ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 17, nº 3 (1963), p. 189-206

<http://www.numdam.org/item?id=ASNSP_1963_3_17_3_189_0>

AN L^p-ESTIMATE FOR THE GRADIENT OF SOLUTIONS OF SECOND ORDER ELLIPTIC DIVERGENCE EQUATIONS

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Introduction. In papers [2] and [3] Boyarskiǐ has studied solutions of two dimensional, first order, uniformly elliptic systems with measurable coefficients. In the simplest case such a system has the form

$$(*) \qquad \begin{aligned} v_y &= a u_x + b u_y \\ - v_x &= b u_x + c u_y . \end{aligned}$$

One of the striking results of Boyarskiǐ's work is the existence of an exponent $p > 2$, depending only on the ellipticity constants of $(*)$, such that if the strong derivatives of a solution (u, v) are in $\mathcal{L}^{p'}$ $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ then they are in $\mathcal{L}^p$. The main tool in the proof is the Calderon-Zygmund Inequality for singular integrals [4]. It is indeed surprising that such a method, which is ultimately based on the case of constant coefficients, can yield results independent of the continuity properties of variable coefficients.

The exponent p has several advantages over the exponent 2, which is the exponent assumed in almost all work on such equations. For example, it immediately implies, by use of the Sobolev lemma, that the solution is Hölder continuous (which is also known from more elementary considerations) and also gives certain geometric properties of homeomorphic solution mappings not derivable in any other way. The best value of the Lebesgue exponent p is unknown and its relation to the best Hölder exponent, which

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Pervenuto in Redazione il 20 Giugno 1963.

(0) This paper has been prepared with the support of the Office of Naval Research, under project Nonr 710 (54) NR-043-041.

is known, remains an interesting unsolved problem. For aesthetic reasons it is natural to hope that the best Hölder exponent follows from the best Lebesgue exponent, via the Sobolev lemma, and several mathematicians have conjectured that this is the case.

Since any elliptic equation of the form

$$(**) \qquad (au_x + bu_y)_x + (bu_x + cu_y)_y = 0$$

is equivalent to a system of the form of $(*)$, the theory also applies to $(**)$. In this paper I extend Boyarskiĭ's result to n-dimensional elliptic equations of divergence structure. In the simplest case such an equation has the form

$$(***) \qquad \sum_{k,\,l=1}^{n} (a_{kl}(x)\, u_{x_l})_{x_k} = 0 \qquad x = (x_1, \dots, x_n).$$

In Theorem 2 I prove there is a number $Q > 2$, depending only on the ellipticity constants of $(***)$ and on n, such that if u has strong derivatives in $\mathcal{L}^{p_1}$ for *some* exponent p_1 in the range $Q' < p_1 < Q$ $\left(\frac{1}{Q} + \frac{1}{Q'} = 1\right)$ then the first derivatives are actually in $\mathcal{L}^p$ for *every* p in the range $Q' < p < Q$ (1). The main tool, as in Boyarskiĭ's proof, is the Calderon-Zygmund Inequality, though its role is largely hidden.

Having such a result, it is tempting to try to prove the Hölder continuity of u by showing that $Q > n$. Such an attempt is doomed. For if it were true, then by the method descent the first derivatives of u would be in $\mathcal{L}^p$ for *every* $p < \infty$. But this is known to be false from simple examples. In fact, in Section 5 I show by means of an example that in each dimension Q tends to 2 as the ellipticity becomes « bad ». From the method of descent it is necessary to show this only in two dimensions. Though this method of proof fails it may still be possible to base a proof of Hölder continuity on the fact that the derivatives are in $\mathcal{L}^p$ (see [5] and [8]).

Besides Theorem 2, I also prove an existence and uniqueness theorem (Theorem 1) for the Dirichlet Problem when the domain is sufficiently smooth. I derive Theorem 2 from Theorem 1, though it is possible to prove Theorem 2 directly, without preliminary results on existence.

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(1) Professor A. P. Calderon has independently proved the same theorem in unpublished work.

1. Notation.

$\mathcal{R}^n$ denotes the real n-dimensional Euclidean space of vectors (or points) $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$. The inner product $[x, y]$ is defined as $\sum_{k=1}^{n} x_k y_k$ and the corresponding norm $|x|$ equals $[x, x]^{1/2}$.

Analogously, $\mathcal{K}^n$ denotes the complex n-dimensional Euclidean space of vectors $\xi = (\xi_1, \dots, \xi_n)$, $\eta = (\eta_1, \dots, \eta_n)$, $(\xi, \eta) = \sum_{k=1}^{n} \xi_k \overline{\eta_k}$ and $|\xi| = (\xi, \xi)^{1/2}$. If $\mathbf{A}$ is an $n \times n$ complex matrix (a_{kl}), then $\mathbf{A}^*$ denotes the conjugate transpose $(\overline{a_{lk}})$ and $\mathbf{A}\xi$ denotes the vector $\left(\sum_{k=1}^{n} a_{1k}\xi_k, \dots, \sum_{k=1}^{n} a_{nk}\xi_k\right)$. $|\mathbf{A}| = \max_{|\xi|=1} |\mathbf{A}\xi|$.

Finally, notations such as $\vec{f}(x)$ denote functions whose domain of definition is contained in $\mathcal{R}^n$ and whose values are in $\mathcal{K}^n$.

2. An alternative statement of the uniform ellipticity condition etc.

Consider the differential operator L defined by the equation

$$Lu(x) = \operatorname{div} \mathbf{A}(x) \operatorname{grad} u(x), \tag{1}$$

where u is a complex valued function and $\mathbf{A}$ is a complex matrix. The definitions of uniform ellipticity of L usually given in the literature are either that there exist positive constants λ_1 and M such that inequalities

$$\begin{aligned} \lambda_1 |\xi|^2 &\leq Re(\mathbf{A}(x)\xi, \xi) \\ |\mathbf{A}(x)| &\leq M \end{aligned} \tag{2}$$

hold for all x under consideration and for all ξ, or else they are obvious restatements of this condition. However, in order to handle the case of non-Hermitian coefficient matrices we need a form of the uniform ellipticity condition which is not quite so obvious.

To this end, rewrite $\mathbf{A}$ for each x as

$$\mathbf{A} = \mathbf{H}_1 + i\mathbf{H}_2 \qquad (i = \sqrt{-1}) \tag{3}$$

where $\mathbf{H}_1 = \frac{1}{2}(\mathbf{A} + \mathbf{A}^*)$ and $\mathbf{H}_2 = \frac{i}{2}(\mathbf{A}^* - \mathbf{A})$, so that $\mathbf{H}_1$ and $\mathbf{H}_2$ are

Hermitian. Using the fact that $|\mathbf{A}^*| = |\mathbf{A}|$, it then follows that

$$\lambda_1 |\xi|^2 \leq (\mathbf{H}_1 \xi, \xi) \leq M |\xi|^2 . \tag{4}$$

Now rewrite $\mathbf{A}$ in the form

$$\mathbf{A} = (\mathbf{H}_1 + c\mathbf{I}) + (i\mathbf{H}_2 - c\mathbf{I}), \tag{5}$$

where c is a non-negative constant and $\mathbf{I}$ is the identity matrix. $\mathbf{H}_1 + c\mathbf{I}$ is Hermitian and from (4) it is clear that

$$(\lambda_1 + c) |\xi|^2 \leq ((\mathbf{H}_1 + c\mathbf{I}) \xi, \xi) \leq (M + c) |\xi|^2 . \tag{6}$$

As for $i\mathbf{H}_2 - c\mathbf{I}$ we have

$$|(i\mathbf{H}_2 - c\mathbf{I}) \xi|^2 = |\mathbf{H}_2 \xi|^2 + c^2 |\xi|^2 \tag{7}$$

and thus

$$|i\mathbf{H}_2 - c\mathbf{I}| \leq (M^2 + c^2)^{1/2} . \tag{8}$$

Define the quantity θ by means of the equation

$$1 - \theta = \min_{c \geq 0} \frac{(M^2 + c^2)^{1/2}}{\lambda_1 + c} . \tag{9}$$

Since the ratio in (9) is less than one for large values of c, we have $1 - \theta < 1$. Setting $\mathbf{H} = \mathbf{H}_1 + c_1\mathbf{I}$, $\mathbf{R} = i\mathbf{H}_2 - c_1\mathbf{I}$, $\lambda = \lambda_1 + c_1$ and $\Lambda = M + c_1$, where c_1 denotes a value of c at which the minimum in (9) is attained, we have

$$\mathbf{A} = \mathbf{H} + \mathbf{R} . \tag{10}$$

Here $\mathbf{H}$ is Hermitian and inequalities

$$\begin{aligned} &\lambda |\xi|^2 \leq (\mathbf{H}\xi, \xi) \leq \Lambda |\xi|^2 && (\lambda > 0) \\ &|\mathbf{R}| \leq (1 - \theta) \lambda && (0 < \theta \leq 1) \end{aligned} \tag{11}$$

hold for all relevant x and for all ξ.

Thus, under the assumption of uniform ellipticity, it is possible to decompose $\mathbf{A}$ into the sum of a positive definite Hermitian matrix $\mathbf{H}$ and a remainder $\mathbf{R}$ which is « smaller » than $\mathbf{H}$. Such a decomposition in turn clearly implies uniform ellipticity and so the two statements are equivalent.

In any differential equation we can assume without loss of generality that $\Lambda = 1$, since we can divide both sides of the equation by Λ. The form (10), (11) of the uniform ellipticity condition with $\Lambda = 1$ is thus perfectly general. In this form the quantity $\theta\lambda$ measures the ellipticity. Note that $\theta\lambda \leq 1$ with equality for the Laplacian.

DEFINITION 1. Let Ω be a bounded domain (open connected set) in $\mathcal{R}^n$. $\mathcal{C}_0^\infty = \mathcal{C}_0^\infty(\Omega)$ stands for the linear space of all complex infinitely continuously differentiable functions with compact support in Ω. $\mathcal{L}^q = \mathcal{L}^q(\Omega)$, $1 \leq q < \infty$, stands for either the Banach space of all complex measurable functions Φ or the space of measurable vector fields Φ defined on Ω with finite q-norm

$$(12) \qquad | \Phi |_q = \left(\int_\Omega | \Phi(x) |^q \, dx \right)^{1/q}.$$

In addition we consider the linear spaces $\mathcal{L}^q_{\text{loc}} = \mathcal{L}^q_{\text{loc}}(\Omega)$ of complex functions or vector fields which are in $\mathcal{L}^q(\Omega')$ for *every* subdomain Ω' whose closure $\bar{\Omega}'$ is contained in Ω.

As is usual, $\mathcal{H}_0^{1,q} = \mathcal{H}_0^{1,q}(\Omega)$ denotes the space of complex functions Φ with strong derivatives in $\mathcal{L}^q$, for which there exists a sequence of functions Φ_k in $\mathcal{C}_0^\infty$ such that

$$(13) \qquad | \Phi_k - \Phi |_q \to 0 \text{ and } | \operatorname{grad} \Phi_k - \operatorname{grad} \Phi |_q \to 0 \text{ as } k \to \infty.$$

Under the norm

$$(14) \qquad \| \Phi \|_q = | \operatorname{grad} \Phi |_q$$

$\mathcal{H}_0^{1,q}$ is a Banach space. We also consider spaces $\mathcal{M}_0^{1,q} = \mathcal{M}_0^{1,q}(\Omega)$, $2 \leq q < \infty$, consisting of those functions in $\mathcal{H}_0^{1,2}$ whose gradients are in $\mathcal{L}^q$. Under the norm

$$(15) \qquad \| \Phi \|_q = | \operatorname{grad} \Phi |_q$$

$\mathcal{M}_0^{1,q}$ is also a Banach space.

DEFINITION 2. We define the bilinear functional

$$(16) \qquad \mathcal{B}_{\mathbf{A}}(\Psi, \Phi) = \int_\Omega (\mathbf{A} \operatorname{grad} \Psi, \operatorname{grad} \Phi) \, dx$$

corresponding to a bounded measurable matrix $\mathbf{A} = \mathbf{A}(x)$. The functions Φ and Ψ will be taken from various spaces depending on the circumstances. For $\mathbf{A}(x) \equiv \mathbf{I}$ we write $\mathcal{B}_{\mathrm{I}} = \mathcal{B}$.

DEFINITION 3. Consider the differential equation

$$Lu = \operatorname{div} \mathbf{A} \operatorname{grad} u = \operatorname{div} \vec{f} + h \tag{17}$$

on Ω, where $\mathbf{A} = \mathbf{A}(x)$ is a complex bounded measurable matrix and $\vec{f} = \vec{f}(x)$ and $h = h(x)$ are in $\mathcal{L}^1_{\mathrm{loc}}$. We say that $u = u(x)$ is a *solution* of equation (17) if u has strong derivatives on Ω and

$$\mathcal{B}_{\mathbf{A}}(u, \Phi) = \int_{\Omega} ((\vec{f}, \operatorname{grad} \Phi) - h\bar{\Phi})\, dx \tag{18}$$

holds for *all* Φ in $\mathcal{C}_0^\infty$.

We shall say that Ω is of class $\mathcal{D}^q$, $2 < q < \infty$, if the equation

$$\Delta u = \operatorname{div} \vec{f} \tag{19}$$

has a unique solution u in $\mathcal{H}_0^{1,q}$ for every $\vec{f}$ in $\mathcal{L}^q$ and

$$\| u \|_q \leq K_q \, | \vec{f} \, |_q \tag{20}$$

holds for some constant K_q independent of $\vec{f}$. This constitutes a condition of regularity on the boundary of Ω, which will hold for any value of q if the boundary is sufficiently smooth. The proof depends on the Calderon-Zygmund Inequality for singular integrals (see [4] and Theorem 15.3′ of [1]).

DEFINITION 4. Let q be a number such that $1 < q < \infty$. Then q' is defined by $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. Further, q^* is defined by means of the equation $\dfrac{1}{q^*} = \dfrac{1}{q} - \dfrac{1}{n}$ if $q < n$ and is defined to be *any* number in the range $1 < q^* < \infty$ if $q \geq n$.

The preceding notation will be used in the remainder of the paper without further comment.

3. Preliminary results.

The results to be derived in this section are generally known but since they are of central importance in the derivation of the main theorems, I include them here.

Let us suppose that $\mathbf{A} = \mathbf{A}(x)$ satisfies the uniform ellipticity condition and that the elements of $\mathbf{A}$ are measurable functions on a bounded domain Ω. Let us further suppose that for the domain Ω

$$(21) \qquad \inf_{\|\Phi\|_{q'}=1} \sup_{\|\Psi\|_{q}=1} | \mathcal{B}_{\mathsf{A}}(\Psi, \Phi) | \geq \frac{1}{K} > 0$$

where q is some fixed number in the range $2 < q < \infty$, K is a positive constant and Φ and Ψ vary over the spaces $\mathcal{H}_0^{1,q'}$ and $\mathcal{H}_0^{1,q}$ respectively. Consider the equation

$$(22) \qquad L^* u = \operatorname{div} \mathbf{A}^* \operatorname{grad} u = \operatorname{div} \vec{g},$$

where $\vec{g} = \vec{g}(x)$ is in $\mathcal{L}^{q'}$. Let $\vec{g}_k = \vec{g}_k(x)$ $k = 1, 2, \ldots$ be a sequence of vector fields in $\mathcal{L}^2$ such that

$$(23) \qquad | \vec{g}_k - \vec{g} |_{q'} \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty .$$

Then it is well known from the $\mathcal{L}^2$-theory that the equations $L^* u = \operatorname{div} \vec{g}_k$ have solutions u_k in $\mathcal{H}_0^{1,2}$. Thus

$$(24) \qquad \mathcal{B}_{\mathsf{A}}(\Psi, u_k) = \int_\Omega (\operatorname{grad} \Psi, \vec{g}_k)\, dx$$

for *all* Ψ in $\mathcal{H}_0^{1,2}$. Since $\mathcal{H}_0^{1,q} \subset \mathcal{H}_0^{1,2} \subset \mathcal{H}_0^{1,q'}$ it follows from assumption (21) and from (24) that

$$(25) \qquad | \operatorname{grad} u_k |_{q'} \leq K \, | \vec{g}_k |_{q'} .$$

Therefore there exists a function u in $\mathcal{H}_0^{1,q'}$ such that

$$(26) \qquad \operatorname{grad} u_k \longrightarrow \operatorname{grad} u \quad \text{weakly in } \mathcal{L}^{q'}.$$

Therefore u solves equation (22), is in $\mathcal{H}_0^{1,q'}$ and from (25) and (26)

$$(27) \qquad \| u \|_{q'} \leq K \, | \vec{g} |_{q'} .$$

Now suppose u is a function in $\mathcal{H}_0^{1,q'}$ and solves equation (22) with $\vec{g} = 0$. Then

$$\mathcal{B}_A(\Psi, u) = 0 \tag{28}$$

for *all* Ψ in $\mathcal{H}_0^{1,q}$ and from (21) we see that $u = 0$. We have thus shown:

Under condition (21), *the uniformly elliptic equation* (22) *is uniquely solvable in* $\mathcal{H}_0^{1,q'}$ *for each* $\vec{g}$ *in* $\mathcal{L}^{q'}$ *and the solution satisfies* (27).

Next consider the equation

$$Lu = \operatorname{div} \mathbf{A} \operatorname{grad} u = \operatorname{div} \vec{f} \tag{29}$$

for $\vec{f}$ in $\mathcal{L}^q$. From the $\mathcal{L}^2$-theory we know a solution in $\mathcal{H}_0^{1,2}$ exists. Therefore

$$\mathcal{B}_A(u, \Phi) = \int_\Omega (\vec{f}, \operatorname{grad} \Phi)\, dx \tag{30}$$

for *all* Φ in $\mathcal{H}_0^{1,2}$. Let $\vec{g}$ be an arbitrary vector field in $\mathcal{L}^2$. Then $L^*\Phi = \vec{g}$ has a solution Φ in $\mathcal{H}_0^{1,2}$ and for this particular Φ, $\mathcal{B}_A(u, \Phi) = \int_\Omega (\operatorname{grad} u, \vec{g})\, dx$.

Therefore

$$\int_\Omega (\operatorname{grad} u, \vec{g})\, dx = \int_\Omega (\vec{f}, \operatorname{grad} \Phi)\, dx. \tag{31}$$

If $|\vec{g}|_{q'} \leq 1$ then $|\operatorname{grad} \Phi|_{q'} \leq K$ from (27). Thus taking the supremum of the absolute value of both sides of (31) over all $\vec{g}$ in $\mathcal{L}^2$ such that $|\vec{g}|_{q'} \leq 1$, we see that $|\operatorname{grad} u|_q \leq K |\vec{f}|_q$. Therefore u is in $\mathcal{M}_0^{1,q}$, is unique in this class since $\mathcal{M}_0^{1,q} \subset \mathcal{H}_0^{1,2}$, and

$$\| u \|_q \leq K |\vec{f}|_q. \tag{32}$$

Under condition (21), *the uniformly elliptic equation* (29) *is uniquely solvable in* $\mathcal{M}_0^{1,q}$ *for each* $\vec{f}$ *in* $\mathcal{L}^q$ *and the solution satisfies* (32).

We have therefore established the existence of transformations T_A and T_{A^*} where T_A is a bounded linear transformation carrying vector fields in $\mathcal{L}^q$ into functions (solutions of (29)) in $\mathcal{M}_0^{1,q}$ and T_{A^*} is a bounded linear transformation carrying vector fields in $\mathcal{L}^{q'}$ into functions (solutions of (22)) in $\mathcal{H}_0^{1,q'}$. We may equivalently consider T_A and T_{A^*} as carrying $\mathcal{L}^q$ and $\mathcal{L}^{q'}$ respectively into themselves by the simple device of replacing the solution by its gradient. No matter how we choose to consider these transformations their norms will be the same. We denote the norms by $\| T_{\mathsf{A}} \|_q$ and $\| T_{\mathsf{A}^*} \|_{q'}$. From equation (31) and a simple argument it follows that

$$\| T_{\mathsf{A}} \|_q = \| T_{\mathsf{A}^*} \|_{q'} \,. \tag{33}$$

Let us now add the further assumptions that

$$\begin{aligned} &(21) \text{ holds for both } \mathbf{A} \text{ and } \mathbf{A}^*, \\ &\Omega \text{ is of class } \mathcal{D}^q. \end{aligned} \tag{34}$$

Then $\mathcal{M}_0^{1,q} = \mathcal{H}_0^{1,q}$. This follows because the equation $\Delta u = \operatorname{div} \operatorname{grad} \Psi$, for Ψ in $\mathcal{M}_0^{1,q}$, has a solution in $\mathcal{H}_0^{1,q}$. Thus $\Delta(u - \Psi) = 0$ and since $u - \Psi$ is in $\mathcal{H}_0^{1,2}$, $u = \Psi$. Furthermore, the transformation T_{A} now also maps $\mathcal{L}^{q'}$ into $\mathcal{L}^{q'}$. From the Riesz Convexity Theorem (see p. 525 and E39, p. 536 of [6]) T_{A} must map $\mathcal{L}^p$ into $\mathcal{L}^p$ for every p in the range $q' \leq p \leq q$ and $\log \| T_{\mathsf{A}} \|_p$ is a convex function of $\dfrac{1}{p}$. It is in fact easy to show that T_{A} maps $\mathcal{L}^p$ into solutions in $\mathcal{H}_0^{1,p}$. Moreover, we have

$$\inf_{\|\Phi\|_{p'}=1} \ \sup_{\|\Psi\|_p=1} | \mathcal{B}_{\mathsf{A}}(\Psi, \Phi) | = \frac{1}{\| T_{\mathsf{A}} \|_p} \qquad (2 \leq p \leq q). \tag{35}$$

For, from the fact that T_{A} maps $\mathcal{L}^p$ into $\mathcal{H}_0^{1,p}$ one infers that the left side of (35) is *greater* than or equal to the right side. On the other hand, from (21) and (32) with q replaced by p one infers that the left side of (35) is *less* than or equal to the right side. Hence, equality holds. We now summarize our results.

LEMMA. 1. *Under assumptions* (34) *the uniformly elliptic equations* (22) *and* (29) *are uniquely solvable in* $\mathcal{H}_0^{1,p}$ *for every* $\vec{g}$ *and* $\vec{f}$ *in* $\mathcal{L}^p$, $q' \leq p \leq q$. *Furthermore* T_{A^*} *and* T_{A} *are bounded linear transformations such that equalities* (33) (*with* q *replaced by* p) *and* (35) *hold. Finally,* $\log \| T_{\mathsf{A}} \|_p$ *is a convex function of* $\dfrac{1}{p}$.

Let us now assume Ω is of class $\mathcal{D}^q$ for some q, $2 < q < \infty$, and take $\mathbf{A} \equiv \mathbf{I}$. Obviously the assumptions of Lemma 1 are fulfilled. Set $K_p = \| T_{\mathsf{I}} \|_p$, $q' \leq p \leq q$. Then $K_p = K_{p'}$ and $K_2 = 1$. From the convexity K_p must then have a minimum at $p = 2$, must be non-decreasing for $p > 2$ and must be continuous in p.

From Lemma 1 we also have

(36) $$\inf_{\|\Phi\|_{p'}=1} \sup_{\|\Psi\|_p=p} |\mathcal{B}(\Psi, \Phi)| = \frac{1}{K_p} \qquad \text{for } 2 \leq p \leq q.$$

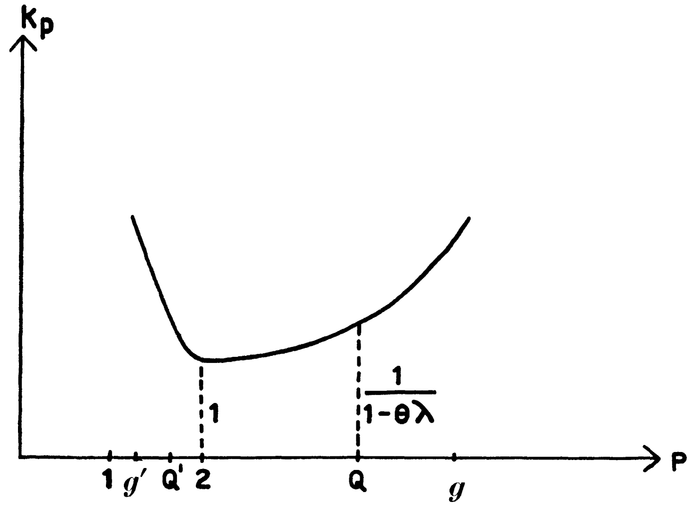

Figure 1

4. Main results.

We are now ready to prove the main results of the paper. These are an existence and uniqueness theorem for solutions in $\mathcal{H}_0^{1,p}$, a theorem on the « true » $\mathcal{L}^p$ class of the gradient of solutions and finally a theorem on the $\mathcal{L}^p$ class of second derivatives of solutions of non-divergence equations in two dimensions.

THEOREM 1. *Consider the differential equation*

(37) $$Lu = \operatorname{div} \mathbf{A} \operatorname{grad} u = \operatorname{div} \vec{f} + h,$$

where $\mathbf{A} = \mathbf{A}(x)$ *is a complex measurable matrix and satisfies the uniform ellipticity condition* (10), (11) *a.e. with* $\Lambda = 1$. If Ω *is a bounded domain of class* $\mathcal{D}^q$ *for some* q, $2 < q < \infty$, *then* (37) *has a unique solution in* $\mathcal{H}_0^{1,p}$ *for every complex vector field* $\vec{f} = \vec{f}(x)$ *in* $\mathcal{L}^p$ *and every complex function* $h = h(x)$ *in* $\mathcal{L}^r$ *with* $r^* \geq p$, *provided*

(38) $$q' \leq Q' < p < Q \leq q.$$

Here $Q > 2$ and depends only on Ω and the ellipticity constant $\theta\lambda$ in such a way that $Q \to q$ as $\theta\lambda \to 1$ and $Q \to 2$ as $\theta\lambda \to 0$. The solutions satisfy

$$|\operatorname{grad} u|_p \leq C\{ |\vec{f}|_p + |h|_r \}, \tag{39}$$

where C is a constant depending only on Ω, $\theta\lambda$, p and r.

PROOF. We may assume without loss of generality that $h = 0$, for the potential of h, $v = v(x)$, satisfies $\Delta v = h$ and from the Sobolev inequality for fractional integrals

$$|\operatorname{grad} v|_{r^*} \leq C\, |h|_r . \tag{40}$$

Thus we can replace the right side of the equation by $\operatorname{div}(\vec{f} + \operatorname{grad} v)$, where $\vec{f} + \operatorname{grad} v$ is a vector field in $\mathcal{L}^p$.

Next rewrite $\mathcal{B}_{\mathsf{A}} = \mathcal{B}_{\mathsf{A}}(\Psi, \Phi)$ as $\mathcal{B}_{\mathsf{A}} = \mathcal{B} + \mathcal{B}_{\mathsf{H-I}} + \mathcal{B}_{\mathsf{R}}$. Hence

$$\sup_{\|\Psi\|_p=1} |\mathcal{B}_{\mathsf{A}}| \geq \sup_{\|\Psi\|_p=1} |\mathcal{B}| - \sup_{\|\Psi\|_p=1} |\mathcal{B}_{\mathsf{H-I}}| - \sup_{\|\Psi\|_p=1} |\mathcal{B}_{\mathsf{R}}| , \tag{41}$$

where $2 \leq p \leq q$, Φ is in $\mathcal{H}_0^{1,p'}$ and Ψ varies over $\mathcal{H}_0^{1,p}$. We then have

$$\begin{aligned} &\sup_{\|\Psi\|_p=1} |\mathcal{B}| \geq \frac{1}{K_p} \|\Phi\|_{p'} \\ &\sup_{\|\Psi\|_p=1} |\mathcal{B}_{\mathsf{H-I}}| \leq (1-\lambda) \|\Phi\|_{p'} \\ &\sup_{\|\Psi\|_p=1} |\mathcal{B}_{\mathsf{R}}| \leq (1-\theta)\lambda \|\Phi\|_{p'} \end{aligned} \tag{42}$$

where the constant K_p is defined at the end of Section 3 and constants θ, λ are defined in (11). Therefore

$$\inf_{\|\Phi\|_{p'}=1} \sup_{\|\Psi\|_p=1} |\mathcal{B}_{\mathsf{A}}(\Psi, \Phi)| \geq \left(\frac{1}{K_p} - 1 + \theta\lambda\right). \tag{43}$$

Now set

$$Q = \sup_E p, \tag{44}$$

where E is the subinterval of $2 \leq p \leq q$ on which $\dfrac{1}{K_p} - 1 + \theta\lambda > 0$. Since $K_2 = 1$ and K_p is continuous in p, $Q > 2$ (see Figure 1). Thus in the interval $2 \leq p < Q$ we have

$$\inf_{\| \Phi \|_{p'}=1} \sup_{\| \Psi \|_{p}=1} | \mathcal{B}_{A} (\Psi, \Phi) | \geq \frac{1}{\widehat{K}_p}, \tag{45}$$

where $\widehat{K}_p = \dfrac{K_p}{1 - K_p (1 - \theta\lambda)}$. Since $\mathbf{A}^*$ has the *same* ellipticity constants as $\mathbf{A}$ inequality (45) also holds for $\mathcal{B}_{A^*}$. The rest is a consequence of Lemma 1.

DEFINITION 5. If Φ is a complex measurable function or a measurable vector field we denote the p-norm of Φ over a sphere of radius R by $| \Phi |_{p;R}$. The center of the sphere is assumed to be known and does not appear in the notation.

THEOREM 2. *Let $u = u(x)$ be a solution on a domain Ω of the equation*

$$Lu = \operatorname{div} \mathbf{A} \operatorname{grad} u = \operatorname{div} \vec{f} + h, \tag{46}$$

where $\mathbf{A} = \mathbf{A}(x)$ is a complex measurable matrix and satisfies the uniform ellipticity condition (10), (11) *a. e. with $\Lambda = 1$. Under this assumption, there exists a number $Q > 2$, which depends only on $\theta\lambda$ and n ($Q \to \infty$ as $\theta\lambda \to 1$ and $Q \to 2$ as $\theta\lambda \to 0$) such that if* grad u *is in $\mathcal{L}^{p_1}_{\mathrm{loc}}$, $\vec{f}$ is in $\mathcal{L}^{p}_{\mathrm{loc}}$ and h is in $\mathcal{L}^{r}_{\mathrm{loc}}$ where*

$$Q' < p_1 < p < Q \quad \textit{and} \quad r^* \geq p, \tag{47}$$

then grad u *is in $\mathcal{L}^{p}_{\mathrm{loc}}$. Moreover, if y is any point in Ω, then over open balls centered at y we have the estimate*

$$| \operatorname{grad} u |_{p;R} \leq C \{ R^{n\left(\frac{1}{p} - \frac{1}{p_1}\right)} | \operatorname{grad} u |_{p_1;2R} + R^{n\left(\frac{1}{p} - 1\right)-1} | u |_{1;2R} + | \vec{f} |_{p;2R} + R^{n\left(\frac{1}{p} - \frac{1}{r}\right)+1} | h |_{r;2R} \}, \tag{48}$$

where p_1 is now any index such that $Q' < p_1 < p < Q$ and C depends only on $\theta\lambda$, p, r, p_1 and n. If in addition $p \geq \dfrac{2n}{n+2}$, then we may replace (48) *by*

$$| \operatorname{grad} u |_{p;R} \leq C \{ R^{n\left(\frac{1}{p} - \frac{1}{2}\right)-1} | u |_{2;2R} + | \vec{f} |_{p;2R} + R^{n\left(\frac{1}{p} - \frac{1}{r}\right)+1} | h |_{r;2R} \}. \tag{49}$$

PROOF. Let $\omega = \omega(x\,;k)$ $k = 2, 3, \ldots$ be an infinitely continuously differentiable function in x such that

$$\tag{50} \begin{aligned} \omega\,(x\,;k) &= 1 \qquad \text{for} \quad |\,x\,| \leq 1 - \frac{1}{k} \\ \omega\,(x\,;k) &= 0 \qquad \text{for} \quad |\,x\,| \geq 1 - \frac{1}{2k}\,. \end{aligned}$$

Let y be a point of Ω with distance d_y from the boundary of Ω and set $\zeta = \zeta\,(x\,;y\,;k) = \omega\left(\dfrac{x-y}{d_y}\,;k\right)$.

We now restrict our considerations to the homogeneous equation and denote the solution by $w = w\,(x)$. Set $z = \zeta w$. Then a brief computation yields

$$\tag{51} Lz = \operatorname{div}\,(w\,\mathbf{A}\,\operatorname{grad}\zeta) + (\operatorname{grad}\zeta,\,\mathbf{A}\,\operatorname{grad}w).$$

If we now set $k = 2$, then equation (51) holds in all space and z vanishes for $|\,x-y\,| > \dfrac{3}{4}\,d_y$.

Any ball, let us say $|\,x-y\,| <$ const., is a domain of class $\mathcal{D}^q$ for *every* q, $2 < q < \infty$. Hence Theorem 1 applies to any ball for p in some interval $Q' < p < Q$, where Q depends only on $\theta\lambda$ and n. Moreover, $Q \to \infty$ as $\theta\lambda \to 1$ and $Q \to 2$ as $\theta\lambda \to 0$.

Assume now that the original solution w has strong derivatives in $\mathcal{L}^{p_1}_{\mathrm{loc}}$ for some index p_1, $Q' < p_1 < Q$. Then from the Sobolev lemma, w itself is in $\mathcal{L}^{p_1^*}_{\mathrm{loc}}$ and z is in $\mathcal{H}^{1,p_1}_0\,(\,|\,x-y\,| < d_y)$. It then follows from Theorem 1 that z is also in $\mathcal{H}^{1,\bar{p}_1}_0\,(\,|\,x-y\,| < d_y)$, where $\bar{p}_1$ is any index less than p_1^* and Q. Hence grad w is in $\mathcal{L}^{\bar{p}_1}\left(\,|\,x-y\,| < \dfrac{1}{2}\,d_y\right)$ and since y is arbitrary grad w is in $\mathcal{L}^{\bar{p}_1}_{\mathrm{loc}}$. But if this is the case, then grad w is in $\mathcal{L}^{\bar{\bar{p}}_1}_{\mathrm{loc}}$ by the same argument. Continuing the argument as long as necessary we find that grad *w is in $\mathcal{L}^{p}_{\mathrm{loc}}$ for every p, $Q' < p_1 \leq p < Q$.*

Now return to the original equation $Lu = \operatorname{div}\vec{f} + h$; define $v = v\,(x)$ to be the unique solution in $\mathcal{H}^{1,p}_0\left(|\,x-y\,| < \dfrac{1}{2}\,d_y\right)$ and set $w = u - v$. From the preceding discussion it is clear that grad u is in $\mathcal{L}^{p}_{\mathrm{loc}}$.

We now turn to the problem of deriving an estimate for the local p-norm of grad u. Assume for the time being that $d_y > 2$ and define

$v = v(x)$ to be the solution of $Lv = \operatorname{div} \vec{f} + h$ which is in the class $\mathcal{H}_0^{1,p}$ $(|x - y| < 2)$. Then from Theorem 1 we get

$$(52) \qquad |\operatorname{grad} v|_{p;2} \leq C\{|\vec{f}|_{p;2} + |h|_{r;2}\}.$$

Set $w = u - v$ and consider equation (51) where the domain is now the ball $|x - y| < 2$. Let p be *any* index in the range $Q' < p < Q$. By applying Theorem 1 to equation (51) we easily infer

$$(53) \qquad |\operatorname{grad} w|_{p;1} \leq C\{|\operatorname{grad} w|_{s;2-\frac{1}{k}} + |w|_{p;2-\frac{1}{k}}\},$$

where s is any index such that $Q' < s < p < Q$ and $s^* \geq p$. C depends only on $\theta\lambda$, p, s, n and k. By means of the Sobolev inequality we can estimate $|w|_{p;2-\frac{1}{k}}$ in terms of $|\operatorname{grad} w|_{s;2-\frac{1}{k}}$ and $|w|_{1;2-\frac{1}{k}}$. Hence

$$(54) \qquad |\operatorname{grad} w|_{p;1} \leq C\{|\operatorname{grad} w|_{s;2-\frac{1}{k}} + |w|_{1;2-\frac{1}{k}}\}.$$

We may now repeat the argument and estimate $|\operatorname{grad} w|_{s;2-\frac{1}{k}}$ in terms of $|\operatorname{grad} w|_{s_1;2-\frac{1}{2k}}$ and $|w|_{1;2-\frac{1}{2k}}$, where s_1 is any index such that $Q' < s_1 < s < p < Q$ and $s_1^* \geq s$. Thus it is clear that after N repetitions of the argument, N depending only on p, p_1 and n we get

$$(55) \qquad |\operatorname{grad} w|_{p;1} \leq C\{|\operatorname{grad} w|_{p_1;2-2^{-N}} + |w|_{1;2}\},$$

where p_1 is any index in the range $Q' < p_1 < p < Q$ and the initial value of k has been chosen equal to 2. If we now choose $p_1 = 2$ we can apply the well known estimate (see [8])

$$(56) \qquad |\operatorname{grad} w|_{2;2-2^{-N}} \leq C|w|_{2;2}.$$

Thus from (55) and (56) we infer

$$(57) \qquad |\operatorname{grad} w|_{p;1} \leq C|w|_{2;2}.$$

We can now derive the desired estimate on grad u. Let p and p_1 be indices such that $Q' < p_1 < p < Q$. Since $u = v + w$ it follows from (52) and (55) that

$$(58) \qquad |\operatorname{grad} u|_{p;1} \leq C\{|\operatorname{grad} u|_{p_1;2} + |u|_{1;2} + |\vec{f}|_{p;2} + |h|_{r;2}\}.$$

If in addition $p \geq \dfrac{2n}{n+2}$ then u and v are in $\mathcal{L}^2$ and it follows from (52) and (57) that

$$|\operatorname{grad} u|_{p\,;1} \leq C\{|u|_{2\,;2} + |\vec{f}|_{p\,;2} + |h|_{r\,;2}\}. \tag{59}$$

The inequalities in the general case of a sphere of radius R follow by performing a similarity transformation.

DEFINITION 6. Consider the differential equation

$$a(x, y)\, u_{xx} + 2b(x, y)\, u_{xy} + c(x, y)\, u_{yy} = f(x, y) \tag{60}$$

on a plane domain Ω. x and y now stand for coordinates. We assume that a, b, c and f are real measurable functions. The equation is said to be uniformly elliptic if

$$\lambda(\xi^2 + \eta^2) \leq a\xi^2 + 2b\xi\eta + c\eta^2 \leq \xi^2 + \eta^2 \qquad (\lambda > 0) \tag{61}$$

holds for almost all (x, y) in Ω and all real (ξ, η). $u = u(x, y)$ is called a solution of equation (60) on Ω, if u has strong second derivatives and satisfies the equation pointwise almost everywhere in Ω.

THEOREM 3. *Let $u = u(x, y)$ be a solution of equation* (60), (61) *on a plane domain Ω. Under this assumption, there exists a number $Q > 2$, which depends only on λ ($Q \to \infty$ as $\lambda \to 1$ and $Q \to 2$ as $\lambda \to 0$) such that if u_{xx}, u_{xy}, u_{yy} are in $\mathcal{L}^{p_1}_{\mathrm{loc}}$ and f is in $\mathcal{L}^{p}_{\mathrm{loc}}$ where*

$$Q' < p_1 < p < Q, \tag{62}$$

then u_{xx}, u_{xy}, u_{yy} are in $\mathcal{L}^{p}_{\mathrm{loc}}$.

PROOF. The proof uses the fortuitous relation that exists between equations of divergence and non-divergence structure in two dimensions.

Set $U = u_x$ and $V = -u_y$. We then have the system

$$\begin{aligned} V_y &= \frac{a}{c}\, U_x + \frac{2b}{c}\, U_y - \frac{f}{c} \\ -V_x &= U_y . \end{aligned} \tag{63}$$

Clearly U is a solution of the uniformly elliptic equation of divergence structure

$$\left(\frac{a}{c}\,U_x + \frac{2b}{c}\,U_y\right)_x + (U_y)_y = \left(\frac{f}{c}\right)_x . \tag{64}$$

Theorem 2 then implies the required result for u_{xx} and u_{yy}. Finally, the first equation in (63) shows that u_{yy} is also in $\mathcal{L}^p_{\mathrm{loc}}$.

REMARKS. By means of the Sobolev lemma Theorem 3 implies the Hölder continuity of grad u (u a solution of equation (60) (61)) if $p > 2$. However a slightly stronger result is already known. If one modifies the work of Finn and Serrin [7] appropriately one can show that grad u is Hölder continuous if only $|f|_{2;R} \leq$ const. R^α for some $\alpha > 0$.

It is clear that Theorems 2 and 3 will continue to hold if the differential equations contain terms of lower order with coefficients in the appropriate $\mathcal{L}^p$ classes.

5. An upper bound for Q in Theorem 2.

Theorem 2 gives a theoretical lower bound for the value of Q. By estimating the value of K_p for the sphere one could in fact give an explicit lower bound. We now give an upper bound for Q by means of an example.

Let x and y be coordinates in the plane. Consider the equation

$$Lu = (au_x + bu_y)_x + (bu_x + cu_y)_y = 0, \tag{65}$$

where

$$\begin{aligned} a &= 1 - (1-\mu^2)\frac{y^2}{x^2+y^2} \\ b &= (1-\mu^2)\frac{xy}{x^2+y^2} \\ c &= 1 - (1-\mu^2)\frac{x^2}{x^2+y^2} \end{aligned} \tag{66}$$

and μ is a fixed constant in the range $0 < \mu < 1$. It is easily seen that at each point (x, y) the eigenvalues of the coefficient matrix are μ^2 and 1. Thus equation (65) is in the form (10) (11) with $\Lambda = 1$, $\lambda = \mu^2$ and $\theta = 1$. $u(x, y) = (x^2 + y^2)^{\frac{\mu-1}{2}} \cdot x$ is a solution of equation (60) and it is easily seen that grad u is in $\mathcal{L}^p_{\mathrm{loc}}$ for $p < \dfrac{2}{1-\mu}$ but that $\displaystyle\int_{x^2+y^2<1} |\,\mathrm{grad}\, u|^{\frac{2}{1-\mu}}\, dxdy = +\infty$

because of the singularity at the origin. Hence we must in general have

$$Q \leq \frac{2}{1 - \sqrt{\theta\lambda}}. \tag{67}$$

(67) is now established only for $n = 2$. We now establish (67) for all dimensions by the method of descent. Let (x, y, z) be a point in n-space where z stands for the remaining $(n - 2)$ coordinates. We extend the given solution u and coefficients by defining $u(x, y, z) = u(x, y)$ etc. Then we have

$$L_{xy} u + \Delta_z u = 0. \tag{68}$$

Equation (63) has the *same* ellipticity constants as the original equation and it is obvious that grad u is *not* in $\mathcal{L}_{\text{loc}}^{\frac{2}{1-\mu}}$. Hence the bound (67) holds in all dimensions.

In particular this shows that Q really does tend to 2 as $\theta\lambda \to 0$. The best value of Q probably depends on n and becomes smaller as n increases and $\theta\lambda$ remains fixed. In any case it is clear from the method of descent that it cannot increase with increasing dimension.

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