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APPLICATION OF THE HOLMGREN-RIESZ TRANSFORM

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1. Introduction.

In several recent works [1, 2, 3] M. A. Bassam has studied the *H-R* transform and its application. In particular he has found it of some use in solving differential equations of the Fuchsian type. In this paper the application of the *H-R* transform to the equation

$$(1.1)\qquad (a_2 x^2 + b_2 x + c_2) y'' + (a_1 x + b_1) y' + c_0 y = 0,$$

where a_1, a_2, b_1, b_2 and c_0 are real constants, is considered in more detail.

In the *H-R* form of (1.1) the exponential function is introduced to remove some undesirable restrictions on the form given by Bassam. Also the range for which the transform is applicable to (1.1) is extended to $R(\alpha) < 0$. A « generalized » Rodriques formula is discussed and some « generalized » formulas for the Laguerre and Legendre functions are given.

Throughout the paper the symbol $\overset{x}{\underset{a}{I}}{}^{\alpha} f(x)$ will represent the *H-R* transform of the real valued function $f(x)$ on the interval $[a, b]$. In some cases, when more meaningful, the operator $\overset{x}{\underset{a}{I}}{}^{\alpha}$ will be replaced by its equivalent, ${}_aD_x^{-\alpha}$. The letters m and n will always denote positive integers.

2. Definitions.

If $f(x)$ is a real valued function of class $C^{(n)}$ on $[a, b]$ and $0 < R(\alpha + n) \leq 1$ then

$$(2.1)\qquad {}_aD_x^{-\alpha} f(x) = \overset{x}{\underset{a}{I}}{}^{\alpha} f(x) = \frac{D_x^n}{\Gamma(\alpha + n)} \int_a^x (x - t)^{\alpha+n-1} f(t)\, dt,$$

$$(2.2)\qquad {}_{-\infty}D_x^{-\alpha} f(x) = \overset{x}{\underset{-\infty}{I}}{}^{\alpha} f(x) = \lim_{a \to -\infty} \overset{x}{\underset{a}{I}}{}^{\alpha} f(x).$$

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3. **THEOREM 1.** If $f(x) = a_2 x^2 + b_2 x + c_2$ and

$$Q(x) = \frac{1}{f(x)} \exp\left(\int \frac{b_1 x + c_1}{f(x)}\, dx\right), \quad f(x) \neq 0,$$

then the differential equation (1.1) has the equivalent H-R form

$$\overset{x}{\underset{a}{I}}{}^{-\alpha} \frac{f^{\alpha}(x)}{Q(x)} D f^{1-\alpha}(x)\, Q(x) \overset{x}{\underset{a}{I}}{}^{\alpha-1} y = 0, \tag{3.1}$$

where

$$a_2 \alpha^2 + (a_2 - b_1)\alpha + c_0 = 0, \tag{3.2}$$

provided $y(x)$ is a real valued function of class $C^{(n)}$ on $[a, b]$ and one of the following conditions is satisfied:

(i) $R(\alpha) \geq 1$

(ii) $\alpha = 0$

(iii) $0 < R(\alpha) < 1$ and $y(a) = 0$.

PROOF. Expanding (3.1) gives

$$f(x) \overset{x}{\underset{a}{I}}{}^{-\alpha} \overset{x}{\underset{a}{I}}{}^{\alpha-2} y + \alpha(2a_2 x + b_2) \overset{x}{\underset{a}{I}}{}^{-\alpha+1} \overset{x}{\underset{a}{I}}{}^{\alpha-2} y + \alpha(\alpha - 1) a_2 \overset{x}{\underset{a}{I}}{}^{-\alpha+2} \overset{x}{\underset{a}{I}}{}^{\alpha-2} y$$

$$+ [(b_1 - 2a_2\alpha)x + c_1 - b_2\alpha] \overset{x}{\underset{a}{I}}{}^{-\alpha} \overset{x}{\underset{a}{I}}{}^{\alpha-1} y + \alpha(b_1 - 2a_2\alpha) \overset{x}{\underset{a}{I}}{}^{-\alpha+1} \overset{x}{\underset{a}{I}}{}^{\alpha-1} y = 0.$$

Hence if one of the conditions i-iii are met the indices can be added so that one gets (1.1). By reversing the argument the equivalence follows.

THEOREM 2. If $f(x)$, $Q(x)$, $y(x)$ and α are defined as is theorem 1, then, if $R(\alpha) < 0$, the differential equation (1.1) has the equivalent H-R form

$$\overset{x}{\underset{a}{I}}{}^{\alpha-1} f^{1-\alpha}(x)\, Q(x)\, D \frac{f^{\alpha}(x)}{Q(x)} \overset{x}{\underset{a}{I}}{}^{-\alpha} Q(x)\, y = 0. \tag{3.3}$$ [1]

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[1] I would like to extend credit to D. R. Myrick for his help in establishing this result.

PROOF: If $R(\alpha) < 0$ (3.3) can be expanded in the same manner as (3.1) to give

$$f(x)[Q(x)y]'' + [(4a_2 - b_1)x + b_2 - c_1][Q(x)y]' + (1-\alpha)(2a_2 + a_2\alpha - b_1)Q(x)y = 0.$$

Hence

$$Q(x)[f(x)y'' + (b_1 x + c_1)y' + c_0 y] = 0$$

so that (1.1) follows. Again the argument can be reversed so that the equivalence is established.

4. General Solution.

By defining the first $n-1$ arbitrary constants of the H-R equation (3.1) as zero, the solution of (1.1) is easily seen to be

$$y = c_1 \overset{x}{\underset{a}{I}}{}^{-\alpha+1} \frac{f^{\alpha-1}(x)}{Q(x)} + c_2 \overset{x}{\underset{a}{I}}{}^{-\alpha+1} \frac{f^{\alpha-1}(x)}{Q(x)} \int^{x} f^{-\alpha}(t)\, Q(t)\, t^{\alpha-n}\, dt, \tag{4.1}$$

where $R(\alpha) \geq 0$, $a \neq -\infty$ and $n = [\alpha] + 1$.

The necessity for letting the constants be zero can be seen by substituting the solution in (3.1). If $a = -\infty$ it follows (see [2]) that the solution is the same with the $t^{\alpha-n}$ factor deleted. Similarly if $R(\alpha) < 0$ and $a \neq -\infty$ the solution of (1.1) is by (3.3)

$$Q(x)y = c_1 \overset{x}{\underset{a}{I}}{}^{\alpha} f^{-\alpha}(x)\, Q(x) + c_2 \overset{x}{\underset{a}{I}}{}^{\alpha} f^{-\alpha}(x)\, Q(x) \int^{x} \frac{f^{\alpha-1}(t)\, t^{\alpha-n}}{Q(t)}\, dt, \tag{4.2}$$

where $n = [\alpha] + 1$.

As before if $a = -\infty$ the solution is the same but with the $t^{\alpha-n}$ factor removed. In both solutions a is of course chosen to meet the continuity conditions on $y(x)$.

5. General solution when α is an integer.

When α is an integer the above results yield the solution of (1.1) in such a simple manner that it deserves special mention. That is, if $\alpha = n$ (1.1) can be written

$$D^n f^n(x) \frac{1}{Q(x)} D f^{-n+1}(y)\, Q(x)\, D^{-n+1} y = 0. \tag{5.1}$$

By defining the first $n-1$ constants of integration as zero the solution is obvious. Similarly if $\alpha = -n$ (1.1) has the equivalent form

$$D^{n+1} f^{n+1}(x)\, Q(x)\, D\, f^{-n}(x) \frac{1}{Q(x)} D^{-n} Q(x)\, y = 0. \tag{5.2}$$

As above, the solution is by inspection. Note that the nonhomogeneous equation could be handled here just as well and that a particular solution could be obtained without the knowledge of either solution of the homogeneous equation.

6. Rodriques formula.

A « generalized » Rodriques formula may be defined as follows:
If $f(x)$ and $Q(x)$ are defined as in theorem 1, and c is a constant, then

$$y = \frac{c}{Q(x)}\, {}_aD_x^{\beta}\, [f^{\beta}(x)\, Q(x)], \quad R(\beta) \geq 0,$$

will be called a Rodriques formula.

It follows at once that if $R(\alpha) \leq 0$ one solution of (1.1) can always be put in the Rodriques form, and if $R(\alpha) > 0$ there is always a solution of an analogous form.

Consider now the following examples.

(A) Laguerre equation. Take the Laguerre equation

$$xy'' + (\mu + 1 - x)\, y' + \beta y = 0. \tag{6.1}$$

By (3.2) $\alpha = -\beta$ sothat the corresponding H-R form is

$$\overset{x}{\underset{a}{I}}{}^{-\beta-1}\, x^{\beta+\mu+1} e^{-x}\, D\, x^{-\beta-\mu} e^{x} \overset{x}{\underset{a}{I}}{}^{\beta}\, e^{-x} x^{\mu} y = 0\,; \ \beta \geq 0 \tag{6.2}$$

and

$$\overset{x}{\underset{a}{I}}{}^{\beta}\, x^{-\beta-\mu} e^{x}\, D\, x^{\beta+\mu+1} e^{-x} \overset{x}{\underset{a}{I}}{}^{-\beta-1}\, y = 0\,; \qquad \beta \leq 0 \tag{6.3}$$

The solution then follows by inspection. In particular consider the solution when $\beta \geq 0$. Choosing $a = 0$ in (6.2) gives

$$y = c_1 x^{-\mu} e^{x}\, {}_0D_x^{\beta}\, e^{-x} x^{\beta+\mu} + c_2 x^{-\mu} e^{x}\, {}_0D_x^{\beta}\, e^{-x} x^{\beta+\mu} \int^{x} e^{t}\, t^{-\mu-n-1}\, dt.$$

Now if $c_1 = \dfrac{1}{\Gamma(\beta+1)}$ and $\beta = n$ the first solution is the Rodriques formula for the Laguerre polynomials. However the solution is not dependent on β being an integer and as a matter of fact yields the Laguerre function for all $\beta \geq 0$ provided $\mu + \beta \neq -m$. That is,

$$L_\beta^\mu(x) = \frac{1}{\Gamma(\beta+1)} x^{-\mu} e^x \, {}_0D_x^\beta \, e^{-x} x^{\beta+\mu} \tag{6.4}$$

when $\beta \geq 0$ and $\beta + \mu \neq -m$.

PROOF. If $\beta = n$ the result is well known. Hence suppose $\beta \neq n$. Then

$$\begin{aligned} {}_0D_x^\beta \, e^{-x} x^{\beta+\mu} &= {}_0D_x^\beta \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+\mu+\beta}}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(k+\mu+\beta+1)\, x^{k+\mu}}{\Gamma(k+\beta+1)\, k!} . \end{aligned}$$

Therefore if $\mu + \beta \neq -m$

$$\frac{1}{\Gamma(\beta+1)} x^{-\mu} e^x \, {}_0D_x^\beta \, e^{-x} x^{\beta+\mu} = \frac{1}{\Gamma(\beta+1)} \sum_{k=0}^{\infty} \sum_{p=0}^{k} \frac{(-1)^p \Gamma(p+\mu+\beta+1) x^k}{\Gamma(k+\beta+1)\, p!\,(k-p)!} .$$

But

$$\sum_{p=0}^{k} (-1)^p \frac{\Gamma(p+\mu+\beta+1)}{\Gamma(p+\beta+1)} \binom{k}{p} = \frac{\Gamma(\mu+\beta+1)\,\Gamma(k-\beta)}{\Gamma(-\beta)\,\Gamma(k+\mu+1)}, \quad k \geq 0.$$

Therefore

$$\begin{aligned} \frac{1}{\Gamma(\beta+1)} x^{-\mu} e^x \, {}_0D_x^\beta \, e^{-x} x^{\beta+\mu} &= \frac{\Gamma(\mu+\beta+1)}{\beta\Gamma(\beta)\,\Gamma(-\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(k-\beta)\, x^k}{\Gamma(k+\mu+1)\, k!} \\ &= -\frac{\sin \pi\beta}{\pi} \Gamma(\mu+\beta+1) \sum_{k=0}^{\infty} \frac{\Gamma(k-\beta)\, x^k}{\Gamma(k+\mu+1)\, k!} \\ &= L_\beta^\mu(x). \end{aligned}$$

(B) Legendre equation. For the Legendre equation,

$$(1-x^2)\, y'' - 2x\, y' + \beta(\beta+1)\, y = 0,$$

$\alpha = \beta + 1$ or $\alpha = -\beta$. Taking $\alpha = \beta + 1$, where $\beta > -1$, gives the *H-R* form

$$\overset{x}{\underset{a}{I}}{}^{-\beta-1}(1-x^2)^{\beta+1} D (1-x^2)^{-\beta} \overset{x}{\underset{a}{I}}{}^{\beta} y = 0.$$

Hence for proper a

$$y = c_1\, {}_aD_x^\beta (1-x^2)^\beta + c_2\, {}_aD_x^\beta (1-x^2)^\beta \int^x (1-t^2)^{-\beta-1} t^{\beta-n}\, dt, \tag{6.5}$$

where the $t^{\beta-n}$ factor is deleted if $a = -\infty$. The solution (6.5) is in a very useful form and suggests the following formulas for the Legendre functions:

$$P_\beta(x) = \frac{(-1)^\beta}{2^\beta \Gamma(\beta+1)} = {}_0D_x^\beta (1-x^2)^\beta\,, \qquad \beta > -1; \tag{6.6}$$

$$Q_\beta(x) = \frac{(-1)^\beta 2^\beta \Gamma(\beta+1)}{\Gamma(2\beta+1)}\, {}_{-\infty}D_x^\beta (1-x^2)^\beta \int_{-\infty}^x (1-t^2)^{-\beta-1}\, dt, \tag{6.7}$$

where $\beta \geq 0$ and $|x| > 1$;

$$Q_\beta(x) = \frac{(-1)^\beta 2^\beta \Gamma(\beta+1)}{\Gamma(2\beta+1)}\, {}_0D_x^\beta (1-x^2)^\beta \int^x (1-t^2)^{-n-1} t^{\beta-n}\, dt, \tag{6.8}$$

where $n = [\beta] + 1$, $\beta \geq 0$ and $|x| < 1$.

That (6.6), (6.7) and (6.8) are formulas for the Legendre functions can be shown as follows. If $\beta = n$ the results are well known. Hence suppose $\beta \neq n$, then

$$ {}_0D_x^\beta (1-x^2)^\beta = (-1)^\beta\, {}_0D_x^\beta \sum_{k=0}^{\infty} (-1)^k \binom{\beta}{k} x^{2\beta-2k}$$

$$= (-1)^\beta \sum_{k=0}^{\infty} (-1)^k \binom{\beta}{k} \frac{\Gamma(2\beta-2k+1)}{\Gamma(\beta-2k+1)} x^{\beta-2k}\,.$$

Therefore

$$\frac{(-1)^\beta}{2^\beta \Gamma(\beta+1)}\, {}_0D_x^\beta (1-x^2)^\beta = \frac{\Gamma(2\beta+1)}{2^\beta [\Gamma(\beta+1)]^2} \sum_{k=0}^{\infty} (-1)^k \binom{\beta}{k}$$

$$\frac{\Gamma(2\beta-2k+1)(\beta+1)\, x^{\beta-2k}}{\Gamma(2\beta+1)\Gamma(\beta-2k+1)} = P_\beta(x).$$

Now to prove (6.7)

$$_{-\infty}D_x^\beta (1-x^2)^\beta \int_{-\infty}^{x} (1-t^2)^{-\beta-1}\,dt$$

$$= \lim_{a\to-\infty} \frac{D_x^n(-1)}{\Gamma(n-\beta)} \int_a^x (x-t)^{n-\beta-1} (t^2-1)^\beta \int_{-\infty}^{t} (y^2-1)^{-\beta-1}\,dy\,dt,$$

where $n=[\beta]+1$.

Then since $|x|>1$ one gets

$$\lim_{a\to-\infty} \frac{D_x^n}{\Gamma(n-\beta)} \int_a^x (x-t)^{n-\beta-1} \sum_{k=0}^{\infty} \sum_{r=0}^{k} (-1)^k \binom{\beta}{k-r} \binom{-\beta-1}{r} \frac{t^{-2k-1}\,dt}{2\beta+2r+1}.$$

Now

$$\sum_{r=0}^{k} \binom{\beta}{k-r} \binom{-\beta-1}{r} \frac{1}{2\beta+2r+1} = \frac{(-1)^k \left(\frac{1}{2}\right)_k}{(2\beta+1)\left(\beta+\frac{3}{2}\right)_k}$$

so that the above limit

$$= \frac{(-1)^{n-\beta-1}}{\Gamma(n-\beta-1)} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} (-1)^r \binom{n-\beta-1}{r} \frac{\left(\frac{1}{2}\right)_k (n-\beta-1-2k)\dots(-\beta-2k) x^{-\beta-2k-1}}{(2\beta+1)\left(\beta+\frac{3}{2}\right)_k (n-\beta-1-2k-r)}.$$

But

$$\sum_{r=0}^{\infty} (-1)^r \binom{n-\beta-1}{r} \frac{1}{n-\beta-1-2k-r} = -\frac{\Gamma(2k+\beta-n+1)\,\Gamma(n-\beta)}{(2k)!}$$

so that the above double series is

$$= (-1)^{-\beta} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \Gamma(\beta+2k+1)\, x^{-\beta-2k-1}}{(2\beta+1)\left(\beta+\frac{3}{2}\right)_k (2k)!}.$$

Hence (6.7)

$$= \frac{2^\beta [\Gamma(\beta+1)]^2}{\Gamma(2\beta+2)} x^{-\beta-1} F\left(\frac{1}{2}\beta+\frac{1}{2}, \frac{1}{2}\beta+1; \beta+\frac{3}{2}; \frac{1}{x^2}\right)$$

$$= Q_\beta(x).$$

Finally to show (6.8) we have, when $|x| < 1$,

$$ {}_0D_x^\beta (1-x^2)^\beta \int_0^x (1-t^2)^{-n-1} t^{\beta-n}\, dt $$

$$ = {}_0D_x^\beta \sum_{k=0}^{\infty} \sum_{r=0}^{k} (-1)^k \binom{\beta}{k-r} \binom{-\beta-1}{r} \frac{x^{2k+\beta-n+1}}{2r+\beta-n+1} $$

$$ = {}_0D_x^\beta \sum_{k=0}^{\infty} (-1)^k \binom{\beta}{k} {}_3F_2 \left[\begin{matrix} -k, \beta+1, \frac{\beta}{2}-\frac{n}{2}+\frac{1}{2}; \\ \frac{\beta}{2}-\frac{n}{2}+\frac{3}{2}, 1+\beta-k; \end{matrix} 1 \right] \frac{x^{2k+\beta-n+1}}{\beta-n+1}, $$

but

$$ {}_3F_2 \left[\begin{matrix} -k, \beta+1, \frac{\beta}{2}-\frac{n}{2}+\frac{1}{2}; \\ \frac{\beta}{2}-\frac{n}{2}+\frac{3}{2}, 1+\beta-k; \end{matrix} 1 \right] = \frac{\left(-\frac{\beta}{2}-\frac{n}{2}+\frac{1}{2}\right)_k k!}{\left(\frac{\beta}{2}-\frac{n}{2}+\frac{3}{2}\right)_k (-\beta)_k} $$

so that the above

$$ = \sum_{k=0}^{\infty} \frac{\left(-\frac{\beta}{2}-\frac{n}{2}+\frac{1}{2}\right)_k \Gamma(2k+\beta-n+2)}{(\beta-n+1)\left(\frac{\beta}{2}-\frac{n}{2}+\frac{3}{2}\right)_k \Gamma(2k-n+2)} x^{2k-n+1}. $$

The above series when multiplied by $\frac{(-1)^\beta 2^\beta \Gamma(\beta+1)}{\Gamma(2\beta+1)}$ gives the desired result. Note that $Q_\beta(x)$ is a series of odd powers when n is even and even powers when n is odd.

(C) Hypergeometric equation. For the hypergeometric equation

$$ x(1-x)y'' + [\gamma-(\mu+\beta+1)x]y' - \mu\beta y = 0 \tag{6.9} $$

$\alpha = \beta$ or $\alpha = \mu$. Hence the equivalent H-R form for $R(\alpha) \geq 0$ is

$$ \overset{x}{\underset{a}{I}}{}^{-\beta} x^{\beta-\gamma+1} (1-x)^{\gamma-\mu} Dx^{\gamma-\beta} (1-x)^{-\gamma+\mu+1} \overset{x}{\underset{a}{I}}{}^{\beta-1} y = 0 \tag{6.10} $$

or

$$ \overset{x}{\underset{a}{I}}{}^{-\mu} x^{\mu-\gamma+1} (1-x)^{\gamma-\beta} Dx^{\gamma-\mu} (1-x)^{-\gamma+\beta+1} \overset{x}{\underset{a}{I}}{}^{\mu-1} y = 0. \tag{6.11} $$

The results (6.10) and (6.11) were obtained in [2] where the solutions were discussed in detail and hence will not be dwelt on here. However by use theorem 2 two additional forms for the hypergeometric equation can be obtained. That is, if $R(\alpha) < 0$ (6.9) has the *H-R* form

$$(6.12) \qquad \overset{x}{\underset{a}{I}}{}^{\beta-1}\, x^{\gamma-\beta}(1-x)^{-\gamma+\mu+1}\, Dx^{\beta-\gamma+1}(1-x)^{\gamma-\mu}\, \overset{x}{\underset{a}{I}}{}^{-\beta}\, Q(x)\, y = 0$$

or

$$(6.13) \qquad \overset{x}{\underset{a}{I}}{}^{\mu-1}\, x^{\gamma-\mu}(1-x)^{-\gamma+\beta+1}\, Dx^{\mu-\gamma+1}(1-x)^{\gamma-\beta}\, \overset{x}{\underset{a}{I}}{}^{-\mu}\, Q(x)\, y = 0,$$

where $Q(x) = x^{\gamma-1}(1-x)^{\mu+\beta-\gamma}$.

Note that if μ or $\beta = -n$, say β, one gets by inspection

$$y = c_1\, x^{1-\gamma}(1-x)^{\gamma+n-\mu}\, D_x^n\, (1-x)^{n+\gamma-1}\, x^{\mu-\gamma}$$

$$+\, c_2\, x^{1-\gamma}(1-x)^{\gamma+n-\mu}\, D_x^n\, x^{n+\gamma-1}(1-x)^{\mu-\gamma} \int^x t^{-\gamma-n}(1-t)^{\gamma-\mu-1}\, dt.$$

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