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ON THE UNION OF TWO GENERALIZED MANIFOLDS

by PAUL A. WHITE (Los Angeles)

The author has made a study in his paper (3) of additive set properties. An i-dimensional property p^i of a space was called additive if it satisfied the following theorem.

Theorem A. *If M_1 and M_2 each have p^i and are compact subsets of the compact space $M_1 \cup M_2$, and if $M_1 \cap M_2$ has p^{i-1}, then $M_1 \cup M_2$ has p^i.* (The symbols « $\cup$ » and « $\cap$ » denote the set theoretic « union » and « intersection », respectively; thus « $+$ » can be reserved for the group operation).

In this paper it is shown that the properties « to be a generalized n manifold with boundary » and « to be an orientable generalized n-manifold with boundary » (see definitions 5 and 7) satisfy a modified form of the additive property.

We shall assume that all sets to be considered are subsets of a fixed compact Hausdorff space S. Since no metric is assumed, we shall use Cech cycles (chains) with coefficients in an arbitrary field G instead of Vietoris cycles (with mod 2 coefficients) as in (3). A knowledge of the Cech theory will be assumed since most of the general definitions and results needed are discussed in chapter 8 of (2) and the specialized ideas in sections 4 and 6 of E. G. Begle's paper (1). The boundary operator will be denoted by « ∂ ».

Definition 1. The closed set $A \subset S$ is *i-lc* (*locally connected*) *at the point* $p \in A$ if for each open set P of S containing p, there exists an open set Q of S such that $p \in Q \subset P$ and such that every i-dimensional cycle on $Q \cap A$ is ~ 0 on $P \cap A$. (A Cech cycle z^n is « on » a set if the nucleus of each cell of each coordinate cycle of z^n intersects the set).

The following definition, which is sometimes more convenient to use, is equivalent to definition 1.

Definition 1'. The closed set $A \subset S$ is *i-lc at* $p \in A$ if corresponding to each open set P of S containing p and covering $\mathfrak{U}$ of S by open sets, there exists an open set Q such that $p \in Q \subset P$, and a covering $\mathfrak{V}$ of S by open sets such that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$ (written $\mathfrak{V} > \mathfrak{U}$) and such that if $z^i(\mathfrak{V})$ is any cycle on $Q \cap A$, then $\pi^{\mathfrak{U}}_{\mathfrak{V}} z^i(\mathfrak{V}) \sim 0$ on $Q \cap A$ ($\pi^{\mathfrak{U}}_{\mathfrak{V}}$ = a simplicial projection from $\mathfrak{V}$ into $\mathfrak{U}$).

Definition 1''. The closed set $A \subset S$ is *i-lc* if it is *i-lc* at each point $p \in A$.

Definition 1'''. The closed set $A \subset S$ is lc^n if it is *i-lc* for all i $(0 \leqq i \leqq n)$.

Definition 2. The closed set $A \subset S$ is *simply-i connected* if each i cycle on A is ~ 0 on A.

Theorem 1. *The property i-lc is additive for each i.*

Corallary 1'. *The property* lc^n *is additive.*

Theorem 2. *The simple-i-connectedness property is additive for each i.*

These theorems were proved in (3) with Vietoris cycles; they are still true in our present more general situation, but the proofs are omitted since they differ only in the mechanical details and not in the essential idea from the first proofs. Similar mechanical details will occur in all the following proofs, and depend on certain lemmas that appear in R. L. Wilder's « Colloquium » (4). I will state them here for reference.

Lemma 1. *If L is a closed subset of S and $\mathfrak{U}$ is a covering of S by open sets, then there exists a covering $\mathfrak{V}$, a refinement of $\mathfrak{U}$, such that if the nucleus of a cell of $\mathfrak{V}$ meets both L and $S - L$, then it meets $F(L)$, the boundary of L.*

Lemma 2. *If z^i is a cycle mod K on M, then the collection $\{\partial z^i(\mathfrak{U})\}$ is an $(i-1)$-cycle on K, which we denote by ∂z^i. Evidently $\partial z^i \sim 0$ on M.*

Lemma 3. *If z^i is a cycle on K such that $z^i \sim 0$ on M, then there exists a cycle z^{i+1} mod K on M such that $\partial z^{i+1} \sim z^i$ on K.*

Lemma 4. *If z^i is a cycle mod K on M such that $\partial z^i \sim 0$ on M, then there exists a cycle γ^i on M such that $z^i \sim \gamma^i$ mod K.*

Lemma 5. *If z^i is a cycle mod K such that $z^i \sim 0$ mod M, then there exists a cycle γ^i mod K on M such that $z^i \sim \gamma^i$ mod K.*

We shall need slightly stronger forms of lemma 1 which we now state and prove.

Lemma 1'. *If L is a closed subset of a closed subset M of S and $\mathfrak{U}$ is a covering of S by open sets, then there exists a covering $\mathfrak{V}$ of S by open sets, a refinement of $\mathfrak{U}$, such that if the nucles of a cell of $\mathfrak{V}$ meets both L and $M - L$ then it meets $F_M(L)$ = boundary of L with respect to M.*

Proof. Let $\mathfrak{U}_1$ be the subcollection of $\mathfrak{U}$ consisting of sets that meet M, then $\mathfrak{U}_1' = \{U_1 \cap M\}$ for all $U_1 \in \mathfrak{U}_1$ is a covering of M by sets open relative to M. By lemma 1 there exists a covering $\mathfrak{V}_1'$ of M with sets open relative to M such that $\mathfrak{V}_1'$ is a refinement of $\mathfrak{U}_1'$ and such that if a cell of $\mathfrak{V}_1'$ meets both L and $M - L$, it will meet the boundary of L relative to M, i.e. $F_M(L)$. Corresponding to each $V_1' \in \mathfrak{V}_1'$ there exists an open set V' of S such that $V' \cap M = V_1'$, also a set $U_1 \in \mathfrak{U}_1$ such that

$V_1' \subset U_1 \cap M \subset U_1$. If we let $V_1 = U_1 \cap V'$, then $V_1' \subset V_1 \subset U_1$; thus $\mathfrak{V}_1' = \{V_1\}$ is a covering of M by open sets of S that is a refinement of $\mathfrak{U}_1$. Let $\mathfrak{U}_2 \subset \mathfrak{U}$ be the sets that meet $S - M$ and let $\mathfrak{V}_2 = \{(S - M) \cap U_2\}$ for all $U_2 \in \mathfrak{U}_2$, then $\mathfrak{V}_2$ is a covering of $S - M$ by open sets of S that is a refinement of $\mathfrak{U}_2$. Let $\mathfrak{V} = \mathfrak{V}_1 \cup \mathfrak{V}_2$, then $\mathfrak{V}$ is a covering of S by open sets that is a refinement of $\mathfrak{U}$ and clearly has the property that if a cell meets both L and $M - L$, then it meets $F_M(L)$.

Lemma 1″. *If A is a closed, and B an arbitrary subset of S, and $\mathfrak{U}$ is a covering of S, then there exists a covering $\mathfrak{V}$, a refinement of $\mathfrak{U}$, such that if the nucleus of a cell of $\mathfrak{V}$ meets both A and B, then it meets $A \cap \bar{B}$, or, more specifically, if $C = A \cup \bar{B}$, then it meets $F_C(A) \subset A \cap \bar{B}$.*

Proof. Let $C = A \cup \bar{B} = A \cup \{(\bar{B} \cap A) \cup [\bar{B} \cap (A - \bar{B} \cap A)]\}$. If a cell is on A and $\bar{B} \cap A$, the conclusion is already satisfied. If a cell is on A and $\bar{B} \cap [A - (\bar{B} \cap A)] = C - A$ then it is on $F_C(A)$ by lemma 1 (if the same choice of $\mathfrak{V}$ is made).

Definition 3. The closed set $A \subset S$ is *i-colc* (*co-locally connected*) *at* $p \in A$, if for every open set P of S containing p there exists an open set Q of S such that $p \in Q \subset P$ and such that any i-cycle on A mod $(S - P) \cap A$ is ~ 0 mod $(S - Q) \cap A$ on A. This definition is equivalent to the following one.

Definition 3′. The closed set $A \subset S$ is *i colc at* $p \in A$ if corresponding to each open set P of S containing p and covering $\mathfrak{U}$ of S by open sets, there exists an open set Q such that $p \in Q \subset P$ and a covering $\mathfrak{V}$ of S by open sets such that $\mathfrak{V} > \mathfrak{U}$ and such that if $z^i(\mathfrak{V})$ is any cycle on A mod $(S - P) \cap A$ then $\pi^{\mathfrak{U}}_{\mathfrak{V}} z^i(\mathfrak{V}) \sim 0$ on A mod $(S - Q) \cap A$.

Definition 3″. The closed set A is *i-colc* if it is i colc at each $p \in A$.

Definition 3‴. The closed set A is said to be lc_n if it is i colc for all i $(0 \leq i \leq n)$.

Theorem 3. *The property i-colc is additive for each i.*

Proof. Let M_1, M_2, and $M = M_1 \cup M_2$ be compact subsets of the compact space S, where M_1 and M_2 are i-colc, and $M_{12} = M_1 \cap M_2$ is $(i - 1)$-colc. Consider $p \in M_1 \cup M_2$. If $p \in M_1 - M_{12}$, we can suppose the open set P of definition 3′ is chosen such that $P \cap M_2 = 0$. The i-colc property for M_1 at p now implies that M is i-colc at p. Similarly if $p \in M_2 - M_{12}$, the i-colc for M_2 at p implies that M is i-colc at p. Finally consider a point $p \in M_{12}$, an open set P such that $p \in P$ and a covering $\mathfrak{U}$ of S by open sets. Let Q be chosen according to definition 3 for the $(i - 1)$ colc property of M_{12} such that any $(i - 1)$-cycle on M_{12} mod $(S - P) \cap M_{12}$ is ~ 0 mod $(S - Q) \cap M_{12}$ on M_{12}. By the i-colc property for M_1 and M_2 according to definition 3′, we can choose open sets R_1 and R_2 and coverings $\mathfrak{V}_1$ and $\mathfrak{V}_2$ such that $p \in R_j \subset Q$, $\mathfrak{V}_j > \mathfrak{U}$ and such that if $z^i(\mathfrak{V}_j)$ is an i-cycle on M_j mod $(S - Q) \cap M_j$, then

$\pi^{\mathfrak{U}}_{\mathfrak{V}_j} z^i(\mathfrak{V}_j) \sim 0$ mod $(S - R_j) \cap M_j$ on M_j $(j = 1,2)$. Let R be an open set such that $p \in R \subset R_j \subset Q$, and let $\mathfrak{V}$ be a refinement of $\mathfrak{U}_j$ $(j = 1,2)$. By lemma 1″ we can suppose that $\mathfrak{V}$ is chosen with the property that if a cell of $\mathfrak{V}$ is on both M_1 and M_2, it is on M_{12}. By applying lemma 1″ again we can require that $\mathfrak{V}$ has the additional property that if one of its cells intersects both $(S - P) \cap M_1$ and $(S - P) \cap M_2$, it intersects $(S - P) \cap M_{12}$. Finally we can suppose $\mathfrak{V}$ is a refinement of $\mathfrak{U}$ with the property that the projection into $\mathfrak{U}$ of any cycle on M mod $(S - P) \cap M$ is the coordinate of a Cech cycle on M mod $(S - P) \cap M$. (The existence of such a refinement for any covering $\mathfrak{U}$ is established in Wilder's « Colloquium »). Now consider any i-cycle $\overline{z}^i(\mathfrak{V})$ on M mod $(S - P) \cap (M_1 \cup M_2)$. By our choice of $\mathfrak{V}$, $\pi^{\mathfrak{U}}_{\mathfrak{V}} \overline{z}^i(\mathfrak{V})$ is the coordinate in $\mathfrak{U}$ of a Cech cycle $z^i = \{z^i(\mathfrak{W})\}$ on M mod $(S - P) \cap M$; i. e. $\pi^{\mathfrak{U}}_{\mathfrak{V}} \overline{z}^i(\mathfrak{V}) = z^i(\mathfrak{U})$. We can write for each $\mathfrak{W}$, $z^i(\mathfrak{W}) = z^i_1(\mathfrak{W}) + z^i_2(\mathfrak{W})$ where $z^i_1(\mathfrak{W})$ is the part of $z^i(\mathfrak{W})$ on M_1 (i. e. formed from cells on M_1 and included in $z^i_1(\mathfrak{W})$ with the same coefficients that they have in $z^i(\mathfrak{W})$) and $z^i_2(\mathfrak{W}) = z^i(\mathfrak{W}) - z^i_1(\mathfrak{W})$; hence $z^i_2(\mathfrak{W})$ is on $M - M_1 \subset \subset M_2$. By hypothesis $\partial z^i(\mathfrak{W}) = \partial z^i_1(\mathfrak{W}) + \partial z^i_2(\mathfrak{W}) = z^{i-1}(\mathfrak{W})$ is a cycle on $(S - P) \cap M$. Let $z^{i-1}(\mathfrak{W}) = z^{i-1}_1(\mathfrak{W}) + z^{i-1}_2(\mathfrak{W})$ where $z^{i-1}_1(\mathfrak{W})$ is the part of $z^{i-1}(\mathfrak{W})$ on $(S - P) \cap M_1$ and $z^{i-1}_2(\mathfrak{W}) = z^{i-1}(\mathfrak{W}) - z^{i-1}_1(\mathfrak{W})$; hence $z^{i-1}_2(\mathfrak{W})$ is on $(S - P) \cap (M - M_1) \subset (S - P) \cap M_2$. Finally let $y^{i-1}(\mathfrak{W}) = \partial z^i_1(\mathfrak{W}) - z^{i-1}_1(\mathfrak{W}) = -\partial z^i_2(\mathfrak{W}) + z^{i-1}_2(\mathfrak{W})$. Since $\partial z^i_1(\mathfrak{W}) - z^{i-1}_1(\mathfrak{W})$ is on M_1 and $-\partial z^i_2(\mathfrak{W}) + z^i_2(\mathfrak{W})$ is on $M - M_1$, it follows that $y^{i-1}(\mathfrak{W})$ is on M_{12}. Furthermore $\partial y^{i-1}(\mathfrak{W}) = -\partial z^{i-1}_1(\mathfrak{W}) = \partial z^{i-1}_2(\mathfrak{W})$; thus $\partial y^{i-1}(\mathfrak{W})$ is on both $(S - P) \cap M_1$ and $(S - P) \cap (M - M_1)$ and is, therefore, on $(S - P) \cap M_{12}$. We have, thus, shown that $y^{i-1}(\mathfrak{W})$ is a cycle on M_{12} mod $(S - P) \cap M_{12}$ for each $\mathfrak{W}$, but we have not yet shown that $y^i = \{y^i(\mathfrak{W})\}$ is a relative Cech cycle. To this end let $\mathfrak{W}_2 > \mathfrak{W}_1$, then since z^i is a Cech cycle on M mod $(S - P) \cap M$, there exists a chain $C^{i+1}(\mathfrak{W}_1)$ on M and a chain $x^i(\mathfrak{W}_1)$ on $(S - P) \cap M$ such that (1) $\partial C^{i+1}(\mathfrak{W}_1) = \pi^1_2 z^i(\mathfrak{W}_2) - z^i(\mathfrak{W}_1) + x^i(\mathfrak{W}_1)$ ($\pi^1_2 =$ a simplicial projection from $\mathfrak{W}_2$ into $\mathfrak{W}_1$). Let $C^{i+1}(\mathfrak{W}_1) = C^{i+1}_1(\mathfrak{W}_1) + C^{i+1}_2(\mathfrak{W}_1)$, where $C^{i+1}_1(\mathfrak{W}_1)$ is the part of $C^{i+1}(\mathfrak{W}_1)$ on M_1, and $C^{i+1}_2(\mathfrak{W}_1) = C^{i+1}(\mathfrak{W}_1) - C^{i+1}_1(\mathfrak{W}_1)$ is on $M - M_1 \subset M_2$. Similarly let $x^i(\mathfrak{W}_1) = x^i_1(\mathfrak{W}_1) + x^i_2(\mathfrak{W}_1)$ where $x^i_1(\mathfrak{W}_1)$ is the part of $x^i(\mathfrak{W}_1)$ on $(S - P) \cap M_1$ and $x^i_2(\mathfrak{W}_1) = x^i(\mathfrak{W}_1) - x^i_1(\mathfrak{W}_1)$ is on $(S - P) \cap (M - M_1) \subset (S - P) \subset M_2$. By taking boundaries in (1) we obtain $0 = \pi^1_2 \partial z^i(\mathfrak{W}_2) - \partial z^i(\mathfrak{W}_1) + \partial x^i(\mathfrak{W}_1) = \pi^1_2 z^{i-1}(\mathfrak{W}_2) - z^{i-1}(\mathfrak{W}_1) + \partial x^i(\mathfrak{W}_1)$. By expanding and algebraic manipulation, this becomes (2) $-\pi^1_2 z^{i-1}_1(\mathfrak{W}_2) + z^{i-1}_1(\mathfrak{W}_1) - \partial x^i_1(\mathfrak{W}_1) = \pi^1_2 z^{i-1}_2(\mathfrak{W}_2) - z^{i-1}_2(\mathfrak{W}_1) + \partial x^i_2(\mathfrak{W}_1)$ which we will denote by $y^{i-1}(\mathfrak{W}_1)$. Since the left hand side is on $(S - P) \cap M_1$ and the right hand side is on $(S - P) \cap M_2$, we conclude that each side, i.e. $y^{i-1}(\mathfrak{W}_1)$, is on $(S - P) \cap M_{12}$. If we expand and rearrange (1), we obtain $\partial C^{i+1}_1(\mathfrak{W}_1) - \pi^1_2 z^i_1(\mathfrak{W}_2) + z^i_1(\mathfrak{W}_1) - x^i_1(\mathfrak{W}_1) =$

$= - \partial C_2^{i+1}(\mathfrak{W}_1) + \pi_2^1 z_2^i(\mathfrak{W}_2) - z_2^i(\mathfrak{W}_1) + x_2^i(\mathfrak{W}_1)$ where the left hand side is on M_1 and the right hand side is on M_2; hence both sides are on M_{12}. That γ^i is a Cech cycle mod $(S - P) \cap M_{12}$ on M_{12} now follows since $\partial(\partial C_1^{i+1}(\mathfrak{W}_2) - \pi_2^1 z_1^i(\mathfrak{W}_2) + z_1^i(\mathfrak{W}_1) - x_1^i(\mathfrak{W}_1)) = - \pi_2^1 \partial z_1^i(\mathfrak{W}_2) + \partial z_1^i(\mathfrak{W}_1) - - \partial x_1^i(\mathfrak{W}_1)$ which by (2) $= - \pi_2^1 \partial z_1^i(\mathfrak{W}_2) + \partial z_1^i(\mathfrak{W}_1) + \pi_2^1 z_1^{i-1}(\mathfrak{W}_2) - - z_1^{i-1}(\mathfrak{W}_1) + y^{i-1}(\mathfrak{W}_1) = \gamma^{i-1}(\mathfrak{W}_1) - \pi_2^1 \gamma^{i-1}(\mathfrak{W}_2) + y^{i-1}(\mathfrak{W}_1)$; i.e. $\gamma^{i-1}(\mathfrak{W}_1) \sim \sim \pi_2^1 \gamma^{i-1}(\mathfrak{W}_2)$ mod $(S - P) \cap M_{12}$ on M_{12}. By the choice of Q, we conclude that $\gamma^i \sim 0$ mod $(S - Q) \cap M_{12}$ on M_{12}; i.e. for every covering $\mathfrak{W}$, $\partial C^i(\mathfrak{W}) = = \bar{\gamma}^{i-1}(\mathfrak{W}) - \gamma^{i-1}(\mathfrak{W})$ where $\bar{\gamma}^{i-1}(\mathfrak{W})$ is a chain on $(S - Q) \cap M_{12}$ and $C^i(\mathfrak{W})$ is a chain on M_{12}. Now let $\gamma_1^i(\mathfrak{W}) = z_1^i(\mathfrak{W}) + C^i(\mathfrak{W})$ and $\gamma_2^i(\mathfrak{W}) = z_2^i(\mathfrak{W}) - - C^i(\mathfrak{W})$, then $\gamma_j^i(\mathfrak{W})$ is a cycle on M_j mod $(S - Q) \cap M_j$ $(j = 1,2)$ for $\partial \gamma_j^i(\mathfrak{W}) = \partial z_j^i(\mathfrak{W}) + (-1)^{j-1} \partial C^i(\mathfrak{W}) = (-1)^{j-1} \gamma^{i-1}(\mathfrak{W}) + z_j^{i-1}(\mathfrak{W}) + + (-1)^{j-1} \bar{\gamma}^{i-1}(\mathfrak{W}) - (-1)^{j-1} \gamma^{i-1}(\mathfrak{W}) = z_j^{i-1}(\mathfrak{W}) + (-1)^{j-1} \bar{\gamma}^{i-1}(\mathfrak{W})$ which is on $(S - Q) \cap M_{12}$ $(j = 1,2)$. In particular $\pi_j^i(\mathfrak{V})$ is a cycle on M_j mod $(S - Q) \cap M_j$; therefore $\pi_{\mathfrak{V}}^{\mathfrak{U}} \gamma_j^i(\mathfrak{V}) \sim 0$ mod $(S - R) \cap M_j$ by the choice of R and $\mathfrak{V}$. Since $z^i(\mathfrak{V}) = z_1^i(\mathfrak{V}) + z_2^i(\mathfrak{V}) = \gamma_1^i(\mathfrak{V}) + \gamma_2^i(\mathfrak{V})$, we conclude that $\pi_{\mathfrak{V}}^{\mathfrak{U}} z^i(\mathfrak{V}) \sim 0$ mod $(S - R) \cap M$ on M. Finally since z^i is a Cech cycle mod $(S - R) \cap M$ on M, $\pi_{\mathfrak{V}}^{\mathfrak{U}} \bar{z}^i(\mathfrak{V}) = z^i(\mathfrak{U}) \sim \pi_{\mathfrak{V}}^{\mathfrak{U}} z^i(\mathfrak{V})$ mod $(S - P) \cap M$ on M; hence $\pi_{\mathfrak{V}}^{\mathfrak{U}} \bar{z}^i(\mathfrak{V}) \sim 0$ mod $(S - R) \cap M$ on M, which by definition 3′ shows that M is i-colc at p.

Corollary 3′. *The property lc_n is additive.*

The following is a slightly generalized form of the definition of a local Betti number (see [1]). It is equivalent to the ordinary definition when $B = 0$.

Definition 4. If B and A are closed subsets of S, then we shall say that *the local Betti number of A at p mod B is the finite positive integer k* (denoted by $R_n(p, A, B)$), if k is the smallest positive integer with the property that corresponding to any open set P such that $p \in P$ there exists an open set Q such that $p \in Q \subset P$ and such that if $z_1^n, z_2^n, \ldots, z_{k+1}^n$ are cycles on A mod $(S - P) \cup B$, then there exists integers $m_1, m_2, \ldots, m_{k+1}$ not all 0 such that $m_1 z_1^n + m_2 z_2^n + \ldots + m_{k+1} z_{k+1}^n \sim 0$ mod $(S - Q) \cup B$ on A.

This is equivalent to the following definition.

Definition 4′. If B and A are closed subsets of S, then $R_n(p,A,B) = k$, if k is the smallest positive integer with the property that corresponding to any open set P with $p \in P$ and covering $\mathfrak{U}$ of S by open sets, there exists an open set Q and covering $\mathfrak{V}$ of S by open sets such that $p \in Q \subset P$, $\mathfrak{V} > \mathfrak{U}$ and such that if $z_1^n(\mathfrak{V}), z_2^n(\mathfrak{V}), \ldots, z_{k+1}^n(\mathfrak{V})$ are cycles on A mod $(S - P) \cup B$, then there exist integers $m_1, m_2, \ldots, m_{k+1}$ not all 0 such that $m_1 \pi_{\mathfrak{V}}^{\mathfrak{U}} z_1^n(\mathfrak{V}) + \ldots + m_{k+1} \pi_{\mathfrak{V}}^{\mathfrak{U}} z_{k+1}^n(\mathfrak{V}) \sim 0$ mod $(S - Q) \cup B$ on A.

In some of the following theorems the following assumptions and notations will be assumed.

As usual M_1, M_2, and $M = M_1 \cup M_2$ will be compact subsets of the compact space S. Also let $F_1 = F(M_1)$, $F_2 = F(M_2)$, $F = F(M)$, and $F_{12} = F(M_{12}) \cap F(M)$, and assume that $F_M(M_1) = F_M(M_2) = M_{12}$ where $F(A)$ and $F_M(A)$ are boundaries of A relative to S and M, resp. Note that $M_{12} \subset F_1 \cap F_2$ and that $F_{12} = M_{12} \cap F$. The latter follows since $F_{12} = [M_{12} \cap \overline{S - M_{12}}] \cap [M \cap \overline{S - M}]$, but $M_{12} \subset M$ implies $\overline{S - M} \subset \overline{S - M_{12}}$; therefore $F_{12} = M_{12} \cap \overline{S - M} \cap M = M_{12} \cap F$. For reference we will call these *assumptions A*.

Theorem 4. *Under the assumptions A if $R_n(p, M_j, F_j) > 0$ for all $p \in M_j$ $(j=1,2)$ and $R_{n-1}(p, M_{12}, F_{12}) \leq 1$ for all $p \in M_{12}$, then $R_n(p,M,F)>0$ for all $p \in M$.*

Proof. If $p \notin M_{12}$, $R_n(p, M_j, F_j) > 0$ implies $R_n(p, M, F) > 0$. If $p \in M_{12}$, then let P be an arbitrary open set of $S \supset p$. Since $R_{n-1}(p, M_{12}, F_{12}) \leq 1$, (1) there exists an open set Q such that $p \in Q \subset P$ and such that for any two Cech cycles z_1^{n-1}, z_2^{n-1} mod $[(S - P) \cap M_{12}] \cup F_{12}$ there exist integers m_1, m_2 not both zero such that $m_1 z_1^{n-1} + m_2 z_2^{n-1} \sim 0$ mod $[(S - Q) \cap M_{12}] \cup F_{12}$. Also since $R_n(p, M_j, F_j) > 0$ (2) there exists an open set $P \supset p$ such that for any open set R such that $p \in R \subset P$, there exist cycles z_j^n on M_j mod $[(S - P) \cap M_j] \cup F_j$, but such that $m z_j^n \sim 0$ mod $[(S - R) \cap M_j]$ (where m is an integer) implies $m = 0$ $(j=1,2)$. Let P be the open set of (2), let Q be the open set from (1) corresponding to P, let R be any arbitrary (but fixed) open set such that $p \in R \subset Q \subset P$, and let z_j^n be the corresponding cycles from (2). By lemma 1″ we can consider that our cycles only have coordinates on a confinal family of coverings with the property that if a cell is on $[(S - P) \cap M_j] \cup F$ and on M_{12}, then it is on $\{[S - P) \cap M_j] \cup F\} \cap M_{12} = [(S - P) \cap M_{12}] \cup F_{12}$; and if a cell is on M_1 and M_2, it is on $M_{12} \subset F_1 \cap F_2$. By lemma 2, $\{\partial z_j^n(\mathfrak{U})\} = z_j^{n-1}$ is a Cech cycle on $[(S - P) \cap M_j] \cup F_j$ $(j = 1,2)$. For each covering $\mathfrak{U}$ of S let $z_j^{n-1}(\mathfrak{U}) = x_j^{n-1}(\mathfrak{U}) + y_j^{n-1}(\mathfrak{U})$ where $x_j^{n-1}(\mathfrak{U})$ is the part of $z_j^{n-1}(\mathfrak{U})$ on $[(S - P) \cap M_j] \cup (F \cap F_j)$ and $y_j^{n-1}(\mathfrak{U}) = z_j^{n-1}(\mathfrak{U}) - x_j^{n-1}(\mathfrak{U})$. Since $F_j \subset F \cup M_{12}$ and $y_j^{n-1}(\mathfrak{U})$ is not on F, it must be on M_{12}. Furthermore $0 = \partial z_j^{n-1}(\mathfrak{U}) = \partial x_j^{n-1}(\mathfrak{U}) + \partial y_j^{n-1}(\mathfrak{U})$, where the first term is on $[(S - P) \cap M_j] \cup F$ and the second is on M_{12}; therefore each is on $[(S - P) \cap M_{12}] \cup F_{12}$. In particular $y_j^{n-1}(\mathfrak{U})$ is a cycle on M_{12} mod $[(S-P) \cap M_{12}] \cup F_{12}$. To show $y_j^{n-1} = \{y_j^{n-1}(\mathfrak{U})\}$ is a Cech cycle, consider a covering $\mathfrak{V} > \mathfrak{U}$. Since z_j^{n-1} is a Cech cycle, there exists a chain $C_j^n(\mathfrak{U})$ on $[(S - P) \cap M_j] \cup F_j$ such that $\partial C_j^n(\mathfrak{U}) = \pi_{\mathfrak{V}}^{\mathfrak{U}} z_j^{n-1}(\mathfrak{V}) - z_j^{n-1}(\mathfrak{U})$. As in the above argument, let $C_j^n(\mathfrak{U}) = A_j^n(\mathfrak{U}) + B_j^n(\mathfrak{U})$ where $A_j^n(\mathfrak{U})$ is on $[(S - P) \cap M_j] \cup F$ and $B_j^n(\mathfrak{U})$ is on M_{12}. This gives $\partial A_j^n(\mathfrak{U}) + \partial B_j^n(\mathfrak{U}) =$

$= \pi^{\mathcal{U}}_{\mathcal{V}} x^{n-1}_j (\mathcal{V}) + \pi^{\mathcal{U}}_{\mathcal{V}} y^{n-1}_j (\mathcal{V}) - x^{n-1}_j (\mathcal{U}) - y^{n-1}_j (\mathcal{U})$. It follows as before that $\partial B^n_j (\mathcal{U}) = \pi^{\mathcal{U}}_{\mathcal{V}} y^{n-1}_j (\mathcal{V}) - y^{n-1}_j (\mathcal{V}) +$ a chain on $[(S-P) \cap M_{12}] \cup F_{12}$; hence $\pi^{\mathcal{U}}_{\mathcal{V}} y^{n-1}_j (\mathcal{V}) \sim y^{n-1}_j (\mathcal{U})$ on M_{12} mod $[(S - P) \cap M_{12}] \cup F_{12}$. This shows that y^{n-1}_j is a Čech cycle on M_{12} mod $[(S - P) \cap M_{12}] \cup F_{12}$, $(j = 1,2)$. There exist integers m_1 and m_2 not both zero such that $m_1 y^{n-1}_1 + m_2 y^{n-1}_2 \sim 0$ mod $[(S - Q) \cap M_{12}] \cup F_{12}$ and let us suppose that $m_1 \neq 0$ for convenience. This implies the existence for each $\mathcal{U}$ of a chain $C^n (\mathcal{U})$ on M_{12} and a chain $\gamma^{n-1} (\mathcal{U})$ on $[(S - Q) \cap M_{12}] \cup F_{12}$ such that $\partial C^n (\mathcal{U}) = m_1 y^{n-1}_1 (\mathcal{U}) + m_2 y^{n-1}_2 (\mathcal{U}) + \gamma^{n-1} (\mathcal{U})$. Let $z^n (\mathcal{U}) = m_1 z^n_1 (\mathcal{U}) + m_2 z^n_2 (\mathcal{U}) - C^n (\mathcal{U})$, then $\partial z^n(\mathcal{U}) = m_1 x^{n-1}_1(\mathcal{U}) + m_2 x^{n-1}_2 (\mathcal{U}) - \gamma^{n-1}(\mathcal{U})$ which is on $[(S-Q) \cap M] \cup F$. By our choice of R and $m_1 \neq 0$, we know that $m_1 z^{n-1}_1 \nsim 0$ mod $[(S - R) \cap M_1] \cup F_1$, i. e. there exists a covering $\mathcal{V}$ such that $m_1 z^{n-1}_1 (\mathcal{V}) \nsim 0$ mod $[(S - R) \cap M_1] \cup F_1$. Now consider any covering $\mathcal{W} > \mathcal{V}$, then for any integer $m \neq 0$ we have $\pi^{\mathcal{V}}_{\mathcal{W}} m z^n (\mathcal{W}) = m m_1 \pi^{\mathcal{V}}_{\mathcal{W}} z^n_1 (\mathcal{W}) + m m_2 \pi^{\mathcal{V}}_{\mathcal{W}} z^n_2(\mathcal{W}) - m \pi^{\mathcal{V}}_{\mathcal{W}} C^n (\mathcal{W})$. It follows that $\pi^{\mathcal{V}}_{\mathcal{W}} m z^n (\mathcal{W}) \nsim 0$ mod $[(S - R) \cap M] \cup F$, for otherwise there exist chains $C^{n+1} (\mathcal{V})$ of M and $\gamma^n (\mathcal{V})$ of $[(S - R) \cap M] \cup F$ such that $\partial C^{n+1} (\mathcal{V}) = m m_1 \pi^{\mathcal{V}}_{\mathcal{W}} z^n_1 (\mathcal{W}) + m m_2 \pi^{\mathcal{V}}_{\mathcal{W}} z^n_2 (\mathcal{W}) - m \pi^{\mathcal{V}}_{\mathcal{W}} C^n(\mathcal{W}) + \gamma^n(\mathcal{V})$. Let $C^{n+1} (\mathcal{V}) = C^{n+1}_1 (\mathcal{V}) + C^{n+1}_2 (\mathcal{V})$ where $C^{n+1}_1 (\mathcal{V})$ is on M_1 and $C^{n+1}_2 (\mathcal{V}) = C^{n+1} (\mathcal{V}) - C^{n+1}_1 (\mathcal{V})$ is on M_2 and let $\gamma^n (\mathcal{V}) = \gamma^n_1 (\mathcal{V}) + \gamma^n_2 (\mathcal{V})$ where $\gamma^n_1 (\mathcal{V})$ is on $[(S - R) \cap M_1] \cup (F \cap M_1) \subset [(S - R) \cap M_1] \cup F_1$ and $\gamma^n_2 (\mathcal{V}) = \gamma^n (\mathcal{V}) - \gamma^n_1 (\mathcal{V})$. It follows that $\partial C^{n+1}_1 (\mathcal{V}) - m m_1 \pi^{\mathcal{V}}_{\mathcal{W}} z^n_1 (\mathcal{W}) - \gamma^n_1 (\mathcal{V}) = - \partial C^{n+1}_2 (\mathcal{V}) + m m_2 \pi^{\mathcal{V}}_{\mathcal{W}} z^n_2 (\mathcal{W}) - m \pi^{\mathcal{V}}_{\mathcal{W}} C^n (\mathcal{W}) + \gamma^n_2 (\mathcal{V})$ where the left side is on M_1 and the right side is on M_2; thus both sides are on $M_{12} \subset F_1$. If we denote this chain by $\gamma^n_{12} (\mathcal{V})$, then we have $\partial C^{n+1}_1 (\mathcal{V}) = m m_1 \pi^{\mathcal{V}}_{\mathcal{W}} z^n_1(\mathcal{W}) + \gamma^n_1 (\mathcal{V}) + \gamma^n_{12} (\mathcal{V})$; thus $m m_1 \pi^{\mathcal{V}}_{\mathcal{W}} z^n_1 (\mathcal{W}) \sim 0$ on M_1 mod $[(S - W) \cap M_1] \cup F_1$. Since z^n_1 is a relative Čech cycle, $\pi^{\mathcal{V}}_{\mathcal{W}} z^n_1 (\mathcal{W}) \sim z^n_1 (\mathcal{V})$ on M_1 mod $[(S-P) \cap M_1] \cup F_1$; therefore $m m_1 z^n_1 (\mathcal{V}) \sim 0$ mod $[(S - R) \cap M_1] \cup F_1$ which implies $m m_1 = 0$, a contradiction since both m and $m_1 \neq 0$. This concludes the proof that $\pi^{\mathcal{V}}_{\mathcal{W}} m z^n (\mathcal{W}) \nsim 0$ on M mod $[(S - R) \cap M] \cup F$ for any integer $m \neq 0$. We have, thus, found an open set $Q \supset p$ and a covering $\mathcal{V}$ of S such that for any $R \subset Q$ and covering $\mathcal{W} > \mathcal{V}$, there exists a cycle $z^n (\mathcal{W})$ on M mod $[(S - Q) \cap M] \cup F$ such that $\pi^{\mathcal{V}}_{\mathcal{W}} m z^n (\mathcal{W}) \nsim 0$ on M mod $[(S-R) \cap M] \cup F$ for all integers $m \neq 0$. This is the statement of the negative of $R_n (p, M, F) = 0$ according to 4′; hence $R_n (p, M, F) > 0$.

Theorem 5. *Under assumptions A, if* $R_n (p, M_j, F_j) \leq 1$ *for all* $p \in M_j$ $(j = 1, 2)$, $R_{n-1} (p, M_{12}, F_{12}) \leq 1$ *and* $R_n (p, M_j, M_j \cap F) = 0$ *for all* $p \in M_{12}$, *then* $R_n (p, M, F) \leq 1$ *for all* $p \in M$.

Proof. According to definition 4′ we must show that for any open set $P \supset p$ and covering $\mathcal{U}$, there exists an open set R such that $p \in R \subset P$ and covering $\mathcal{V} > \mathcal{U}$ such that if $\tilde{z}^n_1 (\mathcal{V})$ and $\tilde{z}^n_2 (\mathcal{V})$ are cycles on M mod

$[(S-P)\cap M]\cup F$, then there exists integers m_1 and m_2 not both 0 such that $m_1\pi^{\mathfrak{U}}_{\mathfrak{V}}\tilde{z}^n(\mathfrak{V})+m_2\pi^{\mathfrak{U}}_{\mathfrak{V}}\tilde{z}^n(\mathfrak{V})\sim 0$ on M mod $[(S-R)\cap M]\cup F$. If $p\notin M_{12}$, then $R_n(p, M_j, F_j)\leqq 1$ implies $R_n(p, M, F)\leqq 1$. If $p\in M_{12}$, let P be arbitrary and choose Q such that $p\in Q\subset P$ according to definition 4 of $R_{n-1}(p, M_{12}, F_{12})\leqq 1$ such that for any two $(n-1)$ — dimensional Cech cycles on M_{12} mod $[(S-P)\cap M_{12}]\cup F_{12}$, there exists a non trivial linear combination that is ~ 0 on M_{12} mod $[(S-Q)\cap M_{12}]\cap F_{12}$. By definition 4′ of $R_n(p, M_j, M_j\cap F_j)=0$, we can choose an open set R such that $p\in R\subset Q$ and a covering $\mathfrak{V}$ that is a normal refinement of $\mathfrak{U}$ with respect to cycles mod $[(S-P)\cap M]\cup F$ and such that if $C^n_j(\mathfrak{V})$ is a cycle mod $[(S-Q)\cap M_j]\cup(F\cap M_j)$, then $\pi^{\mathfrak{U}}_{\mathfrak{V}}C^n_j(\mathfrak{V})\sim 0$ mod $[(S-R)\cap M_j]\cup(F\cap M_j)$. Furthermore we shall assume by lemma 1″ that all cycles are defined only on a confinal family of coverings with the property that any cell on M_1 and M_2 is also on M_{12}, and that any cell on $[(S-P)\cap M_1]\cup(F\cap M_1)$ and on $[(S-P)\cap M_2]\cup(F\cap M_2)$ is also on $[(S-P)\cap M_{12}]\cup F_{12}$.

First consider a cycle $\tilde{z}^n(\mathfrak{V})$ on M mod $[(S-P)\cap M]\cup F$. Since $\mathfrak{V}$ is a normal refinement of $\mathfrak{U}$, $\pi^{\mathfrak{U}}_{\mathfrak{V}}\tilde{z}^n(\mathfrak{V})$ is the coordinate in $\mathfrak{U}$ of a Cech cycle $z^n=\{z^n(\mathfrak{W})\}$ on M mod $[(S-P)\cap M]\cup F$, i. e. $z^n(\mathfrak{U})=\pi^{\mathfrak{U}}_{\mathfrak{V}}\tilde{z}^n(\mathfrak{V})$. For each $\mathfrak{W}$, let $z^n(\mathfrak{W})=z^n_1(\mathfrak{W})+z^n_2(\mathfrak{W})$ where $z^n_1(\mathfrak{W})$ is on M_1 and $z^n_2(\mathfrak{W})=z^n(\mathfrak{W})-z^n_1(\mathfrak{W})$ is on M_2. Since $\partial z^n(\mathfrak{W})=\partial z^n_1(\mathfrak{W})+\partial z^n_2(\mathfrak{W})$ is (a cycle) on $[(S-P)\cap M]\cup F$, we can write $\partial z^n(\mathfrak{W})=\gamma^{n-1}(\mathfrak{W})=\gamma^{n-1}_1(\mathfrak{W})+\gamma^{n-1}_2(\mathfrak{W})$ where $\gamma^{n-1}_1(\mathfrak{W})$ is on $[(S-P)\cap M_1]\cup(F\cap M_1)\subset[(S-P)\cap M_1]\cup F_1$ and $\gamma^{n-1}_2(\mathfrak{W})=\gamma^{n-1}(\mathfrak{W})-\gamma^{n-1}_1(\mathfrak{W})$ is on $[(S-P)\cap M_2]\cup(F\cap M_2)\subset[(S-P)\cap M_2]\cup F_2$. It follows by the usual argument that $\gamma^{n-1}_{12}(\mathfrak{W})=[\partial z^n_1(\mathfrak{W})-\gamma^{n-1}_1(\mathfrak{W})]=-[\partial z^n_2(\mathfrak{W})-\gamma^n_2(\mathfrak{W})]$ is on $M_{12}\subset F_1\cap F_2$; hence $z^n_j(\mathfrak{W})$ is a cycle on M_j mod $[(S-P)\cap M_j]\cup F_j$. Furthermore $\partial\gamma^{n-1}_{12}(\mathfrak{W})=-\partial\gamma^{n-1}_1(\mathfrak{W})=+\partial\gamma^{n-1}_2(\mathfrak{W})$; therefore $\partial\gamma^{n-1}_{12}(\mathfrak{W})$ is on $[(S-P)\cap M_{12}]\cup F_{12}$ and $\gamma^{n-1}_{12}(\mathfrak{W})$ is a cycle on M_{12} mod $[(S-P)\cap M_{12}]\cup F_{12}$. By an entirely analogous argument it follows that $z^n_j=\{z^n_j(\mathfrak{W})\}$ and $\gamma^{n-1}_{12}=\{\gamma^{n-1}_{12}(\mathfrak{W})\}$ are Cech cycles mod $[(S-P)\cap M_j]\cup F_j$ and $[(S-P)\cap M_{12}]\cup F_{12}$, respectively.

Let us suppose for the moment that $\gamma^{n-1}_{12}\sim 0$ mod $[(S-V)\cap M_{12}]\cup F_{12}$; i.e. for each $\mathfrak{W}$ there exist chains $z^n_{12}(\mathfrak{W})$ on M_{12} and $x^{n-1}_{12}(\mathfrak{W})$ on $[(S-V)\cap M_{12}]\cup F_{12}$ such that $\partial z^n_{12}(\mathfrak{W})=\gamma^{n-1}_{12}(\mathfrak{W})+x^{n-1}_{12}(\mathfrak{W})$. Let $C^n_1(\mathfrak{W})=z^n_1(\mathfrak{W})-z^n_{12}(\mathfrak{W})$ and $C^n_2(\mathfrak{W})=z^n_2(\mathfrak{W})+z^n_{12}(\mathfrak{W})$, then $\partial C^n_1(\mathfrak{W})=\gamma^{n-1}_1(\mathfrak{W})-x^{n-1}_{12}(\mathfrak{W})$ where γ_1 is on $[(S-P)\cap M_1]\cup(F\cap M_1)$ and x_{12} is on $[(S-Q)\cap M_{12}]\cup F_{12}$ and $\partial C^n_2(\mathfrak{W})=\gamma^{n-1}_2(\mathfrak{W})+x^{n-1}_{12}(\mathfrak{W})$ where γ_2 is on $[(S-P)\cap M_2]\cup(F\cap M_2)$. This shows that $C^n_j(\mathfrak{W})$ is a cycle on M_j mod $[(S-Q)\cap M_j]\cup(F\cap M_j)$. In particular this is true for $C^n_j(\mathfrak{V})$; hence $\pi^{\mathfrak{U}}_{\mathfrak{V}}C^n_j(\mathfrak{V})\sim 0$ mod $[(S-R)\cap M_j]\cup(F\cap M_j)$ and $\pi^{\mathfrak{U}}_{\mathfrak{V}}(C^n_1(\mathfrak{V})+C^n_2(\mathfrak{V}))\sim 0$ mod $[(S-R)\cap M]\cup F$. Since $C^n_1(\mathfrak{V})+$

$+ C_2^n(\mathfrak{V}) = z_1^n(\mathfrak{V}) + z_2^n(\mathfrak{V}) = z^n(\mathfrak{V})$, we conclude that $\pi_{\mathfrak{V}}^{\mathfrak{U}} z^n(\mathfrak{V}) \sim 0$ mod $[(S - R) \cap M] \cup F$. Finally since z^n is a Cech cycle on M mod $[(S-P) \cap M] \cup F$, $\pi_{\mathfrak{V}}^{\mathfrak{U}} z^n(\mathfrak{V}) \sim z^n(\mathfrak{U}) = \pi_{\mathfrak{V}}^{\mathfrak{U}} \tilde{z}^n(\mathfrak{V})$ mod $[(S - P) \cap M] \cup F$ and $\pi_{\mathfrak{V}}^{\mathfrak{U}} \tilde{z}^n(\mathfrak{V}) \sim 0$ mod $[(S - P) \cap M] \cup F$.

Now let $\tilde{z}_1^n(\mathfrak{V})$ and $\tilde{z}_2^n(\mathfrak{V})$ be any two cycles mod $[(S - P) \cap M] \cup F$. As in the argument of the preceding two paragraphs we can write $z_1^n = z_{11}^n + z_{12}^n$, $z_2^n = z_{21}^n + z_{22}^n$ and obtain cycles γ_{112}^{n-1} and γ_{212}^{n-1} mod $[(S - P) \cap M_{12}) \cup F_{12}$. By our choice of R, there exist integers m_1 and m_2, not both zero, such that $m_1 \gamma_{112}^{n-1} + m_2 \gamma_{212}^{n-1} \sim 0$ on M_{12} mod $[(S - Q) \cap M_{12}] \cup F_{12}$. The argument of the preceding paragraph with $\gamma_{12}^{n-1} = m_1 \gamma_{122}^{n-1} + m_2 \gamma_{212}^{n-1}$ is now applicable. This leads to the conclusion that $\pi_{\mathfrak{V}}^{\mathfrak{U}}(m_1 \tilde{z}_1^n(\mathfrak{V}) + m_2 \tilde{z}_2^n(\mathfrak{V})) = m_1 \pi_{\mathfrak{V}}^{\mathfrak{U}} \tilde{z}_1^n(\mathfrak{V}) + m_2 \pi_{\mathfrak{V}}^{\mathfrak{U}} \tilde{z}_2^n(\mathfrak{V}) \sim 0$ mod $[(S - R) \cap M] \cup F$, wich is the conclusion of the theorem.

Theorem 6. *Under assumptions A, if $R_n(p, M_j, F_j) = 1$ for all $p \in M_j$, $R_{n-1}(p, M_{12}, F_{12}) \leq 1$ and $R_n(p, M_j, M_j \cap F) = 0$ for all $p \in M_{12}$, then $R_n(p, M, F) = 1$ for all $p \in M$.*

Proof. This follows directly from theorem 4 and 5.

Definition 5. A closed set $M \subset S$ (compact) is called a *generalized n-manifold with boundary relative to S* if

a) $\dim M = n$

b) M is lc_{n-1}

c) M is lc^{n-1}

d) $R_n(p, M, F) = 1$ for all $p \in M$

e) $R_n(p, M) = R_n(p, M, 0) = 0$ for all $p \in F =$ the boundary of M relative to S.

Definition 6. A compact space M is called a *generalized closed n-manifold* if it satisfies a, b, and c of definition 5, together with d') $R_n(p, M) = 1$ for all $p \in M$.

Note that this is equivalent to definition 5 where $M = S$, for then $F = 0$ causing condition d to reduce to d' and condition e to be meaningless. This is the definition given by E. G. Begle in [1] without the orientability.

Teorem 7. *The property of having dim n is additive.*

Proof. This follows directly from the « Sum theorem for Dim n » [see 2, p. 30] which tells ut that the union of two closed sets each with dim n also has dim n (regardless of their intersection).

Theorem 8. *Under assumptions A, if M_j is a generalized n-manifold with boundary relative to S $(j = 1, 2)$, M_{12} is a generalized $(n - 1)$-manifold with boundary relative to $\overline{S - M} \cup M_{12}$, $R_n(p, M_j, M_j \cap F) = 0$ for all $p \in M_{12}$ $(j = 1, 2)$, then M is a generalized n-manifold with boundary relative to S.*

Proof. Property a) follows since $dim\ M = n$ is additive. Properties b) and c) follow from theorems 1′ and 3′. Property d) for M_j gives $R_n(p, M_j, F_j) = 1$ for all $p \in M_j$. Property d) for M_{12} gives $R_{n-1}(p, M_{12}, F_{12}) = 1$ for all $p \in M_{12}$ since $F_{12} =$ boundary of M_{12} relative to $\overline{S - M} \cup M_{12}$. To see that the latter statement holds consider $F_{12} = F \cap M_{12}$ and any neighborhood $U \supset p$. Since $p \in F$, there exists $q \in U \cap (\overline{S - M})$, hence $q \in [\overline{S - M} \cup M] - M_{12}$, and $p \in$ boundary of M_{12} relative to $\overline{S - M} \cup M_{12}$. Conversely if $p \in$ boundary of M_{12} relative to $\overline{S - M} \cup M_{12}$, then $p \in M_{12}$ since M_{12} is closed and any neighborhood $U \supset p$ also $\supset q \in [\overline{S - M} \cup M_{12}] - M_{12} \subset \overline{S - M}$. Let $V \subset U$ be a neighborhood of q, then there exists an $r \in V \cap S - M$; thus $p \in \overline{S - M}$ which together with $p \in M_{12} \subset M$ implies $p \in F \cap M_{12} = F_{12}$. Since $R_n(p, M_j, M_j \cap F) = 0$ is assumed, condition d), $R_n(p, M, F) = 1$ for all $p \in M$ follows from theorem 6. Condition e) for M_j and M_{12} gives $R_n(p, M_j) = 0$ for all $p \in F_j$ and $R_{n-1}(p, M_{12}) = 0$ for all $p \in F_{12}$. We must show that $R_n(p, M) = 0$ for all $p \in F \subset F_1 \cup F_2$. If $p \in (F_1 \cup F_2) - F_{12}$, this follows directly from $R_n(p, M_j) = 0$ for all $p \in F_j$. If $p \in F_{12}$, the result follows exactly as in theorem 3 since the condition n-colc at p and $R_n(p, M) = 0$ are so nearly the same. We have now shown that M satisfies all the properties of an n-manifold with boundary relative to F.

If we require that M be imbedded in a compact subset of Euclidean n-space, then the assumptions A follow from the dimensionalities of M_j and M_{12}, i.e. $F_M(M_1) = F_M(M_2) = M_{12}$. Since $F_M(M_j) \subset M_{12}$ in any case, consider $p \in M_{12}$ such that $p \notin F_M(M_1)$ or $F_M(M_2)$; therefore there exist neighborhood U_1 and U_2 of p such that $M \cap U_j \subset M_j$. Choose $U \supset p$ such that $M \cap U \subset M \cap (U_1 \cap U_2) \subset M_{12}$, but it is impossible in Euclidean n-space for the $(n-1)$-dimensional set M_{12} to contain a set like $M \cap U$ wich is open in M_{12} (see theorem IV 3 of [2]).

Theorem 9. *If M_1 and M_2 are generalized n-manifolds with boundaries relative to then n-dimensional space M such that $M_1 \cap M_2$ is the common boundary of M_1 and M_2 and is a generalized closed $(n-1)$-manifold, then M is a generalized closed n-manifold.*

Proof. By hypothesis $F_1 = F_2 = F_M(M_1) = F_M(M_2) = M_{12}$, $F_{12} = F = 0$. Conditions a, b, and c for M follow as in theorem 7. Condition d for M_j and d' for M_{12} give $R_n(p, M_j, F_j) = 1$ for all $p \in M_j$, and $R_{n-1}(p, M_{12}) = 1$ for all $p \in M_{12}$. Also since $F = 0$ the condition $R_n(p, M_j, M_j \cap F) = 0$ for all $p \in M_{12}$ is equivalent to $R_n(p, M_j) = 0$ which is given by condition e) for M_j. Now all the hypothesis of theorem 6 are satisfied and we conclude that $R_n(p, M, F) = R_n(p, M) = 1$ for all $p \in M$, which is condition d') for M.

Definition 7. A generalized n-manifold M with boundary relative to S is called *orientable* if the n dimensional Betti number of M mod $F = p^n(M, F) = 1$ irreducibly (i. e. $p^n(M, F) = 1$; but if $L \subset E$, $L \subset M$, $E \subset F$, are closed sets such that at least one of the last two inclusions is proper, then $p^n(L, E) = 0$).

Definition 8. A generalized closed n-manifold M is called *orientable* if $p^n(M) = 1$ irreducibly (see [1]).

Teorem 10. *Under assumptions A, if M_j is an orientable generalized n-manifold with boundary relative to S $(j = 1, 2)$, M_{12} is an orientable generalized $(n-1)$-manifold with boundary relative to $\overline{S - M} \cup M_{12}$, and $R_n(p, M_j, M_j \cap F) = 0$ for all $p \in M_{12}$ $(j = 1, 2)$, then M is an orientable generalized n-manifold with boundary relative to S.*

Proof. Everything but the orientability follows from theorem 7. To show M is orientable we see that $p^n(M_j, F_j) = 1$ $(j = 1, 2)$ since M_j is orientable and $p^{n-1}(M_{12}, F_{12}) = 1$ since M_{12} is orientable and the observation as in theorem 7 that $F_{12} =$ boundary of M_{12} relative to $\overline{S - M} \cup M_{12}$. Note that $M_j \cap F \subset F_j$, but $M_j \cap F \neq F_j$ $(j = 1, 2)$. To verify this suppose, for example, $M_1 \cap F = F_1$, then $M_{12} \subset F_1$ which implies $M_{12} \subset M_1 \cap F \subset F$. It follows that $F_{12} = M_{12} \cap F = M_{12}$; hence $p^{n-1}(M_{12}, F_{12}) = 0$ which is contrary to hypothesis. This shows that $M_j \cap F$ is a proper subset of F_j; hence $p^n(M_j, M_j \cap F) = 0$ $(j = 1, 2)$. Now $p^n(M, F) = 1$ follows from the analogue of theorem 6 in the large. Actually the same proof could be used where all open sets chosen in the various definitions are taken as the interior of M (which is non-vacuous since $p^n(M_j, F_j) = 1$).

To show the rest of the orientability condition, consider $L \subset M$, $E \subset F$, $E \subset L$, where one of the first two inclusions is proper, and let $L_j = L \cap M_j$, $L_{12} = L_1 \cap L_2 = L \cap M_1 \cap M_2 = L \cap M_{12}$, $F_j = (E \cap M_j) \cup L_{12} \subset (L \cap M_j) \cup L_{12} = L_j$, $E_j \subset (F \cap M_j) \cup M_{12} = F_j$, $E_{12} = E \cap M_{12} \subset F \cap M_{12} = F_{12}$. Note also that $E_1 \cap E_2 \subset L_1 \cap L_2 = L_{12}$, and $L_{12} \subset E_1 \cap E_2$; therefore $E_1 \cap E_2 = L_{12}$. This also shows $E_{12} \subset E \cap M_{12} = E \cap M_1 \cap M_2 \subset E_1 \cap E_2 = L_{12} \subset M_1 \cap M_2 = M_{12}$. We shall first consider the case where L is a proper subset of M and $(M - L) \cap M_{12} \neq 0$, then L_j is a proper subset of M_j $(j = 1, 2)$, for otherwise $L \supset L_j = M_j \supset M_{12}$ for at least one j. Also L_{12} is a proper subset of M_{12} since $L_{12} = L \cap M_{12} \neq M_{12}$ by $(M - L) \cap M_{12} \neq 0$. Using the orientability hypotheses on M_j and M_{12}, we have $p^n(L_j, E_j) = 0$, $p^{n-1}(L_{12}, E_{12}) = 0$ since L_j and L_{12} are proper subsets of M_j and M_{12}, respectively such that $E_j \subset L_j$ and $E_{12} \subset L_{12}$. That $p^n(L, E) = 0$ now follows from a proof that is almost identical with that of theorem 3 (where the sets chosen are all equal to the interior of L).

Finally consider the case where $(M - L) \cap M_{12} = 0$, i. e. $M_{12} = L \cap M_{12}$ or $M_{12} \subset L$; hence $M_{12} = L_{12}$. Since one of the inclusions $L \subset M$, $E \subset F$ must be proper, it follows that one of the four inclusions $L_j \subset M_j$, $E \cap M_j \subset F \cap M_j$ $(j = 1, 2)$ must be proper. Suppose that either $L_1 \subset M_1$ or $E \cap M_1 \subset F \cap M_1$ is proper, which implies, since $L_{12} = M_{12}$, that either

$L_1 \subset M_1$ or $E_1 = (E \cap M_1) \cup L_{12} \subset (F \cap M_1) \cup M_{12} = F_1$ is proper. Suppose that the cycles used only have coordinates on the confinal family guaranteed by lemma 1″ with the property that a cell on L_1 and on L_2 is also on $L_1 \cap L_2 = L_{12}$, and a cell on L_{12} and on $(E \cap M_1)$ is on $L_{12} \cap (E \cap M_1) = L_{12} \cap E \subset M_{12} \cap E = E_{12}$. Since $p^{n-1}(M_{12}, F_{12}) = 1$, it follows that F_{12} is a proper subset of M_{12} ($= L_{12}$ in this case) This in turn implies that $E \cap M_2$ is a proper subset of $E_2 = (E \cap M_2) \cup L_{12}$, for otherwise $L_{12} \subset E \cap M_{12} = E_{12} \subset F_{12}$, wich is a contradiction since F_{12} is a proper subset of $M_{12} = L_{12}$. It follows from the orientability of M_2 relative to F_2 that $p^n(L_2, E \cap M_2) = 0$. Now let us choose any covering $\mathfrak{U}$, then there exists $\mathfrak{V} > \mathfrak{U}$ (both from the above defined confinal family) such that if $C_2^n(\mathfrak{V})$ is any cycle on L_2 mod $E \cap M_2$, then $\pi^{\mathfrak{U}}_{\mathfrak{V}} C_2^n(\mathfrak{V}) \sim 0$ on L_2 mod $E \cap M_2$. Furthermore suppose that $\mathfrak{V}$ is a normal refinement of $\mathfrak{U}$ with respect to cycles on L mod E.

Now let $\tilde{z}^n(\mathfrak{V})$ be any cycle on L mod E, then $\pi^{\mathfrak{U}}_{\mathfrak{V}} \tilde{z}^n(\mathfrak{V}) = z^n(\mathfrak{U})$ is the coordinate on $\mathfrak{U}$ of a Čech cycle $z^n = \{z^n(\mathfrak{W})\}$ on L mod E. For each $\mathfrak{W}$ let $z^n(\mathfrak{W}) = z_1^n(\mathfrak{W}) + z_2^n(\mathfrak{W})$ where $z_1^n(\mathfrak{W})$ is the part of $z^n(\mathfrak{W})$ on L_1 and $z_2^n(\mathfrak{W}) = z^n(\mathfrak{W}) - z_1^n(\mathfrak{W})$ is on L_2. By hypothesis $\partial z^n(\mathfrak{W})$ is on E and we can write $\partial z^n(\mathfrak{W}) = \partial z_1^n(\mathfrak{W}) + \partial z_2^n(\mathfrak{W}) = z_1^{n-1}(\mathfrak{W}) + z_2^{n-1}(\mathfrak{W})$ where $z_1^{n-1}(\mathfrak{W})$ is the part of $\partial z^n(\mathfrak{W})$ on $E \cap M_1 \subset E_1$ and $z_2^{n-1}(\mathfrak{W}) = \partial z^n(\mathfrak{W}) - z_1^{n-1}(\mathfrak{W})$ is on $E \cap M_2$. It follows that $\partial z_1^n(\mathfrak{W}) - z_1^{n-1}(\mathfrak{W}) = -\partial z_2^n(\mathfrak{W}) + z_2^{n-1}(\mathfrak{W}) = z_{12}^{n-1}(\mathfrak{W})$ where the left hand side is on L_1 and the right hand side is on L_2; hence by the choice of our confinal family, both sides $= z_{12}^{n-1}(\mathfrak{W})$ are on L_{12} for each $\mathfrak{W}$. This shows that $z_1^n(\mathfrak{W})$ is a cycle on L_1 mod E_1 for $\partial z_1^n(\mathfrak{W}) = z_1^{n-1}(\mathfrak{W}) + z_{12}^{n-1}(\mathfrak{W})$ where the first term is on $E_1 \cap M$ and the second is on L_{12}; hence both are on $(E \cap M_1) \cup L_{12} = E_1$. By an entirely analogous argument, it follows that $z_1^n = \{z_1^n(\mathfrak{W})\}$ is a Čech cycle on L_1 mod E_1. Since M_1 is a generalized manifold with boundary and by hypothesis either L_1 is a proper subset of M_1 or E_1 is a proper subset of F_1, we have $p^n(L_1, E_1) = 0$. In particular $z_1^n \sim 0$ on L_1 mod E_1, and there exist chains $C^{n+1}(\mathfrak{W})$ on L_1 and $C^n(\mathfrak{W})$ on E_1 for each $\mathfrak{W}$ such that $\partial C^{n+1}(\mathfrak{W}) = z_1^n(\mathfrak{W}) - C^n(\mathfrak{W})$. By taking boundaries on both sides, we see that $\partial z_1^n(\mathfrak{W}) = \partial C^n(\mathfrak{W})$. Also let $C^n(\mathfrak{W}) = C_1^n(\mathfrak{W}) + C_{12}^n(\mathfrak{W})$ where $C_{12}^n(\mathfrak{W})$ is on L_{12} and $C_1^n(\mathfrak{W}) = C^n(\mathfrak{W}) - C_{12}^n(\mathfrak{W})$ is on $E_1 - L_{12} \subset E \cap M_1$, then $\partial C_1^n(\mathfrak{W}) + \partial C_{12}^n(\mathfrak{W}) = \partial C^n(\mathfrak{W}) = \partial z_1^n(\mathfrak{W}) = z_1^{n-1}(\mathfrak{W}) + z_{12}^{n-1}(\mathfrak{W})$. It follows that $\partial C_{12}^n(\mathfrak{W}) - z_{12}^{n-1}(\mathfrak{W}) = -\partial C_1^n(\mathfrak{W}) + z_1^{n-1}(\mathfrak{W})$ where the left hand side is on L_{12} and the right hand side is on $(E \cap M_1)$; thus by the choice of our confinal family both sides $= \gamma_{12}^{n-1}(\mathfrak{W})$, a chain on E_{12}. In particular $\partial C_{12}^n(\mathfrak{W}) = z_{12}^{n-1}(\mathfrak{W}) + \gamma_{12}^{n-1}(\mathfrak{W})$. Now let $C_2^n(\mathfrak{W}) = C_{12}^n(\mathfrak{W}) + z_2^n(\mathfrak{W})$, then $\partial C_2^n(\mathfrak{W}) = \partial C_{12}^n(\mathfrak{W}) + \partial z_2^n(\mathfrak{W}) = z_{12}^{n-1}(\mathfrak{W}) + \gamma_{12}^{n-1}(\mathfrak{W}) + z_2^{n-1}(\mathfrak{W}) - z_{12}^{n-1}(\mathfrak{W}) = \gamma_{12}^{n-1}(\mathfrak{W}) + z_2^{n-1}(\mathfrak{W})$ wich is on $E_{12} \cup (M_2 \cap E) = M_2 \cap E$; thus $C_2^n(\mathfrak{W})$ is a cycle on L_2 mod $(E \cap M_2)$ = a proper closed subset of L_2.

Since this is true for all $\mathfrak{W}$, we can let $\mathfrak{W} = \mathfrak{V}$. By the choice of $\mathfrak{V}$, $\pi^{\mathfrak{U}}_{\mathfrak{V}} C^n_2(\mathfrak{V}) \sim 0$ on L_2 mod $E \cap M_2$; i. e. there exists a chain $C^{n+1}_2(\mathfrak{U})$ on L_2 such that $\partial C^{n+1}_2(\mathfrak{U}) = \pi^{\mathfrak{U}}_{\mathfrak{V}} C^n_2(\mathfrak{V}) +$ (a chain on $E \cap M_2$). Now $\partial(\pi^{\mathfrak{U}}_{\mathfrak{V}} C^{n+}(\mathfrak{V}) + C^{n+1}_2(\mathfrak{U})) = \pi^{\mathfrak{U}}_{\mathfrak{V}} z^n_1(\mathfrak{V}) - \pi^{\mathfrak{U}}_{\mathfrak{V}} C^n(\mathfrak{V}) + \pi^{\mathfrak{U}}_{\mathfrak{V}} C^n_2(\mathfrak{V}) +$ (a chain on $E \cap M_2) = \pi^{\mathfrak{U}}_{\mathfrak{V}} z^n_1(\mathfrak{V}) \quad \pi^{\mathfrak{U}}_{\mathfrak{V}} C^n_1(\mathfrak{V}) - \pi^{\mathfrak{U}}_{\mathfrak{V}} C^n_{12}(\mathfrak{V}) + \pi^{\mathfrak{U}}_{\mathfrak{V}} C^n_{12}(\mathfrak{V}) + \pi^{\mathfrak{U}}_{\mathfrak{V}} z^n_2(\mathfrak{V}) +$ (a chain on $E \cap M_2) = \pi^{\mathfrak{U}}_{\mathfrak{V}} z^n(\mathfrak{V}) +$ (a chain on E); i. e. $\pi^{\mathfrak{U}}_{\mathfrak{V}} z^n(\mathfrak{V}) \sim 0$ on L mod E. Since z^n is a relative Cech cycle, $\pi^{\mathfrak{U}}_{\mathfrak{V}} z^n(\mathfrak{V}) \sim z^n(\mathfrak{U}) = \pi^{\mathfrak{U}}_{\mathfrak{V}} \tilde{z}^n(\mathfrak{V})$ on L mod E; hence $\pi^{\mathfrak{U}}_{\mathfrak{V}} z^n(V) \sim 0$ on L mod E, which shows that $p^n(L, E) = 0$. This shows that M is orientable in any case, and completes the proof of the theorem.

Theorem 11. *If M_1 and M_2 are orientable generalized n manifolds with boundaries relative to the n-dimensional space M, such that $M_1 \cap M_2$ is the common boundary of M_1 and M_2 and is an orientable generalized closed $(n-1)$-manifold, the M is an orientable generalized closed n-manifold.*

Proof. All but the orientability follows from theorem 8, and the orien tability follows from the proof in theorem 9.

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