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POLYGENIC FUNCTIONS IN GENERAL ANALYSIS

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INTRODUCTION - The theory of non-analytic or polygenic functions of a complex variable seems to have been begun by RIEMANN, and has been carried on by KASNER, CALUGARÉANO and others. Unfortunately the treatment of classical polygenic functions does not generalise very well to complex BANACH spaces. For this reason we have had to use a slightly different approach in our theory of polygenic functions of complex BANACH variables.

Due to the inadequacy of the present theory of GATEAUX differential equations, we were unable to give the complete characterization of the differential of polygenic functions, possessing only a continuous GATEAUX differential. We were however able to do this for a certain class of polygenic functions.

Many new problems have been suggested by our present work, and these we have tried to indicate throughout the paper.

Section 1. - *Analytic Functions in General Analysis.*

A. E. TAYLOR (1) has generalised the CAUCHY-RIEMANN equations of classical complex analysis to complex BANACH spaces. For earlier work in analytic functions in general analysis the reader is referred to MICHAL and MARTIN [4] and MARTIN [9]. Before proceeding with our own theory of polygenic functions, we should like to give a brief summary of some of TAYLOR's results.

Let E be a real BANACH space in which there exists a function $[x, y]$ with the following properties; (2)

1) $[x, y]$ is a bilinear function on E^2 to the real numbers,
2) $[x, y]=[y, x]$,
3) $[x, x]\geqq 0$ and $[x, x]=0$ if and only if $x=0$,
4) $||x||^2=[x, x]$.

(1) See TAYLOR [2] - [3].

(2) TAYLOR does not make this restriction on E, but we shall show that it is a necessary one.

From E, we can construct a complex BANACH space in the following way. Let $E(C)$ be the set of all couples $\{x, y\}$, where x, y are elements of E. We define

a) $\{x_1, y_1\}=\{x_2, y_2\}$ if and only if $x_1=x_2$, $y_1=y_2$.
b) $\{x_1, y_1\}+\{x_2, y_2\}=\{x_1+x_2, y_1+y_2\}$,
c) $(a+ib)\{x, y\}=\{ax-by, bx+ay\}$, where a, b are real numbers,
d) $\| \{x, y\}\|=\sqrt{\|x\|^2+\|y\|^2}$.

TAYLOR does not require E to posses the function $[x, y]$, and states that $E(C)$ as defined above forms a complex BANACH space. This however is not true in general for the following reason. One of the postulates that $E(C)$ must satisfy is that

$$\|(a+ib)\{x, y\}\|=\sqrt{a^2+b^2}\,\|\{x, y\}\|. \tag{1.1}$$

But

$$\|(a+ib)\{x, y\}\|=\|\{ax-by, bx+ay\}\|=\sqrt{\|ax-by\|^2+\|bx+ay\|^2}, \tag{1.2}$$

and (1.2) is not in general equal to $\sqrt{a^2+b^2}\,\sqrt{\|x\|^2+\|y\|^2}$. In fact it can easily be shown that (1.1) implies

$$\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2), \tag{1.3}$$

and Von NEUMANN and JORDAN ([3]) have shown that (1.3) implies the existence of a function $[x, y]$ on E^2 to the real numbers with properties 1-4. With the added restriction we have made on E, it can be verified that $E(C)$ is a complex BANACH space ([4]). It might be pointed out that $E(C)$ also possesses a function $[Z, U]$ on $E^2(C)$ to the complex numbers satisfying properties 3-4 and

1′) $[Z, U]$ is complex number valued and is additive and continuous in each place

2′) $\overline{[Z, U]}=[U, Z]$, $[(a+ib)Z, U]=(a+ib)[Z, U]$.

For if $Z=\{x, y\}$, and $U=\{\xi, \eta\}$, we define $[Z, U]=[x, \xi]+[y, \eta]+i([y, \xi]-[x, \eta])$ and this has the required properties.

Let $z=\{x, y\}=\{x, 0\}+i\{y, 0\}$ be any element of $E(C)$. Since there is a simple isomorphism between the elements $\{x, 0\}$ and the elements of E, we write z

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([3]) See Von NEUMANN and JORDAN [8].

([4]) It would be interestring to know what definition should replace d), which would make $E(C)$ a complex BANACH space and place no restiction on the norm of E.

in the form $z=x+iy$. If $E(C)$, $E'(C)$ are two complex BANACH spaces of the above type, a function $f(z)$ on a domain $D(C)\subset E(C)$ to $E'(C)$ can be defined in the usual way. Any function $f(z)$ can then be written in the form $f(z)=f_1(x, y)+if_2(x, y)$.

DEFINITION 1.1. - A function $f(z)$ on $D(C)\subset E(C)$ to $E'(C)$ is said to be analytic if it is continuous throughout $D(C)$, and possesses a unique GATEAUX differential [5] at each point of $D(C)$.

TAYLOR has shown that a necessary and sufficient condition, that $f(z)=$ $=f_1(x, y)+if_2(x, y)$ be analytic in a domain is that

1) $f_1(x, y)$, $f_2(x, y)$ be continuous in the pair (x, y), and possess continuous first GATEAUX differentials at all points of the domain,

2) the equations [6] (TAYLOR's abstract CAUCHY-RIEMANN equations)

$$\delta_\xi^x f_1(x, y)=\delta_\xi^y f_2(x, y), \quad \delta_\xi^y f_1(x, y)=-\delta_\xi^x f_2(x, y) \tag{1.4}$$

be satisfied.

Section 2. - *Polygenic functions.*

DEFINITION 2.1. - By a polygenic function $f(z)=f_1(x, y)+if_2(x, y)$ on $D(C)$ to $E'(C)$ we mean a function for which $f_1(x, y)$, $f_2(x, y)$ are continuous in the pair (x, y) for all points of $D(C)$, and for which the total GATEAUX differentials

$$\lim_{\alpha\to 0}\frac{f_i(x+\alpha\xi,\ y+\alpha\eta)-f_i(x, y)}{\alpha} \qquad (i=1, 2) \tag{2.1}$$

exist, continuous in the pair (x, y), for all points of $D(C)$ and arbitrary ξ, η of E.

We shall write (2.1) in the form $\delta_{\xi\eta}^{xy} f_i(x, y)$. With this notation we see that $\delta_{\xi 0}^{xy} f_i(x, y)=\delta_\xi^x f_i(x, y)$ and $\delta_{0\eta}^{xy} f_i(x, y)=\delta_\eta^y f_i(x, y)$.

THEOREM 2.1. - Let $f(z)=f_1(x, y)+if_2(x, y)$ be a polygenic function on $D(C)$ to $E'(C)$, and let $\Delta z=\xi+i\eta$. If $\tau=\alpha+i\beta$ is a sufficiently small complex number in modulus, then $z+\tau\Delta z$ is in $D(C)$. Further the $\lim\limits_{\tau\to 0}\dfrac{f(z+\tau\Delta z)-f(z)}{\tau}$ exists, continuous in z for all paths of approach of τ to zero, and any chosen Δz in $E(C)$.

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[5] By the GATEAUX differential of $f(z)$, we mean $\lim\limits_{\tau\to 0}\dfrac{f(z+\tau\Delta z)-f(z)}{\tau}$.

[6] The notation $\delta_\xi^x f(x, y)$ means the GATEAUX differential $\lim\limits_{\alpha\to 0}\dfrac{f(x+\alpha\xi, y)-f(x, y)}{\alpha}$. Similarly for $\delta_\xi^y f(x, y)$.

Proof. - Let τ tend to zero along the path $\beta=\alpha\tan\Phi$, then

$$(2.2)\quad \frac{f(z+\tau\Delta z)-f(z)}{\tau}=\frac{1}{1+i\tan\Phi}\left\{\frac{f_1(x+\alpha(\xi-\tan\Phi\cdot\eta),\, y+\alpha(\eta+\tan\Phi\cdot\xi))-f_1(x,y)}{\alpha}+\right.$$
$$\left.+i\,\frac{f_2(x+\alpha(\xi-\tan\Phi\cdot\eta),\, y+\alpha(\eta+\tan\Phi\cdot\xi))-f_2(x,y)}{\alpha}\right\}.$$

But by (2.1), the limit of the right hand side exists continuous in the pair (x, y) and hence the theorem is proven.

The following two things should be noticed. First the limit of (2.2) as τ tends to zero is not in general independent of the angle Φ, and secondly if E is taken to be the space of real numbers our theory does not reduce directly to the classical polygenic theory. In this case however we do obtain the classical theory by restricting Φ to be zero.

DEFINITION 2.2. - Let $f(z)$ be a polygenic function, and let $\tau=\alpha+i\beta$ be a complex number which approaches zero along the path $\beta=\alpha\tan\Phi$. Then

$$(2.3)\qquad \lim_{\tau\to 0}\frac{f(z+\tau\Delta z)-f(z)}{\tau}$$

shall be written $\delta f(z;\,\Delta z,\,\Phi)$ and shall be called the directional GATEAUX differential of $f(z)$.

Under the above hypotheses, it follows from TAYLOR's work ([7]) that

$$f_i(x+s\xi-t\eta,\, y+t\xi+s\eta)-f_i(x,y)=s\delta^x_\xi f_i-t\delta^x_\eta f_i+t\delta^y_\xi f_i+s\delta^y_\eta f_i+\varepsilon_i(s,t)\qquad (i=1,2),$$

where $\lim\limits_{s,t\to 0}\dfrac{\|\varepsilon_i(s,t)\|}{|s|+|t|}=0$. Thus (2.3) takes the form

$$(2.4)\quad \delta f(z;\,\Delta z,\,\Phi)=\frac{1}{1+i\tan\Phi}\{\delta^x_\xi f_1+\delta^y_\eta f_1+i(\delta^x_\xi f_2+\delta^y_\eta f_2)+$$
$$+\tan\Phi(-\delta^x_\eta f_1+\delta^y_\xi f_1+i(\delta^y_\xi f_2-\delta^x_\eta f_2))\},$$

and further (2.4) can be reduced to

$$(2.5)\qquad \delta f(z;\,\Delta z,\,\Phi)=D(f(z),\,\Delta z)+e^{-2i\Phi}P(f(z),\,\Delta z)$$

where

$$(2.6)\quad D(f(z),\,\Delta z)=\frac{1}{2}\{\delta^x_\xi f_1+\delta^y_\eta f_1-\delta^x_\eta f_2+\delta^y_\xi f_2+i(\delta^x_\xi f_2+\delta^y_\eta f_2+\delta^x_\eta f_1-\delta^y_\xi f_1)\}$$

$$(2.7)\quad P(f(z),\,\Delta z)=\frac{1}{2}\{\delta^x_\xi f_1+\delta^y_\eta f_1+\delta^x_\eta f_2-\delta^y_\xi f_2+i(\delta^x_\xi f_2+\delta^y_\eta f_2-\delta^x_\eta f_1+\delta^y_\xi f_1)\}.$$

THEOREM 2.2 - Let $f(z)=f_1(x,y)+if_2(x,y)$ be a polygenic function, and let $D(f(z),\,\Delta z)$, $P(f(z),\,\Delta z)$ be defined by (2.6), (2.7). Then a necessary and

([7]) TAYLOR [2].

sufficient condition that these be simultaneously linear functions of Δz is that $\delta_\xi^x f_i$, $\delta_\eta^y f_i$ $(i=1, 2)$ be linear in their respective increments.

Proof. - Since $\Delta z = \xi + i\eta$ we have for $\eta = 0$ that $\Delta z = \xi$, and

$$P(f(z), \xi) + D(f(z), \xi) = \delta_\xi^x f_1 + i\delta_\xi^x f_2. \tag{2.8}$$

Thus if $P(f(z), \Delta z)$, $D(f(z), \Delta z)$ are both linear in Δz, then the left side of (2.8) is linear in ξ, and this implies $\delta_\xi^x f_i$ are both linear in ξ. Similarly $\delta_\eta^y f_i$ can be shown to be linear in η. This proves the necessity of the condition. The sufficiency can be verified directly.

THEOREM 2.3. - A necessary and sufficient condition that a polygeni function $f(z)$ on $D(C)$ to $E'(C)$ be analytic throughout $D(C)$ is that $P(f(z), \Delta z) = 0$ for all $z \varepsilon D(C)$ and all $\Delta z \varepsilon E(C)$.

The proof of this theorem is obvious.

If $f(z)$ is a polygenic function we have by (2.5)

$$\| \delta f(z; \Delta z, \Phi) - D(f(z), \Delta z) \| = \| P(f(z), \Delta z) \|. \tag{2.9}$$

Thus (2.9) tells us that for fixed z and Δz all the values of $\delta f(z; \Delta z, \Phi)$ lie on a BANACH sphere.

DEFINITION 2.3. - For fixed z_0 and Δz_0, the BANACH sphere (2.9) is called the differential sphere of $f(z)$ at z_0.

We note the following properties of the differential sphere.

I. The differential sphere of $f(z)$ is a point at each point of $D(C)$, if and only if $f(z)$ is analytic in $D(C)$.

This follows theorem 2.3.

II. If all of the differential spheres of $f(z)$ have the same center then the center must be the zero element of $E'(C)$. This is true since $D(f(z), 0) = 0$. Hence if all the differential spheres of $f(z)$ have a common center we must have $D(f(z), \Delta z) = 0$ for all $z \varepsilon D(C)$ and $\Delta z \varepsilon E(C)$. This implies $\delta_\xi^x f_1 = -\delta_\xi^y f_2$, $\delta_\xi^y f_1 = \delta_\xi^x f_2$, which means that $f(z)$ is an analytic function of the complex conjugate of z.

Section 3. - *Theorems on Derivatives and Differentials.*

In sections (1) and (2) of this paper we have only required that $f_i(x, y)$ $(i=1, 2)$ possess continuous total GATEAUX differentials. We shall now make the more restrictive condition that $f_i(x, y)$ possess continuous total FRÉCHET differentials $d_{\xi\eta}^{xy} f_i(x, y)$. This of course implies that the partial FRÉCHET differentials [8] $d_\xi^x f_i(x, y)$, $d_\eta^y f_i(x, y)$ exist continuous in (x, y) and satisfy the relations

$$d_{\xi\eta}^{xy} f_i(x, y) = d_\xi^x f_i(x, y) + d_\eta^y f_i(x, y), \qquad (i=1, 2). \tag{3.1}$$

(8) We use the notation $d_\xi^x f(x, y)$ to denote partial FRÉCHET differentials.

The remaining portion of this section shall be used to prove several isolated theorems which we shall use later on.

THEOREM 3.1. - Let x, y be real variables, and let $f(x, y)$ be a function whose values lie in a BANACH space Z. If $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $\frac{\partial^2 f}{\partial x \partial y}$ exist continuous in the pair (x, y) for $|x-a|<H$ and $|y-b|<K$, then $\frac{\partial^2 f}{\partial y \partial x}$ exists for these intervals and is equal to $\frac{\partial^2 f}{\partial x \partial y}$.

Proof. - By integration we verify that

$$f(x, y)=f(a, y)+\int_a^x \frac{\partial f(u, y)}{\partial u} du, \tag{3.2}$$

$$f(x, y)=f(x, b)+\int_b^y \frac{\partial f(x, v)}{\partial v} dv. \tag{3.3}$$

Calculating $f(a, y)$ and $\frac{\partial f(u, y)}{\partial u}$ from (3.3), (3.2) becomes

$$f(x, y)=f(a, b)+\int_b^y \frac{\partial f(a, v)}{\partial v} dv+\int_a^x \frac{\partial f(u, b)}{\partial u} du+\int_a^x \left\{ \frac{\partial}{\partial u}\int_b^y \frac{\partial f(u, v)}{\partial v} dv \right\} du. \tag{3.4}$$

By theorem 1.8 of MICHAL-ELCONIN [1] we have

$$\frac{\partial}{\partial u}\int_b^y \frac{\partial f(u, v)}{\partial v} dv=\int_b^y \frac{\partial^2 f(u, v)}{\partial u \partial v} dv. \tag{3.5}$$

Using this in (3.4), we can calculate $\frac{\partial^2 f}{\partial y \partial x}$ from the resulting expression, and we find

$$\frac{\partial^2 f}{\partial y \partial x}=\frac{\partial^2 f}{\partial x \partial y}.$$

As an immediate consequence of this theorem we obtain the following theorem.

THEOREM 3.2. - Let x, y be elements of a BANACH space E, and let $f(x, y)$ be a function whose values lie in another BANACH space Z. If the FRÉCHET differentials $d^x_\xi f(x, y)$, $d^y_\eta f(x, y)$ and $d^x_\xi d^y_\eta f(x, y)$ exist continuous in (x, y) for neighborhoods $\|x-a\|<H$, $\|y-b\|<K$, then the FRÉCHET differential $d^y_\eta d^x_\xi f(x, y)$ exists and is equal to $d^x_\xi d^y_\eta f(x, y)$ for all x, y in these neighborhoods.

The proof is a direct application of theorem 3.1, and the fact that [9]

$$d^x_\xi d^y_\eta f(x, y)=\frac{\partial^2}{\partial t \partial s} f(x+t\xi, y+s\eta)\Big|_{t=s=0}.$$

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[9] See MICHAL and ELCONIN [1].

KERNER (10) has shown that if $f(x)$ is a function on an open subset of BANACH space E to a BANACH space Z, and if $f(x)$ has a continuous second FRÉCHET differential $d^x_\xi d^x_\eta f(x)$, then the latter is symmetrical in ξ and η. By means of theorem 3.1, we can extend this theorem to hold for GATEAUX differentials. This is most easily seen by putting $\delta^x_\xi \delta^x_\eta f(x)$ into the form (11)

$$\delta^x_\xi \delta^x_\eta f(x) = \frac{\partial^2}{\partial t \, \partial s} f(x + t\xi + s\eta) \Big|_{t=s=0}. \tag{3.7}$$

THEOREM 3.3. - Let x, y, z, ξ, η, u, v be elements of a BANACH space E and let the functions $\varphi(x, y, z)$, $\Psi(x, y, \xi)$ having values in another BANACH space Σ satisfy the following conditions

a) $\varphi(x, y, z)$, $\Psi(x, y, \xi)$ are continuous in the pair (x, y), and linear in z, ξ respectively, for all (x, y) such that $\sqrt{\|x-u\|^2 + \|y-v\|^2} < \varrho$ and arbitrary z, ξ.

b) $d^{xy}_{z\xi}\varphi(x, y, \eta)$, $d^{xy}_{z\xi}\Psi(x, y, \eta)$ exist continuous in (x, y) for all x, y, z, ξ of (a), and arbitrary η,

c) $d^y_z \varphi(x, y, \xi) = d^x_\xi \Psi(x, y, z)$, for all x, y, z, ξ of (a),

d) $d^x_z \varphi(x, y, \xi) = d^x_\xi \varphi(x, y, z)$, for all x, y, z, ξ of (a),

e) $d^y_z \Psi(x, y, \xi) = d^y_\xi \Psi(x, y, z)$, for all x, y, z, ξ of (a).

Then there exists a unique solution of the total FRÉCHET differential equation

$$d^{xy}_{z\xi} f(x, y) = \varphi(x, y, z) + \Psi(x, y, \xi) \tag{3.8}$$

up to an additive BANACH constant, for all x, y such that $\sqrt{\|x-u\|^2 + y - v\|^2} < \varrho$.

Proof. - Let us consider the real BANACH space $E(R)$ of all couples $Z = \{x, y\}$, with $x, y \in E$ and $\|Z\|$ defined to be $\sqrt{\|x\|^2 + \|y\|^2}$ (12). If in equations (3.8) we consider x, y and z, ξ as couples $Z = \{x, y\}$ and $U = \{z, \xi\}$, then (3.8) takes the form

$$d^Z_U F(Z) = \Phi(Z, U), \tag{3.9}$$

where $\Phi(Z, U)$ is a linear function of U, whose arguments are in $E(R)$ and weose values are in Σ. Conditions *a*) - *e*) imply $d^Z_N \Phi(Z, U) = d^Z_U \Phi(Z, N)$, for $\|Z - A\| < \varrho$, where $N = \{w, \eta\}$ and $A = \{u, v\}$. By theorem 3.2 of MICHAL-ELCONIN [1], (3.9) has unique solution up to an additive BANACH constant, and this implies the same is true for (3.8) for all x, y such that $\sqrt{\|x-u\|^2 + \|y-v\|^2} < \varrho$.

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(10) KERNER [7].

(11) TAYLOR [3].

(12) In $E(R)$ equality, addition and multiplication by real numbers is defined by the ordinary matrix operations. Thus we see $E(R)$ is a real sub-BANACH space of $E(C)$.

Let us define operators P, D by means of the following definitions.

$$D=\frac{1}{2}(d_\xi^x+d_\eta^y+i(d_\eta^x-d_\xi^y)) \tag{3.10}$$

$$P=\frac{1}{2}(d_\xi^x+d_\eta^y-i(d_\eta^x-d_\xi^y)). \tag{3.11}$$

By direct calculation we find that

$$Df(z)=D(f(z),\,\Delta z) \tag{3.12}$$

$$Pf(z)=P(f(z),\,\Delta z). \tag{3.13}$$

DEFINITION 3.1. - Let S denote the class of all polygenic functions $f(z)=f_1(x,y)+if_2(x,y)$ on $D(C)$ to $E'(C)$ such that the FRÉCHET differentials $d_\xi^x d_\eta^y f_i(x,y)$, $d_\xi^x d_\eta^x f_i(x,y)$ and $d_\xi^y d_\eta^y f_i(x,y)$, $(i=1,2)$, exist and are continuous in the pair (x,y).

THEOREM 3-4. - For all polygenic functions $f(z)$ belonging to S, the operators P and D commute. That is

$$PDf(z)=DPf(z). \tag{3.14}$$

This follows from KERNER's theorem and theorem 3.2.

Section 4. - *Characterization of the differential of polygenic functions which belong to S.*

Let $f(z)$ be any polygenic function belonging to S, and let

$$\begin{aligned} H(x,y,\xi,\eta)&=\frac{1}{2}(d_\xi^x f_1+d_\eta^y f_1-d_\eta^x f_2+d_\xi^y f_2),\\ K(x,y,\xi,\eta)&=\frac{1}{2}(d_\xi^x f_2+d_\eta^y f_2+d_\eta^x f_1-d_\xi^y f_1),\\ h(x,y,\xi,\eta)&=\frac{1}{2}(d_\xi^x f_1+d_\eta^y f_1+d_\eta^x f_2-d_\xi^y f_2),\\ k(x,y,\xi,\eta)&=\frac{1}{2}(d_\xi^x f_2+d_\eta^y f_2-d_\eta^x f_1+d_\xi^y f_1). \end{aligned} \tag{4.1}$$

By means of (2.6), (2.7) and (4.1) we obtain

$$\begin{aligned} D(f(z),\,\Delta z)&=H+iK,\\ P(f(z),\,\Delta z)&=h+ik. \end{aligned} \tag{4.2}$$

The following properties ef H, K, h, k can be verified by means of (4.1) and theorem 3.2.

Property I. $H(x, y, \xi, \eta)$, $K(x, y, \xi, \eta)$, $h(x, y, \xi, \eta)$ and $k(x, y, \xi, \eta)$ are all linear in the pair (ξ, η) and continuous in (x, y).

Property II.

a) $H(x, y, \eta, 0)-h(x, y, \eta, 0)=K(x, y, 0, \eta)+k(x, y, 0, \eta)$.

b) $H(x, y, 0, \eta)-h(x, y, 0, \eta)=-(K(x, y, \eta, 0)+k(x, y, \eta, 0))$.

c) $K(x, y, 0, \eta)-k(x, y, 0, \eta)=-(H(x, y, \eta, 0)+h(x, y, \eta, 0))$.

d) $K(x, y, \eta, 0)-k(x, y, \eta, 0)=H(x, y, 0, \eta))+h(x, y, 0, \eta)$.

Property III.

a) $d_\xi^x(H(x, y, \eta, 0)+h(x, y, \eta, 0))=d_\eta^x(H(x, y, \xi, 0)+h(x, y, \xi, 0))$.

b) $d_\xi^y(H(x, y, 0, \eta)+h(x, y, 0, \eta))=d_\eta^y(H(x, y, 0, \xi)+h(x, y, 0, \xi))$.

c) $d_\xi^x(H(x, y, 0, \eta)+h(x, y, 0, \eta))=d_\eta^y(H(x, y, \xi, 0)+h(x, y, \xi, 0))$.

d) $d_\xi^x(K(x, y, \eta, 0)+k(x, y, \eta, 0))=d_\eta^x(K(x, y, \xi, 0)+k(x, y, \xi, 0))$.

e) $d_\xi^y(K(x, y, 0, \eta)+k(x, y, 0, \eta))=d_\eta^y(K(x, y, 0, \xi)+k(x, y, 0, \xi))$.

f) $d_\xi^x(K(x, y, 0, \eta)+k(x, y, 0, \eta))=d_\eta^y(K(x, y, \xi, 0)+k(x, y, \xi, 0))$.

The domain of validity of all these properties is of course the points (x, y) such that $z=x+iy$ lies in $D(C)$.

Let $u+iv$ be a fixed point of $E(C)$. Then the totality of points $x+iy$ such that $\sqrt{\|x-u\|^2+\|y-v\|^2}<\varrho$ is of course a domain $\bar{D}(C)\subset E(C)$. For domains of this type we obtain the following theorem.

THEOREM 4.1. - Let $\varphi_1(z, \Delta z)=H(x, y, \xi, \eta)+iK(x, y, \xi, \eta)$, $\varphi_2(z, \Delta z)=$ $=h(x, y, \xi, \eta)+ik(x, y, \xi, \eta)$ be two polygenic funclions of z, such that H, K, h, and k satisfy properties I - II - III for all points of a domain $\bar{D}(C)$. Then there exists a unique polygenic fonction $f(z)=f_1(x, y)+if_2(x, y)$ (up to an additive BANACH constant), such that

1) $f(z)$ is an element of S,

2) $D(f(z), \Delta z)=\varphi_1(z, \Delta z)$,

3) $P(f(z), \Delta z)=\varphi_2(z, \Delta z)$,

for all $z\,\varepsilon\,\bar{D}(C)$ and arbitrary Δz.

Proof. - Consider the differential equations

$$d_{\xi\eta}^{xy}f_1(x, y)=H(x, y, \xi, 0)+h(x, y, \xi, 0)+H(x, y, 0, \eta)+h(x, y, 0, \eta), \tag{4.3}$$

$$d_{\xi\eta}^{xy}f_2(x, y)=K(x, y, \xi, 0)+k(x, y, \xi, 0)+K(x, y, 0, \eta)+k(x, y, 0, \eta). \tag{4.4}$$

Properties I - II - III and theorem 3.3 ensure solutions of (4.3), (4.4) for $f_1(x, y)$ and $f_2(x, y)$. Further we can obtain the following explicit expressions for the partial FRÉCHET differentials of these solutions.

$$
\begin{aligned}
d^x_\xi f_1 &= H(x, y, \xi, 0) + h(x, y, \xi, 0) \\
d^y_\eta f_1 &= H(x, y, 0, \eta) + h(x, y, 0, \eta) \\
d^x_\eta f_2 &= K(x, y, \eta, 0) + k(x, y, \eta, 0) \\
d^y_\xi f_2 &= K(x, y, 0, \xi) + k(x, y, 0, \xi).
\end{aligned} \tag{4.5}
$$

Obviously $f(z) = f_1(x, y) + if_2(x, y)$ is a polygenic function, and we can easily by means of property II and the linearity of H, K, h and k that

$$D(f(z), \Delta z) = H(x, y, \xi, \eta) + iK(x, y, \xi, \eta) = \varphi_1(z, \Delta z), \tag{4.6}$$

$$P(f(z), \Delta z) = h(x, y, \xi, \eta) + ik(x, y, \xi, \eta) = \varphi_2(z, \Delta z). \tag{4.7}$$

Since $\delta f(z; \Delta z, \varphi) = D(f(z), \Delta z) + e^{-2i\varphi} P(f(z), \Delta z)$, we see that properties I - II - III completely characterize the directional GATEAUX differential of all polygenic functions $f(z)$ on $\bar{D}(C)$ to $E'(C)$ which are elements of S.

Section 5. - *Example of a Polygenic Function.*

Let E be the space of all real valued LEBESGUE measurable functions $f(t)$ on an interval (a, b), such that the LEBESGUE integral of $f^2(t)$ exists for (a, b). Define

1) $[x, y] = \int_a^b x(t)y(t)dt,$

2) $\| x \|^2 = \int_a^b x^2(t)dt.$

Then E forms a suitable space from which we can form the complex BANACH space $E(C)$ as defined in section 1. As an example of a polygenic function in $E(C)$ we take

$$f(z) = \int_a^b \bar{z}(t)dt \tag{5.1}$$

where $z(t) = x(t) + iy(t)$, and $\bar{z}(t) = x(t) - iy(t)$. If $\tau = s + it$, and $\tau \to 0$ along $t = s \tan \varphi$ we have

$$\delta f(z; \Delta z, \varphi) = e^{-2i\varphi} \int_a^b \overline{\Delta z} dt = e^{-2i\varphi} f(\Delta z). \tag{5.2}$$

For this particular function

$$D(f(z), \Delta z) = 0, \tag{5.3}$$

and

$$P(f(z), \Delta z) = f(\Delta z). \tag{5.4}$$

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