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A generalization of lemmas of Marcinkiewicz and Fine with applications to singular integrals

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« A GENERALIZATION OF LEMMAS OF MARCINKIEWICZ AND FINE WITH APPLICATIONS TO SINGULAR INTEGRALS »

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1. Introduction.

Let P be a closed subset of the interval $[0,2\pi]$ and let Q be its complement. Q is then the union of open intervals (a_i, b_i) which we shall call the intervals contiguous to P. We define a distance function D relative to the set P as follows: $D(x) = \varrho(x, P)$ is the distance of the point $x \varepsilon [0,2\pi]$ from the set P. Thus D vanishes on P and is triangular on each contiguous interval.

J. Marcinkiewicz ([2], [3], [5]) has used, as a central argument for proving certain difficult results of the theory of Fourier Series, variants of the following lemma concerning the function D.

LEMMA (Marcinkiewicz)

Suppose
$$I(x) = \int_0^{2\pi} \frac{D(x+t)}{t^2}\, dt, \quad x \varepsilon [0, 2\pi].$$

Then $I(x)$ is finite for almost every $x \varepsilon P$. (2)

In a recent paper on the summability of Walsh-Fourier series, N. J. Fine [1] has used another lemma concerning the distance function.

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(1) Part of the work reported in this paper was done while the first-named author was at the Massachusetts Institute of Technology under contract Nonr-1841 (38) under the Office of Naval Research.

(2) Marcinkiewicz actually used instead of the distance function, the function whose value on each contiguous interval (a_i, b_i) is $b_i - a_i$ and which is otherwise zero. The present modification is due to A. Zygmund.

LEMMA (Fine)

Let $\{h_j\}$ *be a sequence of positive numbers satisfying the two conditions*

(i) $$\sum_{h_j \leq \delta} h_j \leq M\delta \,, \text{ for every } \delta > 0$$

(ii) $$\sum_{h_j > \delta} \frac{1}{h_j} \leq \frac{M}{\delta} \,, \text{ for every } \delta > 0 \,,$$

where M *is an absolute constant.*

$$\text{If } I^* \text{ is defined by } I^*(x) = \sum_{j=1}^{\infty} \frac{D(x + h_j)}{h_j} \,,$$

then $I^*(x)$ *is finite for almost every* $x \,\varepsilon\, P$.

It is our purpose to show that each of these two lemmas is a consequence of a more general one (3) (See Theorem 1 below). With the aid of this theorem, we prove, in the succeding sections, analogues of a theorem of Marcinkiewicz on integrals of « Dini type » and of a theorem of Plessner on the Hilbert transform.

2. The generalized Marcinkiewicz-Fine lemma.

In the following A, A_ϱ, A_λ ... will denote constants (which need not always be the same) depending solely on the indicated parameters.

LEMMA 1. *Let* μ *be a positive measure on the interval* $[0, 2\pi]$ *satisfying the following condition*

a) $$\frac{1}{\tau} \int_0^{\tau} d\mu(t) \leq A \text{ for all } \tau \,, 0 < \tau < 2\pi \,.$$

Then for any $\lambda > 0$ *and for all* $\tau, 0 < \tau < 2\pi, \mu$ *satisfies the conditions*

b) $$\frac{1}{\tau^\lambda} \int_0^{\tau} t^{\lambda - 1} \, d\mu(t) \leq A_\lambda$$

and c) $$\tau^\lambda \int_{\tau}^{2\pi} \frac{d\mu(t)}{t^{\lambda+1}} \leq A_\lambda \,.$$

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(3) That this might be the case was suggested to us by Professor A. Zygmund.

Conversely, if μ satisfies either condition b) or c) for some $\lambda > 0$, then μ satisfies condition a) (4).

Proof: For the sake of simplicity, we suppose that μ has zero mass outside $[0, 2\pi]$.

Then
$$\tau^{\lambda}\int_{\tau}^{2\pi}\frac{d\,\mu(t)}{t^{\lambda+1}} = \tau^{\lambda}\sum_{n=0}^{\infty}\int_{\tau\cdot 2^{n}}^{\tau 2^{n+1}}\frac{d\,\mu(t)}{t^{\lambda+1}}$$

$$\leq \sum_{n=0}^{\infty}\tau^{\lambda}\cdot(\tau\, 2^{n})^{-\lambda-1}\int_{\tau 2^{n}}^{\tau 2^{n+1}}d\,\mu(t)$$

$$\leq \sum_{n=0}^{\infty}2^{-n(\lambda+1)}\,\tau^{-1}\int_{0}^{\tau 2^{n+1}}d\,\mu(t)$$

$$\leq \sum_{n=0}^{\infty}2^{-n\lambda}\cdot 2^{-n}\,\tau^{-1}\cdot A\,\tau\cdot 2^{n+1} = A\sum_{n=0}^{\infty}2^{-n\lambda+1}\leq A_{\lambda}\,.$$

Thus $a)$ implies $c)$.

To prove that $a)$ implies $b)$, we decompose the interval $(0, \tau)$ as the union $\bigcup_{n=0}^{\infty}(\tau\, 2^{-n-1}, \tau\, 2^{-n})$ and proceed as before. In a similar fashion, we can show that $b)$ implies $a)$ and that $c)$ implies $a)$.

We can now state the main theorem, the generalization of the lemmas of Marcinkiewicz and Fine, as follows:

THEOREM 1. *Let μ be a positive measure on $[0, 2\pi]$ satisfying the condition $\frac{1}{\tau}\int_{0}^{\tau}d\,\mu(t)\leq A$, $0<\tau<2\pi$. Let P be a closed subset of $[0, 2\pi]$ and D its distance function. If I_{μ} is defined by*

$$I_{\mu}(x) = \int_{0}^{2\pi}\frac{D(x+t)}{t^{2}}\,d\,\mu(t)\,,$$

then $I_{\mu}(x)$ is finite for almost every $x\in P$.

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(4) This lemma stems from the observation of Professor R. Salem that in the case for which $\lambda = 1$, condition (c) is equivalent to condition (a). He also pointed out to us how our original proof of Theorem 1 below could be simplified.

Before proving this theorem, we wish to make some remarks. Theorem 1 reduces to the lemma of Marcinkiewicz if μ is Lebeague measure (condition a) is trivially true of Lebesgue measure). Similarly, the lemma of Fine is obtained by taking for μ the discrete measure which assigns to the point h_j, the mass h_j, $(j = 1, 2, 3, \ldots)$ and is zero otherwise. In this case, the condition on μ is precisely condition (i) on the sequence $\{h_j\}$. Condition (ii) on the sequence $\{h_j\}$, viz. $\sum_{h_j > \delta} \frac{1}{h_j} \leq \frac{M}{\delta}$, is just the condition $\int_\delta^{2\pi} \frac{d\,\mu(t)}{t^2} \leq \frac{M}{\delta}$, i. e. condition c) on the μ of Lemma 1 $(\lambda = 1)$. Thus the two conditions on the $\{h_j\}$ are equivalent (5). Finally, it should be noted that the behaviour of μ is critical only in a neighborhood of the origin.

Proof of Theorem 1 (6). To prove that $I_\mu(x)$ is finite for almost every $x \in P$, it suffices to show that $\int_P I_\mu(x)\,dx < \infty$.

Let $\Delta_n = (a_n, b_n)$ $(n = 1, 2, 3, \ldots)$ be the contiguous intervals and $|\Delta_n|$ their Lebesgue measure. Since D vanishes on P,

$$I_\mu(x) = \int_0^{2\pi} \frac{D(x+t)\,d\,\mu(t)}{t^2} = \int_{x+t \in Q} \frac{D(x+t)\,d\,\mu(t)}{t^2}$$

where $Q = \bigcup_{n=1}^{\infty} \Delta_n$ is the complement of P.

Thus

$$I_\mu(x) = \int_{t \in Q} \frac{D(t)\,d\,\mu(t-x)}{(t-x)^2}$$

$$= \sum_{n=1}^{\infty} \int_{\Delta_n} \frac{D(t)\,d\,\mu(t-x)}{(t-x)^2}.$$

So

$$\int_P I_\mu(x)\,d\,x = \sum_{n=1}^{\infty} \int_P d\,x \left\{ \int_{\Delta_n} \frac{D(t)\,d\,\mu_t(t-x)}{(t-x)^2} \right\}$$

Now

$$\int_P d\,x \int_{\Delta_n} \frac{D(t)\,d\,\mu_t(t-x)}{(t-x)^2} = \int_{\Delta_n} D(t)\,d\,t \int_P \frac{d\,\mu_x(t-x)}{(t-x)^2}$$

(5) This has already been remarked by Fine in [1].

(6) See footnote (4).

using Fubini's theorem (*). Since $x \in P$ and $t \in \Delta_n \subset Q = P'$, $|t - x| \geq D(t)$ for each fixed t and all $x \in P$.

$$\therefore \int_{\Delta_n} D(t) \left\{ \int_P \frac{d\,\mu_x(t-x)}{(t-x)^2} \right\} d\,t \leq \int_{\Delta_n} D(t) \left\{ \int_{D(t)}^{2\pi} \frac{d\,\mu(s)}{s^2} \right\} d\,t$$

and this last integral, by Lemma 1, is

$$\leq \int_{\Delta_n} D(t) \cdot \frac{A}{D(t)}\, d\,t = A\,|\,\Delta_n\,|\,.$$

$$\therefore \int_P I_\mu(x)\, d\,x \leq \sum_{n=1}^{\infty} A\,|\,\Delta_n\,| = A\,|\,Q\,| < \infty,$$

which concludes the proof.

In the same way, we can prove the following theorem which is due to Marcinkiewicz in the case of Lebesgue measure.

THEOREM 2. *With the notation of Theorem 1, let*

$$I_\mu^\lambda(x) = \int_0^{2\pi} \frac{D^\lambda(x+t)}{t^{\lambda+1}}\, d\,\mu(t) \text{ for any } \lambda > 0\,;$$

then $I_\mu^\lambda(x)$ is finite for almost every $x \in P$. The following is an interesting special case of Theorem 2.

COROLLARY 1: *If $\lambda > 0$,*

$$\sum_1^{\infty} n^{\lambda-1} D\left(x + \frac{1}{n^\lambda}\right) < \infty$$

for almost every $x \in P$.

Proof: We take for μ the discrete measure which assigns to the point $\frac{1}{n^\lambda}$, the mass $\frac{1}{n^{\lambda+1}}$, $(n = 1, 2, 3, \ldots)$ and is otherwise zero. It is simple to check that this measure satisfies the condition of Lemma 1.

In particular if $\lambda = 1$, we see that $\sum_{n=1}^{\infty} D\left(x + \frac{1}{n}\right) < \infty$ for almost every $x \in P$.

(*) The above interchange of order of integration does not follow immediately from the standard version of Fubini's theorem. However, since the integrals are positive, the interchange can be justified by an appropriate limiting argument. Since the argument is standard and the details somewhat lengthy, we omit the justification.

3. Integrals of Marcinkiewicz and Hilbert Transforms.

In this section it is our aim to apply the results of the previous section to some generalizations of an integral of Marcinkiewicz of Dini type and of the Hilbert transform.

We shall assume as before that μ is a positive measure on $[0, 2\pi]$ satisfying the condition

$$\frac{1}{\tau}\int_0^\tau d\,\mu(t) \leq A\,, \quad (0 < \tau < 2\pi)\,. \tag{3.1}$$

Our results are contained in the following two theorems.

THEOREM 3. *Suppose that $F \in L^2([0, 2\pi])$ and is extended periodically. Let μ be a measure satisfying* (3.1). *If F' exists in a set E of positive measure, the integral*

$$\int_0^\pi \frac{[F(x+t) + F(x-t) - 2\,F(x)]^2}{t^3}\, d\,\mu(t) \tag{3.2}$$

is finite almost everywhere in E.

THEOREM 4. *Suppose that $F \varepsilon L([0, 2\pi])$ and is again extended periodically and that the measure μ satisfies* (3.1). *If F' exists in a set E of positive measure, then*

$$\lim_{\varepsilon\to 0}\int_\varepsilon^\pi \left\{\frac{F(x+t) + F(x-t) - 2F(x)}{t^2}\right\} d\,\mu(t) \tag{3.3}$$

exists (*and is finite*) *for almost every $x \in E$.*

Before proceding to the proofs of these theorems, we wish to discuss their background. When $\mu(t) = t$, Theorem 3 reduces to a result of Marcinkiewicz [4]. (7) The assumption that $F'(x)$ exists for $x \varepsilon E$ implies that $F(x+t) + F(x-t) - 2F(x) = o(t)$ for $x \in E$, but this estimate is not sufficient to ensure the convergence of (3.2). Theorem 3, however, does show that this estimate can be improved for almost every $x \in E$.

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(7) See also A. ZYGMUND, «*A Theorem on Generalized Derivatives*» Bull. of A. M. S. 49 (1943) pp. 917-923, esp. p. 919.

Theorem 4 originates in a theorem of Plessner [7] for which $\frac{d\,\mu(t)}{t^2} = \frac{d\,t}{\left(\sin\frac{t}{2}\right)^2}$. In this case, if F is the integral of a function $f \in L^1$, then (3.3) reduces, after integration by parts, to the conjugate function of f. Plessner proved his result by complex variable methods which are of course unavailable in our context. In [3] Marcinkiewicz gave a real-variable proof of Plessner's theorem. One last remark before we pass to the proofs. The existence of the integral (3.3) is more subtle than that of (3.2); for in general (3.3) converges only non-absolutely. In fact, Marcinkiewicz [4] has shown (in the case $\mu(t) = t$) that there exists an $F(x)$ with integrable derivative, for which $\int_0^\pi \frac{|F(x+t) + F(x-t) - 2F(x)|}{t^2}\,d\,t = \infty$ for almost every x. The non-absolute convergence of (3.3) is likely to be the general situation, but we shall not pursue the matter here.

The proofs of the theorems will be split into a number of lemmas. Basic to both proofs is the following decomposition due essentially to Marcinkiewicz [2].

Lemma 2. *Let F be integrable on $[0, 2\pi]$ and extended periodically. Suppose that $F'(x)$ exists at each point of a set $E \subset [0, 2\pi]$ of positive measure. Then given any $\eta > 0$, there exists a closed set P, $P \subset E$, with $|E - P| < \eta$, and a decomposition of F*

$$F(x) = G(x) + B(x) \qquad x \in [0, 2\pi]$$

with the following properties:

1) $G(x)$ has a continuous derivative on $[0, 2\pi]$.

2) $G(x) = F(x)$ for $x \in P$.

3) There is a $\delta > 0$, such that for every $x \in P$ and all t, $|t| < \delta$, $|B(x+t)| \leq AD(x+t)$; D being the distance function relative to P.

Proof. Since F' is measurable, given $\eta > 0$, there is a closed subset P_1 of E with $|E - P_1| < \eta/2$ and on which F' is continuous (by Lusin's theorem). By Egoroff's theorem, there is another closed set $P \subset P_1 \subset E$, $|E - P| < \eta$, such that $\lim_{t\to 0} \frac{F(x+t) - F(x)}{t} = F'(x)$ uniformly for $x \in P$. On P, $|F'| \leq A_\eta$, and since the approach to F' is uniform, there is a $\delta > 0$ such that $|F(x+t) - F(x)| \leq B_\eta |t|$ for all $x \in P$ as soon as $|t| < \delta$.

Let g_1 be a function, continuous on $[0, 2\pi]$ and periodic which coincides with F' on P and let G_1 be the integral of g_1. The function $B_1 = F - G_1$ satisfies uniformly in P the condition

$$\lim_{t \to 0} \frac{B_1(x+t) - B_1(x)}{t} = 0.$$

This implies that $|B_1(x+t) - B_1(x)| = o(|t|)$ for all $x \in P$ as soon as $|t| < \delta$. Thus for all those segments Δ_i of the set Q complementary to P, for which $|\Delta_i| < \delta$ (so for all but a finite number)

$$\max_{x,y \in \Delta_i} |B_1(x) - B_1(y)| = o(\delta_i), \quad \delta_i = |\Delta_i|. \tag{3.4}$$

We now define a function R in the following manner. Choose a polynomial ω satisfying $\omega(0) = 0$, $\omega(1) = 1$, and $\omega'(0) = \omega'(1) = 0$. Both ω and ω' are bounded on $[0, 1]$. If a and b, $a < b$, are the extremities of P in $[0, 2\pi]$, R is defined by the following conditions:

(1) $R(x) = B_1(x)$ for $x \in P$;

(2) if $\Delta_i = (x_i, x_i + \delta_i)$ is a segment contiguous to P,

$$R(x) = B_1(x_i) + [B_1(x_i + \delta_i) - B_1(x_i)]\,\omega\left(\frac{x - x_i}{\delta_i}\right), \quad x_i \leq x \leq x_i + \delta_i;$$

(3) $R(x) = B_1(a)$ $x \leq a$,

$R(x) = B_1(b)$ $x \geq b$.

Clearly $R(x)$ is continuous. On P, $R'(x) = 0$, and since for those intervals Δ_i contiguous to P,

$$\max_{x \in \Delta_i} |R'(x)| \leq A\,|B_1(x_i + \delta_i) - B_1(x_i)|\,\delta_i^{-1} \to 0 \text{ as } \delta_i \to 0$$

by (3.4)), R' is seen to be continuous.

The function $B = B_1 - R$ vanishes on P; moreover, for $x \in \Delta_i$,

$$\begin{aligned} |B(x)| &\leq |B_1(x) - B_1(x_i)| + |R(x) - R(x_i)| \\ &\leq A\,|x - x_i| \text{ as long as } x - x_i < \delta. \end{aligned}$$

With a similar estimate with respect to the other endpoint of Δ_i, we conclude

$|B(x+t)| \leq A\, D(x+t)$ for every $x \in P$ and all t that such that $|t| < \delta$.

Since $F = G_1 + B_1 = G_1 + R + B$, in setting $G = G_1 + R$ we obtain the desired decomposition.

LEMMA 3: *Suppose* $F(x) = \int_0^x f(t)\, dt$ *where* $f \in L^2([0, 2\pi])$. *Let* $M(x)$ *be defined by*

$$M^2(x) = \int_0^\pi \frac{|F(x+t) + F(x-t) - 2F(x)|^2}{t^3}\, d\mu(t).$$

Then $M(x) < \infty$ *for almost every point of* $[0, 2\pi]$.

Proof: It is enough to show that $\int_0^{2\pi} M^2(x)\, dx < \infty$. Let $f(x) \backsim \Sigma a_n e^{inx}$ be the Fourier expansion of f with $\Sigma |a_n|^2 < \infty$ (We suppose that $a_0 = 0$). Then $F(x+t) + F(x-t) - 2F(x) \backsim 4i \Sigma a_n \dfrac{\sin^2 \frac{nt}{2}}{n} e^{inx}$. In applying Fubini's theorem and Parseval's relation, we obtain

$$\int_0^{2\pi} M^2(x)\, dx = \int_0^\pi \left(\int_0^{2\pi} |F(x+t) + F(x-t) - 2F(x)|^2\, dx \right) \frac{d\mu(t)}{t^3}$$

$$= 32\pi \int_0^\pi \Sigma |a_n|^2 \frac{\sin^4 \frac{nt}{2}}{n^2} \frac{d\mu(t)}{t^3}$$

$$= 32\pi\, \Sigma |a_n|^2 \int_0^\pi \frac{\sin^4 \frac{nt}{2}}{n^2 t^3}\, d\mu(t).$$

We conclude the proof by showing that

$$\int_0^\pi \frac{\sin^4 \frac{nt}{2}}{n^2 t^3}\, d\mu(t) \leq A \text{ (independent of } n). \tag{3.5}$$

We split the integral as indicated:

$$\int_0^\pi = \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^\pi = J_1 + J_2 \text{ say}.$$

$$J_1 = \int_0^{\frac{1}{n}} \frac{\sin^4 \frac{n t}{2}}{n^2 t^3} \, d\mu(t) \leq 2^{-4} \int_0^{\frac{1}{n}} \frac{n^4 t^4}{n^2 t^3} \, d\mu(t)$$

$$\leq A n^2 \int_0^{\frac{1}{n}} t \, d\mu(t) \leq A \text{ (by lemma 1 } (b)) .$$

Similarly $J_2 \leq \frac{1}{n^2} \int_{\frac{1}{n}}^{\pi} \frac{d\mu(t)}{t^3} \leq A$, which establishes (3.5).

Thus $\int_0^{2\pi} M^2(x) \, dx \leq A \, \Sigma \, |a_n|^2 = A \|f\|_2^2 < \infty$.

Proof of Theorem 3: Given any $\eta > 0$, it is enough to show that there is a closed set $P \subset E$, $|E - P| < \eta$, for which the integral (3.2) is finite for almost every $x \in P$. Fixing η, we choose P to be that set given by Lemma 2. Thus $|E - P| < \eta$ and F decomposes as the sum $F(x) = G(x) + B(x)$, where G has a continuous derivative on $[0, 2\pi]$ and $|B(x+t)| \leq A \, D(x+t)$ whenever $x \in P$ and $|t| < \delta$. Since $F = G + B$,

$$\int_0^{\pi} \frac{|F(x+t) + F(x-t) - 2F(x)|^2}{t^3} \, d\mu(t) \leq$$

$$2 \int_0^{\pi} \frac{|G(x+t) + G(x-t) - 2G(x)|^2}{t^3} \, d\mu(t)$$

$$+ 2 \int_0^{\pi} \frac{|B(x+t) + B(x-t) - 2B(x)|^2}{t^3} \, d\mu(t) .$$

$$= I_1 + I_2 \text{, (say)} .$$

I_1 is finite for almost every $x \in [0, 2\pi]$ by Lemma 3.

Since $B(x) = 0$ for $x \in P$, $I_2 \leq 4 \int_0^{\pi} \frac{|B(x+t)|^2}{t^3} \, d\mu(t) + 4 \int_0^{\pi} \frac{|B(x-t)|^2}{t^3} \, d\mu(t)$.

Now $\int_0^\pi \frac{|B(x+t)|^2}{t^3}\, d\mu(t) = \int_0^\delta \frac{|B(x+t)|^2}{t^3}\, d\mu(t) + \int_\delta^\pi \frac{|B(x+t)|^2}{t^3}\, d\mu(t)$. Moreover, $\int_0^\delta \frac{|B(x+t)|^2}{t^3}\, d\mu(t) \leq A^2 \int_0^\delta \frac{D^2(x+t)}{t^3}\, d\mu(t) \leq A^2 \int_0^\pi \frac{D^2(x+t)}{t^3}\, d\mu(t) =$

$= A^2 I_\mu^2(x)$ which is finite for almost every x by Theorem 2.

Finally $\int_\delta^\pi \frac{|B(x+t)|^2}{t^3}\, d\mu(t)$ is finite for almost every x since $B \in L^2$ (being the difference of two functions, each of which is in L^2) and $\frac{d\mu(t)}{t^3}$ is of total bounded variation in $[\delta, \pi]$.

Similarly $\int_0^\pi \frac{|B(x-t)|^2}{t^3}\, d\mu(t) < \infty$ for almost every $x \in P$. Thus I_2 is finite for almost every point of P, which concludes the proof.

4. Proof of Theorem 4:

LEMMA 4: *Suppose that* $F(x) = \int_0^x f(t)\, dt$ *where* $f \in L^2(0, 2\pi)$. *We define* $H_\varepsilon(f)$ *by*

$$H_\varepsilon(f)(x) = \int_\varepsilon^\pi \left\{ \frac{F(x+t) + F(x-t) - 2F(x)}{t^2} \right\} d\mu(t) \tag{4.1}$$

Then

(i) $\| H_\varepsilon(f) \|_2 \leq A \| f \|_2$, A independent of ε and f,

(ii) $H_\varepsilon f$ converges in the L_2 norm, as $\varepsilon \to 0$, to a limit which we denote by Hf,

(iii) $\| H(f) \|_2 \leq A \| f \|_2$

Proof: Let $f(x) \backsim \Sigma\, a_n e^{inx}$ be its Fourier development.

Thus (assuming as always that $a_0 = 0$) $F(x+t) + F(x-t) - 2F(x)$ $= 4 i \Sigma\, a_n \frac{\sin^2 \frac{nt}{2}}{n} e^{inx}$; the series converging absolutely.

Therefore $H_\varepsilon(f) = 4i \, \Sigma \, a_n \left(\int\limits_\varepsilon^\pi \frac{\sin^2 \frac{n\,t}{2}}{n\,t^2} \, d\,\mu(t) \right) e^{inx}$.

To establish (i) it suffices to show that

$$\left| \int\limits_\varepsilon^\pi \frac{\sin^2 \frac{n\,t}{2}}{n\,t^2} \, d\,\mu(t) \right| \leq A \text{ , } A \text{ independent of } n \text{ and } \varepsilon \, . \tag{4.2}$$

The integral (4.2) is, however, dominated by

$$\int\limits_0^\pi \frac{\sin^2 \frac{n\,t}{2}}{n\,t^2} \, d\,\mu(t) \tag{4.3}$$

and this last integral is dominated by

$$2^{-2} \int\limits_0^{\frac{1}{n}} \frac{n^2\,t^2}{n\,t^2} \, d\,\mu(t) + \int\limits_{\frac{1}{n}}^\pi \frac{d\,\mu(t)}{n\,t^2} \, .$$

Each of these two integrals is $\leq A$ by Lemma 1 and this proves (i).

Ad (ii). If f is smooth enough, say $C^{(1)}$, it is easy to check that $H_\varepsilon(f)$ converges uniformly and thus in L_2 as $\varepsilon \to 0$. In splitting $f = g + h$, $g \in C^{(1)}$ and $\| h \|_2$ arbitrarily small, and using (i), we conclude that $H_\varepsilon(f)$, $f \in L_2$, is Cauchy; hence converges in L_2 to a limit.

Ad (iii). This is an immediate consequence of (i) and (ii).

The following Lemma is a very well known theorem of Hardy and Littlewood [8]

Lemma 5: *Let* $f \in L_p([0, 2\pi])$; $1 < p$. *Define* f^* *by*

$$f^*(x) = \sup_{|h|>0} \frac{1}{h} \int\limits_x^{x+h} |f(t)| \, dt \, . \tag{4.4}$$

Then $f^* \in L_p$ and $\| f^* \|_p \leq A_p \| f \|_p$.

Lemma 6: *Suppose* f *is bounded on* $[0, 2\pi]$ *and let* $M = \text{ess}_x \sup |f(x)|$, *Define* $\overline{H}(f)$ *by*

$$\overline{H}(f)(x) = \sup_\varepsilon | H_\varepsilon(f)(x) | \tag{4.5}$$

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(8) Cfr. Zygmund [9], page 244.

where $H_\varepsilon(f)$ *is defined by* (4.1); F *being the integral of* f. *Then* $\overline{H}(f) \in L_2$ *and* $\| \overline{H}(f) \|_2 \leq AM$. [9]

Proof: We first regularize f by convoluting it with an *approximation to the identity*: more precisely, choose $\varphi(t) \geq 0$ to have the following properties: (1) $\varphi(t)$ is indefinitely differentiable, (2) $\varphi(t) = 0$ outside $[-1, 1]$, (3) $\int_{-\pi}^{\pi} \varphi(t)\, dt = 1$. Define $\{\varphi_\varepsilon\}$, $\varepsilon > 0$, by

$$\varphi_\varepsilon(t) = 1/\varepsilon\, \varphi\left(\frac{t}{\varepsilon}\right). \tag{4.6}$$

We define then f_ε by

$$f_\varepsilon(x) = f * \varphi_\varepsilon(x) = \int_{-\pi}^{+\pi} f(x-t)\, \varphi_\varepsilon(t)\, dt. \tag{4.7}$$

Assuming, without loss of generality, that $\int_{-\pi}^{+\pi} f(t)\, dt = 0$, so that F is periodic, we define F_ε by $F_\varepsilon(x) = \int_0^x f_\varepsilon(t)\, dt$. We note that f_ε and thus F_ε are infinitely differentiable.

We observe first that $F_\varepsilon = F * \varphi_\varepsilon$; next that for each $\eta > 0$

$$H_\eta(f_\varepsilon) = H_\eta(f) * \varphi_\varepsilon, \tag{4.8}$$

for the operator H_η is essentially a convolution type operator. Now let $\eta \to 0$ (ε fixed). Since f_ε is smooth, $H_\eta(f_\varepsilon) \to H(f_\varepsilon)$ uniformly; but by Lemma 4, $H_\eta(f)$ converges in L_2 to $H(f)$. Thus $H_\eta(f) * \varphi_\varepsilon$ converges uniformly to $H(f) * \varphi_\varepsilon$.

We have obtained

$$H(f_\varepsilon) = H(f) * \varphi_\varepsilon \tag{4.9}$$

for every x.

We next estimate the difference $H_\varepsilon(f) - H(f_\varepsilon)$. Let us introduce the notation $\Delta_t F$ for $F(x+t) + F(x-t) - 2F(x)$. Then

$$\Delta_t F_\varepsilon = F_\varepsilon(x+t) + F_\varepsilon(x-t) - 2F_\varepsilon(x), \text{ so}$$

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(9) This « maximal » theorem is true if $f \in L_2$, in which case M is replaced by $\| f \|_2$. Actually we need it here only for f continuous.

$$H_\varepsilon(f) - H(f_\varepsilon) = \int_\varepsilon^\pi \frac{\Delta_t F}{t^2}\, d\mu(t) - \int_0^\pi \frac{\Delta_t F_\varepsilon}{t^2}\, d\mu(t) =$$

$$(4.10) \qquad = -\int_0^\varepsilon \frac{\Delta_t F_\varepsilon}{t^2}\, d\mu(t) + \int_\varepsilon^\pi \frac{\Delta_t F - \Delta_t F_\varepsilon}{t^2}\, d\mu(t).$$

Since $F_\varepsilon(x) = \int_{-\pi}^{\pi} F(x-s)\,\varphi_\varepsilon(s)\,ds = \int_{-\pi}^{\pi} \varphi_\varepsilon(x-s)\,F(s)\,ds$, and since $\int_{-\pi}^{\pi} \varphi_\varepsilon(s)\,ds = 1$,

$$\Delta_t F_\varepsilon(x) = \int_{-\pi}^{\pi} [F(x+t-s) + F(x-t-s) - 2F(x-s)]\,\varphi_\varepsilon(s)\,ds$$

$$= \int_{-\pi}^{\pi} \left[F\left(x+\frac{t}{2}-s\right) - F\left(x-\frac{t}{2}-s\right)\right]\left[\varphi_\varepsilon\left(s+\frac{t}{2}\right) - \varphi_\varepsilon\left(s-\frac{t}{2}\right)\right] ds.$$

Now $\left| F\left(x+\frac{t}{2}-s\right) - F\left(x-\frac{t}{2}-s\right)\right| \le |t| \max |f| \le |t|\, M$, hence

$$|\Delta_t F_\varepsilon(x)| \le \int_{-\pi}^{\pi} |t|\, M \left| \varphi_\varepsilon\left(s+\frac{t}{2}\right) - \varphi_\varepsilon\left(s-\frac{t}{2}\right)\right| ds.$$

For $|t| \le \varepsilon$, $\varphi_\varepsilon\left(s+\frac{t}{2}\right) - \varphi_\varepsilon\left(s-\frac{t}{2}\right)$ vanishes outside $[-3\varepsilon/2,\ 3\varepsilon/2]$; therefore, if $|t| \le \varepsilon$,

$$|\Delta_t F_\varepsilon(x)| \le |t|\, M \int_{-3\varepsilon/2}^{3\varepsilon/2} \left| \varphi_\varepsilon\left(s+\frac{t}{2}\right) - \varphi_\varepsilon\left(s-\frac{t}{2}\right)\right| ds,$$

$$\le t^2 M \int_{-3\varepsilon/2}^{3\varepsilon/2} \sup_s |\varphi'_\varepsilon|\, ds.$$

Since $\varphi_\varepsilon(s) = \frac{1}{\varepsilon}\,\varphi\left(\frac{s}{\varepsilon}\right)$, $\varphi'_\varepsilon(s) = \frac{1}{\varepsilon^2}\varphi'\left(\frac{s}{\varepsilon}\right)$; thus $\sup|\varphi'_\varepsilon| \leq \frac{A}{\varepsilon^2}$. We finally obtain

$$|\Delta_t F_\varepsilon(x)| \leq A\, t^2/\varepsilon^2 M \int_{-3\varepsilon/2}^{3\varepsilon/2} d\,s \leq A\, t^2/\varepsilon\, M\,, \tag{4.11}$$

for $|t| \leq \varepsilon$.

To estimate $\Delta_t F - \Delta_t F_\varepsilon$, we estimate terms of the form $F(x) - F_\varepsilon(x)$ and $F(x \pm t) - F_\varepsilon(x \pm t)$.

$$|F(x) - F_\varepsilon(x)| = \left|\int_{-\pi}^{\pi} \varphi_\varepsilon(s)\,[F(x) - F(x-s)]\,d\,s\right|$$

$$\leq \int_{-\pi}^{\pi} \varphi_\varepsilon(s)\,|F(x) - F(x-s)|\,d\,s$$

$$\leq \int_{-\pi}^{\pi} \varphi_\varepsilon(s)\,|s|\,M\,d\,s = M\int_{-\varepsilon}^{\varepsilon} |s|\,\varphi_\varepsilon(s)\,d\,s$$

$$\leq \varepsilon\,M \int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(s)\,d\,s = A\,\varepsilon\,M$$

Similarly $|F(x \pm t) - F_\varepsilon(x \pm t)| \leq A\,\varepsilon\,M$. Our final estimate is thus

$$|\Delta_t F - \Delta_t F_\varepsilon| \leq A\,\varepsilon\,M\,. \tag{4.12}$$

Using the estimates (4.11) and (4.12), we obtain

$$|H_\varepsilon(f) - H(f_\varepsilon)| \leq \int_0^{\varepsilon} \frac{|\Delta_t F|}{t^2}\,d\,\mu(t) + \int_{\varepsilon}^{\pi} \frac{|\Delta_t F - \Delta_t F_\varepsilon|}{t^2}\,d\,\mu(t)$$

$$\leq (A/\varepsilon)\,M\int_0^{\varepsilon} d\,\mu(t) + A\,\varepsilon\,M\int_{\varepsilon}^{\pi} \frac{d\,\mu(t)}{t^2}$$

$$\leq A\,M\,.$$

Thus $|H_\varepsilon(f)| \leq |H(f_\varepsilon)| + AM$. By (4.9), $H(f_\varepsilon) = H(f) * \varphi_\varepsilon$; since $|H(f)*\varphi_\varepsilon| \leq A(H(f))^*$ [10], we finally obtain $|H_\varepsilon(f)| \leq AM + A(H(f))^*$. Thus

$$\bar{H}(f) = \sup_\varepsilon |H_\varepsilon(f)| \leq AM + A(H(f))^*. \tag{4.13}$$

By lemmas 4 and 5, $\bar{H}(f) \varepsilon L_2$ and $\|\bar{H}(f)\|_2 \leq AM + A\|Hf\|_2 \leq AM + A\|f\|_2 \leq AM$, which concludes the proof of the lemma.

LEMMA 7: *If f is continuous on $[0, 2\pi]$, then $H_\varepsilon(f) \to H(f)$ at almost every point.*

Proof: It is enough to show that $H_\varepsilon(f)$ converges pointwise almost everywhere for we know from Lemma 4 that it converges to $H(f)$ in the L_2 sense. To do this, we prove: Given $\eta > 0$, $\delta > 0$, there is an $\varepsilon > 0$ and a fixed set F of measure less than η, so that if $\varepsilon_1, \varepsilon_2 < \varepsilon$, then $|H_{\varepsilon_1}(f) - H_{\varepsilon_2}(f)| < \delta$, except possibly in F.

Let K be a fixed constant guaranteed by lemma 6 so that $\int |\bar{H}(f)|^2 dx \leq K^2 (\sup |f|)^2$ for all bounded f. Decompose f as $f = f_1 + f_2$ where f_1 is smooth and $\sup |f_2| < \delta_1 = \frac{\delta}{3K}\sqrt{\eta}$ [11]. By linearity $H_\varepsilon(f) = H_\varepsilon(f_1) + H_\varepsilon(f_2)$; consequently $|H_{\varepsilon_1}(f) - H_{\varepsilon_2}(f)| \leq |H_{\varepsilon_1}(f_1) - H_{\varepsilon_2}(f_1)| + |H_{\varepsilon_1}(f_2)| + |H_{\varepsilon_2}(f_2)|$. f_1 being smooth, $H_\varepsilon(f_1) \to H(f_1)$ uniformly; choose ε so that if $\varepsilon_1, \varepsilon_2 < \varepsilon$, then $|H_{\varepsilon_1}(f_1) - H_{\varepsilon_2}(f_1)| < \delta/3$ everywhere.

In addition, $|H_{\varepsilon_1}(f_2)| \leq \bar{H}(f_2)$; since $\int |\bar{H}(f_2)|^2 dx \leq K^2 (\sup |f_2|)^2 \leq K^2 \delta_1^2 = \frac{\delta^2 \eta}{9}$, $|\bar{H}(f_2)|$ can be $> \delta/3$ only on a set F of measure $< \eta$. Thus $|H_{\varepsilon_1}(f_2)| + |H_{\varepsilon_2}(f_2)| \leq 2\bar{H}(f_2)$ can be $> \frac{2}{3}\delta$ only on F.

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(10) For any g,

$$|g * \varphi_\varepsilon(x)| = \left| \int_{-\pi}^{\pi} g(x-s)\varphi_\varepsilon(s)\,ds \right|$$

$$= \left| \int_{-\varepsilon}^{\varepsilon} g(x-s)\varphi_\varepsilon(s)\,ds \right| \leq \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} |g(x-s)|\,\varphi(s/\varepsilon)\,ds \leq$$

$$\leq A/\varepsilon \int_{-\varepsilon}^{\varepsilon} |g(x-s)|\,ds \leq A\,g^*(x).$$

(11) Here we need the fact that f is continuous so that we can approximate uniformly, i. e. in the sup norm, by smooth functions.

Therefore $| H_{\varepsilon_1}(f) - H_{\varepsilon_2}(f) | < \delta$ except possibly on F as long as $\varepsilon_1, \varepsilon_2 < \varepsilon$. This concludes the proof of the lemma since η and δ are arbitrary.

Conclusion of the proof of Theorem 4. It suffices to show that given an $\eta > 0$, there is a closed set $P \subset E$, $| E - P | < \eta$, so that the limit (3.3) exists (and is finite) for almost every $x \varepsilon P$. Fixing η, we choose P to be that set given by Lemma 2. Thus $| E - P | < \eta$ and F decomposes into the sum $F = G + B$ where G has a continuous derivative on $[0, 2\pi]$ and $| B(x+t) | \leq A | D(x+t) |$ for each $x \varepsilon P$ and all $t, |t| < \delta$. Thus

$$\int_{\varepsilon}^{\pi} \frac{F(x+t) + F(x-t) - 2F(x)}{t^2} d\mu(t)$$

$$= \int_{\varepsilon}^{\pi} \frac{G(x+t) + G(x-t) - 2G(x)}{t^2} d\mu(t) +$$

$$+ \int_{\varepsilon}^{\pi} \frac{B(x+t) + B(x-t) - 2B(x)}{t^2} d\mu(t) = I_{\varepsilon}^{(1)} + I_{\varepsilon}^{(2)} \text{ say}.$$

By Lemma 7, $\lim_{\varepsilon \to 0} I_{\varepsilon}^{(1)}$ exists (and is finite) for almost every point of $[0, 2\pi]$. We conclude the proof in showing that $\lim_{\varepsilon \to 0} I_{\varepsilon}^{(2)}$ exists (and is finite) for almost every $x \varepsilon P$.

Since $B(x) = 0$ for $x \varepsilon P$, $I_{\varepsilon}^{(2)} = \int_{\varepsilon}^{\pi} \frac{B(x+t) + B(x-t)}{t^2} d\mu(t)$.

It is thus sufficient to prove that the integrals

$$\int_0^{\pi} \frac{| B(x \pm t) |}{t^2} d\mu(t)$$

are finite almost everywhere in P. Since $| B(x+t) | \leq A | D(x+t) |$ for each $x \varepsilon P$ and $| t | < \delta$, these integrals are dominated near the origin ($|t| < d$) by $A \int_0^{\pi} \frac{D(x+t)}{t^2} d\mu(t)$ which, by Theorem 1, is finite for almost every $x \varepsilon P$.

Since $B \varepsilon L^1$ (being the difference of two functions in L^1), $\int_{\delta}^{\pi} \frac{| B(x+t) |}{t^2} d\mu(t)$ certainly exists for each x. Thus (3.3) exists almost everywhere in P and hence, almost everywhere in E.

5. Concluding Remarks.

As a result of Theorem 4, we can make the following observation: we have already remarked that Marcinkiewicz has given an example of an absolutely continuous function F for which the integral $\int_0^\pi \frac{|F(x+t)+F(x-t)-2F(x)|}{t^2}\,dt = \infty$ for almost every x. None-the-less, for every partition of the interval $[0, \pi]$ into disjoint intervals Δ_i and for every random choice of ± 1's, we can assert that

$$\sum_{i=1}^{\infty} \pm \int_{\Delta_i} \frac{F(x+t)+F(x-t)-2F(x)}{t^2}\,dt < \infty \text{ for almost every } x \,\varepsilon\, (0, 2\pi).$$

For if we define μ_1 to be the measure equal to Lebesgue measure on the intervals with $+1$ attached to them and 0 otherwise and if μ_2 is similarly defined with respect to the intervals with -1 attached to them, then

$$\Sigma \pm \int_{\Delta_i} \frac{F(x+t)+F(x-t)-2F(x)}{t^2}\,dt =$$

$$\int_0^\pi \frac{F(x+t)+F(x-t)-2F(x)}{t^2}\,d\mu_1(t) -$$

$$\int_0^\pi \frac{F(x+t)+F(x-t)-2F(x)}{t^2}\,d\mu_2(t).$$

Both μ_1 and μ_2 satisfy $\frac{1}{\tau}\int_0^\tau d\mu_i(t) \leq \frac{1}{\tau}\int_0^\tau dt \leq 1$, $(i = 1, 2)$, so by Theorem 4, each of these two last integrals is finite almost everywhere and thus their difference is finite almost everywhere.

In Lemma 4, we considered the $\{H_\varepsilon\}$ as a family of (linear) operators on L_2. We can also consider them as operators on L_p, $1 \leq p < \infty$. In the classical case of the Hilbert transform, M. Riesz has shown that if $f \,\varepsilon\, L_p$, $H(f) = \lim_{\varepsilon \to 0} H_\varepsilon(f)$ is again in L_p and $\| H(f) \|_p \leq A_p \| f \|_p$ for $1 < p < \infty$. There is a substitute result for L_1 for even though $H(f)$ exists almost everywhere in this case, it may not be integrable. Moreover, A. Zygmund has shown that in this case the maximal operator $\overline{H}$ of Lemma 6 is a bounded operator from L_p to L_p for $1 < p < \infty$. The analagous theorems and related questions concerning the more general transforms of this paper will be considered in a future paper.

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