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Some properties of Holmgren-Riesz transform in two dimensions

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SOME PROPERTIES OF HOLMGREN-RIESZ TRANSFORM IN TWO DIMENSIONS [1]

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1. THE TRANSFORM IN THE $u\ v$ - PLANE

1. Preliminaries and Definitions.

Throughout this work the symbols : D_{xy}^{n+m} will be used to represent the derivative operator with respect to x and y of order $n+m$, taken n-times with respect to x and m-times with respect to y ; F_{12} will denote the partial derivatives of F with respect to its first and second variables respectively ; ∇_{xy}^2 will represent the D'Alembert operator « $D_{xx} - D_{yy}$ », and ∇_{xy}^{2n} is the D'Alembert operator of order n. Others used here are conventional symbols.

DEFINITION 1.

If $F(u, v)$ is a real valued function of class $C^{(m+n)}$ with respect to the variables u, v in the region T: $a \leq u \leq b$, $c \leq v \leq d$ and $R\alpha + n > 0$, $R\beta + m > 0$, then

$$(H_1) \qquad \overset{u}{\underset{v}{I}}{}^{\alpha} \overset{v}{\underset{z}{I}}{}^{\beta} F = \frac{D_u^n}{\Gamma(\alpha+n)\ \Gamma(\beta+m)} \int_v^u (u-z)^{\alpha+n-1}$$

$$\left\{ D_v^m \int_z^v (v-t)^{\beta+m-1} F(z, t)\, dt \right\} dz$$

$$= D_{uv}^{n+m} \overset{u}{\underset{v}{I}}{}^{\alpha+n} \overset{v}{\underset{z}{I}}{}^{\beta+m} F$$

([1]) This is a continuation of the work mentioned in [1].

2. Some Properties of (H_1).

It is clear that (H_1) is a repeated form of the $H - R$ transform in one dimension (²) whose lower limits are parameters. Therefore, it is expected to have properties similar to those possessed by the later which are idependent of the lower limits. Thus if $\alpha = \beta = 0$ in (H_1), we would obtain the identity transform

$$\overset{u}{\underset{v}{I^0}} \overset{v}{\underset{z}{I^0}} F = F(u, v), \tag{2.1}$$

and if $\alpha = -n$, $\beta = -m$ we have

$$\overset{u}{\underset{v}{I^{-n}}} \overset{v}{\underset{z}{I^{-m}}} F = D_{uv}^{(n+m)} F(u, v) \tag{2.2}$$

Considering the region of integration as the one bounded by the right

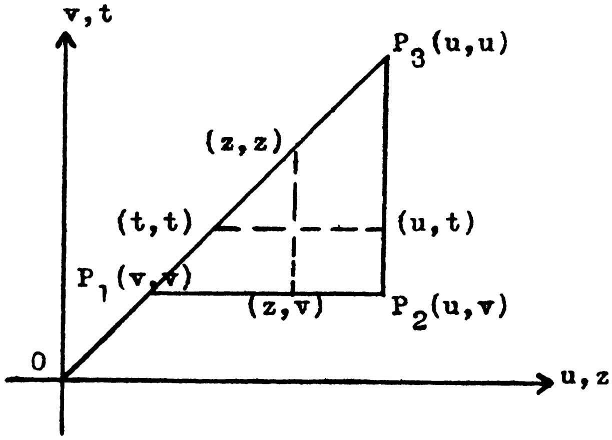

Fig. 1.

traingle $P_1 P_2 P_3$ (Fig. 1), and assuming that $m = n = 0$ in (H_1) then by changing the order of integration we would have

$$\overset{u}{\underset{v}{I^\alpha}} \overset{v}{\underset{z}{I^\beta}} F = \overset{v}{\underset{u}{I^\beta}} \overset{u}{\underset{t}{I^\alpha}} F \tag{2.3}$$

(²) For details see [1].

In this work, our study and investigation will be confined to the case where $\alpha = \beta$ in (H_1).

3. The Extended Form of $\overset{u}{\underset{v}{I}}{}^{\alpha} \overset{v}{\underset{z}{I}}{}^{\alpha} F$.

If $F(u, v) \in C^{(2n)}$ in the region T and $R\alpha > 0$, then

$$(3.1) \qquad \overset{u}{\underset{v}{I}}{}^{\alpha} \overset{v}{\underset{z}{I}}{}^{\alpha} F = \sum_{i=0}^{i=n-1} \frac{D_u^n}{\Gamma(\alpha+i+1)} \overset{u}{\underset{v}{I}}{}^{\alpha+n} (v-u)^{\alpha+i} F_2^{(i)}(u, u)$$

$$+ \sum_{i=0}^{n-1} \sum_{p=0}^{p=i} \frac{(-1)^{p+1} \Gamma(i+1)}{\Gamma(i-p+1)\Gamma(p+1)\Gamma(\alpha+n-p)} \overset{u}{\underset{v}{I}}{}^{\alpha+n} (v-u)^{\alpha+n-p-1} F_{21}^{(n, n-p-1)}$$

$$+ \overset{u}{\underset{v}{I}}{}^{\alpha+n} \overset{v}{\underset{z}{I}}{}^{\alpha+n} F_{uv}^{(n,n)}.$$

To establish this identity, we have by Definition 1

$$\text{(i)} \qquad \overset{u}{\underset{v}{I}}{}^{\alpha} \overset{v}{\underset{z}{I}}{}^{\alpha} F = \frac{D_u^n}{\Gamma^2(\alpha+n)} \int_v^u (u-z)^{\alpha+n-1} \left[D_v^n \int_z^v (v-t)^{\alpha+n-1} F(z, t)\, dt \right] dz$$

Let $v - t = s$, then we have

$$\overset{v}{\underset{z}{I}}{}^{\alpha} F = \frac{1}{\Gamma(\alpha+n)} D_v^n \int_z^v (v-t)^{\alpha+n-1} F(z, t)\, dt$$

$$= \frac{1}{\Gamma(\alpha+n)} D_v^n \int_0^{v-z} s^{\alpha+n-1} F(z, v-s)\, ds$$

$$= \sum_{i=0}^{n-1} \frac{F_2^{(i)}(z, z)(v-z)^{\alpha+i}}{\Gamma(\alpha+i+1)} + \frac{1}{\Gamma(\alpha+n)} \int_z^v F_t^{(n)}(z, t) \, (v-t)^{\alpha+n-1}\, dt$$

Thus (i) can be written as

$$\text{(ii)} \qquad \overset{u}{\underset{v}{I}}{}^{\alpha} \overset{v}{\underset{z}{I}}{}^{\alpha} F = \sum_{i=0}^{i=n-1} \frac{D_u^n}{\Gamma(\alpha+i+1)\Gamma(\alpha+n)} \int_v^u F_2^{(i)}(z, z)(v-z)^{\alpha+i} \, (u-z)^{\alpha+n-1}\, dz$$

$$+\frac{D_u^n}{\Gamma^2(\alpha+n)}\int_v^u (u-z)^{\alpha+n-1}\,dz\int_z^v F_t^{(n)}(z,t)\,(v-t)^{\alpha+n-1}\,dt$$

$$=\sum_{i=0}^{i=n-1}\frac{D_u^n}{\Gamma(\alpha+i+1)}\ \overset{u}{\underset{v}{I}}{}^{\alpha+n}(v-u)^{\alpha+i}F_2^{(i)}(u,u)+K,$$

where K represents the last term in (ii). Let

$$\text{(iii)}\qquad G(z)=\int_z^v F_t^{(n)}(z,t)\,(v-t)^{\alpha+n-1}\,dt,$$

then $G^{(i)}(v)=0$ for $(i=0,1,2,\dots,n-1)$, and consequently K may be written as

$$\text{(iv)}\qquad K=\frac{1}{\Gamma^2(\alpha+n)}\int_v^u (u-z)^{\alpha+n-1}\,G^{(n)}(z)\,dz$$

But from (iii) we find that

$$\text{(v)}\qquad G^{(n)}(z)=\int_z^v F_{12}^{(n,n)}(z,t)\,(v-t)^{\alpha+n-1}\,dt-$$

$$-\sum_{i=0}^{n-1}D_z^i\,F_{21}^{(n,n-i-1)}(z,z)\,(v-z)^{\alpha+n-1}$$

the last term of which may be written in the form

$$\sum_{i=0}^{n-1}\sum_{p=0}^{p=i}\frac{(-1)^p\,\Gamma(i+1)\,\Gamma(\alpha+n)}{\Gamma(i-p+1)\,\Gamma(\alpha+n-p)\,\Gamma(p+1)}\,F_{21}^{(n,n-p-1)}(z,z)\,(v-z)^{\alpha+n-p-1}$$

Therefore (3.1) is obtained by writing the form (v) in (iv) and combining the result with (ii).

The form (3.1) is valid only when $R\alpha>0$, and it has no meaning when $R\alpha\leq 0$, as it contains divergent integrals. If $F_{21}^{(i,j)}(u,u)=0$ for $(i=0,1,\dots n)$, $(j=0,1,\dots,n-1)$, then (3.1) may be defined for $R\alpha\leq 0$ by

$$\text{(3.2)}\qquad \overset{u}{\underset{v}{I}}{}^{\alpha}\ \overset{v}{\underset{z}{I}}{}^{\alpha}\,F=\overset{u}{\underset{v}{I}}{}^{\alpha+n}\ \overset{v}{\underset{z}{I}}{}^{\alpha+n}\,F_{uv}^{(n,n)},$$

the left side of which is defined by the right side which exists for $R\alpha + n > 0$.

4. The Index Law.

The relation

$$\overset{u}{\underset{v}{I}}{}^{\alpha}\,\overset{v}{\underset{z}{I}}{}^{\alpha}\left\{\overset{u}{\underset{v}{I}}{}^{\beta}\,\overset{v}{\underset{z}{I}}{}^{\beta}\,F\right\} = \overset{u}{\underset{v}{I}}{}^{\alpha+\beta}\,\overset{v}{\underset{z}{I}}{}^{\alpha+\beta}\,F \tag{4.1}$$

holds if

(i) $R\alpha > 0$, $R\beta > 0$, and $F(u, v) \in C^{(0)}$ in the region T.

(ii) $R\alpha + n > 0$, $R\beta > 0$, and $F(u, v) \in C^{(2n)}$ in T and $F^{(i,j)}_{21}(u, u) = 0$, $(i = 0, 1, \ldots, n)$, $(j = 0, 1, \ldots, n - 1)$.

(iii) $R\alpha > 0$, $R\beta + m > 0$, and $F(u, v) \in C^{(2m)}$ in T, and $F^{(i,j)}_{21}(u, u) = 0$, $(i = 0, 1, \ldots, m)$, $(j = 0, 1, \ldots, m - 1)$.

(iv) $R\alpha + n > 0$, $R\beta + m > 0$, and $F(u, v) \in C^{(2m+2n)}$ in T and $F^{(i,j)}_{21}(u, u) = 0$, $(i = 0, 1, \ldots, m + n)$, $(j = 0, 1, 2, \ldots, m + n - 1)$.

Case (i)(3). The left hand side of (4.1) may be written as

$$J^* = \overset{u}{\underset{v}{I}}{}^{\alpha}\left\{\overset{v}{\underset{z}{I}}{}^{\alpha}\,\overset{u}{\underset{v}{I}}{}^{\beta}\left(\overset{v}{\underset{z}{I}}{}^{\beta}\,F\right)\right\}.$$

Then by using the property (2.3), this may be made equivalent to the form

$$J^* = \overset{u}{\underset{v}{I}}{}^{\alpha}\,\overset{u}{\underset{v}{I}}{}^{\beta}\left\{\overset{v}{\underset{z}{I}}{}^{\alpha}\,\overset{v}{\underset{z}{I}}{}^{\beta}\,F\right\},$$

which is identically equal to the right hand side of (4.1) according to Theorem 3 [1].

Case (ii). By Definition 1 and (3.2) we have

$$J^{**} = D^{2n}_{uv}\,\overset{u}{\underset{v}{I}}{}^{\alpha+n}\,\overset{v}{\underset{z}{I}}{}^{\alpha+n}\left\{\overset{u}{\underset{v}{I}}{}^{\beta+n}\,\overset{v}{\underset{z}{I}}{}^{\beta+n}\,F^{(n,n)}_{uv}\right\},$$

---

(3) J^*, J^{**} and J^{***} will be used to denote the left hand side of (4.1) in the proof of the first three cases respectively.

and by case (i) we find that

$$J^{**} = D^{2n}_{uv} \underset{v}{\overset{u}{I}}{}^{\alpha+\beta+2n} \underset{z}{\overset{v}{I}}{}^{\alpha+\beta+2n} F^{(n,n)}_{u,v}$$

$$= D^{2n}_{nv} \underset{v}{\overset{u}{I}}{}^{\alpha+\beta+n} \underset{z}{\overset{v}{I}}{}^{\alpha+\beta+n} F$$

$$= \underset{v}{\overset{u}{I}}{}^{\alpha+\beta} \underset{z}{\overset{v}{I}}{}^{\alpha+\beta} F.$$

Case (iii). According to (3.2) we may write

$$\underset{v}{\overset{u}{I}}{}^{\beta} \underset{z}{\overset{v}{I}}{}^{\beta} F = \underset{v}{\overset{u}{I}}{}^{\beta+m} \underset{z}{\overset{v}{I}}{}^{\beta+m} F^{(m,m)}_{uv},$$

and consequently by case (i) we have

$$J^{***} = \underset{v}{\overset{u}{I}}{}^{\alpha+\beta+m} \underset{z}{\overset{v}{I}}{}^{\alpha+\beta+m} F^{(m,m)}_{uv}$$

$$= \underset{v}{\overset{u}{I}}{}^{\alpha+\beta} \underset{z}{\overset{v}{I}}{}^{\alpha+\beta} F.$$

Case (iv). The proof of this case follows from cases (ii) and (iii).

2. The Transform in the xy. - Plane

5. Transformation of the Coordinates.

If in (H_1) we let

(i) $\beta = \alpha$ and $m = n$,

(5.1) (ii) $u = \dfrac{x+y}{\sqrt{2}}, \quad z = \dfrac{\xi+\eta}{\sqrt{2}}$

$$v = \frac{y-x}{\sqrt{2}}, \quad t = \frac{\eta-\xi}{\sqrt{2}}$$

with the Jacobian $\dfrac{\partial (z, t)}{\partial (\xi, \eta)} = 1$, then

$$D_{uv}^{2n} = \frac{(-1)^n}{2^n} \nabla_{xy}^{2n}, \tag{5.2}$$

and by considering the triangular region T (Fig. 2) as the region of integration, then we find that

$$\int_v^u (u - z)^{\alpha+n-1} \, dz \int_v^z (v - t)^{\alpha+n-1} F(z, t) \, dt =$$

$$= \frac{(-1)^{\alpha+n-1}}{2^{\alpha+n-1}} \int_0^x d\xi \int_{y-x+\xi}^{y+x-\xi} [(x-\xi)^2 - (y-\eta)^2]^{\alpha+n-1} F\left(\frac{\xi+\eta}{\sqrt{2}}, \frac{\eta-\xi}{\sqrt{2}}\right) d\eta.$$

Consequently, in the xy - plane

$$\overset{u}{\underset{v}{I^\alpha}} \overset{v}{\underset{z}{I^\alpha}} F = A(\alpha + n) \nabla_{xy}^{2n} \int_0^x d\xi \int_{y-x+\xi}^{y+x-\xi} [(x-\xi)^2 - (y-\eta)^2]^{\alpha+n-1} f(\xi, \eta) \, d\eta$$

where

$$A(\alpha + n) = 1/2^{2(\alpha+n)-1} \Gamma^2(\alpha + n), \tag{5.3}$$

and

$$f(\xi, \eta) = (-1)^\alpha \, 2^\alpha \, F\left(\frac{\xi+\eta}{\sqrt{2}}, \frac{\eta-\xi}{\sqrt{2}}\right). \tag{5.4}$$

Thus we are led to the following

DEFINITION 2.

If $f(x, y)$ is a real valued function of class $C^{(2n)}$ with respect to the variables x, y in the region T and $R\alpha + n > 0$, then

$$I^{2\alpha} f = A(\alpha + n) \nabla_{xy}^{2n} \int_0^x d\xi \int_{y-x+\xi}^{y+x-\xi} E^{\alpha+n-1}(x, y; \xi, \eta) f(\xi, \eta) \, d\eta \tag{H$_2$}$$

where $A(\alpha + n)$ is given by (5.3),

$$E(x, y; \xi, \eta) = (x - \xi)^2 - (y - \eta)^2$$

and $x > 0$.

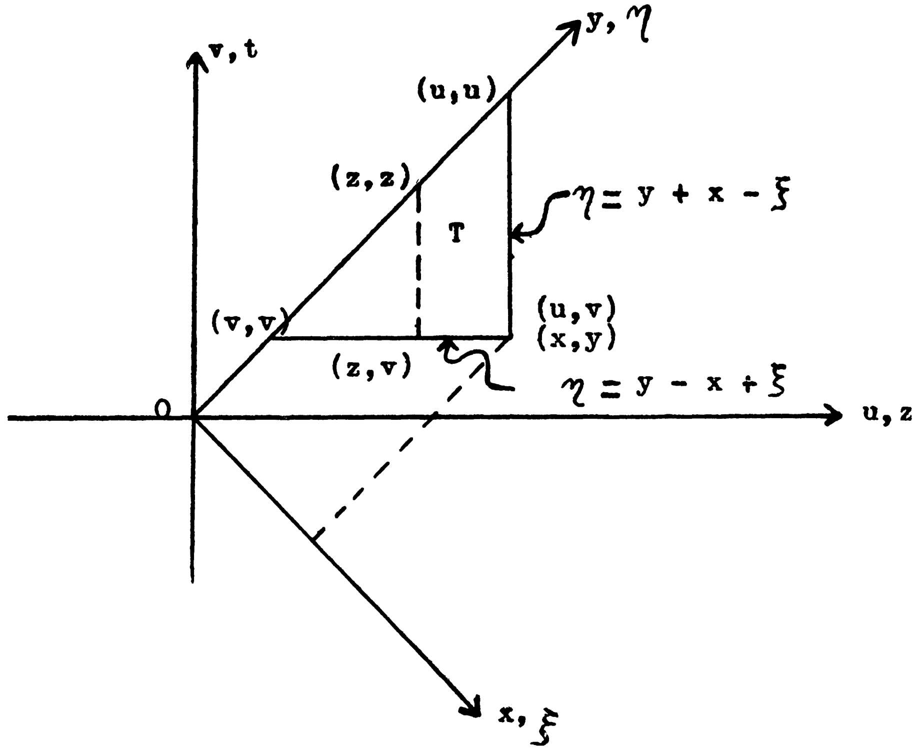

Fig. 2.

6. The Identity and Derivative Properties of (H_2).

If we let $\alpha = 0$, then from (2.1), (5.1) and (5.4) the following identity holds

$$\overset{u}{\underset{v}{I^0}}\ \overset{v}{\underset{z}{I^0}}\ F = I^0 f = f(x, y) \tag{6.1}$$

This identity transform may also be obtained directly from (H_2), for if $\alpha = 0$, $n = 1$, then

$$I^0 f = \frac{1}{2} \nabla^2_{xy} \int_0^x d\xi \int_{y-x+\xi}^{y+x-\xi} f(\xi, \eta)\, d\eta \tag{6.11}$$

$$= \frac{1}{2} (D_{xx} - D_{yy})\, \Phi$$

where Φ represents the double integral. The identity transform easily follows from (6.11) since

$$D_{xx}\,\Phi = \int_0^x [f_2(\xi, y + x - \xi) - f_2(\xi, y - x + \xi)]\,d\xi + 2f(x, y)$$

$$D_{yy}\,\Phi = \int_0^x [f_2(\xi, y + x - \xi) - f_2(\xi, y - x + \xi)]\,d\xi.$$

If in (H_2) α is a negative integer, i.e. $\alpha = -n + 1$, then

$$(6.2) \qquad I^{-(n-1)} f = \frac{1}{2}\nabla^{2n}_{xy}\int_0^x d\xi \int_{y-x+\xi}^{y+x-\xi} f(\xi, \eta)\,d\eta$$

$$= \nabla^{2(n-1)}_{xy} f(x, y).$$

It is to be noted that if $R\alpha > n$ and $f(x, y)$ belongs to the class $C^{(0)}$ in T, then

$$(6.3) \qquad \nabla^{2n}_{xy} I^{2(\alpha+n)} f = I^{2\alpha} f,$$

for we have

$$\nabla^{2n}_{xy} I^{2(\alpha+n)} = A(\alpha + n)\int_0^x d\xi \int_{y-x+\xi}^{y+x-\xi} \nabla^{2n}_{xy} E^{\alpha+n-1}(x, y\,;\,\xi, \eta) f(\xi, \eta)\,d\eta$$

and

$$\nabla^{2n}_{xy} E^{\alpha+n-1} = 2^{2n}[(\alpha + n - 1)(\alpha + n - 2)\ldots(\alpha + 1)]^2 E^{\alpha-1}.$$

7. The Expansion of the Transform by means of Green's Theorem.

If $U(x, y)$ and $V(x, y)$ are two real valued functions of class $C^{(2)}$ with respect to the variables x and y in some region T bounded by a simple closed curve C_T, then according to Green's Theorem we have

$$(7.1) \quad \iint_T (U\,\nabla^2_{\xi\eta} V - V\,\nabla^2_{\xi\eta} U)\,dS = \int_{C_T} [U(V_\xi\,d\eta + V_\eta\,d\xi) - V(U_\xi\,d\eta + U_\eta\,d\xi)].$$

Suppose that $R\alpha > 1$ and

$$V(\xi, \eta) = A(\alpha + 1)\,E^\alpha(x, y\,;\,\xi, \eta)$$

$$U(\xi, \eta) = f(\xi, \eta),$$

and let the region of integration T be the region bounded by the triangle P_1PP_2 (Fig. 3), then the function $E^{\alpha-1}$ vanishes along the sraight lines $\overline{P_1P}$ and $\overline{PP_2}$ which are parts of the boundary curves C_T, and consequently (7.1) becomes

$$A(\alpha)\iint_T f(\xi,\eta)\,E^{\alpha-1}\,dS - A(\alpha+1)\iint_T E^{\alpha}\nabla^2_{\xi\eta}f(\xi,\eta)\,dS = I^{2\alpha}f - I^{2(\alpha+1)}\nabla^2_{xy}f =$$

$$= K_1 x\int_{y+x}^{y-x} f(0,\eta)\,[x^2-(y-\eta)^2]^{\alpha-1}\,d\eta - K_2\int_{y+x}^{y-x} f_\xi(0,\eta)\,[x^2-(y-\eta)^2]^{\alpha}\,d\eta$$

where $K_1 = -1/2^{2\alpha}\Gamma(\alpha+1)\Gamma(\alpha)$, and $K_2 = 1/2^{2\alpha+1}\Gamma^2(\alpha+1)$. Consequently we have

$$(7.2)\quad I^{2\alpha}f = I^{2(\alpha+1)}\nabla^2_{xy}f + A(\alpha+1)\left\{D_x\int_{y-x}^{y+x} f(0,\eta)\,E_0^{\alpha}\,d\eta + \int_{y-x}^{y+x} f_\xi(0,\eta)\,E_0^{\alpha}\,d\eta\right\}$$

where $E_0 = E(x, y; 0, \eta)$.

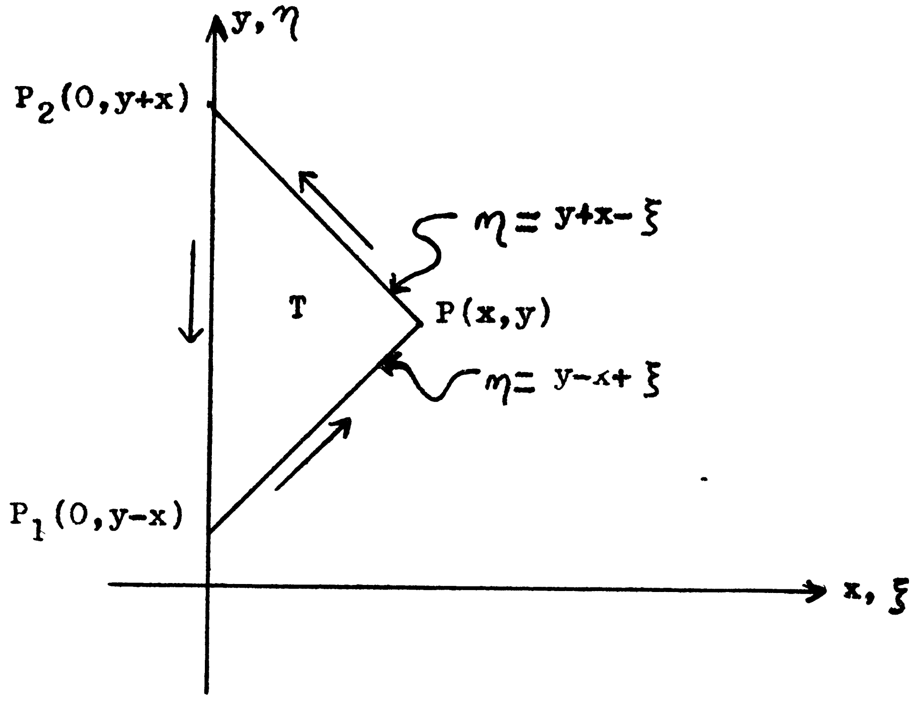

Fig. 3.

Assuming now that $f(x, y)$ is of class $C^{(2n)}$ in T and $R\alpha > 1$, and applying Green's Theorem $(n-1)$-times by the same method used above, then (7.2) may assume the expanded form

$$(7.3)\quad I^{2\alpha} f = I^{2(\alpha+n)} \nabla^{2n}_{xy} f + \sum_{i=1}^{n} A(\alpha+i) \left\{ D_x \int_{y-x}^{y+x} E^{\alpha+i-1} \nabla^{2(i-1)}_{\xi\eta} f(0,\eta)\, d\eta + \right.$$
$$\left. + \int_{y-x}^{y+x} E_0^{\alpha+i-1} D_x \nabla^{2(i-1)}_{\xi\eta} f(0,\eta)\, d\eta \right\}$$

where $\nabla^{2(i-1)}_{\xi\eta} f(0,\eta) = [\nabla^{2(i-1)}_{\xi\eta} f(\xi,\eta)]_{\xi=0}$.

8. Extension of Definition.

The expansion (7.3) is defined for $R\alpha > -1$, but when $R\alpha \leq -1$ it has no menning as it includes integrals which cease to converge for this range of values of α. However, we notice that the existence for $R\alpha > 1$ of the integral

$$\int_{y+x}^{y-x} g(\eta)\, E_0^{\alpha-1}\, d\eta = \frac{\Gamma^2(\alpha)}{2^{2n}\,\Gamma^2(\alpha+n)} \nabla^{2n}_{xy} \int_{y+x}^{y-x} g(\eta)\, E_0^{\alpha+n-1}\, d\eta$$

implies that the same integral exists when $R\alpha + n > 0$ and g is of class $C^{(2n)}$, for if we let $y - \eta = xz$, we find that

$$\nabla^{2n}_{xy} \int_{y+x}^{y-x} g(\eta)\, E_0^{\alpha+n-1}\, d\eta = -\nabla^{2n}_{xy} \left\{ x^{2(\alpha+n)-1} \int_{-1}^{1} g(y-xz)\,(1-z^2)^{\alpha+n-1}\, dz \right\}$$

which clearly shows that the right hand side exists under the conditions imposed on the function g.

This property will permit us to extend the definition of the expansion (7.3) for wider range of values of α such that $R\alpha + n > 0$. Thus for such values of α, (7.3) may be given by the form

$$(8.1)\quad I^{2\alpha} f = I^{2(\alpha+n)} \nabla^{2n}_{xy} f +$$
$$+ A(\alpha+n) \sum_{i=1}^{n} \left\{ D_x \nabla^{2(n-i)}_{xy} \int_{y-x}^{y+x} E_0^{\alpha+n-1} \nabla^{2(i-1)}_{\xi\eta} f(0,\eta)\, d\eta + \right.$$
$$\left. + \nabla^{2(n-i)}_{xy} \int_{y-x}^{y+x} E_0^{\alpha+n-1} D_\xi \nabla^{2(i-1)}_{\xi\eta} f(0,\eta)\, d\eta \right\},$$

for as we have already indicated that when $R\alpha > 1$

$$\int_{y-x}^{y+x} g(\eta)\, E_0^{\alpha+i-1}\, d\eta = \frac{\Gamma^2(\alpha+i)}{2^{2(n-i)}\,\Gamma^2(\alpha+n)} \nabla_{xy}^{2(n-i)} \int_{y-x}^{y+x} g(\eta)\, E_0^{\alpha+n-1}\, d\eta.$$

9. A Property of Equivalence.

It may be interesting to note that the expansion (7.2) is equivalent to the transformed form of (3.1) for $n = 1$. This property may be established as follows : For $n = 1$, (3.1) assumes the form

$$(9.1) \qquad \overset{u}{\underset{v}{I}}{}^{\alpha} \overset{v}{\underset{z}{I}}{}^{\dot{\alpha}} F = \overset{u}{\underset{v}{I}}{}^{\alpha+1} \overset{v}{\underset{z}{I}}{}^{\alpha+1} F_{uv}^{(1,1)} - \frac{1}{\Gamma(\alpha+1)} \overset{u}{\underset{v}{I}}{}^{\dot{\alpha}+1} (v-u)^{\alpha} F_2(u,u)$$

$$+ \frac{Du}{\Gamma(\alpha+1)} \overset{u}{\underset{v}{I}}{}^{\alpha+1} (v-u)^{\alpha} F(u,u).$$

Now let the terms of the right hand side of (9.1) be denoted by I_1, I_2 and I_3 respectively. Then by applying the transformation (5.1) and using the assumption (5.4) we find that

$$F_{zt}^{(1,1)}(z,t) = \frac{(-1)^{\dot{\alpha}+1}}{2^{\dot{\alpha}+1}} \nabla_{\xi\eta}^2 f(\xi,\eta),$$

and consequently

$$(9.11) \qquad I_1 = I^{2(\alpha+1)} \nabla_{xy}^2 f.$$

Also we have

$$I_2 = \frac{-1}{\Gamma^2(\alpha+1)} \int_{\frac{y-x}{\sqrt{2}}}^{\frac{y+x}{\sqrt{2}}} \left(\frac{y-x}{\sqrt{2}} - z\right)^{\dot{\alpha}} \left(\frac{y+x}{\sqrt{2}} - z\right)^{\alpha} F_2(z,z)\, dz.$$

If we let $\sqrt{2}\, z = \eta$, and use the relation

$$F_2\left(\frac{\xi+\eta}{\sqrt{2}}, \frac{\eta-\xi}{\sqrt{2}}\right) = \frac{1}{(-1)^{\alpha+1}\, 2^{\alpha+\frac{1}{2}}} [f_\xi(\xi,\eta) - f_\eta(\xi,\eta)]$$

which is obtained from (5.4), then we find that

$$(9.12) \qquad I_2 = A(\alpha+1) \left\{ \int_{y-x}^{y+x} E_0^{\alpha} f_\xi(0,\eta)\, d\eta - \int_{y-x}^{y+x} E_0^{\alpha} f_\eta(0,\eta)\, d\eta \right\}.$$

As to the last term of (9.1) we have

$$I_3 = \frac{D_x + D_y}{\sqrt{2}\,\Gamma^2(\alpha+1)} \int\limits_{\frac{y-x}{\sqrt{2}}}^{\frac{y+x}{\sqrt{2}}} \left(\frac{y-x}{\sqrt{2}} - z\right)^\alpha \left(\frac{y+x}{\sqrt{2}} - z\right)^\alpha F(z, z)\, dz$$

which, after making the substitution $\sqrt{2}\, z = \eta$ and taking the derivative with respect to y, may be written as

$$I_3 = A(\alpha+1) \left\{ D_x \int\limits_{y-x}^{y+x} E_0^\alpha f(0, \eta)\, d\eta + \int\limits_{y-x}^{y+x} E_0^\alpha f_\eta(0, \eta)\, d\eta \right\}. \tag{9.13}$$

Adding the results (9.11), (9.12) and (9.13), then under the transformation mentioned above we have

$$\overset{u}{\underset{v}{I^\alpha}}\, \overset{v}{\underset{z}{I^\alpha}}\, F = I_1 + I_2 + I_3$$

$$= I^{2\alpha} f$$

10. Some Particular Cases.

(i) If $f(x, y) \equiv g(x)$ and $R\alpha + n > 0$, then

$$I^{2\alpha} g = \overset{x}{\underset{0}{I^{2\alpha}}}\, g. \tag{10.1}$$

For we have

$$I^{2\alpha} g = A(\alpha+n)\, \nabla_{xy}^{2n} \int\limits_0^x g(\xi)\, d\xi \int\limits_{y-x+\xi}^{y+x-\xi} E^{\alpha+n-1}\, d\eta$$

and by applying the transformation $2(x-\xi)\, z = \eta - y + x - \xi$ to the integral we find that

$$I^{2\alpha} g = \frac{1}{\Gamma[2(\alpha+n)]} D_x^{2n} \int\limits_0^x g(\xi)(x-\xi)^{2(\alpha+n)-1}\, d\xi$$

$$= \overset{x}{\underset{0}{I^{2\alpha}}}\, g$$

(ii) It is clear that if $g(x) \equiv 1$ and $R\alpha + n > 0$, then

$$I^{2\alpha} 1 = \frac{x^{2\alpha}}{\Gamma(2\alpha + 1)}$$

(iii) The extended forms of the transform are of significant value with regard to their relationship with the boundary value problems of the partial differential equations of hyperbolic type. For example if we let in (8.1), $\alpha = 0$, $n = 1$ and $u(x, y) \equiv f(x, y)$, then we obtain

$$u(x,y) = I^2 \nabla^2_{xy} u + \frac{1}{2} D_x \int_{y-x}^{y+x} u(0, \eta)\, d\eta + \frac{1}{2} \int_{y-x}^{y+x} u_\xi(0, \eta)\, d\eta,$$

from which we may conclude that the solution of the second order partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = F(x, y),$$

with the conditions

$$u(0, y) = g(y)$$

$$u_x(0, y) = h(y),$$

can be written as

$$u(x, y) = I^2 F + \frac{1}{2}[g(x + y) + g(y - x)] + \frac{1}{2} \int_{y-x}^{y+x} h(\eta)\, d\eta,$$

which is the D'Alembert solution of the Cauchy problem for this particular case.

11. The Index Law.

The relation

$$I^{2\alpha} I^{2\beta} f = I^{2(\alpha+\beta)} f$$

holds if

(i) $R\alpha > 0$, $R\beta > 0$, and $f(x, y) \in C^{(0)}$ in the region T.

(ii) $R\alpha + n > 0$, $R\beta > 0$, and $f(x, y) \in C^{(2n)}$ in T, and

$$\nabla^{2i}_{xy} f(0, y) = 0, \quad D_x \nabla^{2i}_{xy} f(0, y) = 0, \quad (i = 0, 1, \dots, n - 1).$$

(iii) $R\alpha > 0$, $R\beta + m > 0$, and $f(x, y) \in C^{(2m)}$ in T and

$$\nabla^{2j}_{xy} f(0, y) = 0, \quad D_x \nabla^{2j}_{xy} f(0, y) = 0, \quad (j = 0, 1, \dots, m - 1).$$

(iv) $R\alpha + n > 0$, $R\beta + m > 0$, and $f(x, y) \in C^{2(m+n)}$ in T and

$$\nabla_{xy}^{2k} f(0, y) = 0, D_x \nabla_{xy}^{2k} f(0, y) = 0, \quad (k = 0, 1, \dots, m + n - 1).$$

Case (i). The validity of this case follows after applying the transformation (5.1) to both sides of (4.1)-(i).

Case (ii). By (7.3)

$$I^{2\beta} f = I^{2(\beta+n)} \nabla_{xy}^{2n} f.$$

Thus we have

$$I^{2\alpha} I^{2\beta} f = \nabla_{xy}^{2n} I^{2(\alpha+n)} I^{2(\beta+n)} \nabla_{xy}^{2n} f$$

$$= \nabla_{xy}^{2n} I^{2(\alpha+\beta+2n)} \nabla_{xy}^{2n} f, \text{ (by case (i))}.$$

Consequently,

$$I^{2\alpha} I^{2\beta} f = I^{2(\alpha+\beta+n)} \nabla_{xy}^{2n} f$$

$$= I^{2(\alpha+\beta)} f$$

Case (iii). By (8.1) ad Case (i) we have

$$I^{2\alpha} I^{2\beta} f = I^{2\alpha} I^{2(\beta+n)} \nabla_{xy}^{2n} f$$

$$= I^{2(\alpha+\beta+n)} \nabla_{xy}^{2n} f$$

$$= I^{2(\alpha+\beta)} f.$$

Case (iv). The proof of this case follows from cases (ii) and (iii).

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(*) *ERRATA*. In this paper the following corrections should be observed:

Page	*Line*				
7	2	replace	$(x - 2)$	by	$(x - z)$
8	5	»	$= \frac{(-1)^{n+1}}{\Gamma(\alpha)}$	»	$+ \frac{(-1)^{n+1}}{\Gamma(\alpha)}$
8	6	»	$p = m + 1$	»	$p = m + \imath$
9	3	»	$\alpha + 1$	»	$\alpha + i$
12	12	»	$R \leqq 0$	»	$R\,\alpha \leqq 0$
13	14	»	$(x - a)^{x-1}$	»	$(x - a)^{\alpha-1}$
16	8	»	$\int_a^x f(\zeta) \dots$	»	$\int_a^t f(\zeta) \dots$
16	10	»	$\int_a^x (x - t)^{\alpha-1} \dots$	»	$\int_\zeta^x (x - t)^{\alpha-1} \dots$
16	11	»	$x - t$	»	$x - \zeta$
17	1	»	(the last term) replace β	»	$\beta + m$
22	11	»	$w = \sum_{i=1}^{3} \alpha_i + 1 = 0$	»	$w - \sum_{i=1}^{3} \alpha_i + 1 = 0$
22	15	insert	(G) below (F).		
23	12	replace	$\int_z^u$	by	$\int_z^v$