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ALMOST-PERIODIC SOLUTIONS OF PARABOLIC SYSTEMS

By C. Foias and S. Zaidman

Introduction.

Begining with Muckenhoupt [9] and Bochner [3], the almost-periodic character of the solutions of partial differential equations was proved especially for hyperbolic conservative equations. In latter time, work on this subject was done by Amerio [1], Bochner [5], and the second of the authors [15].

The non — conservative, or more general cases were considered:

Firstly, in the joint work of Bochner and von Neumann [6[, where for a large class of partial differential or more general operational equations, it was proved that all the solutions whose range is relatively compact in some Hilbert space are also almost-periodic.

Secondly, more recently, in a paper by the second author [16] it was proved that all the bounded solutions of the inhomogeneous telegraph equation with known term almost-periodic, are also almost-periodic.

The present paper concerns the parabolic systems of the form

$$u_t (X , t) = L\, u(X , t) + f (X , t)$$

where $u (X , t) = (u_1 (X , t) , \dots , u_N (X , t))$, $X = (x_1 , \dots , x_n)$ is a point in a *bounded* —n-dimensional domain Ω, $f (X , t) = (f_1 (X , t) , \dots , f_N (X , t))$, L is a strongly elliptic operator as defined by Vishik [13], for which $Re\,(L\,u , u) \leq 0$ if $u \in D_L$, the scalar product (,) being considered in the complex Hilbert space $L^2 (\Omega) = L^2 (\Omega) \times \dots \times L^2 (\Omega)$, N-times.

Though for parabolic equations and systems, solutions are usually considered only for $t \geq \alpha$, and it is not always possible to extend these solutions to all real t, in our case it is an obvious fact that there exist solutions which are defined for all $t \in (-\infty, +\infty)$, as for the homogeneous equation $u_t(X, t) = Lu(X, t)$, and also for the non-homogeneous one $u_t(X, t) = L\, u(X, t) + f(X, t)$. For such solutions it is possible to obtain results analogous to those for hyperbolic equations, concerning their almost-periodicity.

Our main theorems are the following:

1) *If* $u(X, t) \in D_L$ *is a strong solution on* $-\infty < t < +\infty$, *of the equation* $u_t(X, t) = L\, u(X, t)$, *where* $Re\,(Lu, u) \leq 0$, $u \in D_L$, *and if its range is bounded in* $L^2(\Omega)$, *than* $u(X, t)$ *is almost-periodic from* $t \in (-\infty, +\infty)$ *to* $L^2(\Omega)$.

2) *Let* $f(X, t)$ *be almost-periodic from* $t \in (-\infty, +\infty)$ *to* $L^2(\Omega)$. *Then, a strong solution* $u(X, t)$, *on* $-\infty < t < +\infty$, *of the equation* $u_t(X, t) = L\, u(X, t) + f(X, t)$, *whose range is relatively compact in* $L^2(\Omega)$, *is an almost-periodic solution from* $-\infty < t + \infty$ *to* $L^2(\Omega)$.

3) *Moreover, if* $f(X, t)$ *is restricted to belong in a certain subspace of* $L^2(\Omega)$ *which will be defined below, then the condition for* $u(X, t)$ *in* 2) *to have a relatively compact range, may be replaced by the boundness of this range.*

In this case, a spectral condition on $f(X, t)$ will ensure the existence of almost periodic solutions for the inhomogeneous equation.

Some ideas of the proofs.

For 1) they are based essentially on some new results for stronlgy continuous semi-groups of contractions in Hilbert spaces, due to Sz.-Nagy and the first author, [11], [12].

For 2), the proof is an adaptation to our case of the method used by J. Favard in his first theorem on almost-periodic differential systems [7].

Some difficulties which appeared in our case and are due to the « non-correctness on the left » for parabolic systems, lead us to make some change in the method of Favard.

For 3) the method of proofs is not essentially new, since it has already been used in similar problems by Amerio [1], Bochner [5], and the second author [16].

Finally we note that our methods are much more general, being not restricted especially to the parabolic systems with strongly elliptic right side.

§ 1. The parabolic systems with strongly elliptic right side were considered firstly by V. E. Lyance in [8]. We shall now give here the complete definitions.

Let $L\left(X, \frac{\partial}{\partial x}\right)$ be a differential operator, of the form

$$L\left(X, \frac{\partial}{\partial x}\right) = \left\| L_{ij}\left(X, \frac{\partial}{\partial x}\right)\right\|, \quad i, j = 1, \dots, N.$$

Here

$$L_{ij}\left(X, \frac{\partial}{\partial x}\right) = (-1)^m \sum_{|k|=2m} a_{ij}^{(k)}(X)\, D^k + T_{ij}\left(X, \frac{\partial}{\partial x}\right)$$

where $(k) = (k_1, \dots, k_n)$, $X = (x_1, \dots, x_n)$, $|k| = k_1 + \dots + k_n$,

$$D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}, \quad T_{ij}\left(X, \frac{\partial}{\partial x}\right) = \sum_{|k|<2m} a_{ij}^{(k)}(X)\, D^k.$$

Let $A^{(k)}(X)$ be defined by the matrix $\|a_{ij}^{(k)}(X)\|$, $k = 2m$, i, $j = 1 \dots N$ and $C^{(k)}(X) = (1/2)\,(A^{(k)}(X) + [A^{(k)}(X)]^*)$, $|k| = 2m$. $L\left(X, \frac{\partial}{\partial x}\right)$ is said to be a strongly elliptic operator if the matrix $(-1)^m \sum_{|k|=2m} C^{(k)}(X)\, \xi_1^{k_1} \dots \xi_n^{k_n}$ is a positive definite one, for every choise of real numbers $(\xi_1, \dots, \xi_n)$, $\sum_1^n \xi_i^2 > 0$.

Let Ω be a bounded domain in the n-dimensional euclidian space, R^n, and let S be its $(n-1)$-dimensional boundary. We denote by $H = L^2(\Omega)$ the complex Hilbert space $L^2(\Omega)\, X \dots L^2(\Omega)$ (N-times). The scalar product is then defined usually by

$$(u, v) = \int_\Omega (u_1(X)\, \overline{v_1(X)} + \dots + u_N(X)\, \overline{v_N(x)})\, dx_1 \dots dx_n.$$

The preliminary domain of definition for $-L\left(X, \frac{\partial}{\partial x}\right)$ is composed of vectors $(u_1(X), \dots, u_N(X))$, where $u_i(X) \in C^{(2m)}(\overline{\Omega})$ and $D^r u_p(X)_S = 0$, for $|r| = r$, $r = 0, 1, \dots, m-1$.

In his work [13], Vishik proved that.

The operator $L\left(X, \frac{\partial}{\partial x}\right)$ admits an extension A, which is a closed linear operator with dense domain in H. It was also proved that, if $L\left(X, \frac{\partial}{\partial x}\right)$ satisfies in Ω the so called E-condition (which is valid in particular for $L\left(X, \frac{\partial}{\partial x}\right)$ non-depending on X), then A and A^* are upper semi-bounded operators, i. e. there exists a certain real β, such that $Re\,(A\,x\,,\,x) \leq \beta\,(x\,,\,x)$, $Re\,(A^*\,y\,,\,y) \leq \beta\,(y\,,\,y)$ for every $x \in D_A\,, y \in D_{A^*}$.

Moreovor, using the boundness of Ω, it is proved in [13] that the resolvent $(\lambda\,I\text{-}A)^{-1}$ is a compact operator in H, for every $\lambda \in \varrho\,(A)$.

In the present paper we shall suppose that $\beta = 0$, that is $Re\,(Ax\,,\,x) \leq 0$, $Re\,(A^*y\,,\,y) \leq 0$, for every $x \in D_A$, $y \in D_{A^*}$. Such an operator is called « dissipative ».

Let now $u\,(X\,,\,t)$ be a vector-function from $-\infty \leq a < t < b \leq +\infty$, to D_A, for which $u_t(X\,,\,t) = Au\,(X\,,\,t)$, the derivative in respect of the time-variable t, being strong in H. Such a function will be called a strong solution for the parabolic system considered.

Let now u_0 be an element of D_A : Lyance proved in [8] using the Hille-Yosida theorem on the generation of strongly continuous semi-groups, that there exists a strong solution $u\,(X\,,\,t)$, defined for $t \geq 0$, such that $\|u\,(X\,,\,t) - u_0\|_H \to 0$, if $t \to 0$.

For certain $u_0 \in D_A$ these solutions may be extendend to be defined for all $t \in (-\infty\,,\,+\infty)$. In this paper we shall consider only such solutions. Their existence is obvious due to the discretness of the spectrum of A.

Let now $f\,(X\,,\,t)$ be a function from $-\infty < t < +\infty$ to H. Suppose that is has a strongly continuous time derivative. It may be proved, using a result of Phillips [10], that for every $u_0 \in D_A$ there exists a strong solution of the non-homogeneous equation

$$u_t\,(X\,,\,t) = A\,u(X\,,\,t) + f\,(X\,,\,t)$$

which is defined for $t \geq 0$ and $u\,(X\,,\,0) = u_0 \in D_A$.

Sometimes this solution may be extended tó the whole real axe, and we shall throughout this paper consider only such solutions.

A continuous function from $t \in (-\infty, +\infty)$ to H is called almost-periodic (after Bochner [4]) if every sequence $(h_n)_1^\infty$ contains a subsequence $(h_{n_k})_1^\infty$, such that the sequence of functions from $t \in (-\infty, +\infty)$ to H, $\{f\,(X\,,\,t + h_{n_k})\}_1^\infty$, is strongly convergent in H, uniformly in respect of $t \in (-\infty, +\infty)$.

We can now ennounce our first two theorems:

THEOREM 1. *If* $u\,(X\,,\,t)$ *from* $t \in (-\infty, +\infty)$ *to* D_A *is a strong solution of the equation* $u_t\,(X\,,\,t) = A\,u(X\,,\,t)$, *defined for* $-\infty < t < +\infty$, *and if his range is a bounded set in* H, *then* $u(X\,,\,t)$ *is almost periodic from* $t \in (-\infty, +\infty)$ *to* H.

THEOREM 2. *Let $f(X, t)$ be an almost periodic function from $t \in (-\infty, +\infty)$ to H, and suppose moreover that it has a strongly continuous time-derivative. Let $u(X, t)$ be a strong solution from $-\infty < t < +\infty$ to D_A of the inhomogeneous equation*

$$u_t(X, t) = A\, u(X, t) + f(X, t)$$

Then if the range of $u(X,t)$ is relatively compact in H, for $-\infty < t < +\infty$, $u(X, t)$ is almost-periodic from $t \in (-\infty, +\infty)$ to H.

We shall give the last theorems latter, because we don't possess yet all the necessary.

§ 2. The proof of these theorems is essentially depending of the following.

Decomposition theorem. Let A be a linear operator in the separable Hilbert space $\mathcal{H}$, such that: $Re\,(Ax, x) \leq 0$ for $x \in D_A$ and $(A - I)^{-1}$ is a compact operator. Then A is the generator of a strongly continuous semi-group of contractions $\{T_t\}$, $t \geq 0$ where

$$(*) \qquad \begin{aligned} T_t\, x &= T_t\, Qx + \sum_{n=1}^{\infty} e^{i\alpha_n t}\, P_n\, x \\ T_t\, Q\, x &= Q\, T_t\, x \to 0\,, \quad t \to +\infty \end{aligned}$$

(for all $x \in \mathcal{H}$).

$(P_n)_1^{\infty}$ being a sequence of mutually ortogonally projections in $\mathcal{H}$, on finite dimensional subspaces, $Qx = x - \sum_{n=1}^{\infty} P_n\, x$, $x \in \mathcal{H}$, α_n are real numbers such that $i\alpha_n$ are proper values of A, and $|\alpha_n| \to \infty$ as $n \to \infty$.

Proof of the decomposition theorem.

Put $Tx = (A + I)\,(A - I)^{-1} x$; it is a linear operator which is defined for all $x \in \mathcal{H}$. In fact, if $y = (A - I)^{-1}\, x$, we have

$$\begin{aligned} \|T x\|^2 &= ((A + I)\, y\,,\ (A + I)\, y) = (Ay\,, Ay) + 2\, Re\,(Ay\,, y) + (y\,, y) = \\ &((A - I)y, (A - I)\, y) + 4\, Re\,(Ay, y) = \|x\|^2 + 4 Re\,(Ay\,, y) \leq \|x\|^2; \end{aligned}$$

that is $\|T\| \leq 1$, i. e. T is a contraction.

Now, I is not a proper value for T: in fact this would imply $Tx = x$ for a certain $x \neq 0$, that is

$$(A + I)\, y = (A - I)\, y\,, \quad y = (A - I)^{-1}\, x, \quad y = -\,y\,, \quad y = 0\,, \quad x = 0.$$

In [11] — § 11 it is shown that for such a contraction there exists a strongly continuous semi-group of contractions $\{T_t\}_{t\geq 0}$, such that its generator is $A' = (T + I)\,(T - I)^{-1} = I + 2\,(T - I)^{-1}$. But from $T = (A + I)\,(A - I)^{-1}$ it results also $A = (T - I)\,(T - I)^{-1} = I + 2\,(T - I)^{-1}$, this shows that the domain $D_A = D_{A'}$ is dense in $\mathcal{H}$, and A is the generator of T_t.

Now for the proof of the decomposition (*), we shall need some results of the paper [12], which are given here in a convenable form.

A strongly continuous semi-group of contractions $\{T_t\}_{t\geq 0}$ is completely non-unitary, if $\inf_{0\leq t<+\infty} \{\|T_t\,x\|,\ \|T_t^*\,x\|\} < \|x\|$, for all $x \in \mathcal{K}$.

If $\{T_t\}$ is an arbitrary strongly continuous semi-group of contractions in $\mathcal{H}$, A its generator, and $T = (A + I)\,(A - I)^{-1}$ its *«cogenerator»*, then (Th. 3 and its following lemma in [12]), there exists an orthogonal projection Q commuting with $T_t\,(t \geq 0)$ and T such that: *a*) the restriction $\{T_t^Q\}_{t\geq 0}$ of $\{T_t\}_{t\geq 0}$ to $Q\mathcal{H}$ is completely non-unitary; *b*) the restriction $\{T_t^{I-Q}\}_{t\geq 0}$ of $\{T_t\}_{t\geq 0}$ to $(I - Q)\,\mathcal{H}$ is the restriction on $(0\,,\,\infty)$ of an unitary group.

The cogenerators of $\{T_t^Q\}_{t\geq 0}$ and $\{T_t^{I-Q}\}_{t\geq 0}$ are the restrictions T^Q and T^{I-Q} of T to $Q\,\mathcal{H}$ and $(I - Q)\,\mathcal{H}$.

Firstly we shall prove that the completely non-unitary part of $\{T_t\}_{t\geq 0}$ is convergent to 0 as $t \to +\infty$, in the weak operator topology of $\mathcal{H}$.

The th. 4 in [12], gives immediately that if $\{T_t\}_{t\geq 0}$ is a completely non-unitary semi-group of contractions, in $\mathcal{H}$, then, there exists an isometric linear application L from $\mathcal{H}$ in a direct countable sum $\sum_{n=1}^{\infty} \oplus\, L^2(-\infty, +\infty)$ such that

$$(T_t\,x\,,\,y) = \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} e^{its}\,(L\,x)_n\,(s)\,\overline{(L\,y)_n\,(s)}\,ds$$

for all $x\,,\,y \in \mathcal{H}$.

Then, in our case, if $x_0\,,\,y_0$ are arbitrary in $\mathcal{H}$, we have for all $t \geq 0$, $(T_t\,Q\,x_0\,,\,y_0) = (T_t\,Q^2\,x_0\,,\,y_0) = Q\,T_t\,Q\,x_0\,,\,y_0) = (T_t\,Q\,x_0\,,\,Q\,y_0) =$

$$\sum_{n=1}^{\infty}{}' \int_{-\infty}^{+\infty} e^{its}\,(L^Q\,Q\,x_0)_n\,(s)\,\overline{(L^Q\,Q\,y_0)_n\,(s)}\,ds.$$

Using the fact that this series is absolutely and uniformly (in $t \in (-\infty, +\infty)$) convergent, and also that the functions $(L^Q\,Q\,x)_n\,(s)$

$\overline{(L^Q Qy_0)_n (s)}$ are in $L^1(-\infty, +\infty)$ it follows by the classical Riemann-Lebesgue theorem that $T_t Qx_0$ is convergent to θ as $t \to +\infty$, in the weak topology of $\mathcal{H}$. But $(A - I)^{-1}$ is compact; hence $Q T_t (A-I)^{-1} x_0 = = (A - I)^{-1} Q T_t x_0$ is convergent to θ as $t \to +\infty$, in the strong topology of $\mathcal{H}$. In this manner, $Q T_t y \to \theta$ strongly as $t \to +\infty$, for all $y \in (A - I)^{-1} (\mathcal{H}) = D_A$. But we have seen that D_A is dense in $\mathcal{H}$, and $\| Q T_t \| \leq 1$, $t \geq 0$; one deduces that $Q T_t x \to \theta$ as $t \to +\infty$, strongly, for every $x \in \mathcal{H}$. The second relation in (x) is proved.

Concernig the first relation in (*), let us remark firstly that by the T. 4 of [11], T^{I-Q} is an unitary operator, such that if $\{K_\lambda\}$ is his spectral measure, then

$$T_t^{I-Q} = \int\limits_{|\lambda|=1} \exp\left(t \frac{\lambda+1}{\lambda-1}\right) dK_\lambda$$

But $T^{I-Q} - I^{I-Q}$ is the restriction to $(I-Q)\ (\mathcal{H})$ of $T - I = 2(A-I)^{-1}$; thus $T^{I-Q} - I_{I-Q}$ is compact. It follows that, excluding 0, the spectrum of $T^{I-Q} - I^{I-Q}$ is discret and formed only by proper values, the corresponding spaces of proper vectors being of finite dimension. Now, $T^{I-Q} = I^{I-Q} + (T^{I-Q} - I^{I-Q})$ so that $\sigma(T^{I-Q}) = 1 + \sigma(T^{I-Q} - I^{I-Q})$. It results that excluding $\lambda = 1$, the spectrum of the unitary operator T^{I-Q} is formed by a sequence $\{\lambda_n\} = \{e^{i\theta_n}\}$, with $\theta_n \to 0$. The corresponding $K_{\{\lambda_n\}}$ are of finite dimension.

Put $P_n = K_{\{\lambda_n\}}\ (I - Q)$, $n = 1, 2, \ldots$ and $i\alpha_n = (\lambda_n + 1)\ (\lambda_n - 1)^{-1}$ for $K_{\{\lambda_n\}} \neq 0$ (and hence $\lambda_n \neq 1$); in fact $\lambda_n = 1$, would imply $K_{\{\lambda_n\}} = K_{\{1\}} = 0$, 1 being not a proper value for T. Then α_n are real numbers and if $P_n \neq 0$, $|\alpha_n| \to \infty$.

Hence we have:

$$T_t x = T_t Q x + T(I - Q) x = T_t Qx + T_t^{I-Q} (I - Q) x =$$

$$= T_t Qx + \int\limits_{|\lambda|=1} \exp\left(t \frac{\lambda+1}{\lambda-1}\right) d K_\lambda (I - Q) x =$$

$$= T_t Q x + \sum_{n=1}^{\infty} \exp\left(t \frac{\lambda_{n+1}}{\lambda_{n-1}}\right) K_{\{\lambda_n\}} (I - Q)x = T_t Qx + \sum_{n=1}^{\infty} e^{it\alpha_n} P_n x$$

and (*) is established.

It remains to show that if $P_n \neq 0$, then $i\alpha_n$ is a proper value for. A. For this remark that form (*) it follows that $T_t P_n = e^{iant} P_n$, and hence $A P_n x = \lim\limits_{t \to 0} \frac{1}{t} (T_t - I) P_n x = i \alpha_n P_n x$, and the theorem is proved.

§ 3. We are now able to give the.

Proof of Theorem 1. Let $u(X, t)$ a bounded strong solution, defined for $t \in (-\infty, +\infty)$ of the equation: $u_t(X, t) = A\,u(X, t)$. Let $u_0(X) = u(X, 0) \in D_A$. Due to the fact that $Re\,(Ax, x) \leq 0$, and $Re\,(A^* x, x) \leq 0$, it results immediately (see for ex. Lyance [8]), that $(A - I)^{-1}$ is a bounded operator, defined in all. H. Moreover in Vishik [13] is proved that $(A - I)^{-1}$ is also compact in H. Let $\{T_t\}$ $(t \geq 0)$, be the strongly continuous semi-group in H, whose generator is A. As is well known, for $t \geq T_0$, the function $u_1(X, t) = T(t - T_0)\,u(X, T_0)$ is a strong solution of $u_{1,t}(X, t) = Au_1(X, t)$. But $u(X, t)$ is also a strong solution of the same equation, and $u(X, T_0) = u_1(X, T_0)$. Then their difference $u_0(X, t) = u(X, t) - u_1(X, t)$ satisfies

$$u_{0,t}(X, t) = A\,u_0(X, t), \text{ and } u_0(X, T_0) = \theta$$

But $d/dt\ (u_0(X, t), u_0(X, t)) = 2\,Re\,(A\,u, u) \leq 0$

$$\|u_0(X, t)\|^2 - \|u_0(X, T_0)\|^2 \leq 0; \quad \|u(X, t)\|^2 = 0 \quad \text{for } t \geq T_0.$$

Therefore the solution on the whole real axis, $u(X, t)$, admits for every $t \geq T_0$, T_0 real, the representation:

$$u(X, t) = T(t - T_0)\,u(X, T_0).$$

Consider the projection Q which was defined in the decomposition theorem. Then $u(X, t) = Q\,u(X, t) + (I - Q)\,u(X, t)$, and, for $t \geq T_0$

$$u(X, t) = T(t - T_0)\,Q\,u(X, T_0) + T(t - T_0)\,(I - Q)\,u(X, T_0)$$

Therefore, for $t \geq T_0$

$$u(X, t) = T(t - T_0)\,Q\,u(X, T_0) + \sum_{n=1}^{\infty} e^{i(t-T_0)a_n}\,P_n\,u(X, T_0)$$

But, by the hypotesis, $u(X, t)$ is bounded for $-\infty < t < +\infty$; hence, there exists a constant K, non-dependig of T_0, such that

$$\|Q\,u(X, t)\| = \|T(t - T_0)\,Q\,u(X, T_0)\| \leq K, \quad \text{for } t \geq T_0.$$

Let now $v(X, t) = (A - I)^{-1}\,Q\,u(X, t)$. Let $\{T_n\}_1^{\infty}$ be a sequence of real numbers, converging to $-\infty$. The sequence $\{Q\,u(X, T_n)\}_1^{\infty}$ is bounded in H; due to the compactness of $(A - I)^{-1}$, the sequence $\{v(X, T_n)\}_1^{\infty}$

is relatively compact in QH and contains a subsequence $\{v(X, T_{nk})\}_1^\infty$ which is strongly convergent to $v(X) \in QH$. We have, for $t \geq T_0$,

$$v(X, t) = (A - I)^{-1} Q\, u(X, t) = (A - I)^{-1} T(t - T_0)\, Q\, u(X, T_0) =$$

$$= T(t - T_0)(A - I)^{-1} Q\, u(X, T_0) = T(t - T_0)\, v(X, T_0).$$

Let $t \in (-\infty, +\infty)$ be fixed, and k_0 sufficiently great to ensure $t > T_{n_k}$, for $k \geq k_0$. Then

$$v(X, t) = T(t - T_{n_k})\, v(X, T_{n_k}) = T(t - T_{n_k})\,[v(X, T_{n_k}) - v(X)] +$$

$$+ T(t - T_{n_k})\, v(X)$$

$$\| v(X, t) \| \leq \| v(X, T_{n_k}) - v(X) \| + \| T(t - T_{n_k})\, v(X, \|$$

If $k \to +\infty$, $\| v(X, T_{n_k}) - v(X) \| \to 0$ and also $\| T(t - T_{n_k}) v(X) \| \to 0$ due to the fact established in the decomposition theorem that

$$T_t\, x \to \theta \quad \text{for} \quad t \to +\infty, \quad \text{if} \quad x \in Q\,H.$$

We obtain $v(X, t) = \theta$ for every t real. That is, $(A - I)^{-1} Q\, u(X, t) = \theta$ and $Q\, u(X, t) = \theta$ for every real t.

Hence, for every $t \geq T_0$, $u(X, t) = \sum_{n=1}^{\infty} e^{i(t-T_0)\alpha_n} P_n\, u(X, T_0)$, and for $t \geq 0$, $u(X, t) = \sum_{n=1}^{\infty} e^{it\alpha_n} P_n\, u(X, 0)$.

But for all $T_0 < 0$, $\sum_{n=1}^{\infty} e^{i(t-T_0)n} P_n\, u(X, T_0)$ is an almost periodic function from $-\infty < t < +\infty$ to H (due to the obviously uniform convergence of this series on the whole real axis).

For $t \geq 0$, this function coincides with $u(X, t) = \sum_{n=1}^{\infty} e^{it\alpha_n} P_n\, u(X, 0)$.

But, if two almost-periodic functions are identical for $t \geq 0$, they are identical everywhere. It results

$$u(X, t) = \sum_{n=1}^{\infty} e^{i\alpha_n t} P_n\, u(X, 0), \quad \text{for} \quad t \in (-\infty, +\infty),$$

and the theorem is completely proved.

We pass to the.

Proof of Theorem 2. Let $u(X, t)$ be a strong solution of the inhomogeneous system: $u_t(X, t) = A\,u(X, t) + f(X, t)$, $f(X, t)$ being almost-periodic from $-\infty < t < +\infty$ to H. Suppose moreover that the range of $u(X, t)$ is relatively compact in H. But

$$u(X, t) = Q\,u(X, t) + (I - Q)\,u(X, t)$$

and these projections are respectively strong solutions on the real axis, of the equations

$$(*) \qquad (d/dt)\,Q\,u(X, t) = A\,Q\,u(X, t) + Q f(X, t)$$

$$(**) \qquad (d/dt)(I - Q)\,u(X, t) = A\,(I - Q)\,u(X, t) + (I - Q) f(X, t).$$

Using the decomposition theorem we observe that AQ is the generator of te completely non-unitary part of $\{T_t\}$, and $A\,(I - Q)$ is the generator of the unitary-(almost-periodic) part of T_t.

Obviously $Q f(X, t)$ and $(I - Q) f(X, t)$ are almost-periodic functions, from $-\infty < t < +\infty$ to $Q\,H$ and respectively $(I - Q)\,H$.

The function $f(X, t)$ is supposed to have strongly continuous derivative in t; hence, by a theorem due to Phillips [10], the solutions of the equation (**) are given by the formula

$$(I - Q)\,u(X, t) = T_t^{I-Q}(I - Q)\,u(X, 0) + \int_0^t T^{I-Q}(t - s)\,(I - Q) f(X, s)\,ds$$

valid for all real t. Here T_t^{I-Q} is an unitary group, and even an strongly almost-periodic one (that is, for every $x \in (I - Q)\,H$, $T_t^{I-Q}x$ is an almost-periodic function).

It has, been proved in a previous paper of the second author [14 — lemme 2.3], that $T^{I-Q}(-s)(I - Q) f(X, s)$ is an almost periodic function. How $(I - Q)\,u(X, t)$ is relatively-compact, and hence bounded for $-\infty < t < +\infty$, it results that

$$T^{I-Q}(t) \int_0^t T^{I-Q}(-s)\,(I - Q)\,f(X, s)\,ds = z(t)$$

is bounded for $-\infty < t < +\infty$; hence, the function

$$T^{I-Q}(-t)\,z(t) = \int_0^t T^{I-Q}(-s)\,(I - Q) f(X, s)\,ds$$

is also bounded $-\infty < t < +\infty$. Using a recent result of Amerio [2] one obtains that

$$\int_0^t T^{I-Q}(-s)(I-Q)f(X,s)\,ds$$

is an almost-periodic function. Then, using again the lemma 2.3 in [14] we have that $z(t)$ is almost-periodic. Therefore $(I-Q)u(X,t)$ is almost-periodic; in the proof we used only the boundness of $(I-Q)u(X,t)$; a remark useful for the following theorems.

Now consider the function $Qu(X,t)$, which is relatively compact, and is strong solution on the whole real axis of the equation (*), $A\,Q$ being the generator of the completely non-unitary part $\{T_t^Q\}$ of $\{T_t\}$. To prove the almost-periodicity of $Q\,u(X,t)$ we adapt to our case a procedure used by Favard in his first theorem on almost-periodic differential systems [7].

For this we use the criterium of Bochner for almost-periodicity. Let (h_n) be an arbitrary sequence of real numbers. It contains a subsequence denoted also by (h_n) for which $Qf(t+h_n)$ is uniformly convergent on $-\infty < t < +\infty$ and $Q\,u(X,h_n)$ is strongly convergent in QH, due to the almost-periodicity of $Qf(X,t)$ and to the compactness of the closure of $Q\,u(X,t)$. We assert that $Qu(X,t+h_n)$ in also uniform convergent on $-\infty < t < +\infty$. In fact, if that would be not true, there would exists a sequence $|t_p| \to \infty$, an $\alpha > 0$ and two subsequences $(h_{n_p})_1^\infty$ and $(h_{m_p})_1^\infty$ from $(h_n)_1^\infty$, such that

$$\| Q\,u(X,t_p+h_{n_p}) - Q\,u(X,t_p+h_{m_p}) \| > \alpha, \quad p = 1,2,\ldots$$

Now, due to the relatively compactness of $Qu(X,t)$, and by a repeated use of the diagonal procedure, we may choose a subsequence $(p') \subset (p)$ such that the sequences

$$\{ Q\,u(X,-N+t_{p'}+h_{n_{p'}}) \}_1^\infty, \quad \{ Q\,u(X,-N+t_{p'}+h_{m_{p'}}) \}_1^\infty$$

are strongly convergent in QH, for every $N = 1,2,\ldots$ to $\omega_1(-N)$ and $\omega_2(-N)$ respectively, $N = 1,2,\ldots$ The set $\omega_1(-1), \omega_1(-2), \omega_1(-N), \ldots, \omega_2(-1), \omega_2(-2) \ldots \omega_2(-N), \ldots$ is relatively compact, being contained in the closure of the range of $Q\,u(X,t)$.

Now, consider the functions $Q\,u\,(X\,,t+t_{p'}+h_{n_{p'}})$ and $Q\,u\,(X\,,t+t_{p'}+h_{m_{p'}})$. They are strong solutions on the whole axis of the equation (*), where $Q\,f\,(X\,,t)$ is replaced respectively by $Q\,f(X\,,t+t_{p'}+h_{n_{p'}})$ and $Q\,f\,(X\,,t+t_{p'}+h_{m_{p'}})$, and the initial conditions, (for $t=0$) are respectively $Q\,u\,(X\,,t_{p'}+h_{n_{p'}})$ and $Qu\,(X\,,t_{p'}+h_{m_{p'}})$.

Using the theorem of Phillips, cited above, (and the unicity on the right of the solution) we may write these solutions, for $t>-N$, by the formulas

$$Qu\,(X\,,t+t_{p'}+h_{n_{p'}})=T^Q\,(t+N)\,Q\,u\,(X\,,-N+t_{p'}+h_{n_{p'}})+$$
$$+\int_{-N}^{t}T^Q\,(t-s)\,Q\,f\,(X\,,s+t_{p'}+h_{n_{p'}})\,ds$$

$$Q\,u\,(X\,,t+t_{p'}+h_{m_{p'}})=T^Q\,(t+N)\,Q\,u(X\,,-N+t_{p'}+h_{m_{p'}})+$$
$$+\int_{-N}^{t}T^Q\,(t-s)\,Q\,f\,(X\,,s+t_{p'}+h_{m_{p'}})\,ds$$

Let now $p'\to\infty$. Using again the criterium of Bochner, we may suppose that the sequences $f\,(X\,,s+h_{n_{p'}}+t_{p'})$ and $f\,(X\,,s+h_{m_{p'}}+t_{p'})$ are uniformly convergent on the real axis $-\infty<s<+\infty$, and is easy to see that their limits $l'\,(X\,,s)$, $l''\,(X\,,s)$ are the same $=f^*\,(X\,,s)$.

One obtains, for $t\geq-N$,

$$\lim_{p'\to\infty}Q\,u\,(X\,,t+t_{p'}+h_{n_{p'}})=T^Q\,(t+N)\,\omega_1\,(-N)+\int_{-N}^{t}T^Q\,(t-s)f^*\,(X\,,s)\,ds$$

$$\lim_{p'\to\infty}Q\,u\,(X\,,t+t_{p'}+h_{m_{p'}})=T^Q\,(t+N)\,\omega_2\,(-N)+\int_{-N}^{t}T^Q\,(t-s)f^*\,(X\,,s)\,ds$$

Hence, their difference, for $t=0$, is convergent to

$$\lim_{p'\to\infty}[Qu(X,t_{p'}+h_{n_{p'}})-Q\,u(X,t_{p'}+h_{m_{p'}})]=T^Q\,(N)\,[\omega_1\,(-N)-\omega_2\,(-N)]$$

It results, for $N=1,2,\dots,$ that

$$\|\,T^Q\,(N)\,[\omega_1\,(-N)-\omega_2\,(-N)]\,\|\geq\alpha$$

But the sequence $\omega_1(-N) - \omega_2(-N)$ contains a subsequence convergent to $\omega_0 \in Q\,H$. We have, for this sequence $(N_k)_1^\infty$, the relation:

$$\| T^Q(N_k)\,\omega_0^{N_k} \| = \| T^Q(N_k)\,(\omega_0^{N_k} - \omega_0) + T^Q(N_k)\,\omega_0 \| \leq$$
$$\leq \| \omega_0^{N_k} - \omega_0 \| + \| T^Q(N_k)\,\omega_0 \|$$

where $\omega_0^{N_k} = \omega_1(N_k) - \omega_2(-N_k)$. Obviously $\| \omega_0^{N_k} - \omega_0 \|$ is convergent to O and $\| T^Q(N_k)\,\omega_0 \|$ is also convergent to O, due to the fact that $\omega_0 \in Q\,H$ and $T^Q(t)\,x \to 0$ for $t \to +\infty$, $x \in Q\,H$, by the decomposition theorem.

This contradicts the fact that

$$\| T^Q(N_k)\,\omega_0^{N_k} \| \geq \alpha$$

and hence $Q\,u(X, t)$ is almost-periodic. Our theorem is completely proved.

§ 4. A function $u(t)$ from $-\infty < t < +\infty$ to H will be said weak almost-periodic, if for every $v \in H$, the scalar function $(u(t), v)$ in almost-periodic in the usual sense. We derive from the Theorem 2 the following result on bounded solutions of the inhomogeneous equation $u_t(X, t) = A\,u(X, t) + f(X, t)$ with almost-periodic $f(X, t)$.

THEOREM 3. *Let $f(X,t)$ be an almost-periodic function from $-\infty<t<+\infty$ to H, and supcose moreover that it has a strongly continuous time-derivative; Let $u(X, t)$ be a strong solution from $-\infty < t < +\infty$ to D_A of the inhomogeneous equation*

$$u_t(X, t) = A\,u(X, t) + f(X, t).$$

Then, if the range of $u(X, t)$ is bounded in H for $-\infty < t < +\infty$, then $u(X, t)$ is weak almost-periodic from $-\infty < t < +\infty$ to H.

Proof of the theorem 3. We have only to consider the projection $Q\,u(X, t)$ corresponding to the completely-non unitary part of the semigroup $\{T_t\}_{t \geq 0}$; in fact, for the projection $(I - Q)\;u(X, t)$, as we have alreay observed in the proof of the preceding theorem, boundness implies (strong) almost-periodicity, *via* Amerio's result on the integrals of almost-periodic functions in Hilbert spaces.

If $Qu(X, t)$ is bounded, $v(X, t) = (A - I)^{-1}\,Q\,u(X, t)$ is relatively compact; moreover, it verifies the equation

$$v_t(X, t) = Av(X, t) + (A - I)^{-1}\,Q\,f(X, t);$$

in the strong sense. Using the Theorem 2 it results that $v(X, t)$ is almost-periodic. But $Q\,u(X, t) = (A - I)\,v(X, t)$ is weak almost periodic. In fact, let firstly $u_0 \in D_{A^*}$; we have

$$(Q\,u(X, t), \quad u_0(X)) = ((A - I)\,v(X, t), \quad u_0(X)) =$$

$$= (v(X, t), \ (A - I)^*\,u_0(X)) = (v(X, t), \varphi_0(X)),$$

which is obviously almost-periodic.

How D_{A^*} is dense in H and $Q\,u(X, t)$ is bounded in H, our result is now immediately.

Suppose now that $f(X, t)$ is such an almost-periodic function from $-\infty < t < +\infty$ to H, that $Q f(X, t)$ is $= \theta$. Then is valid the following.

THEOREM 4. *Let $f(X,t)$ be an almost-periodic function from $-\infty<t<+\infty$ to H, with a strongly continuous time derivative, and $Q f(X, t) = \theta$, for each t. Let $u(X, t)$ be a strong solution on $-\infty < t < +\infty$ of te equation*

$$u_t(X, t) = A\,u(X, t) + f(X, t).$$

Then, if the range of $u(X, t)$ is bounded in H for $-\infty < t < +\infty$, then $u(X, t)$ is (strong) almost-periodic from $-\infty < t < +\infty$ to H.

The result is obvious from the preceeding consideration beeause a bounded strong solution on $-\infty < t < +\infty$ of the completely non-unitary part

$$Q\,u_t(X, t) = A\,Q\,u(X, t)$$

is identical θ).

In this case we can give also a sufficient condition on the «spectrum» of $(I - Q)\,f(X, t) \equiv f(X, t)$, which will ensure the existence of almost-periodic solutions of the inhomogeneous equation.

The corresponding theorem is the following:

THEOREM 5. *Let $f(X, t)$ be an almost-periodic function from $-\infty < t < +\infty$ to H, with a strongly continuous time-derivative, and $Q f(X, t) = \theta$. Let $f(X, t) \sim \sum_{1}^{\infty} A_n(X)\,e^{i\lambda_n t}$, $(\lambda_n)_1^\infty$ are its Fourier exponents. Suppose moreover that the real numbers $(\alpha_n)_1^\infty$ from the decomposition theorem, are not limit points of the set $(\lambda_n)_1^\infty$. Then every strong solution*

$u(X, t)$ of the equation

$$u_t(X, t) = A\, u(X, t) + f(X, t)$$

for which $u(X, 0) = u_0(X) \in D_A \cap (I - Q) H$, is almost-periodic.

Proof. Such a solution is given by the formula

$$u(X, t) = T^{I-Q}(t)\, u_0(X) + \int_0^t T^{I-Q}(t - s)(I - Q) f(X, s)\, ds.$$

This formula is valid for $-\infty < t < +\infty$. The first term in the right side is almost-periodic. The second term may be written as

$$T^{I-Q}(t) \int_0^t T^{I-Q}(-s)\, f(X, s)\, ds$$

By an argument already used in the paper [16] of the second author, it results that $T^{I-Q}(-s)(I - Q) f(X, s)$ is an almost-periodic function, for which the spectrum has not O as a limit point. By an obvious vector-extension of a theorem due to Favard, it results that the integral

$$\int_0^t T^{I-Q}(-s)(I - Q) f(X, s)\, ds$$

is almost-periodic, and hence $T^{I-Q}(t) \int_0^t T^{I-Q}(-s)\, f(X, s)\, ds$ is almost-periodic. The theorem is proved.

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