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# SOME FORMULAS IN A RIEMANNIAN SPACE

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In this paper a number of theorems and formulas involving two arbitrary affine connections in a Riemannian space $V_n$ have been established by imposing certain conditions on the affine connections. In section I it has been assumed that the covariant derivatives of the metric tensor of the $V_n$ with respect to the affine connections are the same while in section 2 the torsions of the affine connections have been taken to be the same.

1. Let $\Gamma^i_{jk}$ and $L^i_{jk}$ be the coefficients of two arbitrary affine connections in a Riemannian space $V_n$ with metric tensor $g_{ij}$ and let a comma and a semicolon denote the covariant derivatives of the $g_{ij}$'s with respect to the two connections. Then

$$g_{ij,k} - g_{ij;k} = - g_{is} (\Gamma^s_{jk} - L^s_{jk}) - g_{js} (\Gamma^s_{ik} - L^s_{ik})$$

Putting $T^i_{jk} = \Gamma^i_{jk} \sim L^i_{jk}$, it follows that

$$g_{ij,k} = g_{ij;k}$$

if and only if

$$g_{is} T^s_{jk} + g_{js} T^s_{ik} = 0 \tag{I. 1}$$

Hence we have the following theorem:

THEOREM 1. The covariant derivatives of the $g_{ij}$'s with respect to two affine connections with coefficients $\Gamma^i_{jk}$ and $\Gamma^i_{jk} + T^i_{jk}$ are the same if and only if the tensor $T^i_{jk}$ satisfies (I. 1).

As an example it is easy to verify that the above result holds with respect to the coefficients of affine connections

$$\Gamma^t_{jk} \qquad \text{and} \qquad \Gamma^t_{jk} \pm g^{st} (g_{jk,s} - g_{ks,j})$$

This example is an application of Sen's sequence (3) which is defined as follows:

$$\text{Put } a = \Gamma_{ij}^{t}, \ a^* = \Gamma_{ij}^{t} + g^{mt} g_{im,j}, \qquad a' = \Gamma_{ji}^{t}$$

Then it is known that for every affine connection $a$ there exist uniquely two others $a^*$ and $a'$ which are respectively called the associate and the conjugate of $a$ having the property

$$a^{**} = a'' = a$$

In particular, $a$ is self-associate if $a = a^*$ and self-conjugate if $a = a'$. Now if we construct the sequence

$$\text{(I. 2)} \qquad a_1 = a, a_2 = a^*, a_3 = a^{*\prime}, a_4 = a^{*\prime *}, a_5 = a^{*\prime *\prime}, \ldots\ldots$$

then the sequence is a finite cyclic sequence of twelve terms and it is Sen's sequence. In the sequence if we put

$$\alpha = g^{mt} g_{im,j}, \qquad \alpha_c = g^{mt} g_{jm,i}, \qquad \gamma = g^{mt} g_{ij,m} = \gamma_c$$

$$\beta = g^{mt} g_{is} (\Gamma_{mj}^{s} - \Gamma_{jm}^{s}), \qquad \beta_c = g^{mt} g_{js} (\Gamma_{mi}^{s} - \Gamma_{im}^{s})$$

and suppose that $a$ is self-conjugate, then we have

$$a_1 = a_{12} = a, \qquad a_2 = a_{11} = a' + \alpha, \qquad a_3 = a_{10} = a + \alpha_c$$

$$a_4 = a_9 = a + \alpha - \gamma, \qquad a_5 = a_8 = a + \alpha_c - \gamma, \qquad a_6 = a_7 = a + \alpha + \alpha_c - \gamma$$

It follows that

$$\text{(I. 3)} \qquad a_1 - a_5 = a_2 - a_6 = g^{st} (g_{jk,s} - g_{ks,j})$$

Further, if the covariant derivatives of the $g_{ij}$'s with respect to two affine connections are the same and if one of them is self-associate then the other is also self-associate, because the covariant derivatives of the $g_{ij}$'s must vanish. Now when $a_1 = a_2$ is self-associate, then $a_7 = a_8$ is also self-associate. Hence we have the following theorem:

THEOREM 2. If $a_1$ be coefficients of a self-conjugate affine connection, then the $g_{ij}$'s have the same covariant derivatives with respect to the pairs of affine connections $(a_1, a_5)$ and $(a_2, a_6)$ of Sen's sequence. And if $a_1$ be self-associate then the $g_{ij}$'s have the same covariant derivatives with respect to the pair $(a_1, a_7)$.

Further, let $\Gamma^t_{ij}$ and $\Gamma^t_{ij}+T^t_{ij}$ be the coefficients of two affine connections. Then their associates are

$$\Gamma^{*t}_{ij} \qquad \text{and } \Gamma^{*t}_{ij} - g^{mt} g_{is} T^s_{mj}$$

and their conjugates are

$$\Gamma^t_{ji} \qquad \text{and } \Gamma^t_{ji} + T^t_{ji}$$

It follows immediately from Theorem 1 that

THEOREM 3. If the $g_{ij}$'s have the same covariant derivatives with respect to two arbitrary affine connections, then the same is true with respect to their associates and conjugates.

Now, let $\Gamma^t_{ij}$ and $L^t_{ij}$ be the coefficients of two affine connections and

$$\Delta^t_{ij} = \frac{1}{2}(\Gamma^t_{ij} + L^t_{ij}), \quad T^t_{ij} = \Gamma^t_{ij} - L^t_{ij}$$

Also, let $\Gamma^t_{ijk}$, $L^{t'}_{ijk}$ and $\Delta^t_{ijk}$ denote curvature tensors formed with $\Gamma^t_{ij}$, $L^t_{ij}$ and $\Delta^t_{ij}$ respectively.

Then it is known that

$$\text{(I. 4)} \qquad \Delta^t_{ijk} - \frac{1}{2}(\Gamma^t_{ijk} + L^t_{ijk}) = \frac{1}{4}(T^t_{sk} T^s_{ij} - T^t_{sj} T^s_{ik})$$

Now if the $g_{ij}$'s have the same covariant derivatives with respect to $\Gamma^t_{ij}$ and $L^t_{ij}$ then by (I. 1)

$$T^t_{sk} T^s_{ij} = g^{mt} g_{sp} T^p_{mk} g^{ns} g_{iq} T^q_{nj} = g^{mt} g_{iq} T^n_{mk} T^q_{nj}$$

Therefore

$$g_{ht}(T^t_{sk} T^s_{ij} - T^t_{sj} T^s_{ik}) = g_{ht} g^{mt} g_{iq} (T^n_{mk} T^q_{nj} - T^n_{mj} T^q_{nk}) = g_{it}(T^t_{sj} T^s_{hk} - T^t_{sk} T^s_{hj})$$

Or

$$\Delta_{hijk} - \frac{1}{2}(\Gamma_{hijk} + L_{hijk}) = \frac{1}{4} g_{ht}(T^t_{sk} T^s_{ij} - T^t_{sj} T^s_{ik}) = \frac{1}{4} g_{it}(T^t_{sj} T^s_{hk} - T^t_{sk} T^s_{hj})$$

Hence

$$\text{(I. 5)} \qquad \Delta_{hijk} - \frac{1}{2}(\Gamma_{hijk} + L_{hijk}) = -\left[\Delta_{ihjk} - \frac{1}{2}(\Gamma_{ihjk} + L_{ihjk})\right]$$

Thus we have the following theorem:

THEOREM 4. If the $g_{ij}$'s have the same covariant derivatives with respect to $\Gamma^t_{ij}$ and $L^t_{ij}$ and if $\Delta^t_{ij} = \frac{1}{2}(\Gamma^t_{ij} + L^t_{ij})$, then the curvature tensors formed with them satisfy the relation (I. 5).

This result is easily verified in the case when $\Gamma^t_{ij}$ and therefore $L^t_{ij}$, $\Delta^t_{ij}$ are self-associate. For in this case the curvature tensors are skew in the first two indices (4).

As before, suppose that the covariant derivatives of the $g_{ij}$'s with respect to $\Gamma^t_{ij}$ and $L^t_{ij}$ are the same. Forming the second covariant derivatives it is seen that

(I. 6) $$g_{ij,kl} - g_{ij;kl} = -[g_{sj,k}\, T^s_{il} + g_{is,k}\, T^s_{jl} + g_{ij,s}\, T^s_{kl}]$$

where

$$\Gamma^t_{ij} - L^t_{ij} = T^t_{ij} = -g^{mt} g_{i\,\cdot}\, T^n_{mj}$$

Therefore

by (I. 6) $g^{ij}(g_{ij,kl} - g_{ij;kl}) = g^{ms} g_{sj,k}\, T^j_{ml} + g^{ms} g_{is,k}\, T^i_{ml} - g^{ij} g_{ij,s}\, T^s_{kl} =$

$= g^{ij}[g_{sj,k}\, T^s_{il} + g_{si,k}\, T^s_{jl} + g_{ij,s}\, T^s_{kl} - 2\, g_{ij,s}\, T^s_{kl}] = -g^{ij}(g_{ij,kl} - g_{ij;kl}) - 2\, g^{ij} g_{ij,s}\, T^s_{kl}$

Therefore

$$g^{ij}(g_{ij,kl} - g_{ij;kl}) = -g^{ij} g_{ij,s}\, T^s_{kl}$$

Interchanging $k$ and $l$ and subtracting

$$g^{ij}[(g_{ij,kl} - g_{ij,lk}) - (g_{ij;kl} - g_{ij;lk})] = g^{ij} g_{ij,s}(T^s_{lk} - T^s_{kl})$$

Finally using Ricci's identity

(I. 7) $$g^{ij}[(g_{it}\Gamma^t_{jkl} + g_{jt}\Gamma^t_{ikl}) - (g_{it} L^t_{jkl} + g_{jt} L^t_{ikl})] = g^{ij} g_{ij,s}[(\Gamma^s_{lk} - L^s_{lk}) - (\Gamma^s_{kl} - L^s_{kl})]$$

Let us further suppose that $\Gamma^t_{ij}$ and $L^t_{ij}$ are both self-associate or both self-conjugate. Then the right hand side of (I. 7) vanishes. We have therefore

$$g^{ij}[(\Gamma_{ijkl} + \Gamma_{jikl}) - (L_{ijkl} + L_{jikl})] = 0$$

whence

(I. 8) $$g^{ij}(\Gamma_{jikl} - L_{ijkl}) = 0$$

As said before, this result is obvious when both the affine connections are self-associate.

Hence we have the following theorem :

THEOREM 5. If the $g_{ij}$'s have the same covariant derivatives with respect to two self-conjugate affine connections with coefficients $\Gamma_{ij}^{t}$, $L_{ij}^{t}$ and if $\Gamma_{ijkl}$, $L_{ijkl}$ be the corresponding covariant curvature tensors, then (I. 8) holds.

2. The torsion of an affine connection with coefficients $\Gamma_{ij}^{t}$ is defined to be the tensor $\frac{1}{2}(\Gamma_{ij}^{t} - \Gamma_{ji}^{t})$ (2). It follows that two arbitrary affine connections with coefficients $\Gamma_{ij}^{t}$ and $\Gamma_{ij}^{t} + T_{ij}^{t}$ have the same torsion if and only if $T_{ij}^{t}$ is symmetric in $i$ and $j$. It is now easy to see that if two affine connections have the same torsion the same is true of their conjugates. E. g., in Sen's sequence each of the pairs $(a_1, a_6)$, $(a_2, a_9)$, $(a_4, a_{11})$ and therefore their conjugates $(a_{12}, a_7)$, $(a_3, a_8)$, $(a_5, a_{10})$ have the same torsion.

Again, let $a = \Gamma_{ij}^{t}$, $b = L_{ij}^{t}$ be the coefficients of two affine connections and $a - b = T_{ij}^{t}$. Their associates $a^*$ and $b^*$ will have the same torsion if the tensor

$$T_{ij}^{t} + g^{mt}(g_{im,j} - g_{im;j}) = - g^{mt} g_{is} T_{mj}^{s}$$

is symmetric in $i, j$ i. e., if

$$g_{is} T_{mj}^{s} = g_{js} T_{mi}^{s} \tag{2. 1}$$

Putting $g_{is} T_{mj}^{s} = T_{imj}$ we have the following theorem:

THEOREM 6. Let $a$ and $b$ have the same torsion; then their associates will also have the same torsion if the tensor $T_{ijk}$ is symmetric in all the indices.

Let $e_0 = (a\ ,b\ ) = \frac{1}{2}(a + b) + \quad (a - b)$, $\bar{e}_0 = (b\,, a) = \frac{1}{2}(a + b) + (b - a)$

$$e_1 = (e_0, \bar{e}_0) = \frac{1}{2}(a + b) + 2(a - b),\ \bar{e}_1 = \frac{1}{2}(a + b) + 2(b - a)$$

Similarly for $e_2$, $\bar{e}_2$ etc.
Then

$$e_r = \frac{1}{2}(a + b) + 2^r(a - b),\ \bar{e}_r = \frac{1}{2}(a + b) + 2^r(b - a)$$

Therefore

$$e_r - \bar{e}_r = 2^r(a - b) \tag{2. 2}$$

It follows that if $a$ and $b$ have the same torsion then the same is true of $e_r$ and $\bar{e}_r$.

Further, we have the following theorem:

THEOREM 7. If the associates of $a$ and $b$ have the same torsion, the same is true of the associates of $e_r$ and $\overline{e}_r$.

Let $\Gamma_{ij}^t$ and $L_{ij}^t$ have the same torsion. Then applying the condition that $T_{ij}^t$ is symmetric in $i, j$, we obtain from (I. 4) the cyclical property, namely

$$(2.\ 3)\quad \Delta_{ijk}^t - \frac{1}{2}(\Gamma_{ijk}^t + L_{ijk}^t) + \Delta_{jki}^t - \frac{1}{2}(\Gamma_{jki}^t + L_{jki}^t) + \Delta_{kij}^t - \frac{1}{2}(\Gamma_{kij}^t + L_{kij}^t) = 0$$

This result is obvious if $\Gamma_{ij}^t$ and therefore $L_{ij}^t$, $\Delta_{ij}^t$ are self-conjugate.

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