K}$ contains a point $(x, y(x)) \in P$, and
2. $\bar{K}$ is closed with respect to uniform convergence on every compact set in $(-\infty, \infty)$.

A theorem on existence of a minimum of $I[y]$ in $\bar{K}$ is given. The usual condition of strong growth of $F(x, y, d, p)$ with respect to the variable d guarantees that a minimizing sequence of functions possesses a convergent subsequence. However, the convergence is uniform on every compact set in $(-\infty, \infty)$, and additional hypotheses are needed to guarantee uniform convergence on $(-\infty, \infty)$. An example is given of a class $\bar{K}$ and a function $F(x, y, d, p)$ which satisfy the hypotheses of the theorem.

This problem will be discussed in the framework of the direct methods of the calculus of variations. In particular our approach will be in the spirit of Tonelli's book and paper [8, 9] on the extrema of the ordinary form of the integrals of the calculus of variations, and the subsequenct extensions to infinite intervals by Cinquini [3] and Faedo [4, 5]. The recent extensions of Tonelli's work by Turner [10] will also be employed. However, since these authors only required uniform convergence on compact sets, and the functionals considered in this paper are not lower semicontinuous with respect to this mode of convergence, a different analysis is needed.

2. Preliminary Definitions and Lemmas.

The following lemma states some important properties of the convolution integral. The proofs may be found in [6] and [7].

LEMMA 2.1. If $f(x)$ and $g(x)$ are in $L^1(-\infty, \infty)$ then the convolution integral $f * g = \int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt$ exists for almost all x, is in $L^1(-\infty, \infty)$, and satisfies the following properties

a) $f * g = g * f$

b) $(f + g) * h = f * h + g * h$, and

c) $(f * g) * h = f * (g * h)$,

where $h(x)$ is in $L^1(-\infty, \infty)$.

Given a real valued function $f(x)$, $a \leq x \leq b$, we denote by $\overset{b}{\underset{a}{\mathrm{V}}}(f)$ the total variation of $f(x)$ on $[a, b]$.

DEFINITION: The function $f(x)$, $-\infty < x < \infty$, is said to be of bounded variation $\left(\text{denoted by } \overset{\infty}{\underset{-\infty}{\mathrm{V}}}(f) < \infty\right)$ if there exists a number $M > 0$ such thal $\overset{b}{\underset{a}{\mathrm{V}}}(f) < M$ for every finite interval $[a, b]$.

If a function $f(x)$, $-\infty < x < \infty$, is absolutely continuous on every finite interval, and satisfies $f'(x) \in L^1(-\infty, \infty)$, then we have the relationship $\overset{\infty}{\underset{-\infty}{\mathrm{V}}}(f) = \int_{-\infty}^{\infty} |f'(x)|\, dx$.

A sufficient condition to pass from uniform convergence on every compact set in $(-\infty, \infty)$ to uniform convergence on $(-\infty, \infty)$ in the class of functions of bounded variation is given in the next lemma. Examples may be constructed to show it is not a necessary condition.

LEMME 2.2 Let $f_n(x), n = 1, 2, \ldots$ be of bounded variation in $-\infty < x < \infty$ and suppose $\lim_{n \to \infty} f_n(x) = f(x)$ uniformly on every compact set in $(-\infty, \infty)$, where $\overset{\infty}{\underset{-\infty}{\mathrm{V}}}(f) < \infty$. If given arbitrary $\varepsilon > 0$ there exists a positive integer N and a constant $k > 0$ such that $\underset{|x| \geq k}{\mathrm{V}}(f_n) < \varepsilon$ for $n > N$, then $\lim_{n \to \infty} f_n(x) = f(x)$ uniformly on $(-\infty, \infty)$.

PROOF : By hypothesis and since $\overset{\infty}{\underset{-\infty}{\mathrm{V}}}(f) < \infty$, given $\varepsilon > 0$ we may choose $k > 0$ such that $\underset{|x| \geq k}{\mathrm{V}} (f) < \varepsilon$ and $\underset{|x| \geq k}{\mathrm{V}} (f_n) < \varepsilon$ for n greater than some positive integer N_1. Since the convergence is uniform on $[-k, k]$ there exists a positive integer N_2 such that $n > N_2$ and $-k \leq x \leq k$ implies $|f_n(x) - f(x)| < \varepsilon$. Thus for $n > N = \max(N_1, N_2)$ and $k \leq x < \infty$ we have $|f_n(x) - f(x)| < 3\varepsilon$ and a similar result holds for $-\infty < x \leq -k$, hence the convergence is uniform on $(-\infty, \infty)$.

We shall now give several definitions concerning semiregular functions, then state without proof a result due to Turner [10] which characterizes semiregular positive seminormal functions. This result extends an analogous statement proved by Tonelli for functions of class C^1.

Let $f(x, y, d, p)$ be a real valued function on E^{3n+1}, where x is real, y, d and p are in E^n.

DEFINITION : The function $f(x, y, d, p)$ is said to be semiregular positive if $f(x, y, d, p)$ is convex in d and p for every (x, y).

DEFINITION : The function $f(x, y, d, p)$ is said to be semiregular positive seminormal if it is semiregular positive and for no points $x_0, y_0, d_0, d_1, p_0, p_1$ with $(d_1, p_1) \neq (0, 0)$ is it true that $\frac{1}{2}[f(x_0, y_0, d_0 + \lambda d_1, p_0 + \gamma p_1) + f(x_0, y_0, d_0 - \lambda d_1, p_0 - \gamma p_1)] = f(x_0, y_0, d_0, p_0)$ for all real λ and γ.

LEMMA 2.3. The function $f(x, y, d, p)$ is semiregular positive seminormal if and only if for every $(x_0, y_0, d_0, p_0) \in E^{3n+1}$ and $\varepsilon > 0$, there are constants $\delta > 0$, $\nu > 0$ and a linear function $z(d, p) = a + b \cdot d + c \cdot p$ such that for all (x, y) with $|x - x_0| < \delta$, $|y - y_0| < \delta$

a) $f(x, y, d, p) \geq z(d, p) + \nu |(d, p) - (d_0, p_0)|$ for all d and p, and
b) $f(x, y, d, p) \leq z(d, p) + \epsilon$ when $|d - d_0| < \delta$ and $|p - p_0| < \delta$.

3. Theorems on Lower Semicontinuity.

This section considers sufficient conditions for the functional $I[y] = \int_{-\infty}^{\infty} F[x, y(x), y'(x), p(x)]\, dx$ to be lower semicontinuous at an element of K, the class of functions defined in § 1, where $F(x, y, d, p)$ is a real valued continuous function defined on E^4 and $p(x)$ is one of the convolution integrals previously mentioned. We first consider the case $p(x) = y' * y'$.

THEOREM 3.1. Let $y_0(x)$, $-\infty < x < \infty$, be in K and suppose that:

a) $F[x, y(x), y'(x), p(x)] \geq 0$ for every $-\infty < x < \infty$, and every $y(x) \in K$.

b) For almost every x_0 at which $d_0(x_0) = y_0'(x_0)$ and $p_0(x_0) = (y_0' * y_0')(x_0)$ exist the following holds: for every $\varepsilon > 0$ there exists $\delta > 0$ and a linear function $z(d, p) = r + b \cdot d + c \cdot p$ such that for every $(x, y) \in E^2$ with $| x - x_0 | < \delta$ and $| y - y_0(x_0) | < \delta$ we have

i) $F(x, y, d, p) \geq z(d, p)$ for all d and p, and

ii) $F(x, y, d, p) \leq z(d, p) + \varepsilon$ for all d, p such that

$$| d - d_0(x_0) | < \delta \text{ and } | p - p_0(x_0) | < \delta.$$

If $y_n(x)$, $n = 1, 2, \dots$, $-\infty < x < \infty$, is any sequence of elements of K possessing uniformly bounded total variation and satisfying $\lim_{n \to \infty} y_n(x) = y_0(x)$ uniformly on $(-\infty, \infty)$, then $I[y_0] \leq \liminf_{n \to \infty} I[y_n]$.

PROOF: Let $y_n(x)$, $n = 1, 2, \dots$, $-\infty < x < \infty$, be such a sequence, then under the hypotheses given, the sequence is equicontinuous. This follows from the fact that the convergence is uniform on $(-\infty, \infty)$, all the functions are of bounded variation, and that a sequence of continuous functions, uniformly convergent on a closed interval, is equicontinuous.

Choose any k such that $0 < k < \infty$ and define for $y(x) \in K$

$$I_k[y] = \int_{-k}^{k} F[x, y(x), y'(x), y' * y']\, dx.$$

Let $\varepsilon > 0$ be chosen; then there exists $\tau > 0$ such that

$$\left| \int_E F(x, y_0, d_0, p_0)\, dx \right| < \varepsilon$$

for every measurable set E contained in $[-k, k]$ such that $m(E) < \tau$, where m denotes Lebesgue measure. Further, $y_0'(x) \in L^1(-\infty, \infty)$ gives $p_0(x) \in L^1(-\infty, \infty)$ and by Lusin's theorem there exists a non-empty closed set B contained in $[-k, k]$ such that $d_0(x)$ and $p_0(x)$ exists and are continuous on B, and in addition, $m([-k,k]-B)<\tau$ and for each $\bar{x}\in B$ hypothesis b) is satisfied. Thus for every $\bar{x}\in B$ there exists a $\bar{\delta}$ and a linear function $\bar{z}(d,p)=\bar{r}+\bar{b}\cdot d+\bar{c}\cdot p$ satisfying properties b.i) and b.ii).

We assert that a finite set of non-overlapping closed intervals $[\alpha_s,\beta_s]$, $s=1,\dots,N$, whose union is contained in $[-k,k]$, can be constructed such that

a) Their union forms a covering of B

b) With each interval $[\alpha_s,\beta_s]$, $s=1,\dots,N$, is associated a constant δ_s and a linear function $r_s+b_s\cdot d+c_s\cdot p$ such that hypotheses b.i) and b.ii) are satisfied for all $x\in[\alpha_s,\beta_s]$ with $x_0=x_s=\frac{1}{2}(\alpha_s+\beta_s)$ and $\delta=\delta_s$, and

c) $m\left(\bigcup_{s=1}^{N}[\alpha_s,\beta_s]-B\right)<\sigma<\frac{\varepsilon}{R}$ where $R=\max[|a_s|,|b_s|,|c_s|,s=1,\dots,N]$ and σ satisfies the following: for every measurable set $H\subset[-k,k]$ with $m(H)<\sigma$ we have $\int_H|d_0(x)|\,dx<\frac{\varepsilon}{R}$, and $\int_H|p_0(x)|\,dx<\frac{\varepsilon}{R}$.

The construction is an extension of a construction of Tonelli [9], where he proves a similar theorem for the ordinary form of the integral of the calculus of variations. See also Cesari [1] and Turner [10].

Letting $E_s=[\alpha_s,\beta_s]-B$, $s=1,\dots,N$, we have

$$I_k[y_0]=\int_{-k}^{k}F(x,y_0,d_0,p_0)\,dx\le\int_B F(x,y_0,d_0,p_0)\,dx+\varepsilon$$

$$=\sum_{s=1}^{N}\left\{\int_{\alpha_s}^{\beta_s}[r_s+b_s\cdot d_0+c_s\cdot p_0]\,dx-\int_{E_s}[r_s+b_s\cdot d_0+c_s\cdot p_0]\,dx\right\}+(1+2k)\,\varepsilon$$

$$\le\sum_{s=1}^{N}\int_{\alpha_s}^{\beta_s}[r_s+b_s\cdot d_0+c_s\cdot p_0]\,dx+(4+2k)\,\varepsilon.$$

By hypothesis $y_0(x)\in K$, hence let $\int_{-\infty}^{\infty}|y_0'(x)|\,dx=L<\infty$, and by the assumptions on the sequence $\{y_n(x)\}$ we know there exists a positive constant S such that $\overset{\infty}{\underset{-\infty}{\mathrm{V}}}(y_n)=\int_{-\infty}^{\infty}|y_n'(x)|\,dx<S$, $n=1,2,\dots$. Let $P>L+S$

and $M \geq 0$ be such that

$$|y_n(x) - y_0(x)| < \delta, \qquad -\infty < x < \infty,$$

for $n \geq M$ where

$$0 < \delta < \min\left\{\frac{\delta_1}{2}, \ldots, \frac{\delta_N}{2}, \frac{\varepsilon}{2RN}, \frac{\varepsilon}{2RNP}, \varepsilon\right\}.$$

Then

$$I_k[y_n] = \int_{-k}^{k} F(x, y_n, d_n, p_n)\, dx \geq \sum_{s=1}^{N} \int_{\alpha_s}^{\beta_s} F(x, y_n, d_n, p_n)\, dx$$

$$\geq \sum_{s=1}^{N} \int_{\alpha_s}^{\beta_s} [r_s + b_s \cdot d_n + c_s \cdot p_n]\, dx$$

and therefore,

$$I_k[y_n] - I_k[y_0] \geq \sum_{s=1}^{N} \left\{ \int_{\alpha_s}^{\beta_s} [r_s + b_s \cdot d_n + c_s \cdot p_n]\, dx - \right.$$

$$\left. - \int_{\alpha_s}^{\beta_s} [r_s + b_s \cdot d_0 + c_s \cdot p_0]\, dx \right\} - (4 + 2k)\,\varepsilon$$

$$= \sum_{s=1}^{N} b_s [y_n(\beta_s) - y_0(\beta_s) - y_n(\alpha_s) + y_0(\alpha_s)]$$

$$+ \sum_{s=1}^{N} c_s \left[\int_{\alpha_s}^{\beta_s} p_n\, d_x - \int_{\alpha_s}^{\beta_s} p_0\, d_x \right] - (4 + 2k)\,\varepsilon$$

$$\geq \sum_{s=1}^{N} c_s \left[\int_{\alpha_s}^{\beta_s} y_n' * y_n'\, dx - \int_{\alpha_s}^{\beta_s} y_0' * y_0'\, dx \right] - (5 + 2k)\,\varepsilon \quad \text{for} \quad n \geq M.$$

By commutativity of the convolution operation, the expression

$$\int_{\alpha_s}^{\beta_s} y_0' * y_n'\, dx - \int_{\alpha_s}^{\beta_s} y_n' * y_0'\, dx$$

may be inserted in the bracketed expression above. In addition, the convolution operation is distributive, and certain interchanges of limits of integration are justified since $y_0'(x)$ and $y_n'(x)$ are in $L^1(-\infty, \infty)$, $n = 1, 2, \dots$, hence we can write

$$I_k[y_n] - I_k[y_0] \geq$$

$$\geq \sum_{s=1}^{N} c_s \left\{ \int_{-\infty}^{\infty} [y_n'(t) + y_0'(t)] \left[\int_{\alpha_s}^{\beta_s} [y_n'(x-t) - y_0'(x-t)]\, dx \right] dt \right\} - (5+2k)\varepsilon$$

$$= \sum_{s=1}^{N} c_s \left\{ \int_{-\infty}^{\infty} [y_n'(t)+y_0'(t)]\, [y_n(\beta_s - t) - y_0(\beta_s - t) - y_n(\alpha_s - t) + y_0(\alpha_s - t)]\, dt \right\} - (5+2k)\varepsilon$$

$$\geq -(6+2k)\varepsilon$$

for $n \geq M$.

Since $y_0 \in K$, we have $I[y_0] < \infty$. By hypothesis, $F(x, y_0, d_0, p_0) \geq 0$, hence for arbitrary $\bar{\varepsilon} > 0$ we may choose k large enough so that

$$I[y_0] - \frac{\bar{\varepsilon}}{2} \leq I_k[y_0].$$

Now let $\varepsilon = \dfrac{\bar{\varepsilon}}{2(6+2k)}$ and then for M sufficiently large

$$I[y_0] \leq I[y_k] + \bar{\varepsilon}$$

for $n > M$ which is the desired conclusion.

COROLLARY: If all the hypotheses of Theorem 3.1 are satisfied with the exception that for *a*) and *b*) are substituted

a') $F(x, y, d, p) \geq 0$, $-\infty < x, y, d, p < \infty$, and

b') $F(x, y, d, p)$ is semiregular positive seminormal, then

$$I[y_0] \leq \liminf_{n \to \infty} I[y_n].$$

PROOF: This assertion follows from the fact that by Lemma 2.3, the condition of semiregular positive seminormality is a condition stronger than *b*) of Theorem 3.1.

EXAMPLE: The following example shows that one cannot substitute the condition

$y_n(x)$ converse uniformly to $y_0(x)$ on every compact set in $-\infty < x < \infty$

for the condition

$y_n(x)$ converge uniformly to $y_0(x)$ in $-\infty < x < \infty$

in the hypotheses of Theorem 3.1.

Consider

$$f(x, y, d, p) = \frac{(p + 4)^2}{1 + \alpha^2 x^2}$$

and for $-\infty < x < \infty$ let $y_0(x) = 0$, and for $n = 1, 2, \ldots,$

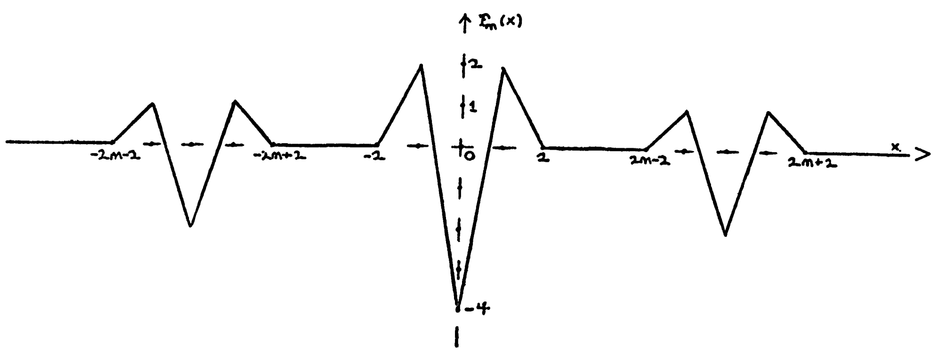
$$y_n(x) = \begin{cases} 1 - |x - n| & \text{if } |x - n| < 1 \\ 1 - |x + n| & \text{if } |x + n| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $y_n(x)$, $n = 1, 2, \ldots,$ converge in the first sense given above, but not the second, and $f(x, y, d, p)$ and all the functions concerned satisfy the remaining hypotheses of Theorem 3.1.

Computation of $p_n(x) = y'_n * y'_n$, $n = 0, 1, 2, \ldots$ gives the following:

$$p_0(x) = 0 \qquad -\infty < x < \infty,$$

and the graph of $p_n(x)$ for $n \geq 2$ is as follows:



The following estimate can be made:

$$\int_{-2n-2}^{2n+2} \frac{[p_n(x)+4]^2}{1+\alpha^2 x^2}\,dx < \frac{72}{\alpha^2}\left[1-\frac{\arc\tan\alpha}{\alpha}\right] + \frac{72}{1+\alpha^2} + \frac{200}{1+(2n-2)^2\alpha^2}.$$

Therefore,

$$I[y_n] - I[y_0] = \int_{-\infty}^{\infty} \frac{[p_n(x)+4]^2}{1+\alpha^2 x^2}\,dx - \int_{-\infty}^{\infty} \frac{[p_0(x)+4]^2}{1+\alpha^2 x^2}\,dx < \Delta_1 + \Delta_2$$

where

$$\Delta_1 = \frac{72}{\alpha^2}\left[1-\frac{\arctan\alpha}{\alpha}\right] + \frac{72}{1+\alpha^2} - \frac{32}{\alpha}\arctan 2\alpha$$

$$\Delta_2 = \frac{200}{1+(2n-2)^2\alpha^2} - \frac{32}{\alpha}[\arctan(2n+2)\alpha - \arctan(2n-2)\alpha].$$

Evidently we have $\Delta_2 \to 0$ as $n \to \infty$ for every α, while $\lim_{\alpha\to+\infty} \alpha^2 \Delta_1 = -\infty$. Hence for a sufficiently large value of α we have $\Delta_1 < 0$.

Corresponding to such a value of α, choose n sufficiently large so that $\Delta_2 < \frac{1}{2}|\Delta_1|$, and this gives

$$I[y_n] - I[y_0] < \Delta_1 + \Delta_2 < \frac{1}{2}\Delta_1 < 0.$$

But this proves that $I[y]$ is not lower semicontinuous with respect to the weaker mode of convergence.

An important extension of the concept of lower semicontinuity occurs when all the hypotheses of Theorem 3.1 are satisfied with the exception that $y_0(x)$, $-\infty < x < \infty$, is not in K in the sense that

$$I[y_0] = \int_{-\infty}^{\infty} F[x, y_0(x), y_0'(x), y_0' * y_0']\,dx$$

does not exist. The following theorem considers this important case.

THEOREM 3.2. Suppose all the hypotheses of Theorem 3.1 are satisfied with the exception that $y_0(x)$, $-\infty < x < \infty$ satisfies

1) $y_0(x)$ is absolutely continuous in every finite interval,
2) $y_0'(x)$ is in $L^1(-\infty, \infty)$, and
3) $I[y_0] = +\infty$.

Then given any sequence $y_n(x)$, $n = 1, 2, \dots$, $-\infty < x < \infty$, of elements of K possessing uniformly bounded total variation and converging uniformly on $(-\infty, \infty)$ to $y_0(x)$, and given any $H > 0$ there exists an $M > 0$ such that $I[y_n] > H$ for $n \geq M$.

PROOF: Given $H > 0$, we can find a $\tau > 0$ small enough and $k > 0$ large enough so that

a) there exists a closed set $B \subset |-k, k|$ with $m([-k, k] - B) < \tau$ and $d_0(x)$ and $p_0(x)$ exist and are continuous on B. Furthermore, every $x \in B$ satisfies the hypothesis *b*) of Theorem 3.1 with $x = x_0$, and

b) a covering of B consisting of non-overlapping closed intervals $[\alpha_s, \beta_s]$, $s = 1, \dots N$, can be constructed as in Theorem 3.1 for which we have

$$\sum_{s=1}^{N} \int_{\alpha_s}^{\beta_s} F(x, y_0, d_0, p_0)\, dx > H + 1.$$

Therefore,

$$I_k[y_n] - \sum_{s=1}^{N} \int_{\alpha_s}^{\beta_s} F(x, y_0, d_0, p_0)\, dx$$

$$\geq \sum_{s=1}^{N} \int_{\alpha_s}^{\beta_s} F(x, y_n, d_n, p_n)\, dx - \sum_{s=1}^{N} \int_{\alpha_s}^{\beta_s} F(x, y_0, d_0, p_0)\, dx$$

$$\geq \sum_{s=1}^{N} \int_{\alpha_s}^{\beta_s} (r_s + b_s \cdot d_n + c_s \cdot p_n)\, dx - \sum_{s=1}^{N} \int_{\alpha_s}^{\beta_s} F(x, y_0, d_0, p_0)\, dx.$$

But for any $\varepsilon > 0$ by the previous theorem there exists an $M > 0$ such that for $n > M$

$$I_k[y_n] - \sum_{s=1}^{N} \int_{\alpha_s}^{\beta_s} F(x, y_0, d_0, p_0)\, dx \geq \sum_{s=1}^{N} \int_{\alpha_s}^{\beta_s} (r_s + p_s \cdot d_0 + c_s \cdot p_0)\, dx - \varepsilon$$

$$- \sum_{s=1}^{N} \int_{\alpha_s}^{\beta_s} F(x, y_0, d_0, p_0)\, dx.$$

But

$$\sum_{s=1}^{N} \int_{\alpha_s}^{\beta_s} F(x, y_0, d_0, p_0)\, dx \leq \sum_{s=1}^{N} \int_{\alpha_s}^{\beta_s} [r_s + b_s \cdot d_0 + c_s \cdot p_0]\, dx + 2k\varepsilon,$$

and since $F(x, y_n, d_n, p_n) \geq 0$,

$$I[y_n] \geq I_k[y_n] \geq H + 1 - (1 + 2k)\varepsilon$$

for $n > M$ and arbitrary $\varepsilon > 0$.

REMARK: The conclusion of the above theorem holds when the condition of semiregular positive seminormality is satisfied and $F(x, y, d, p) \geq 0$, $-\infty < x, y, d, p < \infty$.

The extension of the results of the previous two theorems to the case $p(x) = y' * g$, where $g(x)$, $-\infty < x < \infty$, is given and satisfies $g(x) \in L^1(-\infty, \infty)$ and $\int_{-\infty}^{\infty} |g(x)|\, dx \neq 0$, follows easily. No essential changes in the proof are required.

The previous two results also apply in the case $p(x) = |y'| * |y'|$. Proceeding as in Theorem 3.1 one arrives at the following inequality for n large enough and $\varepsilon > 0$, $\infty > k > 0$ arbitrary:

$$I_k[y_n] - I_k[y_0] \geq \sum_{s=1}^{N} c_s \left\{ \int_{-\infty}^{\infty} (|y_n'| + |y_0'|) \left[\bigvee_{\alpha_s - t}^{\beta_s - t} (y_n) - \bigvee_{\alpha_s - t}^{\beta_s - t} (y_0) \right] dt \right\} - (5 + 2k)\varepsilon.$$

Since the total variation is lower semicontinuous in the class of functions of bounded variation on $(-\infty, \infty)$, and the $y_n(x)$ converge uniformly to $y_0(x)$ on $(-\infty, \infty)$, it can be shown that given any $\varepsilon > 0$, there exists an $M > 0$ such that

$$\bigvee_{\alpha_s - t}^{\beta_s - t} (y_n) - \bigvee_{\alpha_s - t}^{\beta_s - t} (y_0) \geq -\varepsilon$$

for $n > M$ and any value of α_s, β_s and t. The proof of the theorem now proceeds as before and leads to the desired lower semicontinuity. The case where $I[y_0]$ does not exist may be treated in a similar fashion and analogous results hold for the case $p(x) = |y'| * |g|$ where $g(x)$, $-\infty < x < \infty$, is given and satisfies $g(x) \in L^1(-\infty, \infty)$ and $\int_{-\infty}^{\infty} |g(x)|\, dx \neq 0$.

The case of vector valued functions may be considered as follows: let $F(x, y, d, p)$ be a real valued continuous function defined in E^{3n+1}, where x is real and y, d, p are in E^n. Let K be the class of functions $y = y(x) = [y_1(x), \dots, y_n(x)]$, $-\infty < x < \infty$, satisfying

1) $y_i(x)$ is absolutely continuous in every finite interval, $i = 1, \dots, n$,

2) $y_i'(x)$ is in $L^1(-\infty, \infty)$, $i = 1, \dots, n$, and

3) $F[x, y(x), y'(x), p(x)]$ is L-integrable on $(-\infty, \infty)$, where $p(x)$ is one of the following convolution integrals:

a) $p(x) = (y_1' * y_1', \dots, y_n' * y_n')$,

b) $p(x) = (|y_1'| * |y_1'|, \dots, |y_n'| * |y_n'|)$,

c) $p(x) = (y_1' * g_1, \dots, y_n' * g_n)$ where $g(x) = [g_1(x), \dots, g_n(x)]$, $-\infty < x < \infty$, is given with $g_i(x) \in L^1(-\infty, \infty)$, $i = 1, \dots, n$, or

d) $p(x) = (|y_1'| * |g_1|, \dots, |y_n'| * |g_n|)$ with $g(x)$ given as above.

Assuming K is non-empty, theorems of lower semicontinuity in K with respect to uniform convergence on $(-\infty, \infty)$ may be stated and proved. The statements and the proofs are similar to those of Theorems 3.1 and 3.2 and will not be given.

4. Theorems on Existence of a Minimum.

In this section is given a theorem on existence of a minimum of $I[y] = \int_{-\infty}^{\infty} F[x, y(x), y'(x), p(x)]\, dx$ in the class $\bar{K}$ defined in § 1, where $p(x)$ is one of the convolution integrals mentioned in § 1. We will consider the case $p(x) = y' * y'$, then mention extensions to the other values of $p(x)$ as well as for vector valued functions.

THEOREM 4.1. Consider $I[y] = \int_{-\infty}^{\infty} F[x, y(x), y'(x), p(x)]\, dx$, where $p(x) = y' * y'$, and suppose that for all $(x, y) \in E^2$:

a) $F(x, y, d, p) \geq \psi(x)$ where $\psi(x)$ is continuous and in $L^1(-\infty, \infty)$. The inequality is assumed to hold for all values of d and p.

b) $F(x, y, d, p)$ is semiregular positive (see § 2).

c) There exists a function $\Omega(s)$, defined in $0 \leq s < \infty$, such that for all s, $\Omega(s) \geq k^2$, $k \neq 0$ and constant, $\lim_{s \to \infty} \Omega(s) = \infty$, and furthermore, $F(x, y, d, p) - \psi(x) \geq |d| \, \Omega(|d|)$. The inequality is assumed to hold for any value of p.

d) For any value of d, there exists a constant $L > 0$ such that

$$F(x, y, d, p) - F(x, y, d, \bar{p}) \leq L \,|\, p - \bar{p} \,|.$$

Furthermore, suppose that given a class of curves $\bar{K}$ the following is satisfied:

e) Given any $y = y(x)$, $-\infty < x < \infty$, in $\bar{K}$, there exist constants $M > 0$ and $a > 0$, with M not depending on y, and a function $\Phi_{y,a}(x)$, which is L-integrable on $|x| \geq a$, such that

$$F[x, \bar{y}(x), \bar{y}'(x), \bar{p}(x)] - F[x, y(x), y'(x), p(x)] \geq M(\,|\,\bar{y}'(x)\,| - |\,y'(x)\,|\,) - \Phi_{y,a}(x)$$

holds for $|x| \geq a$ and any other $\bar{y} = \bar{y}(x)$, $-\infty < x < \infty$, in $\bar{K}$.

f) There exists at least one curve $y = y(x)$, $-\infty < x < \infty$, in $\bar{K}$ and a constant N such that $I[y] - \int_{-\infty}^{\infty} \psi(x)\, dx \leq N$, where N satisfies $NL < k^2 M$.

Then $I[y]$ possesses an absolute minimum in $\bar{K}$.

PROOF: If we define $F_1(x, y, d, p) = F(x, y, d, p) - \psi(x)$ and $I_1[y] = \int_{-\infty}^{\infty} F_1[x. y(x), y'(x), p(x)]\, dx$, then by hypotheses *a*), *b*) and *c*) we have

i) $i_1 = \inf_{y \in \bar{K}} I_1[y] \geq 0$,

ii) $F_1(x, y, d, p) \geq 0$ for $-\infty < x, y, d, p < \infty$, and

iii) $F_1(x, y, d, p)$ is semiregular positive seminormal.

Let $[y_n(x)]$, $n = 1, 2, \ldots, -\infty < x < \infty$, be a minimizing sequence of elements of $\bar{K}$ satisfying $i_1 \leq I[y_n] \leq i_1 + \frac{1}{n}$. By the assumptions given for the class $\bar{K}$, there exists a positive constant Q such that for each $n = 1, 2, \ldots$, there exists an x_n such that

$$|\,x_n\,| \leq Q, \quad |\,y_n(x_n)\,| \leq Q, \quad \text{and} \quad [x_n, y_n(x_n)] \in P.$$

Since $y_n'(x) \in L^1(-\infty, \infty)$ we may write for all x and $n = 1, 2, \ldots$

$$|\,y_n(x)\,| - Q \leq |\,y_n(x)\,| - |\,y_n(x_n)\,| \leq |\,y_n(x) - y_n(x_n)\,| = \left| \int_{x_n}^{x} y_n'(t)\, dt \right|$$

$$\leq \int_{-\infty}^{\infty} |\,y_n'(x)\,|\, dx \leq \frac{1}{k^2} I_1[y_n] \leq \frac{1}{k^2}\left[i_1 + \frac{1}{n}\right] < \frac{1}{k^2}(i_1 + 1).$$

Hence the $[y_n(x)]$, $n = 1, 2, \ldots, -\infty < x < \infty$, are equibounded.

In view of hypothesis c), the usual argument of the direct methods (see for instance, Tonelli [9]) enables us to assert that the $y_n(x)$, $n = 1, 2, \ldots$, are equiabsolutely continuous in $(-\infty, \infty)$. Therefore, the hypotheses of the Ascoli-Arzela theorem are satisfied, and there exists a subsequence which converges uniformly on every compact set in $(-\infty, \infty)$ to an absolutely continuous function $y_0(x)$, $-\infty < x < \infty$. To avoid added indices, denote the subsequence by $y_n(x)$, $n = 1, 2, \ldots$, $-\infty < x < \infty$.

Since the set P is compact and the convergence is uniform on P, there exists a point $[x_0, y_0(x_0)] \in P$. By lower semicontinuity of the total variation we have for any $t > 0$

$$\int_{-t}^{t} | y_0'(x) | \, dx = \bigvee_{-t}^{t} (y_0) \leq \liminf_{n \to \infty} \bigvee_{-t}^{t} (y_n) \leq \liminf_{n \to \infty} \int_{-\infty}^{\infty} | y_n'(x) | \, dx \leq \frac{1}{k^2}(i_1 + 1).$$

The last expression is independent of the choice of t, and hence we may assert that $y_0'(x)$ is in $L^1(-\infty, \infty)$. The following analysis shows that $I[y_0]$ exists.

Given $\varepsilon > 0$ there exists $A > 0$ such that $\bigvee_{|x| \geq A/2} (y_0) < \varepsilon$. Define

$$\bar{y}_n(x) = \begin{cases} y_n(x) & \text{if} \quad |x| \leq A, \\ y_0(x) + [y_n(A) - y_0(A)] & \text{if} \quad x > A, \\ y_0(x) + [y_n(-A) - y_0(-A)] & \text{if} \quad x < -A, \qquad n = 1, 2, \ldots, \end{cases}$$

hence

$$\bar{y}_n'(x) = \begin{cases} y_n'(x) & \text{if} \quad |x| < A, \\ y_0'(x) & \text{if} \quad |x| > A. \end{cases}$$

Furthermore $\bar{y}_n(x)$, $n = 1, 2, \ldots$, $-\infty < x < \infty$, are absolutely continuous in every finite interval, $\bar{y}_n'(x) \in L^1(-\infty, \infty)$, and $\lim_{n \to \infty} \bar{y}_n(x) = y_0(x)$ uniformly in $(-\infty, \infty)$. Suppose $I_1[y_0]$ does not exist, then by Theorem 3.2 given any $H > 0$ there exists a positive integer M such that $n \geq M$ implies $I_1[\bar{y}_n] > H$.

For $|t| \leq A/2$ and $|x| \leq A/2$ we have $|x - t| \leq A$, and

$$p_n(x) - \bar{p}_n(x) = \int_{|t| \geq A/2} [y_n'(x - t)\, y_n'(t) - \bar{y}_n'(x - t)\, \bar{y}_n'(t)] \, dt.$$

Therefore

$$\int_{-A/2}^{A/2} | p_n(x) - \bar{p}_n(x) | \, dx \leq \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} | y_n'(x-t) | \, | y_n'(t) | \, dt +$$

$$+ \int_{|t| \geq A/2} | y_0'(t) | \left[\int_{-A/2-t}^{A/2-t} | \bar{y}_n'(u) | \, du \right] dt$$

$$\leq \frac{1}{k^4} (i_1 + 1)^2 + \varepsilon \left(\frac{i_1 + 1}{k^2} + \varepsilon \right).$$

By hypothesis d) we have

$$0 \leq \int_{-A/2}^{A/2} F_1 [x, \bar{y}_n(x) \bar{y}_n'(x), \bar{p}_n(x)] \, dx \leq L \left[\frac{1}{k^4} (i_1 + 1)^2 + \varepsilon \left(\frac{i_1 + 1}{k^2} + \varepsilon \right) \right]$$

$$+ \int_{-A/2}^{A/2} F [x, \bar{y}_n(x), \bar{y}_n'(x), p_n(x)] \, dx$$

$$\leq L \left[\frac{1}{k^4} (i_1 + 1)^2 + \varepsilon \left(\frac{i_1 + 1}{k^2} + \varepsilon \right) \right] + (i_1 + 1).$$

But the right hand expression is independent of n and the choice of A, therefore the $I_1 [\bar{y}_n]$, $n = 1, 2, \ldots$, are uniformly bounded which is a contradiction.

Since $\bar{K}$ was assumed to be closed with respect to uniform convergence on every compact set in $(-\infty, \infty)$, we may assert that $y_0(x) \in \bar{K}$, and hence $I_1 [y_0] \geq i_1$. We may now employ hypothesis e), and choose A sufficienthy large so that in addition $A/2 \geq a$, and $\int_{|x| \geq A/2} | \Phi y_{0,a}, (x) | dx < \varepsilon$.

Neglecting the term of order ε^2, a modification of the above estimate leads to the following inequality

$$\int_{-A/2}^{A/2} | p_n(x) - \bar{p}_n(x) | \, dx \leq \frac{1}{k^2} \left(i_1 + \frac{1}{n} \right) \int_{|x| \geq A/2} | y_n'(x) | \, dx + \frac{\varepsilon}{k^2} (i_1 + 1)$$

$$n = 1, 2, \ldots .$$

By hypothesis d) it follows that

$$\int_{-A/2}^{A/2} \{F_1[x, \bar{y}_n(x), \bar{y}_n'(x), \bar{p}_n(x)] - F_1[x, \bar{y}_n(x), \bar{y}_n'(x), p_n(x)]\}\, dx$$

$$= \int_{-A/2}^{A/2} \{F_1[x, \bar{y}_n(x), \bar{y}_n'(x), \bar{p}_n(x)] - F_1[x, y_n(x), y_n'(x), p_n(x)]\}\, dx$$

$$\leq L \int_{-A/2}^{A/2} |\bar{p}_n(x) - p_n(x)|\, dx \leq \frac{L}{k^2}\left(i_1 + \frac{1}{n}\right) \int_{|x| \geq A/2} |y_n(x)|\, dt + \frac{L}{k^2}(i_1 + 1)\, \varepsilon$$

$$n = 1, 2, \dots .$$

By hypothesis e) we have

$$\int_{|x| \geq A/2} \{F_1[x, y_n(x), y_n'(x), p_n(x)] - F_1[x, y_0(y), y_0'(x), p_0(x)]\}\, dx$$

$$\geq M \int_{|x| \geq A/2} [\,|y_n'(x)| - |y_0'(x)|\,]\, dx - \int_{|x| \geq A/2} \Phi_{y_0, a}(x)\, dx$$

$$\geq M \int_{|x| \geq A/2} |y_n'(x)|\, dx - \varepsilon(M + 1).$$

Since the $\bar{y}_n(x)$, $n = 0, 1, 2, \dots$, are absolutely continuous in $-\infty < x < \infty$, $\bar{y}_n'(x) \in L^1(-\infty, \infty)$, and $\lim_{n \to \infty} \bar{y}_n(x) = y_0(x)$ uniformly on $-\infty < x < \infty$, by the proof of Theorem 3.1 we can assert that for n large enough,

$$\int_{-A/2}^{A/2} F_1[x, \bar{y}_n(x), \bar{y}_n'(x), \bar{p}_n(x)]\, dx - \int_{-A/2}^{A/2} F_1[x, y_0(x), y_0'(x), p_0(x)]\, dx > -\varepsilon.$$

Therefore, we arrive at the following estimate:

$$\frac{1}{n} = i_1 + \frac{1}{n} - i_1 \geq \int_{-\infty}^{\infty} F_1[x, y_n(x), y_n'(x), p_n(x)]\, dx - \int_{-\infty}^{\infty} F_1[x, y_0(x), y_0'(x), p_0(x)]\, dx$$

$$\geq M \int_{|x| \geq A/2} |y_n'(x)|\, dx - \varepsilon(M + 1) + \int_{-A/2}^{A/2} F_1[x, \bar{y}_n(x), \bar{y}_n'(x), p_n(x)]\, dx$$

$$-\int_{-A/2}^{A/2} F_1\,[x, y_0(x), y_0'(x), p_0(x)]\,dx$$

$$\geq M \int_{|x| \geq A/2} |\,y_n'(x)\,|\,dx - \varepsilon\,(M+1) + \int_{-A/2}^{A/2} F_1\,[x, \bar{y}_n(x), \bar{y}_n'(x), \bar{p}_n(x)]\,dx$$

$$-\int_{-A/2}^{A/2} F_1\,[x, y_0(x), y_0'(x), p_0(x)]\,dx - \frac{L}{k^2}\left(i_1 + \frac{1}{n}\right) \int_{|x| \geq A/2} |\,y_n'(x)\,|\,dx - \frac{L}{k^2}\,(i_1+1)\,\varepsilon$$

$$\geq \left(M - \frac{L}{k^2}\,i_1\right) \int_{|x| \geq A/2} |\,y_n'(x)\,|\,dx - \varepsilon\left[M + 2 + \frac{L}{k^2}\,(i_1+1)\right] - \frac{L}{nk^4}\,(i_1+1).$$

By hypothesis *f*) there exists a $y = y(x)$ in $\bar{K}$ such that

$$I_1\,[y] = I\,[y] - \int_{-\infty}^{\infty} \psi(x)\,dx \leq N < \frac{k^2 M}{L},$$

and it follows that

$$M - \frac{L}{k^2}\,i_1 = M - \frac{L}{k^2} \inf_{y \in K} I_1\,[y] > 0.$$

Hence we have

$$\int_{|x| \geq A/2} |\,y_n'(x)\,|\,dx \leq \frac{\varepsilon\left[M + 2 + \frac{L}{k^2}\,(i_1+1)\right] + \frac{1}{n}\left[\frac{L}{k^4}\,(i_1+1) + 1\right]}{M - \frac{L}{k^2}\,i_1}$$

and since $\varepsilon > 0$ was arbitrary, for n large enough the right-hand expression can be made as small as desired. Since the choice of A did not depend on the $y_n(x)$, $n = 1, 2, \ldots$ we can assert by Lemma 2.2 that the $[y_n(x)]$ converge uniformly on $(-\infty, \infty)$ to $y_0(x)$.

Since $F_1(x, y, d, p)$ is semiregular positive seminormal the hypotheses of the corollary of Theorem 3.1 are satisfied, and

$$I_1\,[y_0] \leq \underline{\lim}_{n \to \infty}\, I_1\,[y_n] \leq \underline{\lim}_{n \to \infty} \left[i_1 + \frac{1}{n}\right] = i_1\,.$$

But $y_0(x) \in \overline{K}$ implies $I_1[y_0] \geq i_1$ and therefore $I_1[y_0] = i_1$. Thus $y_0(x)$, $-\infty < x < \infty$, gives $I_1[y]$, and hence gives $I[y]$, an absolute minimum in $\overline{K}$.

REMARK: It should be noted that conditions of the type $NL < k^2N$ in hypothesis *f*) are found in a different setting in [1, 2]. The following is an example of a class $\overline{K}$ and an integral $I[y]$ satisfying the hypotheses of Theorem 4.1.

EXAMPLE: We wish to minimize

$$I[y] = \int_{-\infty}^{\infty} \left[|y'(x)| (|y'(x)| + 1) + \left| \frac{y' * y'}{4 + x^2} \right| \right] dx.$$

In the class $\overline{K}$ of all curves $y = y(x)$, $-\infty < x < \infty$, satisfying the hypotheses given in § 1 and satisfying

i) $|y'(x)| \leq D$, where D is a given constant with $1 < D < \infty$, and the inequality holds for all x where derivatives (one sided or two sided) are defined,

ii) $y(0) = 0$, and

iii) $y(2) = 1$.

We will show hypotheses *a*) through *f*) of Theorem 4.1 are satisfied, hence $I[y]$ possesses an absolute minimum in $\overline{K}$, as follows:

a) $F(x, y, d, p) = |d|(|d| + 1) + \dfrac{|p|}{4 + x^2} \geq 0$, hence let $\psi(x) = 0$.

b) $F(x, y, d, p)$ is semiregular positive seminormal.

c) $F(x, y, d, p) \geq |d|(|d| + 1)$, hence let $\Omega(s) = |s| + 1$ and let $k^2 = 1$.

d) $F(x, y, d, p) - F(x, y, d, \bar{p}) = \dfrac{|p| - |\bar{p}|}{4 + x^2} \leq \dfrac{1}{4} |p - \bar{p}|$, hence let $L = \dfrac{1}{4}$.

e) Every $y(x)$ in $\overline{K}$ passes through $P = [(0, 0)]$.

f) $F(x, \bar{y}(x), \bar{y}'(x), \bar{p}(x)) - F(x, y(x), y'(x), p(x)) = (|\bar{y}'(x)| - |y'(x)|)$
$(|\bar{y}'(x)| + |y'(x)| + 1) + \dfrac{|\bar{p}(x)| - |p(x)|}{4 + x^2} \geq (|\bar{y}'(x)| - |y'(x)|) - \dfrac{|p(x)|}{4 + x^2}$.

Furthermore $|p(x)| = \left| \displaystyle\int_{-\infty}^{\infty} y'(x - t)\, y'(t)\, dt \right| \leq D \displaystyle\int_{-\infty}^{\infty} |y'(t)|\, dt < \infty$ since $y(x) \in \overline{K}$. Thus let $M = 1$ and let $\Phi_{y,a}(x) = \dfrac{D \int_{-\infty}^{\infty} |y'(t)|\, dt}{4 + x^2}$ with $a > 0$ and arbitrary.

$$g)\ \text{Consider}\ y(x) = \begin{cases} 0 \text{ if } x \leq 0, \\ x \text{ if } 0 \leq x \leq 1, \\ 1 \text{ otherwise,} \end{cases}$$

and $y(x) \in \bar{K}$. Furthermore

$$I(y) = 2 + \int_0^1 \frac{xdx}{4+x^2}\,dx + \int_1^2 \frac{2-x}{4+x^2}\,dx \sim 2.21.$$

Hence we may let $N = 3$, and $NL = 3/4 < k^2 M = 1$.

COROLLARY. Suppose all the hypotheses of Theorem 4.1 are satisfied with the exception that for $e)$ is substituted

$e')$ There exist constants $M > 0$ and $a > 0$ such that for $|x| \geq a$ and all y_1, y_2, d_1, d_2, p_1, and p_2 the following holds:

$$F(x, y_1, d_1, p_1) - F(x, y_2, d_2, p_2) \geq M(|d_1| - |d_2|) - \Phi(x),$$

where $\Phi(x)$ is L-integrable on $|x| \geq a$.
Then $I[y]$ possesses an absolute minimum in $\bar{K}$.

The extension of Theorem 4.1 and the corollary to the case $p(x) = |y'| * |y'|$ follows immediately. The statements and proofs are the same and will not be given. For the case $p(x) = y' * g$ a modification is required.

THEOREM 4.2. Consider $I[y] = \int_{-\infty}^{\infty} F[x, y(x), y'(x), p(x)]\,dx$ with $p(x) = y' * g$ where $g = g(x)$, $-\infty < x < \infty$, is given, $g(x) \in L^1(-\infty, \infty)$ and $\int_{-\infty}^{\infty} |g(x)|\,dx \neq 0$. Suppose that hypotheses $a)$ through $e)$ of Theorem 4.1 are satisfied and

$f)$ The constants M and L satisfy $M - L\int_{-\infty}^{\infty} |g(x)|\,dx > 0$. Then $I[y]$ possesses an absolute minimum in $\bar{K}$.

PROOF: No changes are made in the construction of a minimizing subsequence $y_n(x) \in \bar{K}$, $n = 1, 2, \dots$, $-\infty < x < \infty$, which converges uniformly on every compact set in $-\infty < x < \infty$ to $y_0(x)$, $-\infty < x < \infty$. Similarly, it can be shown that $y_0(x) \in \bar{K}$.

Given $\varepsilon > 0$, we choose the constant A, and define the functions $\bar{y}_n(x)$, $n = 0, 1, 2, \ldots, -\infty < x < \infty$, as in the proof of Theorem 4.1. Then for $|x| \leq A/2$, a similar computation gives

$$\int_{-A/2}^{A/2} | p_n(x) - \bar{p}_n(x) | \, dx \leq \int_{-A/2}^{A/2} \left[\int_{|t| \geq A/2} | y_n'(t) | \, | g(x-t) | \, dt \right] dx$$

$$\leq \int_{-\infty}^{\infty} | g(x) | \, dx \left[\int_{x \geq A/2} | y_n'(x) | \, dx \right].$$

The proof now follows that of Theorem 4.1, and leads to the following inequality:

$$\frac{1}{n} \geq \left[M - L \int_{-\infty}^{\infty} | g(x) | \, dx \right] \int_{x \geq A/2} | y_n'(x) | \, dx - \varepsilon \left[M + 2 - L \int_{-\infty}^{\infty} | g(x) | \, dx \right].$$

By hypothesis $f)$ the expression in the first bracket is positive, and the proof proceeds as in Theorem 4.1. Hence we may assert that $y_0(x)$ gives $I[y]$ an absolute minimum in $\bar{K}$.

The extension to the case $p(x) = | y' | * | g |$ follows from the above, and the statement and the proof of the corresponding theorem is the same. The statements of the above existence theorems and their proofs for the cases of vector valued functions (as defined in § 3) are analogous and will not be given.

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