ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

L. NIRENBERG

On elliptic partial differential equations

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 13, nº 2 (1959), p. 115-162

<http://www.numdam.org/item?id=ASNSP_1959_3_13_2_115_0>

ON ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

by L. NIRENBERG (New York) (*)

Outline.

This series of lectures will touch on a number of topics in the theory of elliptic differential equations. In Lecture I we discuss the fundamental solution for equations with constant coefficients. Lecture 2 is concerned with Calculus inequalities including the well known ones of Sobolev. In lectures 3 and 4 we present the Hilbert space approach to the Dirichlet problem for strongly elliptic systems, and describe various inequalities. Lectures 5 and 6 comprise a self contained proof of the well known fact that « weak » solutions of elliptic equations with sufficiently « smooth » coefficients are classical solutions.

In Lectures 7 and 8 we describe some work of Agmon, Douglis, Nirenberg [14] concerning estimates near the boundary for solutions of elliptic equations satisfying boundary conditions. This work is based on explicit formulas, given by Poisson kernels, for solutions of homogeneous elliptic equation with constant coefficients in a half space.

Throughout, for simplicity we treat one equation in one unknown. The material will on the whole be self contained, though of course not all proofs can be included. However, we shall attempt to indicate those of the main results.

(*) Questo ciclo di conferenze è stato tenuto a Pisa dal 1⁰ al 10 settembre 1958, e ha fatto parte del corso del C. I. M. E. che ha avuto per tema: « Il principio di minimo e sue applicazioni alle equazioni funzionali ». Tale corso si è svolto in collaborazione con la Scuola Narmale Superiore e l'Istituto Matematico dell'Università di Pisa. In questi Annali saranno successivamente pubblicati i corsi di conferenze tenuti dai professori C. B. Morrey e L. Bers.

Lecture I. The Fundamental Solution.

I would like to start with a few general and somewhat unrelated comments. In studying differential equations one is usually interested in obtaining *unique* solutions by imposing suitable boundary or initial conditions, the kind depending on the so - called «type» of the equation - elliptic, hyperbolic, etc. However, the type classification for general equations has not been carried out, and in many cases it is not known what boundary conditions to impose. Indeed for equations that change type — and we are all familliar with the initial work in this field due to Professor Tricomi — the nature of the boundary conditions is far from obvious.

Thus if one considers an arbitrary equation without regard to type it is a natural question to ask whether there exist solutions at all. In fact there are occasions when one simply wants some solutions. Such occur often in differential geometry. Take a well known case: to introduce isothermal coordinates with respect to a given Riemannean metric on a two dimensional manifold. This reduces to a local problem of finding nontrivial solutions of a differential equation in a neighborhood of a point.

Another question is: are there solutions in the large of a given equation. For the preceding this is answered by uniformization theory for Riemann surfaces.

In this talk we will consider for some special cases the question: For a given differential operator L are there solutions of $Lu = f$ for « well behaved » functions f. Of course equations with analytic coefficients always have local solutions, obtained for instance by power series expansions (Cauchy-Kowalewski).

Recently Hans Lewy [1] exhibited an equation with C^∞ coefficients having no solutions ewen locally. Since it is easy to describe, we present it:

In 3-space with coordinates x, y, t, set $z = x + iy$, write the Cauchy-Riemann operator as $\dfrac{\partial}{\partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$, and consider the differential equation

$$Lu = \left(\frac{\partial}{\partial \bar{z}} + i\,z\,\frac{\partial}{\partial t}\right)u = \frac{\partial \psi(t)}{\partial t}$$

where the right hand side is a continuous real function of t alone which, for convenience, is written as a derivative of a real function ψ.

THEOREM: *If there is a continuously differentiable solution u of the equation in a neighborhood of the origin, then $\psi(t)$ is real analytic.*

Thus for any non-analytic ψ there is no solution near the origin. (The proof may be easily modified to show that there are also no « generalized solutions »).

Proof: If we integrate $\frac{\partial u}{\partial \bar{z}} d\,\theta$ over a circle $|z|^2 = s \geq 0$, $z = s^{1/2} e^{i\theta}$, we establish easily the identity

$$\int_0^{2\pi} \frac{\partial u}{\partial \bar{z}} d\,\theta = \frac{\partial}{\partial s} \int_0^{2\pi} z\,u\,d\,\theta\,.$$

Now set $\zeta = s + i\,t$ and $U(\zeta) = \int z\,u\,d\,\theta$. Integrating the equation for u over the circle we find that U satisfies

$$\left(\frac{\partial}{\partial s} + i\frac{\partial}{\partial t}\right) U = 2\pi \frac{d\psi}{dt}$$

or

$$\left(\frac{\partial}{\partial s} + i\frac{\partial}{\partial t}\right) (U + 2\,\pi\,i\,\psi) = 0\,.$$

It follows that $V(\zeta) = U + 2\,\pi\,i\,\psi$ is a holomorphic function of $\zeta = s + i\,t$ in a domain near the origin with re $\zeta = s > 0$. But on $s = 0$ the function U, i. e. the real part of V, vanishes, and therefore V can be continued analytically across $s = 0$. Hence ψ is analytic.

In [1] Lewy also constructs a function F such that the equation $Lu = F$ has no « smooth » solution in the neighborhood of any point. Lewy also conjectures that there are *homogeneous* equations with C^∞ coefficients having no solutions in the neighborhood of any point.

The simplest class of differential operators L of arbitrary type, for which one might expect solutions u of

$$Lu = f \tag{1.1}$$

to exist, for all well behaved functions f, are operators with constant coefficients. In the last few years a considerable study has been made of general differential operators with constant coefficients. (See Ehrenpreis [2], Hörmander [3], Malgrange [4]. Solutions of (1.1) can be found, at least locally, if one knows that a fundamental solution E of $L\,E = \delta$ (the Dirac δ function) exists. This is a (possibly generalized) function E such that

$$E * Lu = u$$

for all C^∞ functions u with compact support. We shall denote the class of such functions by C_0^∞. Here $*$ denotes convolution. Then if f is in C_0^∞ the function $u = E * f$ is a solution of (1.1).

Malgrange [4] and Ehrenpreis [2] proved the existence of a fundamental solution for any differential operator with constant coefficients. However it is not difficult to construct one explicitily, as Hörmander, and also Trêves [5], have shown, and we shall now describe such a construction.

First we fix our

NOTATION: We consider functions $u(x)$ of n variables $x = (x_1, \dots, x_n)$ and denote the differentiation vector by $D = (D_1, \dots, D_n)$, $D_i = \partial/\partial x_i$. The letters β, γ, μ, ν will denote vectors $\beta = (\beta_1, \dots, \beta_n)$ with non-negative integral coefficients β_i, and we set $|\beta| = \Sigma \beta_i$. Otherwise for any vector $\xi = (\xi_1, \dots, \xi_n)$, $|\xi|$ will represent its Euclidean length $|\xi|^2 = \Sigma |\xi_i|^2$, and $\xi \cdot \eta = \Sigma \xi_i \eta_i$. We write

$$\xi^\beta = \xi_1^{\beta_1} \dots \xi_n^{\beta_n}, \qquad D^\beta = D_1^{\beta_1} \dots D_n^{\beta_n};$$

for convenience we shall also, on occasion, express a general m^{th} order partial derivative of a function u by $D^m u \cdot C_0^\infty$ will denote the class of C^∞ functions with compact support.

We consider now a differential operator L of order k with constant coefficients, which we may write as a polynomial in D of order k.

$$L = L(D).$$

In constructing the fundamental solution let us first argue in a heuristic manner. Introduce the Fourier transform of the function $u(x)$

$$\tilde{u}(\xi) = \int e^{-ix\cdot\xi} u(x)\, dx,$$

integration being over the entire n-space. Then

$$\widetilde{L(D)u} = L(i\xi)\, \tilde{u}(\xi).$$

So if $u = E * Lu = \int E(x-y)\, Lu(y)\, dy$ then

$$\tilde{u} = \tilde{E}(\xi)\, L(i\xi)\, \tilde{u}(\xi)$$

or

$$\tilde{E} = \frac{1}{L(i\xi)},$$

or

$$(1.2) \qquad E(x) = (2\pi)^{-n} \int \frac{e^{ix\cdot\xi}}{L(i\xi)}\, d\xi \, .$$

Problem : give formula (1.2) *a meaning.*

In attempting to do this (and there are many ways) there are two difficulties that occur. The first is the non-integrability at infinity, duo to the fact that we are integrating over the full n-space. The second difficulty is caused by the real roots ξ of the polynomial $L(i\xi)$.

The first difficulty is easily overcome. It essentially expresses the fact that is general E is a distribution, i. e. a finite derivative of a continuous function. Instead of constructing E directly we shall construct the fundamental solution E_N of the operator $(1-\Delta)^N L = (1 - \sum_i D_i^2)^N L(D)$. We shall construct a fundamental solution E_N having continuous derivatives up to any given order, by taking N sufficiently large. We may then take, in the distribution sense,

$$(1.3) \qquad E = (1-\Delta)^N E_N \, ,$$

i. e. for f in C_0^∞ the function

$$u = E_N * (1-\Delta)^N f$$

is a solution of $Lu = f$.

Thus we consider, for $p(\xi) = 1 + \sum \xi_j^2$

$$(1.4) \qquad E_N = (2\pi)^{-n} \int \frac{e^{ix\cdot\xi}}{p^N(\xi)\, L(i\xi)}\, d\xi \, .$$

Taking N large eliminates the first difficulty, i. e. the trouble at infinity.

Now to handle the second difficulty. We may assume, after a possible rotation of coordinates, that the coefficient of D_n^k in $L(D)$ is $\neq 0$, say unity. Consider $L(i\xi)$ as a polynomial in ξ_n. We shall first integrate in (1.4) with respect to the variable ξ_n, keeping $\xi' = (\xi_1, \dots, \xi_{n-1})$ fixed, however we shall move the line of integration from the real line to a parallel line lying in the complex ξ_n plane.

For fixed real ξ' there are k roots ξ_n of $L(i\xi)$. In the strip $|\Im m\, \xi_n| \leq \frac{1}{2}$ in the complex ξ_n plane there is therefore a line parallel to the real axis whose distance from any root is at least $(2k+2)^{-1}$, as one easily sees. Let us a choose one such line $\Im m\, \xi_n = c(\xi')$ whose distance to any root is at least $(4k+4)^{-1}$. The choice of $c(\xi')$ depends on ξ', but it is easy to see that $c = c(\xi')$ may be chosen so as to be continuous except on a set of ξ' of $(n-1)$-dimensional measure zero.

Setting $\eta = \eta(\xi') = (0, \ldots, c(\xi'))$ we now take as definition

$$E_N = (2\pi)^{-n} \int \frac{e^{ix\cdot(\xi + i\eta(\xi'))}}{p^N(\xi + i\eta)\, L(i(\xi + i\eta))}\, d\xi \tag{1.4$'$}$$

where integration is first with respect to ξ_n.

Since

$$|p(\xi + i\eta(\xi'))| \geq \frac{3}{4} \quad \text{and} \quad |L(i(\xi + i\eta))| \geq (4k+4)^{-k}$$

we see that E_N has derivatives up to any given order, if N is large enough.

We have finally to verify that for $u \in C_0^\infty$

$$u = E_N * (1-\Delta)^N Lu \equiv \int E_N(x-y)(1-\Delta)^N Lu(y)\, dy.$$

Setting $(1-\Delta)^N L(D) = L_N(D)$, the right hand side equals

$$(2\pi)^{-n} \iint \frac{e^{i(x-y)\cdot(\xi+i\eta)}}{L_N(i(\xi + i\eta))}\, d\xi\, L_N(D)\, u(y)\, dy.$$

Since u has compact support its Fourier transform $\tilde{u}(\xi)$ can be extended to complex vectors ξ as an entire analytic function, and since $u \in C^\infty$ the derivatives of $\tilde{u}$ die down faster that any power of $|\xi|$ as we go to infinity in a strip $|\mathfrak{Im}\, \xi| <$ constant. Thus, interchanging the order of integration in the above, we find that it equals

$$(2\pi)^{-n} \int \frac{e^{ix(\xi+i\eta)}}{L_N(i(\xi + i\eta))}\, L_N(i(\xi + i\eta))\, \tilde{u}(\xi + i\eta)\, d\xi =$$

$$= (2\pi)^{-n} \int e^{ix(\xi+i\eta)}\, \tilde{u}(\xi + i\eta)\, d\xi.$$

Because of the behaviour of $\tilde{u}$ of infinity we may shift the line of integration of the ξ_n parallel to itself and find that this expression

$$= (2\pi)^{-n} \int e^{ix\cdot\xi}\, \tilde{u}(\xi)\, d\xi = u(x).$$

Thus the function E_N defined by (1.4)′ is a fundamental solution for the operator L_N. The desired fundamental solution of Lu then is given by (1.3).

One sees easily that the fundamental solution E_N given by (1.4)′ has exponential growth in the x_n variable.

For further important work on fundamental solutions for equations with constant coefficients we refer to Hörmander [6].

Consider now elliptic differential operators with constant coefficients. These are operators L whose leading part L' — consisting of the terms of highest order — satisfy

$$L'(\xi) \neq 0 \quad \text{for real} \quad \xi \neq 0 .$$

We shall have need later of the fundamental solution for a homogeneous elliptic operator with constant coefficients, i. e. $L' = L$. For such, of course, the fundamental solution first constructed by Herglotz is well behaved at infinity. We shall use the following form of it, given in F. John's book [7].

$$(1.5) \qquad E(x) = -\frac{1}{(2\pi i)^n (k+q)!} \Delta^{\frac{n+q}{2}} \int_{|\xi|=1} \frac{(x\cdot\xi)^{k+q}}{L(\xi)} \log \frac{x\cdot\xi}{i}\, d\,\omega_\xi$$

where integration is over the full unit sphere with $d\,\omega_\xi$ as the element of area, q is a non-negative integer of the same parity n, i. e. $q+n$ is even, and the principal branch of the logarithm is taken with the plane slit along the negative real axis.

From (1.5) we obtain as a special case, for $L = \Delta$ power, the following identity which is due to F. John and used extensively in [7], representing the δ function in terms of plane waves: For u in C_0^∞

$$(1.6) \qquad u = -\frac{1}{(2\pi i)^n q!} \Delta^{\frac{n+q}{2}} \left[\int_{|\xi|=1} (x\cdot\xi)^q \log \frac{x\cdot\xi}{i}\, d\,\omega_\xi * u \right].$$

In [7] John derives (1.6) from the known expression for the fundamental solution for a power of the Laplacean, and then derives (1.5) from (1.6). This may be done as follows. Suppose $K(x\cdot\xi)$ satisfies

$$L\,K(x\cdot\xi) = (x\cdot\xi)^q \log\frac{x\cdot\xi}{i},$$

then a fundamental solution of the operator L is given by

$$-\frac{1}{(2\pi i)^n q!} \Delta^{\frac{n+q}{2}} \int_{|\xi|=1} K(x\cdot\xi)\, d\,\omega_\xi .$$

But such a K is easily found. If we set $x \cdot \xi = \sigma$ then $K(\sigma)$ satisfies

$$L(\xi)\left(\frac{d}{d\sigma}\right)^k K(\sigma) = \sigma^q \log \sigma/i\,,$$

a solution of which is

$$K(\sigma) = \frac{1}{L(\xi)}\frac{q!}{(k+q)!}\sigma^{k+q}\left(\log\frac{\sigma}{i} + c_{k,q}\right),$$

with $c_{k,q}$ an appropriate constant. If we insert this into the above expression for the fundamental solution of L we obtain the expression

$$-\frac{1}{(2\pi i)^n (k+q)!}\Delta^{\frac{n+q}{2}}\int\limits_{|\xi|=1}\frac{(x\cdot\xi)^{k+q}}{L(\xi)}\left(\log\frac{x\cdot\xi}{i} + c_{k,q}\right)$$

which differs from (1.5) only by the term involving $c_{k,q}$. But this term is a polynomial of degree $k-n$ which is therefore a solution of $Lv=0$, and so may be ignored.

It should also be possible to derive (1.5) from the heuristic formula (1.2). (1.5) aserts that

$$-\frac{1}{(2\pi i)^n (k+q)!}\int\limits_{|\xi|=1}\frac{(x\cdot\xi)^{k+q}}{L(\xi)}\log\frac{x\cdot\xi}{i}\,d\omega_\xi \tag{1.7}$$

is a fundamental solution for the operator $\Delta^{\frac{n+q}{2}} L$. Let us attempt to derive this expression from the corresponding expression of (1.2):

$$(-1)^{\frac{n+q}{2}}(2\pi)^{-n}\int\frac{e^{ix\cdot\xi}}{|\xi|^{n+q}L(i\xi)}\,d\xi\,. \tag{1.8}$$

Arguing heurisitically again let us modify the expression by introducing polar coordinates in the ξ space

$$\xi = \varrho\,\eta\,,\qquad \varrho = |\xi|\,,\qquad |\eta| = 1\,.$$

Then (1.8) becomes

$$(-1)^{n+q+k}(2\pi)^{-n}\int\limits_{|\eta|=1}\int\limits_0^\infty\frac{e^{i\varrho x\cdot\eta}}{L(\eta)}\varrho^{-1-q-k}\,d\varrho\,d\omega_\eta\,. \tag{1.8$'$}$$

Let us now write the heuristic expression

$$\int_0^\infty e^{i\varrho x\cdot\eta}\,\varrho^{-1-q-k}\,d\,\varrho \tag{1.9}$$

as a well defined contour integral

$$\frac{1}{2\pi i}\int_{\mathcal{C}} e^{i\varrho x\cdot\eta}\,\varrho^{-1-q-k}\,(\log(-\varrho)+c) \tag{1.9$'$}$$

where the contour $\mathcal{C}$ is a curve which goes from $+\infty$ in the complex ϱ plane, encircles the origin counterclockwise and returns to $+\infty$ along the real axis, the branch for the logarithm is the same as above, and the constant c is chosen so that

$$\int_{\mathcal{C}} e^{i\varrho}\,\varrho^{-1-q-k}\left(\log(-\varrho)+c-\frac{i\pi}{2}\right)d\varrho=0\,.$$

The expressions (1.9)$'$ may be evaluated explicity, and on insertion into (1.8)$'$, yields the expression (1.7). We leave the calculation to the reader.

Lecture II. Calculus Inequalities.

A priori estimates play a central role in the theory of partial differential equations. They are of various kinds — pointwise estimates for derivatives of solutions and their modulus of continuity, and estimates of, say, L_p norms of solutions and their derivatives — and it is naturally important to understand the relationships between these various estimates.

For instance, the well known results of Sobolev assert that if the m'th order derivatives $D^m u$ of a function $u(x_1,\dots,x_n)$ (with compact support) are in L_r, $1<r<\infty$ then lower order derivatives $D^j u$, $j<m$ belong to L_p for some p, or, if r is sufficiently high, the $D^j u$ are bounded and satisfy a Hölder condition with a certain exponent α.

Since we shall often make use of it, let us recall here the notion of

HÖLDER CONTINUITY. A function $f(x)$ defined on a set S in a Euclidean space satisfies a Hölder condition there with exponent α, $0<\alpha<1$, if

$$[f]_\alpha=[f]_\alpha^S=\underset{x,y\in S}{e.u.b}\,\frac{|f(x)-f(y)|}{|x-y|^\alpha} \tag{2.1}$$

is finite. It is Hölder continuous (exponent α) in a domain if it satisfies a Hölder condition with exponent α in every cempact subset of the domain.

This lecture is concerned with calculus inequalities relating integral and pointwise estimates of functions and their derivatives. The recent important result of de Giorgi [11] on the differentiability of solutions of regular variational problems seems in fact to be based on a calculus inequality asserting that certain integral estimates imply Hölder continuity. We shall consider functions $u(x)$ defined in n-dimensional Euclidean space and belonging to L_q, and whose derivatives of order m belong to L_r, $1 \leq q$, $r \leq \infty$. We shall present interpolative inequalities for the L_p and Hölder norms $[\ ]_\alpha$ of derivatives $D^j u$, $0 \leq j < m$, for the maximal range of p and α. Our inequalities are a combination of, and include, those usually called of Sobolev type (which hold also for fractional derivatives, and rather straightforward proofs of which may be found in [8]), and familiar interpolative inequalities such as

$$M_1^2 \leq \text{constant } M_0 \cdot M_2$$

where M_i is $e.u.b.$ of the L_p norms of the derivatives of order i of a function u, $i = 0, 1, 2$. The proofs use only first principles and are entirely elementaty. (No attempt will be made here to obtain best constants). The inequalities is this section were presented at the Int'e Congress in Edinburgh August 1958, where we learned that almost equivalent results had also been proved by E. Gagliardo.

In this lecture we shall use the following

NOTATION: For $-\infty < \dfrac{1}{p} < \infty$ we defins the norms and seminorms $|u|_p$ for functions $u(x)$ defined in a domain $\mathcal{D}$ in n-dimensional spaces:

For $p > 0$

$$|u|_p = \text{the } L_p \text{ norm of } u \text{ in } \mathcal{D}.$$

$$= \left(\int_{\mathcal{D}} |u|^p \, dx \right)^{\frac{1}{p}}.$$

For $p < 0$ set $s = [-n/p]$, $-\alpha = s + n/p$ and define

$$|u|_p = e.u.b.[D^s u]_\alpha^{\mathcal{D}} \qquad \text{if } \alpha > 0,$$

$$|u|_p = e.u.b.|D^s u| \qquad \text{if } \alpha = 0,$$

where $e.u.b.$ is taken with respect to all partial derivatives D^s of order s, and over points in $\mathcal{D}$.

We define $|D^j u|_p$ as the maximum of the $|\ |_p$ norms of all j-th order derivatives of u.

We shall express our result for functions u defined in the entire n-space E^n. Extension to other domains will be described briefly in the remarks after the theorem.

THEOREM: *Let u belong to L_q in E^n and its derivatives of order m, $D^m u$, belong to L_r, $1 \leq q, r \leq \infty$. For the derivatives $D^j u$, $0 \leq j < m$, the following inequalities hold*

$$|D^j u|_p \leq \text{constant}\, |D^m u|_r^a\, |u|_q^{1-a}, \tag{2.2}$$

where

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q},$$

for all a in the interval

$$\frac{j}{m} \leq a \leq 1 \tag{2.3}$$

(the constant depending only on n, m, j, q, r, a), with the following exceptional cases

1. *If $j = 0$, $rm < n$, $q = \infty$ then we make the additional assumption that either u tends to zero at infinity or $u \in L_{\tilde{q}}$ for some finite $\tilde{q} > 0$.*

2. *If $1 < r < \infty$, and $m - j - n/r$ is a non negative integer then* (2.2) *holds only for a satisfying $j/m \leq a < 1$.*

We shall not give a complete proof of the theorem here but shall indicate the main steps. First some comments.

1. The value of p is determined simply by dimensional analysis.

2. For $a = 1$ the fact that u is contained in L_q does not enter in the estimate (2.2), and the estimate is equivalent to the results of Sobolev (note that we permit r to be unity).

3. That j/m is the smallest possible value for a may be seen by taking $u = \sin \lambda x_1 \zeta(x)$ where ζ is in C_0^∞: For large λ we have $|u|_q = 0(1)$, $|D^j u|_p = 0(\lambda^j)$, $|D^m u|_r = 0(\lambda^m)$ where no 0 can be replaced by o.

4. It will be clear from the proof that the result holds also for u defined in a product domain

$$-\infty < x_s < \infty,\ 0 < x_t < \infty\ :\ s = 1, \dots, k:\ t = k+1, \dots, n,$$

and hence for any domain that can be mapped in a one-to-one way onto such a domain by a sufficiently «nice» mapping.

5. For a bounded domain (with « smooth » boundary) the result holds if we add to the right slde of (2.1) the term

$$\text{constant} \mid u \mid_{\tilde{q}} .$$

for any $\tilde{q} > 0$. The constants then depend also on the domain.

6. Similar estimates hold for the L_p norms of $D^j u$ on linear subspaces of lower dimension, for suitable p.

7. Similar interpolation inequalities also hold for fractional derivatives, but their proof is not so elementary.

The theorem, in its full generality should be useful in treating nonlinear problems. We mention in particular that from (2.2) for $a = j/m$, $q = \infty$ it follows that the set of functions u which are bounded and have derivatives of order m belonging to L_r forms a Banach Algebra. For $r = 2$ this is called the Schauder ring.

The proof of the theorem is elementary and contains in particular an elementary proof for the Sobolev case $a = 1$. In order to prove (2.2) for any given j one has only to prove it for the extreme values of a, j/m and unity. (If Case 2 holds some additional remark has to be made.) For in general there is a simple

Interpolation Lemma : if $-\infty < \lambda \leq \mu \leq \nu < \infty$ *then*

$$\mid u \mid_{\frac{1}{\mu}} \leq c \mid u \mid_{\frac{1}{\lambda}}^{\frac{\nu-\mu}{\nu-\lambda}} \cdot \mid u \mid_{\frac{1}{\nu}}^{\frac{\mu-\lambda}{\nu-\lambda}}$$

where c is independent of u.

The lemma is easily proved; for $\lambda > 0$ it is merely the usual interpolation ineqnality for L_p norms.

Let us turn now to the proof of the theorem, or at least to the main points. Consider first the Sobolev case, $a = 1$. It suffices to consider the case $j = 0$, $m = 1$, from which the general result may then be derived. If $r > n$ (2.2) asserts that u satisfies a certain Hölder condition, and an elementary proof due to Morrey has long been known. We shall sketch it here for functions defined in a general domain $\mathcal{D}$.

Definition : A domain $\mathcal{D}$ is said to have the strong cone property if there exist positive constants d, λ and a closed solid right spherical cone V of fixed opening and height such that any points P, Q in $\overline{\mathcal{D}}$ (the closure of $\mathcal{D}$) with

$$\mid P = Q \mid \leq d$$

are vertices of cones V_P, V_Q lying in $\bar{\mathcal{D}}$ which are conrguent to V and have the following property: the volume of the intersection of the sets: V_P, V_Q, and the two spheres with conters P, Q and radius $|P - Q|$, is not less than $\lambda \, |P - Q|^n$.

We now prove the assertion

If u has first derivatives in L_r, $r > n$, in a domain $\mathcal{D}$ having the strong cone property, then for points P, Q in $\mathcal{D}$ with $|P - Q| \leq d$, we have

$$\frac{|u(P) - u(Q)|}{|P - Q|^{1-\frac{n}{r}}} \leq \text{constant}\, |D u|_r$$

where the constant depends only on d, λ, V, n and r.

(From this follows easily an estimate for $[u]_{1-\frac{n}{r}}$, depending on the domain).

Proof: Set $s = |P - Q|$ and let $S_P\,(S_Q)$ be the intersection of $V_P\,(V_Q)$ with the sphere about $P\,(Q)$ radius s. Set $S_P \cap S_Q = S$. If R is a point in S we have, on integrating with respect to R over S,

$$\text{Volume of } S \cdot |u(P) - u(Q)| \leq \int_S |u(P) - u(R)|\, dR + \\ + \int_S |u(R) - u(Q)|\, dR .$$

Because of the strong cone property the left hand side is not less than

$$\lambda\, s^n \, |u(P) - u(Q)| .$$

The first term on the right may be estimated as follows. Introducing polar coordinates ϱ, η, about P, where η is a unit vector, we find easily that the first term in the right is bounded by

$$\int_{S_P} \varrho^{n-1}\, d\,\omega_\eta\, d\,\varrho \int_0^\varrho \left|\frac{\partial u}{\partial \varrho}\right| d\,\varrho \leq \text{constant } s^n \int_{S_P} \left|\frac{\partial u}{\partial \varrho}\right| \frac{d\,x}{\varrho^{n-1}}$$

(where $d\,\omega$ is the element of area on the unit sphere, and $d\,x$ is the element of volume)

$$\leq \text{constant } s^n \left(\int_{S_P} \left|\frac{\partial u}{\partial \varrho}\right|^r d\,x\right)^{\frac{1}{r}} \left(\int_{S_P} \varrho^{(1-n)\frac{r}{r-1}}\, d\,x\right)^{\frac{r-1}{r}}$$

by Hölder's inequality,

$$\leq \text{constant} \cdot s^{n+1-\frac{n}{r}} \left(\int_{S_P} \left| \frac{\partial u}{\partial \varrho} \right|^r d\,x \right)^{\frac{1}{r}}.$$

A similar estimate holds for the term $\int_S |u(R) - u(Q)|\, d\,R$, and the result follows.

We return now to functions defined in the full n-space.

Suppose $r < n$. We shall prove a stronger formulation of (2.2), namely

$$|u|_{\frac{nr}{n-r}} \leq \frac{r}{2} \frac{n-1}{n-r} \prod_i \left| \frac{\partial u}{\partial x_i} \right|_r^{\frac{1}{n}}. \tag{2.4}$$

For $1 < r < n$ (2.4) follows from the special case $r = 1$, as one readily verifies, by simply applying the inequality for $r = 1$ to the function $V = |u|^{\frac{n-1}{n-r}r}$ and using Hölder's inequality in a suitable way. Thus it suffices to prove (2.4) for the case $r = 1$.

$$|u|_{\frac{n}{n-1}} \leq \frac{1}{2} \prod_i \left| \frac{\partial u}{\partial x_i} \right|_1^{\frac{1}{n}}. \tag{2.4$'$}$$

We shall prove (2.4)$'$ here for $n = 3$. One sees easily that

$$|u(x)| \leq \frac{1}{2} \int_i \left| \frac{\partial u}{\partial x_i} \right| d\,x_i \qquad i = 1, 2, 3.$$

where $\int_i$ denotes integration along the full line through x parallel to the x^i, axis. Thus

$$|2\,u(x)|^{\frac{3}{2}} \leq \left(\int_1 \left| \frac{\partial u}{\partial x_1} \right| d\,x_1 \right)^{\frac{1}{2}} \cdot \left(\int_2 \left| \frac{\partial u}{\partial x_2} \right| d\,x_2 \right)^{\frac{1}{2}} \cdot \left(\int_3 \left| \frac{\partial u}{\partial x_3} \right| d\,x_3 \right)^{\frac{1}{2}}.$$

Integrating with respect to x_1 then x_2, and then x_3 we find with the aid of Schwarz's inequality

$$\int_1 |2\,u(x)|^{\frac{3}{2}} d\,x_1 \leq \left(\int_1 \left| \frac{\partial u}{\partial x_1} \right| d\,x_1 \right)^{\frac{1}{2}} \cdot \left(\iint_{12} \left| \frac{\partial u}{\partial x_2} \right| d\,x_2\, d\,x_1 \right)^{\frac{1}{2}} \cdot \left(\iint_{13} \left| \frac{\partial u}{\partial x_3} \right| d\,x_3\, d\,x_1 \right)^{\frac{1}{2}}.$$

$$\iint_{21} |2u|^{\frac{3}{2}} dx_1 dx_2 \leq \left(\iint_{21} \left| \frac{\partial u}{\partial x_1} \right| dx_1\, dx_2 \right)^{\frac{1}{2}} \cdot \left(\iint_{12} \left| \frac{\partial u}{\partial x_2} \right| dx_2\, dx_1 \right)^{\frac{1}{2}} \cdot \left(\iiint \left| \frac{\partial u}{\partial x_3} \right| dx \right)^{\frac{1}{2}},$$

and finally

$$\iiint |2u|^{\frac{3}{2}}\, d\,x \leq \left(\iiint \left|\frac{\partial\, u}{\partial\, x_1}\right| d\,x\right)^{\frac{1}{2}} \left(\iiint \left|\frac{\partial\, u}{\partial\, x_2}\right| d\,x\right)^{\frac{1}{2}} \left(\iiint \left|\frac{\partial\, u}{\partial\, x_3}\right| d\,x\right)^{\frac{1}{2}}.$$

that is, (2.4)$'$.

For general n the inequality is proved in the same way with the aid of Hölder's inequality.

Suppose finally, for $j = 0$, $m = 1$, that $r = n$; this is the exceptîonal case 2. We claim that

$$|\,u\,|_p \leq \text{costant}\ |\,D\,u\,|_n^{\frac{1-q}{p}}\ |\,u\,|_p^{\frac{q}{p}} \quad 0 < q \leq p < \infty$$

where the constant depends only n, q and p. It suffices to show this for large p and this is easily done by appliyng (2.4)$'$ to the function $v = |\,u\,|^{p(1-1/n)}$, and using Hölder's inequality in a judicious manner.

Let us now consider the other extreme case $a = j/m$. It suffices to consider the case $j = 1$, $m = 2$, the general case may then be proved by induction on m. We claim that the following holds

$$(2.5) \qquad |\,D\,u\,|_p \leq c\,|\,D^2\,u\,|_r^{\frac{1}{2}}\ |\,u\,|_q^{\frac{1}{2}} \ \text{for}\ \frac{2}{p} = \frac{1}{r} + \frac{1}{q};\ 1 \leq q\,,\, r \leq \infty\,,$$

with c an absolute constant. Incidentally, as Ungar pointed out, we may permit q to be any positive number, but I shall confine myself to the case cited, in fact to the case q finite, $1 < r < \infty$. The general case may be obtained by a slightly different argument, or just by letting q tend to ∞, and r tend to 1 or ∞ in (2.5).

Inequality (2.5) follows from the corrisponding inequality in one dimension

$$(2.6) \qquad \int |\,u_x\,|^p\, d\,x \leq c^p \left(\int |\,u_{xx}\,|^r\, d\,x\right)^{\frac{p}{2r}} \left(\int |\,u\,|^q\, d\,x\right)^{\frac{p}{2q}}, \quad \frac{2}{p} = \frac{1}{r} + \frac{1}{q}\,,$$

which holds for the full, or half-infinite line (with c an absolute constant), by integrating with respect to the other variables and appliyng Hölder's inequality.

Our proof of (2.6), though elementary, is slightly tricky. Peter Ungar has found another slightly longer proof which furnishes a better value for c.

The proof is based on a simple lemma which we leave as an exercise.

LEMMA: *On an interval λ, whose lenght we also denote by λ, we have*

$$(2.7)\qquad \int_\lambda |u_x|^p\,dx \leq \bar{c}^p\,\lambda^{1+p-\frac{p}{r}}\left(\int_\lambda |u_{xx}|^r\,dx\right)^{\frac{p}{r}} + \bar{c}^p\,\lambda^{-\left(1+p-\frac{p}{r}\right)}\left(\int_\lambda u^q\,dx\right)^{\frac{p}{q}}$$

with $\bar{c}$ an absolute constant.

We shall prove that for any interval $L: 0 \leq x \leq L$ the following inequality holds

$$(2.8)\qquad \int_0^L |u_x|^p\,dx \leq 2\,\bar{c}^p\left(\int_0^\infty |u_{xx}|^r\,dx\right)^{\frac{p}{2r}}\left(\int_0^\infty |u|^q\,dx\right)^{\frac{p}{2q}}.$$

(2.6) follows easily from (2.8).

In proving (2.8) we may suppose that $|u_{xx}|_r = 1$. We shall cover the interval L by a finite number of successive intervals $\lambda_1, \lambda_2, \dots$, each one having as initial point the end point of the preceding. For k a fixed positive integer, choose first the interval $\lambda: 0 \leq x \leq \frac{L}{k}$, and consider (2.7) for this interval. If the first term on the right of (2.7) is greater than the second set $\lambda_1 = \lambda$; we then have

$$\int_{\lambda_1} |u_x|^p\,dx \leq 2\,\bar{c}^p\left(\frac{L}{k}\right)^{1+p-\frac{p}{r}},$$

since $|u_{xx}|_r = 1$. If however the second term of (2.7) is the greater extend the interval λ (keeping its left endpoint fixed) until the two terms of the right of (2.7) become equal. Since $1 + p - \frac{p}{r} > 0$ equality of these two terms must occur for a finite value of λ. Let λ_1 be the resulting interval. We then have

$$\int_{\lambda_1} |u_x|^p\,dx \leq 2\,\bar{c}^p\left(\int_{\lambda_1} |u_{xx}|^r\,dx\right)^{\frac{p}{2r}}\left(\int_{\lambda_1} |u|^q\,dx\right)^{\frac{p}{2q}}.$$

Starting at the end point of λ_1 repeat this process, keeping k fixed, choosing $\lambda_2, \lambda_3, \dots$, until L is covered. There are clearly at most k such intervals λ_j. If we now sum our estimates for $\int_{\lambda i} |u_x|^p\,dx$ we find, with the aid of Hölder's inequality (recall that $\frac{p}{2r} + \frac{p}{2q} = 1$) that

$$\int_0^L |u_x|^p \, dx \leq 2 \bar{c}^p \left(\frac{L}{k}\right)^{1+p-\frac{p}{r}} \cdot k + 2 \bar{c}^p \left(\int_0^\infty |u_{xx}|^r \, dx\right)^{\frac{p}{2r}} \cdot$$

$$\cdot \left(\int_0^\infty |u|^q \, dx\right)^{\frac{p}{2q}} .$$

If we now let $k \to \infty$ the first term on the right of the preceding tends to zero, because $r > 1$, and we obtain (2.8), completing the proof of (2.5).

Lecture III. The Dirichlet Problem.

We consider now elliptic differential operators, confining ourselves for simplicity to a single equation for one unknown. Let $L(x, D)$ be a partial differential operator with complex valued coefficients, and let L' be the part of highest order. L is elliptic if there are no real characteristics, i. e.,

$$L'(x, \xi) \neq 0 , \qquad \text{real} \quad \xi \neq 0 . \tag{3.1}$$

It is easily seen that for more than two variables, $n > 2$, ellipticity implies that the order k of L is even. In treating the Dirichlet problem we shall assume that $k = 2m$ is even and that the operator is strongly elliptic, i. e. that (efter possibly multiplying by a suitable complex function)

$$\mathcal{R}e \, L'(x, \xi) \neq 0 , \qquad \text{real} \quad \xi \neq 0 . \tag{3.2}$$

The Dirichlet problem consists of finding a solution in a domain $\mathcal{D}$ of

$$Lu = f \qquad \text{in} \quad \mathcal{D}$$

$$\left(\frac{\partial}{\partial \vec{n}}\right)^j u = \Phi_j \qquad \text{on} \quad \dot{\mathcal{D}} , \qquad j = 0, \ldots, m-1 ,$$

where $\partial / \partial \vec{n}$ represents differentiation normal to the boundary. Here f and Φ_j are given functions in $\mathcal{D}$ and $\dot{\mathcal{D}}$ respectively.

We shall describe here the Hilbert space approach to the Dirichlet problem, which is based on some form of the projection theorem, and is related to the classical method of minimizing the Dirichlet integral. In its present form the existence theory is mainly due to Gårding, Vishik, Browder and others; we refer the reader to [9] and [C] for expositions and references. This and the following lecture comprise a brief description of [9]. The theory is based on a single L_2 inequality. Gårding's inequality, expressing the positive definiteness of the Dirichlet integral associated with the differential operator.

Since this approach to the Dirichlet problem requires considerable differentiability assumptions on the coefficients we shall assume for simplicity that they are of class C^∞ in $\overline{\mathcal{D}}$ and that the boundary $\dot{\mathcal{D}}$ is sufficiently smooth. We shall also assume $\mathcal{D}$ to be bounded. Furthermore if the Φ_j are sufficiently smooth we may subtract from u a function having the same Dirichlet data as u, so we shall consider the case where the Φ_j vanish

$$
\begin{aligned}
&Lu = f \qquad \text{in } \mathcal{D} \\
&\left(\frac{\partial}{\partial \vec{n}}\right)^j u = 0 \qquad \text{on } \dot{\mathcal{D}}, \qquad j = 0, \dots, m-1.
\end{aligned}
\tag{3.3}
$$

The Hilbert space approach yields at first « generalized solutions » of (3.3) which we must define. A function u which belongs, say, to L_2 in every compact subdomain of $\mathcal{D}$ is a « weak » solution of $Lu = f$ if

$$
(u, L^* \varphi) = (f, \varphi) \tag{3.4}
$$

for every φ which belongs to $C_0^\infty(\mathcal{D})$, i. e. is of the class C^∞ and has compact support in $\mathcal{D}$. Here (,) denotes the L_2 scalar product, and L^* is the formal adjoint of L. In addition to the L_2 norm we also introduce the Hilbert spaces $H_j(\overset{\circ}{H}_j)$, j a non-negative integer. These are the closures in the norm (using the notation of Lecture I)

$$
\| u \|_j = \left[\sum_{|\beta| \le j} \int_{\mathcal{D}} | D^\beta u |^2 \, dx \right]^{\frac{1}{2}}
$$

of the spaces $C^\infty(\mathcal{D})\ C_0^\infty(\mathcal{D})$. The associated norm and spaces relative to a subdomain $\mathcal{C}$ will be denoted by $\|\ \|_j^{\mathcal{C}}$, $H_j^{\mathcal{C}}$, $\overset{\circ}{H}{}_j^{\mathcal{C}}$. Clearly $H_0 = \overset{\circ}{H} = L_2$.

We remark that for $j > i$ $H_j \subset H_i$ and the set $\| u \|_j \le$ constant is compact in H_i.

Following Sobolev and Friedrichs we say that a function u in $\mathcal{D}$ has strong derivatives in L_2 up to order j in $\mathcal{D}$ $(\overline{\mathcal{D}})$ if u belongs to $H_j^{\mathcal{C}}(H_j^{\mathcal{D}})$ for every compact subdomain $\mathcal{C}$ of $\mathcal{D}$. With the aid of the results of the preceding lecture we see that a function in H_j is continuous if $2j > n$. Functions in $\mathring{H}_m$ satisfy the boundary conditions of (3.3) in a generalized sense.

We now formulate the

GENERALIZED DIRICHLET PROBLEM: *Given f in H_0 find a weak solution u in $\mathring{H}_m$ of $Lu = f$.*

Using the notation of Lecture 1 we may write the operator L in the form

$$L = \sum_{|\beta|,|\gamma| \leq m} D^\beta a_{\beta,\gamma} D^\gamma .$$

If u is a weak solution in $\mathring{H}_m$ we may then carry out some partial integration in equation (3.4) and write it as

$$(3.4)' \qquad B[u, \varphi] = \Sigma (-1)^{|\beta|} (a_{\beta,\gamma} D^\gamma u, D^\beta \varphi) = (f, \varphi), \varphi \quad \text{in } C_0^\infty(\mathcal{D}).$$

$B[u, v]$ is linear in u, antilinear in v and satisfies, by Schwarz'inequality

$$(3.5) \qquad | B[u, v] | \leq \text{constant} \| u \|_m \cdot \| v \|_m .$$

We shall assume the strong ellipticity (3.2) to hold uniformly, i. e. for some positive constant c_0

$$\mathcal{R}e \, (-1)^m \sum_{|\beta|,|\gamma|=m} a_{\beta,\gamma}(x) \xi^\beta \xi^\gamma \geq c_0 | \xi |^{2m}, \qquad \xi \text{ real},$$

for all x in $\overline{\mathcal{D}}$. Our main result is

THEOREM: *For $\overline{C}$ sufficiently large the generalized Dirichlet problem for the equation $(L + \overline{C}) u = f$ admits a unique solution. For the equation $Lu = f$ we have the Fredholm alternative.*

The L_2 estimate on which the theorem is based is

GÅRDING'S INEQUALITY: *There exist constants $c > 0$ and C such that*

$$(3.6) \qquad \mathcal{R}e \, B[\varphi, \varphi] = \mathcal{R}e \, (L\varphi, \varphi) \geq c \| \varphi \|_m^2 - C \| \varphi \|_0^2$$

holds for every φ in $C_0^\infty(\mathcal{D})$.

This will be proved in the next lecture. It is clear from (3.5) that the inequality extends also to functions in $\mathring{H}_m$, and it follows from (3.6) that the only solution in $\mathring{H}_m$ of $(L + C) u = 0$ is $u = 0$.

Let us now prove the theorem. Suppose first that the operator is symmetric, i. e. $B[\varphi, \varphi]$ is real. and that the constant C in (3.6) vanishes — which we may achieve by considering $L + C$ in place of L. It follows from (3.5), (3.6) (with $C = 0$) that $B[u, v]$ serves as an alternative scalar product in the Hilbert space $\overset{\circ}{H}_m$; the norms $B[u, u]$ and $\| u \|_m$ are equivalent. We see that the antilinear functional (f, φ) defined for all φ in $\overset{\circ}{H}_m$ satisfies

$$|(f, \varphi)| \leq \|f\|_0 \|\varphi\|_0 \leq \|f\|_0 \|\varphi\|_m \leq c^{-\frac{1}{2}} \|f\|_0 B[\varphi, \varphi]^{\frac{1}{2}}$$

and is therefore a bounded functional. By the well known representation theorem there exists therefore a function u in the Hilbert space $\overset{\circ}{H}_m$ such that

$$(f, \varphi) = B[u, \varphi];$$

u is then the solution of the Dirichlet problem, and we have proved the first part of the theorem with $\overline{C} = C$. To prove the second part we write the equation $Lu = f$ in the form $(L + C) = Cu + f$ or

$$u = C(L + C)^{-1} u + (L + C)^{-1} f.$$

Since $(L + C)^{-1}$ maps H_0 boundedly into $\overset{\circ}{H}_m$ it is completely continuous in H_0, by a previous remark, and from the Riesz theory for completely continuous operators we derive the second part of the theorem.

Suppose now that $B[\varphi, \varphi]$ is not symmetric. If we add $C(\varphi, \varphi)$ to B so that it satisfies

$$B[\varphi, \varphi] \geq c \|\varphi\|_m^2, \qquad \varphi \in \overset{\circ}{H}_m,$$

then we may still rely on a generalized representation theorem due to Lax and Milgram. We conclude the lecture with this

REPRESENTATION THEOREM: *Let $B(x, y)$ be a form defined for pairs of vector x, y in a Hilbert space H* (norm $\| \ \|$), *which is linear in x, antilinear in y, and satisfies*

$$(3.7) \qquad |B(x, y)| \leq \text{constant} \|x\| \cdot \|y\| .$$

Suppose that for some positive constant c the inequality

$$(3.8) \qquad |B(x, x)| \geq c \|x\|^2$$

holds for every x in H. Then every bounded antilinear functional $F(x)$ admits the representation

$$F(x) = B(v, x) = \overline{B(x, w)} .$$

For fixed elements v, w which are unique.

Proof: For any fixed element v, $B(v, x)$ is a bounded antilinear functional of x and therefore admits the representation

$$B(v, x) = (y, x)_H$$

for some element y, where $(\ ,\ )_H$ denotes the scalar product in H. This defines a mapping $y = Av$ which is clearly linear. Letting $x = v$ and applying (3.8) we find that

$$c \mid v \mid^2 \leq \mid B(v, v) \mid \leq (y, v)_H \leq \| y \| \cdot \| v \|,$$

or

$$\| v \| \leq c^{-1} \| y \| .$$

It follows that the operator A has a bounded inverse and that its range is closed. Furthermore the v corresponding to any y is unique. To see that the range of A is the whole space H suppose that z is orthogonal to it. Then we have $B(v, z) = 0$ for all v. From (3.8) it follows, by setting $v = z$, that $z = 0$. Thus A maps onto the entire space, and therefore every antilinear functional $F(x)$ being of the form $(y, x)_H$ admits the representation $F(x) = B(v, x)$. The other representation is proved in a similar way.

Lecture IV. A Priori Estimates.

Before proving Gårding's inequality let us make some general remarks about a priori estimates. Consider a differential equation $Lu = f$ of order k and assume that the solution has been made unique by some auxiliary conditions. One wants to study the inverse operator — to see, for instance, to what class of functions the solution belongs, if f belonge to a given class. For this problem, and also for the existence theory, a priori inequalities play a basic role. Let us suppose that the auxiliary conditions are homogeneous, then a typical a priori estimate would assert that for some norm $\|\ \ \|$

$$\| D^\beta u \| \leq \text{constant} \; \| Lu \| \qquad \mid \beta \mid \leq K .$$

For instance, if we know that the equation has a solution of class C^K for all f of class C^j then indeed, by a simple application of the closed graph theorem, we would have

$$\| D^\beta u \| \leq \text{constant} \; \| Lu \| , \qquad \mid \beta \mid \leq K - j ,$$

with $\| \quad \|$ the usual norm in C^j. In general if Lu has finite $\| \quad \|$ norm we will not obtain such an inequality for $K = k$, rather $K < k$; that is we cannot estimate individually all derivatives entering in L. However I believe that elliptic equations can be characterized as those for which one can estimate all derivatives, i. e.

$$(4.1) \qquad \| D^\beta u \| \leq \text{constant} \| Lu \| , \qquad | \beta | \leq k ,$$

for a wide class of norms (this is stated as a conviction not a theorem).

Consider now an elliptic equation $Lu = f$ with suitable homogeneous boundary conditions. Most a priori estimates are just of the type (4.1) or, if one does not assume uniqueness, of the form

$$(4.2) \qquad \| D^\beta u \| \leq \text{constant} (\| Lu \| + \| u \|) \qquad | \beta | \leq k .$$

Indeed much of the theory of elliptic equations is concerned with proving such estimates for various norms $\| \quad \|$, and proving analogous estimates for functions with no boundary restrictions:

$$(4.2)' \qquad \| D^\beta u \|^{\mathcal{A}} \leq \text{constant} (\| L \|^{\mathcal{D}} + \| u \|^{\mathcal{D}}) \qquad | \beta | \leq k .$$

Here $\mathcal{A}$ is any compact subdomain of $\mathcal{D}$, and the norm $\| \quad \|^{\mathcal{A}}$ is considered only for functions defined in $\mathcal{A}$.

A word of caution: The estimate do not hold for the most obious norm that one would try, namely the maximum (or C^0) norm nor in general for C^j norms, however they do hold for $C^{j+\alpha}$ norms, $0 < \alpha < 1$, and for many integral norms.

We quote some immediate consequence of (4.2), (4.2)'.

1. If f and the coefficients of L are in C^∞ then a solution of $Lu = f$ is also in C^∞. This follows fairly easily from (4.2)'.

2. Solutions of $Lu = 0$ with bounded norm $\| \quad \|$ form a compact family. This follows from (4.2)' and the

Calculus Lemma: *The set* $\| u \| + \| Du \|$ constant *is compact in the space with* $\| \quad \|$ *as norm.*

This lemma holds for a wide class of norms.

3. The set of solutions of $Lu = 0$ satisfying the boundary conditions (so that (4.2) holds) is finite dimensional. This follows with the aid of the Calculus Lemma.

I would like to describe briefly a general recipe for proving such estimates. This consists of several steps:

1. In case of (4.2)' prove it first for equations with constant coefficients and only highest order terms, and for functions of compact support.

In case (4.2), prove it also for such equations and for functions defined in a half space, vanishing near infinity, and satisfiyng (on the planar boundary) the boundary conditions. These are also assumed to have constant coefficients (i. e. to be translation invariant).

2. Now eliminate the hypothesis of compact support.

3. Extend the estimate to variable coefficients as follows: with the aid of a partitions of unity write the function u as a sum of functions u_i with small support, in each of which the leading coefficients are close to constants, and treat the variation from constant as an error term, using the results of Step 2 and the following lemma which may also be used in the proof of Step 2.

Calculus Lemma: For appropriate constants c_1, c_2

$$\| D^j u \| \leq c_1 \| D^m u \|^{j/m} \| u \|^{1-j/m} + c_2 \| u \| , \tag{4.3}$$

where for functions of compact support we may take $c_2 = 0$ *and* c_1 *independent of the support of* u .

This holds for a wide class of norms.

In case the support of u_i touches the boundary, make a local change of variable to flatten out the boundary so that Steps 1 and 2 can be applied.

The main step here is Step 1. We remark that in Step 3 we rely on (at least) the continuity of the leading coefficients of L, or on the fact that they differ little from constants in small domains. Because of this one does not obtain in this way the more refined estimates required for treating nonlinear problems, such as those in Bers, Nirenberg [10], de Giorgi [11], or Nash [12].

The norms for which such estimates are easiest to derive are the L_2 norms for functions and their derivatives, and we shall illustrate the recipe for these by proving Gårding's inequality in its general form.

Consider a quadratic integral form defined for C^∞ functions with compact support in a bounded domain $\mathcal{D}$

$$B[u, u] = \sum_{|\beta|, |\gamma| \leq m} (c_{\beta,\gamma} D^\beta u , D^\gamma u) \tag{4.4}$$

and suppose that the (complex valued) coefficients $c_{\beta,\gamma}$ are continuous in $\overline{\mathcal{D}}$. *A necessary and sufficient condition for the existence of positive constants* c, C *so that the inequality*

$$\| u \|_m^2 \leq c \, \Re e \, B[u, u] + C \| u \|_0^2 \tag{4.5}$$

holds for all $u \in C_0^\infty(\mathcal{D})$ *is that for some positive constant* c_0

$$(4.6) \qquad \mathcal{R}e \sum_{|\beta|,|\gamma|=m} c_{\beta,\gamma}\, \xi^\beta \xi^\gamma \geq c_0 |\xi|^{2m} \qquad \text{for all real } \xi .$$

Here the notations of Lecture 3 is used.

Proof: We prove first the sufficiency, following our recipe. The Calculus Lemma (4.3) will be used in the form: For every $\varepsilon > 0$ there is a constant $C(\varepsilon)$ such that for every C^∞ function u with compact support

$$(4.7) \qquad \| u \|_{m-1}^2 \leq \varepsilon \| u \|_m^2 + C(\varepsilon) \| u \|_0^2 .$$

This is contained in our inequalities of Lecture 2, but is most easily proved with the aid of Fourier transforms.

We consider now the different steps in proving (4.5), the Step 2 of the recipe does not occur here since our functions have compact support.

1. Suppose that the $c_{\beta,\gamma}$ are constant and vanish unless $|\beta| = |\gamma| = m$. We introduce the Fourier transform of u

$$\tilde{u}(\xi) = \int e^{-ix\cdot\xi} u(x)\, dx .$$

By Parseval's theorem we have

$$\begin{aligned} \mathcal{R}e\, B[u, u] &= (2\pi)^{-n}\, \mathcal{R}e \sum \int c_{\beta,\gamma}\, \xi^\beta \xi^\gamma |\tilde{u}(\xi)|^2 d\xi \\ &\geq (2\pi)^{-n} c_0 \int |\xi|^{2m} |\tilde{u}^2|\, d\xi \\ &\geq c_0' \| u \|_m^2 \end{aligned}$$

proving (4.5) for this special case.

We now consider the variable coefficient case and break Step 3 into two parts.

2. Suppose that the support of u is sufficiently small, contained, say, in a small sphere about the origin. Then accorfing to the preceding inequality we have

$$c_0' \| u \|_m^2 \leq \mathcal{R}e\, B[u, u] + \mathcal{R}e \sum_{|\beta|=|\gamma|=m} \int (c_{\beta,\gamma}(0) - c_{\beta,\gamma}(x))\, D^\beta u\, D^\gamma \bar{u}\, dx -$$

$$- \mathcal{R}e \sum_{|\beta|+|\gamma|<2m} \int c_{\beta,\gamma}(x)\, D^\beta u\, D^\gamma \bar{u}\, dx .$$

If now the support of u is so small that $c_{\beta,\gamma}$ has small oscillation there we see that the second term on the right may be bounded by

$$\frac{1}{2} c_0' \| u \|_m^2 .$$

The third term is trivially bounded by constant $\| u \|_m \| u \|_{m-1}$.

Thus we find that

$$\frac{1}{2} c_0' \| u \|_m^2 \leq \Re e\, B[u, u] + \text{constant} \| u \|_m \| u \|_{m-1}$$

from which follows the inequality

$$\| u \|_m^2 \leq \text{constant}\, \Re e\, B[u, u] + \text{constant} \| u \|_{m-1}^2 .$$

(4.5) now follows with the aid of (4.7).

3. Consider now the general case. Construct a partition of unity in $\overline{\mathcal{D}}$,

$$1 \equiv \sum_1^N \omega_j^2 , \qquad \omega_j \in C_0^\infty ,$$

with the support of each ω_j as small as desired. Then

$$\Re e\, B[u, u] = \Re e \sum \int c_{\beta,\gamma} D^\beta u\, D^\gamma \bar{u}\, dx =$$

$$= \Re e \sum \sum_j \int \omega_j^2 c_{\beta,\gamma} D^\beta u\, D^\gamma \bar{u}\, dx =$$

$$= \Re e \sum \sum \int c_{\beta,\gamma} D^\beta (\omega_j u)\, D^\gamma (\omega_j \bar{u})\, dx + 0(\| u \|_m \cdot \| u \|_{m-1})$$

$$\geq \text{constant} \sum_{|\beta| \leq m} \sum_j \int | D^\beta (\omega_j u) |^2\, dx + + 0(\| u \|_m \cdot \| u \|_{m-1})$$

by the preceding Case 2,

$$\geq \text{constant} \| u \|_m^2 + 0(\| u \|_m \cdot \| u \|_{m-1}) ,$$

and the desired result now follows easily with the aid of (4.7).

We see that the constants c, C in (4.5) depend on c_0, an upper bound for the $|c_{\beta,\gamma}|$, and on the modulus of continuity of the leading $c_{\beta,\gamma}$ with $|\beta| = |\gamma| = m$, and finally on the domain $\mathcal{D}$.

Now for the proof of the nacessity of (4.6). Suppose that (4.5) holds and that the left hand side of (4.5) vanishes for some real $\xi = \xi'$, $|\xi|' = 1$, and some point in $\overline{\mathcal{D}}$, say the origin. Following the ergument in Step 2 in the proof of sufficiency we see that the inequality

$$(4.5)' \qquad \| u \|_m^2 \leq \text{constant} \left(\Re e \sum_{|\beta|, |\gamma| = m} \int c_{\beta,\gamma}(0)\, D^\beta u\, D^\gamma \bar{u}\, dx + \| u \|_0^2 \right)$$

holds for all $C^\infty u$ with support in some fixed neighborhood U about the origin and in $\mathcal{D}$. Set $u = e^{i\lambda\xi' \cdot x} \zeta(x)$ for real λ, where $\zeta(x)$ is a fixed real C^∞ function with support in U and in $\mathcal{D}$. One sees readily that as $\lambda \to \infty$ the left hand side of (4.5)′ is $0(\lambda^{2m})$ and not $o(\lambda^{2m})$ while the right hand side is $0(\lambda^{2m-1})$, so that (4.5)′ does not hold.

Gårding's inequality (4.5) is at one end of a whole spectrim of interesting and useful inequalities making different requirements on u at the boundary, Gårding's inequality making the maximal requirement — that all derivatives of u of order less than m vanish at the boundary. At the other end of the spectrum is the inequality of Aronszajn [13] involving no boundary conditions whatsoever.

Aronszajn considers a number of differential operators $L_j(x, D)$, $j = 1, \dots, N$ of order m, with coefficients continuous in the closure of a bounded domain $\mathcal{D}$, and solves the following problem: Under what conditions can one assert that for all C^∞ functions u in $\mathcal{D}$ the inequality

$$(4.8) \qquad \| u \|_m^2 \leq \text{constant} \left(\Sigma \| L_j u \|_0^2 + \| u \|_0^2 \right)$$

holds, with the constant independent of u? He gives necessary and sufficient conditions:

(a) the operator $\Sigma L_j L_j^*$ is elliptic, here L_j^* is the formal adjoint of L_j.

(b) At any boundary point x of $\dot{\mathcal{D}}$, if $\vec{n}$ is the unit normal to $\dot{\mathcal{D}}$ and $\xi \neq 0$ is any real vector tangent to $\dot{\mathcal{D}}$ then the polynomials in τ, $L_j'(x, \xi + \tau \vec{n})$ have no common complex root τ. Here L_j' is the leading part of L_j.

An example of Aronzsajn's inequality is the following; for functions $u(x, y)$ in a bounded domain in the plane

$$\int | u_{xy} |^2\, dx\, dy \leq \text{constant} \int \left(| u_{xx} |^2 + | u_{yy} |^2 + | u |^2 \right) dx\, dy .$$

Even this simple example is not trivial to prove.

Since the report of Aronszajn a number of people have coonsidered the problem of proving (4.5) for various quadratic forms (4.4) and under various differential boundary conditions. For one operator L_j Agmon, Douglis, Nirenberg [14], (in a fortheoming paper which will be discussed later) have characterized these differential boundary conditions which are $m/2$ in number and for which (4.8) holds. Schechter [15] has treated N operators and general boundary conditions. Aronszajn, in unpublished work, has treated the general problem (4.5). Also Hörmander and Agmon [16] have solved the general problem for (4.5) and general differential boundary conditions. The proofs follow the recipe outlined above, the main step being the first, for functions in a half space.

We conclude the lecture with a result that will be used in proving the differentiability at the boundary of solutions of elliptic equations. In the following Σ_R denotes the hemisphere $|x| < R$, $x_n \geq 0$. We shall denote the variable x_n by t, $(x_1, \dots, x_{n-1})$ by x and $(x_1, \dots, x_n)$ by (x, t).

Lemma: Let u be a weak solution of a differential equation (of order k) with, for simplicity, C^∞ coefficients,

$$Lu = \sum_{|\beta| \leq k-j-1} D^\beta f_j \tag{4.9}$$

in the interior of a hemisphere Σ_R, where f_β are given functions, and assume that the plane $t = 0$ is nowhere characteristic, in fact that the coefficient a of D_t^k in L does not vanish. If for every $\delta > 0$ the functions f_β, $D^\beta u$ for $|\beta| \leq j$, $D_x D^j u$ belong to L_2 in $\Sigma_{R-\delta}$ then also the function $D_t^{j+1} u$ has this property.

For $j \geq k-1$ there is nothing to prove, as we may solve for the function $D_t^{j+1} u$ from the differential equation (4.9) operated on by D_t^{j+1-k}. Thus we suppose $j < k-1$.

The proof makes use of a well known formula giving explicitly a smooth extension of a function v defined in a half space $t > 0$ to a function defined in the full space:

$$v_N(x, t) = v(x, t) \qquad t > 0 \tag{4.10}$$

$$v_N(x, t) = \sum_{j=1}^{N} \lambda_j v(x, -jt) \qquad t < 0$$

with the λ_j chosen so that

$$\sum_j (-j)^k \lambda_j = 1, \qquad k = 0, \dots, N-1,$$

We observe that,

$$\| v_N \|_k \leq \text{constant} \; \| v \|_k \qquad k = 0, \dots, N-1 .$$

Here the norm on the left is over the full space while on the right it is over the half space $t > 0$.

Proof of the Lemma: Choose a fixed $\delta > 0$, let $\zeta(x, t)$ be a fixed C^∞ function with support in $|x|^2 + t^2 < R^2$ and which equals one in $\Sigma_{R-\delta}$, and set $\zeta a u = v$. If we can prove that $D_t^{j+1} v$ belongs to L_2 then, since $a \neq 0$ it follows easily that $D_t^{j+1} u$ is in L_2 in $\Sigma_{R-\delta}$. From our assumptions we see that v is a weak solution of a differential equation of the form

$$\text{(4.11)} \qquad D_t^k v = \sum_{s+|\gamma| \leq k-j-1} D_t^s D_x^\gamma v_{s,\gamma}$$

where the $v_{s,\gamma}$ belong so L_2, and that derivatives $D_x D^j v$ and v itself belong to L_2.

For N sufficiently large we now extend the functions $v, v_{s,\gamma}$ to negative t, defining v_N by (4.10) and $v_{s,\gamma,N}$ by

$$v_{s,\gamma,N}(x, t) = v_{s,\gamma}(x, t), \qquad t > 0$$

$$v_{s,\gamma,N}(x, t) = \sum_{j=1}^{N} \lambda_j (-j)^{k-s} v_{s,\gamma}(x, -jt), \qquad t < 0 .$$

One may then verify that the equation

$$D_t^k v_N = \Sigma D_t^s D_x^\gamma v_{s,\gamma,N}$$

holds in the entire space in the weak sense, and that the $v_{s,\gamma,N}$, the derivatives $D_x D^j v_N$ and v_N itself belong to L_2.

Let us now take Fourier transforms with respect to x and t, and write $(\xi_1, \dots, \xi_{n-1}) = \xi$, $\xi_n = \tau$. Denoting the trasform of a function f by $\tilde{f}$ we find that

$$(\widetilde{4.11}) \qquad (i\tau)^k \tilde{v}_N = \Sigma (i\tau)^s (i\xi)^\gamma \tilde{v}_{s,\gamma,N},$$

with $\tilde{v}_{s,\gamma,N}$, $\tilde{v}_N$ and $|\xi| (|\xi|^j + |\tau|^j) \tilde{v}_N$ belonging to L_2 in the (ξ, τ) space.

To conclude the proof we have to show that $\tau^{j+1} \tilde{v}_N$ belongs to L_2. To this end write

$$\text{(4.12)} \qquad |\tau|^{j+1} \tilde{v}_N = \frac{|\tau|^{j+1}}{\tau^{2k} + |\xi|^{2k} + 1} \tau^{2k} \tilde{v}_N + \frac{|\tau|^{j+1}}{\tau^{2k} + |\xi|^{2k} + 1} (|\xi|^{2k} + 1) \tilde{v}_N .$$

We shall show that each term on the right belongs to L_2. From $(\widetilde{4.11})$ we find that the first term on the right is bounded by

$$\frac{|\tau|^{k+j+1}}{\tau^{2k}+|\xi|^{2k}+1} \Sigma |\tau|^s |\xi|^\gamma |v_{s,\gamma,N}| .$$

Since $s+|\gamma| \leq k-j-1$ it follows that the factor of $v_{s,\gamma,N}$ is uniformly bounded, and hence that this term belongs to L_2, since the $v_{s,\gamma,N}$ do.

The second term on the right of (4.12) is bounded by

$$c\,(|\xi|\,(|\xi|^j+|\tau|^j)+1)\,|\tilde{v}_N|$$

with c an absolute constant, and hence belongs also to L_2, by an earlier remark.

This completes the proof of the Lemma.

Lecture V. The Differentiability of Weak Solutions of Elliptic Equations

In this and the next lecture we shall present a self contained proof of the well known result that solutions of elliptic equations with C^∞ coefficients are of class C^∞.

Many proofs exist in the literature including proofs for more general classes of equations, see Hörmander [17], Malgrange [18] The proof here seems rather straigtforward; it is based essentially on a proof given by Lax [19] and is closely related to proofs given in lectures by Bers [20] and Schwartz [21] (see also [9]). We confine ourselves as before to a single equation (not necessarily strongly elliptic) although the argument extends also to systems.

Differentiability Theorem: If u is a locally square integrable weak solution of the elliptic equation $Lu=f$, and $f \in C^\infty$ then $u \in C^\infty$.

Remark: If u is a distribution solution then $u = \Delta^k v$ for some continuous v (here Δ is the Laplace operator), and v is then a weak solution of $L\Delta^k v = f$. The Theorem holds therefore for this case also.

The proof consists in showing that u has L_2 derivatives of all orders in every compact subdomain. That $u \in C^\infty$ then follows from the Sobolev estimates proved in Lecture 2. However since we only need a very simple case of the Sobolev lemmas we present a separate proof of it here.

Lemma (Sobolev): In a «smooth» domain $\mathcal{D}$ if u has L_2 derivatives up to order s in $\overline{\mathcal{D}}$ for $s > n/2$, then u is continuous in $\mathcal{D}$.

In fact

$$\max |u| \leq K \left(\int \sum_{j=0}^{s} | D^j u |^2 \, dx \right)^{1/2}, \qquad s > n/2 .$$

Proof: The first assertion follows easily from the inequality. To prove the inequality let x_0 be an inner point in $\mathcal{D}$ (for simplicity take $x_0 = 0$) and suppose there is a sphere about x_0 in $\mathcal{D}$ with radius R. Let furthermorl $\zeta(r)$ be a function in C^∞, equal to 1 for $0 \leq r \leq R/2$, and vanishing for $r \geq R$. By integration along any radius from $x_0 = 0$, and by repeated partial integration we see that

$$u(0) = - \int_0^R 1 \cdot (\zeta u)_r \, dr = \text{const} \int_0^R r^{s-1} \left(\frac{\partial}{\partial r} \right)^s (\zeta u) \, p \, r .$$

integrating over the unit sphere (with area Ω) of radial directions one finds

$$| \Omega u(0)) | = | \text{const} \int r^{s-n} \left(\frac{\partial}{\partial r} \right)^s (\zeta u) \, dx | \leq$$

$$\leq \text{const} \left(\int \left| \left(\frac{\partial}{\partial r} \right)^s (\zeta u) \right|^2 dx \right)^{1/2} \left(\int r^{2(s-n)} \, dx \right)^{1/2}$$

using Schwarz inequality. For $s > n/2$ the last integral is finite.

If the boundary of $\mathcal{D}$ is such that at any point in $\overline{\mathcal{D}}$ there exists a cone with a fixed opening and length contained in $\overline{\mathcal{D}}$ then the same proof holds; instead of integrating over the full sphere of radial directions, we merely integrate over the directions lying in the cone.

The proof of the Differentiability Theorem consists mainly of a series of simple lemmas of calculus concerned with a special situation) that of periodic functions, and this lecture will confined to these calculus statements.

We consider *periodic* functions $u \in C^\infty$ with period 2π in each x_j. For such functions the Fourier series

$$u = \sum_{\xi} u_\xi \, e^{ix \cdot \xi}, \quad \xi = (\xi_1 \cdot \ldots \cdot \xi_n)$$

($\xi_j =$ integer) converges uniformly.

By Parseval's equality we have the following estimate for each non-negative s

$$\text{(5.1)} \qquad \text{canstant} \sum_{\xi} (1 + |\xi|^2)^s |u_\xi|^2 \leqq \int \sum_{j=0}^{s} |D^j u|^2 \, dx \leqq \text{constant} \sum_{\xi} (1 + |\xi|^2)^s |u_\xi|^2$$

where the integral is taken over a period cube.

For *any* integer s we introduce the following scalar product and norm, differing slightly from our previous notation,

$$(u, v)_s = (2\pi)^n \sum_{\xi} (1 + |\xi|^2)^s u_\xi \overline{v_\xi}$$

$$\| u \|_s^2 = (u, u)_s .$$

We write $(u, u)_0 = (u, u)$ and proceed with the

Calculus :

1. $\| u \|_s$ is inocreasing in s. Furthermore for $t_1 < s < t_2$ and any $\varepsilon > 0$ there is a constant $C(\varepsilon)$ such that

$$\text{(5.2)} \qquad \| u \|_s \leq \varepsilon \| u \|_{t_2} + C(\varepsilon) \| u \|_{t_1} .$$

Proof: For any $\sigma \geq 0$, $\sigma^s \leq \varepsilon \sigma^{t_2} + C(\varepsilon) \sigma^{t_1}$.

2. Set $\varphi = (1 - \Delta)^t u$, $\psi = (1 - \Delta)^t v$, se that $\varphi = \sum_{\xi} u_\xi (1 + |\xi|^2)^t e^{ix \cdot \xi}$. From this we find

$$\text{(5.3)} \qquad \| u \|_s = \| \varphi \|_{s-2t} = \| (1 - \Delta)^t u \|_{s-2t}$$

$$\text{(5.4)} \qquad (u, v)_s = (u, (1 - \Delta)^t v)_{s-t} = ((1 - \Delta)^t u, v)_{s-t} .$$

As a consequence we have

Lemma: If $\omega \in C^\infty$, then

$$\text{(5.5)} \qquad (\omega u, v)_t = (u, \overline{\omega} v)_t + 0(\| u \|_t \| v \|_{t-1} + \| u \|_{t-1} \| v \|_t) .$$

Proof: Assume $t \leq 0$. Using (5.4), (5.3), (5.1), and partial integration, we find

$$(\omega u, v)_t = (\omega (1 - \Delta)^{-t} \varphi, \psi) = ((1 - \Delta)^{-t} \varphi, \overline{\omega} \psi)$$

$$= (\varphi, (1 - \Delta)^{-t} (\overline{\omega} \psi)) =$$

$$= (\varphi, \overline{\omega} (1 - \Delta)^{-t} \psi) + 0(\| \varphi \|_{-t} \| \psi \|_{-t-1} + \| \varphi \|_{-t-1} \| \psi \|_{-t})$$

$$= (u, \overline{\omega} v)_t + 0(\| u \|_t \| v \|_{t-1} + \| u \|_{t-1} \| v \|_t) .$$

In the case $t \geq 0$ the proof is similar.

3. *Schwar(t)z's inequality :*

$$| (u , v)_s | \leq \| u \|_{s+t} \| v \|_{s-t} . \quad (5.6)$$

(Clear !)

In fact

$$\| u \|_{s+t} = \text{l.u.b.} \frac{| (u , v)_s |}{\| v \|_{s-t}} . \quad (5.7)$$

Proof: According to (5.6) the left side of (5.7) is not smaller than the right side. If however we set $v = (1 - \Delta)^t u$, then, by (5.4)

$$\frac{(u , v)_s}{\|v\|_{s-t}} = \frac{(u , (1 - \Delta)^t u)_s}{\| (1 - \Delta)^t u \|_{s-t}} = \frac{(u , u)_{s+t}}{\| u \|_{s+t}} = \| u \|_{s+t} ,$$

proving (5.7).

We can now form Hilbert espace H_s by completing C^∞ functions in the norms $\| \ \|_s$. For $s \geq 0$ these agree with our previous definitions. Obviously $H_s \subset H_t$ for $s > t$. All the previous results hold for functions with the appropriate norms finite, for instance (5.7). We may regard H_s as given by a formal Fourier series with finite $\| \ \|_s$ norm.

We remark that the scalar product

$$(u , v)$$

is defined, by extension, for any functions $u \in H_s$, $v \in H_{-s}$, and that any bounded linear functional $f(u)$ defined on H_s may be represented in the form

$$f(u) = (u , v)$$

with $v \in H_{-s}$; this follows immediately from the Fourier series representation, so that we may regard H_{-s} as dual to H_s.

Though we shall not use this, we remark that the closed unit ball $\| u \|_s \leq 1$ in H_s is compact in H_t for $s > t$.

We continue with the calculus.

4. Consider any differential operator L of order k with C^∞ coefficients. *Claim :*

$$\| L u \|_s \leq \text{const} \| u \|_{s+k} . \quad (5.8)$$

More precisely

$$\| L u \|_s \leq c K \| u \|_{s+k} + c K' \| u \|_{s+k-1} , \quad (5.9)$$

where $c = c(k , n)$, K is a bound for the leading coefficients, and K' is a bound for all coefficients and their derivatives up to order $| s |$.

Proof: Since obviously $\| D^j u_s \| \leq \text{const} \| u \|_{s+j}$ it suffices, in order to prove (5.9), to show that if $a \in C^\infty$ then

$$\| a\, u \|_s \leq c\, k' \| u \|_s + c\, k'' \| u \|_{s-1} \tag{5.10}$$

where k' and k'' are bounds for $| a |$ and $| D^j a | \, (j \leq | s |)$ respectively.

Proof of (5.10): Consider first the case $s < 0$. Set $\varphi = (1 - \Delta)^s u$ $\psi' = (1 - \Delta)^s au$ then we have, by (5.4), and partial integration,

$$\| a\, u \|_s^2 = \| \psi \|_{-s}^2 = (a\, u \,, \psi) = (a\, (1 - \Delta)^{-s} \varphi \,, \psi) \,.$$

Integrating the last by parts $(-s)$ times we find it is not greater than

$$c\, k \| \varphi \|_{-s} \| \psi \|_{-s} + c\, k' \| \psi \|_{-s} \| \varphi \|_{-s-1} \,.$$

So dividing by $\| \psi \|_{-s}$ we have, with the aid of (5.3),

$$\| \psi \|_{-s} = \| a\, u \|_s \leq c\, k \| \varphi \|_{-s} + c\, k' \| \varphi \|_{-s-1}$$

$$= c\, k \| u \|_s + c\, k' \| u \|_{s-1} \text{ by } (5.3).$$

In case $s \geq 0$ we have

$$\| a\, u \|_s^2 = (a\, u \,, (1 - \Delta)^s\, a\, u) \,,$$

and may integrate by parts as above.

So L can be extended to all of H_s and maps it boundedly into H_{s-k}. This operation of L agrees with that of L acting on u, regarded as a distribution.

Technical Lemma: Suppose ω is a C^∞ real function, then

$$(L\, (\omega^2 u) \,, L\, u)_s = \| L(\omega\, u) \|_s^2 + 0\, (\| u \|_{s+k} \| u \|_{s+k-1}) \,. \tag{5.11}$$

Proof:

$$(L\, (\omega^2 u) \,, L\, u)_s = (\omega\, L\, (\omega\, u) \,, L\, u)_s + 0\, (\| u \|_{s+k} \| u \|_{s+k-1}) \text{ by } (5.8)$$

$$= (L\, (\omega\, u) \,, \omega\, L\, u)_s + 0\, (\| u \|_{s+k} \| u \|_{s+k-1}) \text{ by } (5.5)$$

$$= (L\, (\omega\, u) \,, L\, (\omega\, u))_s + 0\, (\| u \|_{s+k} \| u \|_{s+k-1}) \text{ by } (5.8)$$

To conclude this lecture we consider

Difference Quotients: For a given vector h let

$$u^h = \frac{u\, (x + h) - u\, (x)}{| h |}$$

be the difference quotient. One verifies easily : $\| u(x+h) \|_s = \| u(x) \|_s$,

$$\| u^h \|_s \leq \| u \|_{s+1} . \tag{5.12}$$

Furthermore : If $u \in H_s$, $u^h \in H_s$ and $\| u^h \|_s \leq k$ for each h, then $\| u \|_{s+1} \leq k$.

Corollary : If $u \in H_s$, $\| u^h_s \| \leq k$ for each h, then $u \in H_{s+1}$ and $\| u \|_{s+1} \leq k$.

Proof; Let $u = \sum_\xi u_\xi e^{ix\cdot\xi}$, and let $u_N = \sum_{|\xi|\leq N} u_\xi e^{ix\cdot\xi}$. One finds $\| u_N \|_{s+1} \leq k$. Q. E. D.

Lecture VI. Proof of the Differentiability Theorem.

Let now L be an elliptic operator of order k. In the periodic case we prove the Differentiability Theorem in the form

DIFFERENTIABILITY THEOREM: *If* $u \in H_s$, $L u \in H_{s-k+1}$, *then* $u \in H_{s+1}$. *So it follows that if* $u \in H_s$ *and* $L u \in H_{t-k}$, *then* $u \in H_t$.

The non-periodic case is easily reduced to this as follows; We prove successively that u has L_2 first order derivatives, then second order derivatives, then second order derivatives, and so on.

To carry out this reduction let ζ be a C^∞ function defined in a neighborhood of a point and with compact support. Let $v = \zeta u$ and extend v and the coefficients of L as periodic functions. So

$$L v = L(\zeta u) = f + g$$

where $f = \zeta L u$, $g = L(\zeta u) - \zeta L u$; g contains only derivatives of u up to order $k-1$, and so, as is easily seen with aid of (5.8) has finite $\| \; \|_{1-k}$ norm.

So $L v \in H_{1-k}$, therefore $v \in H_1$ and so u has L_2 derivatives in a neighborhood of the point. Using this one repeats the argument for a smaller neigborhood, and sees that $L v \in H_{2-k}$, so $v \in H_2$, and so on.

The proof of the Differentiability Theorem in the periodic case follows easily, in turn, from the following estimate which is the analogue of Gårding's inequality.

Basic Estimate : For any s, s_0

$$\| u \|_{s+k} \leq \text{constant} \; \| L u \|_s + \text{constant} \; \| u \|_{s_0} . \tag{6.1}$$

Postponing the proof of (6.1) let us prove the Differentiability Theorem. Consider a difference quotient u^h. If L^h represents the operator obtained by replacing each coefficient in L by its difference quotient we see that $Lu^h =$

$= (Lu)^h - L^h u(x+h)$ Thus we have, from (6.1)

$$\| u^h \|_s \leq \text{constant} \| L u^h \|_{s-k} + \text{constant} \| u^h \|_{s-1}$$

$$\leq \text{constant} \| (L u)^h \|_{s-k} + \text{constant} \| L^h u (x+h) \|_{s-k}$$

$$+ \text{constant} \| u \|_s \qquad \text{by (5.12)}$$

$$\leq \text{constant} \| L u \|_{s-k+1} + \text{constant} \| u \|_s \text{ by (5.12), (5.8)}$$

The desired result follows from the corollary after (5.12).

Proof of the Basic Estimate: Clearly only $s_0 < s + k$ is of interest. The proof consists of several steps, following our recipe, and the proof of Gårding's inequality in Lecture 4.

1. L has constant coefficients with leading terms only. Then

$$\| L u \|_s^2 = (2\pi)^n \sum_\xi | u_\xi |^2 | L(\xi) |^2 (1 + |\xi|^2)^s$$

$$\geq \text{constant} \sum_\xi | u_\xi |^2 | \xi |^{2k} (1 + |\xi|^2)^s$$

while

$$\| u \|_{s_0}^2 = (2\pi)^n \sum_\xi | u_\xi |^2 (1 + |\xi|^2)^{s_0}.$$

Hence

$$\| L u \|_s^2 + \| u \|_{s_0}^2 \geq \text{constant} \sum_\xi | u_\xi |^2 (1 + |\xi|^2)^{s+k} =$$

$$= \text{constant} \| u \|_{s+k}^2.$$

2. L has variable coeffieients with leading coefficients differing from constant value by less than ε, ε sufficiently small: Let L_0 be the operator with these constant coefficients. By case 1. we have

$$\| u \|_{s+k} \leq \text{constant} \| L_0 u \|_s + \text{constant} \| u |_{s_0}$$

$$\leq \text{constant} (\| L u \|_s + \| (L_0 - Lu) \|_s) + \text{constant} \| u \|_{s_0}$$

$$\leq \text{constant} \| L u \|_s + \text{constant}\, \varepsilon \| u \|_{s+k} + \text{constant} \| u \|_{s}$$

$$\leq \text{constant} \| L u \|_s + \text{constant}\, \varepsilon \| u \|_{s+k} + \frac{1}{2} \| u \|_{s+k} + \text{constant} \| u \|_{s_0}$$

by (5.2),

from which (6.1) follows.

Note: If u has its support in a small set then the leading coefficients, being continuous, difrer little from constant values. So (6.1) holds in that case.

3. General case: Intruduce a partition of unity over the closed period cube

$$1 = \sum_j \omega_j^2$$

with each ω_j having its support in a small region.

$$\| u \|_{s+k}^2 = (u , u)_{s+k} = (\sum_j \omega_j^2 u , u)_{s+k}$$

$$= \sum_j (\omega_j u , \omega_j u)_{s+k} + 0 (\| u \|_{s+k} \| u \|_{s+k-1}) \text{ by } (5.5) ,$$

$$\leq \text{constant} \sum_j (L \omega_j u, L\omega_j u_s) + \text{constant} \sum_j \| \omega_j u \|_{s_0}^2 + 0 (\| u \|_{s+k} \| u \|_{s+k-1})$$

(since $\omega_j u$ has its support in a small set)

$$= \text{constant } (L (\sum_j \omega_j^2 u)_0 \, L u)_s + 0 (\| u \|_{s+k} \| u \|_{s+k-1}) , \text{ by } (5.11),$$

$$\leq \text{constant} \| L u \|_s^2 + \frac{1}{2} \| u \|_{s+k}^2 + c \| u \|_{s+k-1}^2$$

$$\leq \text{constant} \| L u \|_s^2 + \frac{3}{4} \| u \|_{s+k}^2 + c \| u \|_{s_0}^2 - , \text{ by } (5.2)$$

from which (6.1) follows.

We conclude this lecture with some remarks concerning the differentiability near the boundary of the solution of the Dirichlet problem obtained in Lecture 3. Using the notation of that lecture we recall that the solution of $L u = f$ belonged to the space H_m. Since the discussion is local we may assume that the boundary is given by $x_n = 0$ and that we have a solution of the equation in the hemisphere Σ_R (see Lecture 4). It suffices, by Sobolev, to show that the solution u has derivatives of all orders in L_2 in $\Sigma_{R-\delta}$, for f in C^∞. We shall merely indicate the first step — the proof that u has derivatives of order $(m + 1)$ in L_2 (see [9]) — and shall use the notation of the lemma at the end of Lecture 4. By that lemma it suffices to prove that derivatives of the form $D_x D^m u$ belong to L_2 in $\Sigma_{R-\delta}$ for any $\delta > 0$.

This we do with the aid of difference quotients as above. For fixed $\delta > 0$ let $\zeta (x, t)$ be a C^∞ function with support in $|x|^2 + t^2 < R^2$, and which equals one in $\Sigma_{R-\delta}$. Since the function u satisfies

$$B [u , \varphi] = (f, \varphi)$$

for all φ in $\overset{\circ}{H}_m(\Sigma_R)$ it follows that the function $v = \zeta u$ satisfies

$$B[v, \varphi] \leq \text{constant} \| \varphi \|_{m-1}$$

for such φ. If we nov form difference quotients v^h as above with h parallel to the boundary $t = 0$ we find easily that

$$B[v^h, \varphi] \leq \text{constant} \| \varphi \|_m$$

for φ in $\overset{\circ}{H}_m(\Sigma_R)$. Setting $\varphi = v^h$ and applying Gårding's inequality for strongly elliptic operators of Lecture 3 we obtain a bound for

$$\| v^h \|_m$$

which is independent of h, and it follows easily that the derivatives $D_x D^m v$ are in L_2, hence that $D_x D^m u \in L_2$ in $\Sigma_{R-\delta}$.

Lecture VII. A Priori Estimates Near the Boundary.

In the remainig time we shall discuss briefly the derivation of estimates near the boundary for solutions of elliptic equations in, for simplicity, a bounded domain $\mathcal{D}$. This material is taken from a paper by Agmon, Douglis, Nirenberg [14] which is concerned with both Schauder and L_p estimates near the boundary for solutions satisfying general boundary condition. As remarked in Lecture 4 the estimates for $p = 2$ are special cases of more general results. We wish also to draw attention to a paper [22] by Hörmander concerned with equations $L u = 0$ with constant coefficients in a half space, and solutions satisfying a number of boundary conditions $B_j u = 0$ described by differential operators B_j with constant coefficients. Hörmander characterizes all such systems for which the solutions belong to C^∞ on the boundary, and also those for which the solutions are analytic at the boundary.

Since the time is limited we shall restrict ourselves here to the Dirichlet problem for a single elliptic equations of order $2m$

$$(7.1) \qquad L u = f \qquad \text{in } \mathcal{D}$$

$$\left(\frac{\partial}{\partial \vec{n}}\right)^{j-1} u = \varphi_s \qquad \text{on } \dot{\mathcal{D}}, j = 1, \dots, m,$$

where $\vec{n}$ represents the unit normal to the boundary.

The operator L will be required to satisfy a certain condition.

Condition on L: *If* $L'(x, D)$ *is the leading part of* L, *we require that for every pair of independent real vectors* ξ^1, ξ^2 *the polynomial in* τ

$$L'(x, \xi^1 + \tau \xi^2)$$

have exactly m roots on either side of the real τ *axis.*

In three or more dimensions this condition is automatically satisfied. We see however that the operator $\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)^2$ in two dimensions violates the condition. (This operator and others in two dimensions come under the theory when treated as a system).

In describing the estimates we make use of the following norms and seminorms. In lecture 2 we already met the Hölder norm, for $0 < \alpha < 1$

$$[u]_\alpha = [u]_\alpha^{\mathfrak{D}} = \underset{P, Q \epsilon \mathfrak{D}}{\text{l. u. b.}} \frac{| u(P) - u(Q) |}{| P - Q |^\alpha} .$$

For functions of classe C^k in $\overline{\mathcal{D}}$ we also introduce (differing from the notation in Lecture 2)

$$[u]_k = \text{l. u. b.} \, | D^k u |$$

$$| u |_k = \sum_{j=0}^{k} [u]_j$$

where the l. u. b. is taken over all derivatives of order k, and all points in $\mathcal{D}$. In addition, for functions in C^k with Hölder oontinuous (exponent α) derivatives of order k, we introduce

$$[u]_{k+\alpha} = \max [D^k u]_\alpha ,$$

$$| u |_{k+\alpha} = | u |_k + [u]_{k+\alpha} ,$$

where the max, is taken over all derivatives of order k. The space of functions with finite $| \ |_{k+\alpha}$ norm is denoted by $C^{k+\alpha}(\overline{\mathcal{D}})$. (We also use the notation of Lecture 3 and 4).

For functions φ defined on the (smooth) boundary $\dot{\mathcal{D}}$ of $\mathcal{D}$ we also have analogous norms, defined in a tather obvious way en terms of local coordinates, and which we denote in the same way.

We now summarize the results without specifying the exact smoothness conditions on the boundary; k will denote a non-negative integer. The integral estimates will be stated only for $p = 2$.

L_2 *Estimates: If* $u \in H_{2m}$ *and satisfies, for simplicity, homogeneous Dirichlet data,* i. e. $\varphi_j = 0$, *and if* $L u \in H_k$, *and the coefficients of* L *belong to* C^k, *then* u *belongs to* H_{2m+k} *and*

$$(7.2) \qquad \| u \|_{2m+k} \leq \text{constant} \; (\| L u \|_k + \| u \|_0) .$$

Similar results hold for equations in integral, or variational, form.

Schauder Estimates: If for some positive $\alpha < 1$, $u \in C^{2m+\alpha}(\overline{\mathcal{D}})$, $Lu \in C^{k+\alpha}(\overline{\mathcal{D}})$, $\varphi_j \in C^{2m+k+1-j+\alpha}$, *and the coefficients of* L *belong to* $C^{k+\alpha}$, *then* $u \in C^{2m+k+\alpha}$ *and*

$$(7.3) \qquad | u |_{2m+k+\alpha} \leq \text{constant} \; ([L u]_{k+\alpha} + \sum_j | \varphi_j |_{2m+k+1-j+\alpha} + | u |_0) \cdot$$

Similar results hold for equations in variational form. From these one may derive, dor instance, the following result for solutions of $L u = 0$, *under suitable smoothness assumptions on the coəfficients:*

If $u \in C^{m-1+\alpha}(\overline{\mathcal{D}})$ *and* $\varphi_j \in C^{m-j+k+\alpha}$, $j = 1, \ldots, m$, *then* $u \in C^{m-1+k+\alpha}(\overline{\mathcal{D}})$ *and*

$$(7.4) \qquad | u |_{m-1+k+\alpha} \leq \text{constant} \; (\Sigma \, | \varphi_j |_{m-j+k+\alpha} + | u |_0) .$$

The constants in the above are independent of u. In case of uniqueness of the solution in the class considered the terms $\| u \|_0$ or $| u |_0$ may be dropped. Miranda [23], using results of Agmon [25], has recently proved an extended maximum principle for solutions of strongly elliptic equations (7.1) in two dimensions, which asserts that (7.4) holds for $k = \alpha = 0$. I believe that this holds true in general for operators satisfying the condition on L.

The Schauder estimates have a number of useful consequences. With their aid one may prove the existence of solutions of strongly elliptic equations having merely Hölder continuous coefficients. In particular, with the aid of (7.4) one may solve such equations with the given Φ_j in class $C^{m-j+\alpha}$.

In addition one can also solve the Dirichlet problem for a wide class of equations which are not strongly elliptic. Futhermore, and this is perhaps the most useful feature of the estimates, with their aid one may prove local parturbation theorems for nonlinear elliptic squations. For example if $F_\lambda(x, u, \ldots, D^{2m} u) = 0$ is a nonlinear equation depending on a parameter λ and «smoothly» on all variables, such that for $\lambda = 0$ the function u_0 is a solution with, say, zero Dirichlet data, and if the « first variation » of F at u_0 is a linear elliptic operator L which is invertible (i. e. for which the Dirichlet problem (7.1) has one and only one solution) then for $|\lambda|$ sufficiently small there exists a unique solution u_λ of the nonlinear equation with zero Dirichlet data. The estimates also yield differentiability theorems at the boundary for solutions of nonlinear elliptic equations.

The estimates are derived following the « recipe » of Lecture 4, the main step being the first one. That is, one considers equations (7.1) in a half space $x_n > 0$, for operators L with constant coefficients and only highest order terms, and C^∞ functions u in $t \geq 0$ vanishing outside some sphere. This system is then treated with the aid of explicitly constructed Poisson kernels, which will be described in the next lecture, with which one solves the system (7.1) with $f = 0$. With the aid of the explicit representations for u and its derivatives so obtained, the desired estimates for this constant coefficient case are then obtained with the aid of certain potential theoretic results.

I would like to describe these results, which I believe should prove useful for other problems. Since we are operating in a half space $x_n > 0$ it is convenient to rename the coordinates, set $(x_1, \dots, x_{n-1}) = x$, $x_n = t$, $(x_1, \dots, x_n) = (x, t) = P$.

We consider integral transforms of functions $f(x)$ into functions $u(x, t)$, $t > 0$. Let $K(x,t)$ be a kernel defined in the half space $t \geq 0$ and homogeneous of degree $1 - n$.

$$K(P) = \frac{\Omega(x/|P|, t/|P|)}{|P|^{n-1}}$$

here $|P| = (|x|^2 + t^2)^{1/2}$; Assume that Ω is continuous on the half sphere $|P| = 1$, $t \geq 0$ and assume also (this condition can be weakened considerably) that Ω has continuous first derivatives on the half sphere which, together with Ω itself, are bounded in absolute value by $\varkappa$. In addition we make the basic assumption

$$\int_{|x|=1} \Omega(x, 0)\, d\omega_x = 0 .$$

Here integration is over the unit sphere $|x| = 1$, with $d\omega_x$ as element of area.

Consider the transformation

$$u(x, t) = \int K(x - y, t) f(y)\, dy \qquad t > 0,$$

integration being over the entire y space. Denote the L_p norm of f by $|f|_{L_p}$, and that of $u(x, t)$ in x, for any fixed t, by $|u|_{L_{p,t}}$.

THEOREM: 1. *For* $0 < \alpha < 1$

$$[u]_\alpha \leq c \cdot \varkappa\, [f]_\alpha$$

where c *depends only on* α *and* n. *Here the norms refer to the half space* $t > 0$ *for* u, *and the plane* $t = 0$ *for* f.

2. *For every* $t > 0$ *and* $1 < p < \infty$,

$$|u|_{L_{p,t}} \leq c\, \varkappa\, |f|_{L_p}$$

where c *depends only on* p *and* n.

3. $$\left[\iint |u(x, t)|^2\, dx\, dt\right]^{1/2} \leq c\, x\, \langle f \rangle_{-1/2}$$

where c *is an absolute constant. Here* $\langle f \rangle_{-1/2}$ *is defined in terms of the Fourier transform* $\widetilde{f}(\xi)$ *of* f *by*

$$\langle f \rangle_{-1/2} = \int |\xi|^{-1}\, |\widehat{f}(\xi)|^2\, d\,\xi]^{1/2}.$$

There is an L_p analogue of 3, which is however more complicated to state.

We call Part 1 of the theorem a result of Privaloff type. It is a simple extension of classical results of Hölder, Giraud and others, to which it reduces if we set $t = 0$. Part 2, a result of Riesz type, is a straighforwatd extension of recent results of Calderon and Zygmund [24], to which it reduces if we set $t = 0$. For the special case of the Hilbert transform for $n = 2$ it is due to Riesz, and in fact it is proved by reduction to the Riesz result with the aid of a device of [24]. Part 3, is proved with the aid of Fourier transforms — one shows that the Fourier transform $\widehat{K}(\xi, t)$ of $K(x, t)$ with respect to the x variables is bounded in absolute value by constant $(1 + t\,|\xi|)^{-1}$, from which the result follows easily. Part 3 plays an essential role in the derivation of the L_2 estimates.

Lecture VIII. The Boundary Value Problem in a Half Space; The Poisson Kernels.

In this lecture we shall show how to solve explicity the elliptic system (7.1) with constant coefficients for the special case of a half space. Making a slight change of notation we shall consider the space to be

$n+1$ dimensional, with the first n coordinates denoted by $x=(x_1, \dots, x_n)$ and the last coordinate by t. In the half space $t>0$ we consider for simplicity the homogeneous equation, with $D_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$, $D_t = \frac{\partial}{\partial t}$,

$$L(D_x, D_t)\, u = 0 \tag{8.1}$$

where L is an elliptic operator of order $2m$ with only highest order terms, satisfying the « condition on L » of the previous lecture, i. e. for fixed real $\xi = (\xi_1, \dots, \xi_n) \neq 0$ the polynomial $L(\xi, \tau)$ has exactly m roots τ on each side of the real axis.

On $t=0$ we prescribe the derivatives

$$D_t^{j-1} u = \Phi_j(x) \qquad j = 1, \dots, m \tag{8.2}$$

with the Φ_j in C_0^∞, for simplicity.

The solution will be given in terms of kernels $K_j(x, t)$, $j = 1, \dots, m$, the Poisson kernels,

$$u(x, t) = \Sigma_j \int K_j(x-y, t)\, \Phi_j(y)\, dy = \Sigma K_j * \Phi_j, \tag{8.3}$$

where $*$ denotes convolution. Our construction of the K_j is an extension of the construction given by Agmon [25] in two dimensions, $n=1$, but it is based on the Fritz John identity (1.6) of Lecture 1: For $\Phi(x)$ in C_0^∞

$$\Phi = -\frac{1}{(2\pi i)^n q!} \Delta^{(n+q)/2} \left[\int_{|\xi|=1} (x\cdot\xi)^q \log \frac{x\cdot\xi}{i}\, d\omega_\xi * u \right], \tag{8.4}$$

where q is a non-negative integer of the same parity as n, Δ is the Laplacean, and the principal branch of the lagarithm is taken with the plane slit along the negative real exis.

First some preliminaries. For fixed real $\xi \neq 0$ denote by $\tau_k^+ = \tau_k^+(\xi)$, $k = 1, \dots, m$, the roots τ with positive imaginary parts of $L(\xi, \tau) = 0$, and set

$$M^+(\xi, \tau) = \pi_k (\tau - \tau_k^+(\xi)) = \sum_{p=0}^{m} a_p^+(\xi)\, \tau^{m-p}.$$

The coefficients a_p^+ are analytic in ξ for real $\xi \neq 0$, and homogeneous of degree p. With M^+ we associate the polynomials (in τ)

$$M_j^+(\xi, \tau) = \sum_{p=0}^{j-1} a_p^+(\xi)\, \tau^{j-1-p}, \qquad j = 1, \dots, m. \tag{8.5}$$

The following relations are easily verified.

$$\frac{1}{2\pi i}\int_{\gamma} \frac{M^{+}_{m+1-j}(\xi\,,\tau)}{M^{+}(\xi\,,\tau)}\,\tau^{k-1}\,d\tau = \delta^{k}_{j}\,, \qquad 1 \leq j\,,\ k \leq m \tag{8.6}$$

where γ is a rectifiable Jordan contour in the complex τ plane enclosing all the roots $\tau^{+}(\xi)$ in its interior; δ^{k}_{j} is the Kronecker delta.

We can now writhe down the

Poisson Kernels: For $j - 1 \geq n$

$$K_j(x\,,t) = \frac{\beta_j}{2\pi i}\int_{|\xi|=1} d\omega_\xi \left[\int_{\gamma} \frac{M^{+}_{m+1-j}(\xi\,,\tau)}{M^{+}(\xi\,,\tau)}(x\cdot\xi + t\tau)^{j-1-n}\log\frac{x\cdot\xi+t\tau}{i}\,d\tau\right], \tag{8.7}$$

for $j - 1 < n$

$$K_j(x\,,t) = \frac{\beta_j}{2\pi i}\int_{|\xi|=1} d\omega_\xi \left[\int \frac{M^{+}_{m+1-j}(\xi\,,\tau)}{M^{+}(\xi,\tau)(x\cdot\xi + t\tau)^{n-j+1}}\,d\tau\right], \qquad j = 1\,,\dots,m\,. \tag{8.7$'$}$$

Here

$$\beta_j = -\frac{1}{(2\pi i)^n\,(j-1-n)!} \qquad \text{if } j-1 \geq n\,,$$

$$\beta_j = (-1)^{n-j+1}\frac{(n-j)!}{(2\pi i)^n} \qquad \text{if } j-1 < n\,, \tag{8.8}$$

and γ is a Jordan contour in $\mathfrak{Im}\,\tau > 0$ enclosing all the roots τ of $M^{+}(\xi,\tau)$ *for all* $|\xi| = 1$, ξ real.

Before proving that these formulas represent Poisson kernels we observe, with the aid of the identities

$$\frac{\mu!}{(\lambda+\mu)!}\left(\frac{d}{dz}\right)^{\lambda}\left[z^{\lambda+\mu}\left(\log\frac{z}{i} + c_{\lambda,\mu}\right)\right] = z^{\mu}\log\frac{z}{i}\,, \qquad \mu\,,\lambda \geq 0\,,$$

$$\frac{(-1)^{1+\mu}}{(\mu+\lambda)!\,(-1-\mu)!}\left(\frac{d}{dz}\right)^{\lambda}\left(z^{\lambda+\mu}\log\frac{z}{i}\right) = z^{\mu}\,, \quad \mu<0,\ \lambda+\mu\geq 0 \tag{8.9}$$

for λ, u integers, and $c_{\lambda,\mu}$ some appropriate constants, that we may represent the functions K_j in the form — with q a non-negative integer having the same parity as n —

$$K_j = \Delta_x^{(n+q)/2}\,K_{j,q}(x\,,t) \tag{8.10}$$

where, for $j-1 \geq n$,

$$(8.10)' \qquad K_{j,q} = \frac{\beta_j (j-1-n)!}{2\pi i (j-1+q)!} \cdot$$

$$\cdot \int_{|\xi|=1} d\omega_\xi \left[\int_\gamma \frac{M^+_{m+1-j}(x\cdot\xi + t\tau)^{j-1+q}}{M^+} \left(\log \frac{x\cdot\xi + t\tau}{i} + c_{n+q,\, j-1-n} \right) d\tau \right],$$

and for $j-1 < n$

$$(8.10)'' \qquad K_{j,q} = \frac{(-1)^{n-j}}{2\pi i (j-1+q)!\,(n-j)!} \cdot$$

$$\cdot \int_{|\xi|=1} d\omega_\xi \left[\int_\gamma \frac{M^+_{m+1-j}(x\cdot\xi + t\tau)^{j-1+q}}{M^+} \log \frac{x\cdot\xi + t\tau}{i}\, d\tau \right].$$

It is easily seen that $K_{j,q}$ and all its derivatives up to order $j+q$ are continuous in the closed half space $t \geq 0$.

We now prove that the kernels K_j given by (8.7), (8.7)′ are indeed Poisson kernels. By inspection we see that the K_j are analytic solutions of $Lu = 0$ for $t > 0$. Hence u defined by (8.3) is a solution. Setting

$$(8.3)_j \qquad u_j = K_j * \Phi_j .$$

we shall show that u_j belongs to C^∞ in $t \geq 0$ and that

$$(8.11) \qquad D_t^{k-1} u_j = \delta_j^k\, \Phi_j(x) \qquad \text{for } t=0,\ k=1,\dots,m.$$

Consider any partial derivative of order s of u_j. Choosing an integer q of the same parity as n, and such that $q \geq s-j+2$ we have, for $t>0$

$$(8.12) \qquad D^s u_j = D^s \int \Delta_x^{(n+q)/2} K_{j,q}(x-y\,,t)\, \Phi_j(y)\, dy$$

$$= \int D^s K_{j,q}(x-y\,,t)\, \Delta_y^{(n+q)/2}\, \Phi_j(y)\, dy$$

after partial integration, recalling that $\Phi_j \in C_0^\infty$. Since, as remarked above, $D^s K_{j,q}$ is continuous in the closed half space $t \geq 0$ it follows that $D^s u_j$ can be extended as a continuous function in the entire closed half space $t \geq 0$. Since s is arbitrary we have proved that $u_j \in C^\infty$ in $t \geq 0$.

To verify (8.11) choose q sufficiently large so that $q \geq j - k + 1$, $j = 1, \ldots, m$. Using (8.12) we have, for $t = 0$,

$$(8.13) \qquad D_t^{k-1} u_j = \int \Delta_y^{(n+q)/2} \Phi_j(y) \cdot D_t^{k-1} K_{j,q}(x-y, 0)\, dy$$

$$= \int \Delta_x^{(n+q)/2} \Phi_j(x-y)\, D_t^{k-1} K_{j,q}(y, 0)\, dy,$$

after a change of variable.

Assume first that $k \neq j$. Using (8.10)′, (8.10)″ we find, for $t = 0$, and appropriate constants c', c''

$$D_t^{k-1} K_{j,q}(y, 0) = c' \int\limits_{|\xi|=1} \int\limits_{\gamma} \frac{M^+_{m+1-j}}{M^+} \tau^{k-1}\, d\tau\, (y \cdot \xi)^{j-k+q} \left(\log \frac{y \cdot \xi}{i} + c''\right) d\omega_\xi = 0$$

by (8.6). Thus (8.11) is proved for $k \neq j$.

Now suppose $k = j$. If $j - 1 > n$ we have, using (8.9) (8.10)′ and (8.6), for some constant c'

$$(8.14)' \qquad D_t^{j-1} K_{j,q}(y, 0) =$$

$$= \frac{\beta_j (j-1-n)!}{2\pi i\, q!} \int\limits_{|\xi|=1} d\omega_\xi \left[(y \cdot \xi)^q \left(\log \frac{y \cdot \xi}{i} + c'\right) \int\limits_{\gamma} \frac{M^+_{m+1-j}}{M^+} \tau^{j-1}\, d\tau \right]$$

$$= \frac{\beta_j (j-1-n)!}{q!} \int\limits_{|\xi|=1} (y \cdot \xi)^q \log \frac{y \cdot \xi}{i} + \psi_q(y)$$

where $\psi_q(y)$ is a homogeneous polynomial of degree q.

Similarly if $j - 1 < n$ we find, using (8.10)″ and (8.6)

$$(8.14)'' \qquad D_t^{j-1} K_{j,q}(y, 0) = \frac{(-1)^{n-j} \beta_j}{(n-j)!\, q!} \int\limits_{|\xi|=1} (y \cdot \xi)^q \log \frac{y \cdot \xi}{i}\, d\omega_\xi + \psi_q(y)$$

where again ψ_q denotes a homogeneous polynomial of degree q.

From (8.13), (8.14)′, (8.14)″ we find, after inserting the value of β_j from (8.8), and rechanging variables, that

$$(8.15) \qquad D_t^{j-1} u_j(x, 0) = -\frac{1}{(2\pi i)^n\, q!} \Delta_x^{(n+q)/2} \int \Phi_j(y) \int\limits_{|\xi|=1} ((x-y) \cdot \xi)^q \cdot$$

$$\cdot \log \frac{(x-y) \cdot \xi}{i}\, d\omega_\xi\, dy.$$

Here we have used the fact that

$$\int \Delta_y^{(n+q)/2} \Phi_j(y) \cdot \psi_q(x-y)\, dy = \int \Phi_j(y) \cdot \Delta_x^{(n+q)/2} \psi_q(x-y)\, dy = 0$$

since ψ_q is a polynomial of degree q and is therefore annihilated by $\Delta_x^{(n+q)/2}$. By John's identity (8.4) the right side of (8.15) equals $\Phi_j(x)$, and the proof that the K_j are Poisson kernels is complete.

We remark that the functions $K_{j,q}$ are actually analytic in $t \geq 0$ except at the origin, and that for $s \geq j+q$, $D^s K_{j,q}$ is homogeneous of degree $j-1+q-s$. It follows from our proof above that if $s = n+q+j-k \geq 0$ then

$$(8.16) \qquad | D_x^s D_t^{k-1} K_{j,q}(x, t) | \leq \text{constant} \cdot \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}, \qquad k \neq j.$$

Furhermore we see that because of the reproducing properties (8.11) of the K_j we may assert that

$$D_t^{j-1} K_j(x, 0) = 0 \qquad \text{for } x \neq 0,$$

or

$$(8.16)' \qquad D_t^{j-1} K_j(x, t) \leq \text{constant} \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}.$$

With the aid of (8.16), (8.16)′ it is not difficult to establish the following *Extended Maximum Principle: The solution* (8.3) *of the Dirichlet problem* (8.1), (8.2) *satisfies*

$$\text{l. u. b.} | D^{m-1} u(x, t) \leq \text{constant l. u. b.} | D^{m-1} u(x, 0) |$$

where the least upper bounds are taken with respect to all derivatives of order $m-1$ *and, on the left, with respect to all* (x, t) *in the half space, on the right with respect to all* x.

This is an analogue of a special case of Miranda's extended maximum principle of [23].

BIBLIOGRAPHY

[1] H. LEWY, *An example of a smooth linear partial differential equation without solution.* Annals of Math. 66 (1957) p. 155-158.

[2] L. EHRENPREIS, *Solutions of some problems of division I, II.* American Journal of Math 76 (1954) p. 883-903, 77 (1955) p. 286-292. *The division problem for distributions.* Proc. Nat. Acad. Sci. 41-10 (1955) p. 756-758.

[3] L. HÖRMANDER, *On the theory of general partial differential operators.* Acta Math. 94 (1955) p. 160-248.

[4] B. MALGRANGE, *Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolutions.* Annales de L'Inst. Fourier 6 (1955-6) p. 271-355.

[5] F. TRÊVES, *Solutions élémentaire d'équations aux dérivées partielles dépendent d'un paramètre,* C. R. Acad. Sci. Paris 242 (1956) p. 1250-1252.

[6] L. HÖRMANDER, *Local and global properties of fundamental solutions.* Math. Scandinavica 5 (1957) p. 27-39. *On the division of distributions by polynomials.* Arkiv. för Mat. 3 No. 53 (1958) p. 555-568.

[7] F. JOHN, *Plane waves and spherical means applied to partial differential equations.* Interscionce, New York, 1955.

[8] N. DU PLESSIS, *Some theorema about the Riesz fractional integral.* Trans. Amer. Math. Soc. 80 (1955) p. 124-134.

[9] L. NIRENBERG, *Remarks on strongly elliptic partial differential equations.* Comm. Pure Appl. Math. 8 (1955) p. 649-675.

[10] L. BERS, L. NIRENBERG, (a) *On a representation theorem for linear elliptic systems with discontinuous coefficients and its applications.* (b) *On linear and nonlinear elliptic boundary value problems in the plane.* Atti de Convegno Inter. sul. Equazioni alle derivate Parziali, Trieste, 1954 (published 1955).

[11] DE GIORGI, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari.* Mem. della Accad. delle Scienze di Torino. Ser. 3, Vol. 3 (1957) p. 25-43.

[12] J. NASH, *Cantinuity of solutions of parabolic and elliptic equations.* Amer. Journ. Math. 80 (1958) p. 931-954

[13] N. ARONSZAJN, *On coercive integro differential quadratic forms.* Conference on Partial Differential Equations, University of Kansas, 1954, Technical Report No. 14, p. 94-106.

[14] S. AGMON, A. DOUGLIS, L. NIRENBERG *Estimates near the boundary for sotutions of elliptic partial differential equations satisfying general boundary conditions I.* To appear in Comm. Pure Appl. Math.

[15] M. SCHECHTER, *Integral inequalities for partial differential operators and functions satisfying general boundary conditions.* To appear in Comm. Pure Appl. Math. Vol. 12, No. 1 (1959).

[16] S. AGMON, *The coerciveness problem for integro differential forms,* Journal d'Analyse Math. 6 (1958) p. 184-223.

[17] L. HÖRMANDER, *On the interior regularity of the solutions of partial differential equations.* Comm. Pure Appl. Math. 11 (1958) p. 197-218.

[18] B. MALGRANGE, *Sur une classe d'opérateurs différentiels hypoelliptiques.* Bull. Soc. Math. France. 85, 3 (1957) p. 283-306.

[19] P. D. LAX, *On Cauchy's problem for hyperbolic equations and the differentiability of solutions of elliptic equations.* Comm. Pure Appl. Math. 8 (1955) p. 615 633.

[20] L. BERS, *Elliptic partial differential equations.* Lecture notes of the Seminar in Applied Mathematics, University of Colorado, June 1957.

[21] L. SCHWARTZ, *Ecuaciones diferenciales parciales elipticas,* Lectures at Bogota Colombia, 1956.

[22] L. HÖRMANDER, *On the regularity of the solutions of boundary problems.* Acta Math 99 (1958) p. 225-264.

[23] C. MIRANDA, *Teorema del massimo modulo e teorema di esistenza e di unicità per il problema di Dirichlet relativo alle equazioni ellittiche in due variabili.* Annali di Mat. Pura ed Appl. Ser. 4, Vol. 46 (1958) p. 265-312.

[24] A. P. CALDERON, A. ZYGMUND, *On singular integrals,* Amer. Journal Math. 79 (1956) p. 289-309.

[25] S. AGMON, *Multiple layer potentials and the Dirichlet problem for higher order elliptic equations in the plane I.* Comm. Pure Appl. Math. 10 (1957) p. 179-239.

ADDITIONAL GENERAL BIBLIOGRAPHY

[A] C. MIRANDA, *Equazioni alle derivate parziali di tipo ellittico,* Springer, Berlin 1955.

[B] *Transactions of the Symposium on Partial Differential Equations.* Berkeley California, 1955, published in Comm. Pure Appl. Math., Vol. 9, No. 3 (1956).

[C] E. MAGENES, G. STAMPACCHIA, *I problemi al contorno per le equazioni differenziali lineari di tipo ellittico.* Annali della Scuola Norm. Sup. di Pisa Ser. 3, Vol. 12, Fasc. 3 (1958) p. 297-358.