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CURVE-SPACE TOPOLOGIES ASSOCIATED WITH VARIATIONAL PROBLEMS

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In the calculus of variations it is usual to say that a curve Γ gives a weak relative minimum to an integral

$$J(C) \equiv \int_a^b F(y(t), \dot{y}(t))dt$$

in a class K of curves if there is a representation

$$y^i = \gamma^i(t), \qquad a \leqq t \leqq b, \qquad (i = 1, \ldots, n)$$

of Γ and a positive ε such that $J(C) \geqq J(\Gamma)$ for all curves C of the class K which have representations

$$C: \quad y^i = y^i(t), \qquad a \leqq t \leqq b$$

such that

$$|y^i(t) - \gamma^i(t)| < \varepsilon$$

for $a \leqq t \leqq b$ and

$$|\dot{y}^i(t) - \dot{\gamma}^i(t)| < \varepsilon$$

except at corners.

While there can be no doubt of the usefulness of this definition, it would seem esthetically more desirable to develop an appropriate definition of weak neighborhood, and then to say that Γ gives a weak relative minimum to $J(C)$ in the class K if $J(C) \geqq J(\Gamma)$ for all curves C of the class K which lie in a sufficiently small weak neighborhood of Γ. The purpose of this note is primarily to discuss certain definitions of weak neighborhoods and the interrelations between them. However, to avoid conflict with the usual topological terminology, we shall speak of our neighborhoods as « first-order » neighborhoods, rather than calling them « weak » neighborhoods.

1. - Notation. — Henceforward y will denote a ν-tuple (or vector) $(y^1, \dots, y^\nu)$, and so will γ. Likewise $\dot{y}(t)$ will denote [1] $(\dot{y}^1(t), \dots, \dot{y}^\nu(t))$. The length of y will be denoted by $|y|$:

$$|y| = [(y^1)^2 + \dots + (y^\nu)^2]^{\frac{1}{2}}.$$

A system of continuous functions $y = y(t)$ (i. e., $y^i = y^i(t)$), $a \leqq t \leqq b$ will be called a representation of a continuous curve C. The concepts of change of parameter on a curve and of the length $\mathfrak{L}(C)$ of a rectifiable curve C need no elucidation here. Every rectifiable curve can be represented by equations $y = y(t)$ in which the functions $y^i(t)$ are absolutely continuous. Henceforth we assume that all representations mentioned are absolutely continuous and, unless the contrary is stated, all curves mentioned are non-degenerate. Moreover, if C is rectifiable it can be represented in terms of arc length as parameter: $y = y(s)$, $0 \leqq s \leqq \mathfrak{L}(C)$. A curve is of class D' if it has a representation $y = y(t)$, $a \leqq t \leqq b$ in which the functions $y(t)$ are continuous and the derivatives $y'(t)$ are continuous except at a finite number of discontinuities of the first kind (simple finite jumps), and $|y'(t)|$ is bounded from zero. Henceforth, in discussing curves of class D' we restrict ourselves without further mention to representations of the type just mentioned.

2. - Definitions of First-Order Neighborhoods. — The idea to which we desire to give precise expression is that a curve C is in a small first-order neighborhood of a curve Γ if a correspondence between the points of the curves exists under which corresponding points are near each other and the directions or derivatives at corresponding points also differ only by little. A first attempt would be as follows. The curve C is in the first-order ε-neighborhood of Γ if the curves have respective representations

$$y = y(t), \qquad a \leqq t \leqq b, \tag{2.1}$$

$$y = \gamma(t), \qquad a \leqq t \leqq b \tag{2.2}$$

such that

$$|y(t) - \gamma(t)| < \varepsilon, \qquad a \leqq t \leqq b \tag{2.3}$$

and

$$|\dot{y}(t) - \dot{\gamma}(t)| < \varepsilon \text{ for almost all } t. \tag{2.4}$$

(If C and Γ were of class D' the words « for almost all t » could be replaced by « except at corners » without changing the content of the definition). However, this is merely a new and more complicated definition of an ε-neighborhood

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[1] With CARATHÉODORY, we denote by $\dot{y}(t)$ the function which is $y'(t)$ when $y'(t)$ is defined and finite, and which is zero elsewhere.

of order zero (defined by suppressing (2.4)); inequality (2.4) introduces no restrictions. For simplicity, we suppose C and Γ of class D'. Let (2.3) hold. The function $|\dot{y}(t)-\dot{\gamma}(t)|$ is bounded, say less than $M\varepsilon$. Introduce the new parameter $\tau=Mt$; the curves C, Γ then have the respective representations

$$\left\{\begin{array}{ll} y=y_1(\tau)\equiv y(\tau/M), & Ma\leqq\tau\leqq Mb, \\ y=\gamma_1(\tau)\equiv\gamma(\tau/M), & Ma\leqq\tau\leqq Mb. \end{array}\right. \tag{2.5}$$

Then

$$|\dot{y}_1(\tau)-\dot{\gamma}_1(\tau)|=|\dot{y}/M-\dot{\gamma}/M|=|\dot{y}-\dot{\gamma}|/M<\varepsilon,$$

so in the representations (2.5) inequality (2.4) holds. Since (2.5) continues to hold, we see that by this definition C is in the first-order ε-neighborhood of Γ if and only if it is in the zero-order ε-neighborhood of Γ.

It follows that if we are to have a useful definition inequality (2.4) must be altered so as to neutralize the effect of such changes of parameter. Two devices suggest themselves readily. The ε in inequality (2.4) may be replaced by $\varepsilon|\dot{\gamma}|$, or it may be replaced by $\varepsilon/(b-a)$. We shall see that these two devices lead to very different notions of first-order neighborhoods.

Still another self-suggesting device is to require that the least positive angle $\vartheta(t)$ $(0<\vartheta<\pi)$ between the vectors $\dot{y}$ and $\dot{\gamma}$ be small. This is clearly equivalent to demanding that $2\sin\vartheta/2$ be small; and by elementary trigonometry we have

$$2\sin\vartheta/2=\left|\frac{\dot{y}}{|\dot{y}|}-\frac{\dot{\gamma}}{|\dot{\gamma}|}\right|. \tag{2.6}$$

Our six definitions are as follows.

The curve C lies in the first-order ε-neighborhood $U_\varepsilon^i(\Gamma)$ $(i=1,\dots,6)$ if C and Γ have respective representations (2.1), (2.2) *for which $|\dot{\gamma}|$ and $|\dot{y}|$ are almost everywhere positive, inequality* (2.3) *holds, and there is an $\varepsilon_1<\varepsilon$ such that the corresponding condition below is satisfied for almost all t in $[a, b]$.*

(2.7) [*for* $U_\varepsilon^1(\Gamma)$] $\quad |\dot{y}-\dot{\gamma}|\leqq\varepsilon_1|\dot{\gamma}|$

(2.8) [*for* $U_\varepsilon^2(\Gamma)$] $\quad \left|\frac{\dot{y}}{|\dot{y}|}-\frac{\dot{\gamma}}{|\dot{\gamma}|}\right|\leqq\varepsilon_1$

(2.9) [*for* $U_\varepsilon^3(\Gamma)$] $\quad \left|\frac{\dot{y}}{|\dot{y}|}-\frac{\dot{\gamma}}{|\dot{\gamma}|}\right|\leqq\varepsilon_1$ *and* $|\log(|\dot{y}|/|\dot{\gamma}|)|\leqq\varepsilon_1$

(2.10) [*for* $U_\varepsilon^4(\Gamma)$] $\quad |\dot{y}-\dot{\gamma}|\leqq\varepsilon_1/(b-a)$

(2.11) [*for* $U_\varepsilon^5(\Gamma)$] $\quad \int_a^b|\dot{y}-\dot{\gamma}|\,dt<\varepsilon$

(2.12) [*for* $U_\varepsilon^6(\Gamma)$] $\quad |\mathfrak{L}(C)-\mathfrak{L}(\Gamma)|<\varepsilon.$

The definition of $U_\varepsilon^3(\Gamma)$ is a mild rephrasing of one given by TONELLI [2] for curves of class C'.

It is readily seen that if any of the inequalities (2.3) or (2.7)-(2.12) with the exception of (2.10) is satisfied and a new parameter $\tau=\tau(t)$, $\tau'>0$ is introduced on both C and Γ, the inequality continues to hold almost everywhere in the new representation. It is also evident that each neighborhood is determined by Γ and ε alone, and does not depend on any particular choice of representation of Γ. Furthermore

$$\text{(2.13)} \qquad \textit{if } \delta<\varepsilon \textit{ then } U_\delta^i(\Gamma)\subset U_\varepsilon^i(\Gamma).$$

3. - **Metrics.** — The neighborhood-systems U_ε^2, U_ε^3, U_ε^5 and U_ε^6 have the virtue of leading at once to a definition of distance. For we can define the distance $d_2(C,\Gamma)$ to be the greatest lower bound of numbers ε such that C is in $U_\varepsilon^2(\Gamma)$. The usual postulates for a distance are easily shown [3] to hold. The neighborhood $U_\varepsilon^2(\Gamma)$ appears in terms of this metric as the sphere in curve-space consisting of all curves C with $d_2(C,\Gamma)<\varepsilon$, so these neighborhoods must satisfy the HAUSDORFF postulates for neighborhoods. A like statement holds for U_ε^3, U_ε^5 and U_ε^6. Later we shall show that $U_\varepsilon^4(\Gamma)$ is identical with $U_\varepsilon^5(\Gamma)$, so that the U_ε^4 neighborhoods also yield a metric $d_4(C,\Gamma)\equiv d_5(C,\Gamma)$.

If we apply the same process to the topology U_ε^1, we obtain a « regular écart » instead of a metric. It is then true [4] that there is a metric leading to the same notion of limit, but we shall not use this. Nevertheless it is our duty to show that the sets $U_\varepsilon^1(\Gamma)$ actually constitute a set of neighborhoods in the sense of HAUSDORFF. Clearly Γ is in $U_\varepsilon^1(\Gamma)$ for all positive ε. If Γ_1 and Γ_2 are distinct, by use of (2.3) alone we can find a positive ε such that $U_\varepsilon^1(\Gamma_1)$ and $U_\varepsilon^1(\Gamma_2)$ are disjoint. Let now C be in $U_\varepsilon^1(\Gamma)$; then (2.7) holds. We must show that there is a positive δ such that $U_\delta^1(C)\subset U_\varepsilon^1(\Gamma)$. Choose δ so small that

$$\text{(3.1)} \qquad \delta\leqq\varepsilon-\max|y(t)-\gamma(t)|, \qquad \delta\leqq(\varepsilon-\varepsilon_1)/2(1+\varepsilon_1).$$

By (2.7), $|\dot y|\leqq(1+\varepsilon_1)|\dot\gamma|$. If C_1 is any curve in $U_\delta^1(C)$, it has a representation $y=\eta(t)$, $a\leqq t\leqq b$ with $|\dot\eta|>0$ almost everywhere and

$$|\eta-y|<\delta,$$
$$|\dot\eta-\dot y|\leqq\delta_1|\dot y|, \qquad \delta_1<\delta.$$

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(2) L. TONELLI: *Fondamenti di Calcolo delle Variazioni*, vol. II, p. 171.

(3) In fact, as Mr. A. D. WALLACE commented, this is exactly the FRÉCHET metric for the (possibly discontinuous) curves in 2ν-space defined by equations

$$z^i=y^i(t), \qquad z^{\nu+i}=\dot y^i(t)/|\dot y(t)|, \qquad (i=1,\dots,\nu).$$

(4) ALINE FRINK: *Distance Functions and the metrization problem.* Bull. Amer. Math. Soc. 43 (1937), pp. 133-142; in particular, p. 138.

With the above estimate for $|\dot{y}|$ we have

$$\begin{aligned} |\eta-\gamma| &\leqq |\eta-y|+|y-\gamma|<\varepsilon \\ |\dot{\eta}-\dot{\gamma}| &\leqq |\dot{\eta}-\dot{y}|+|\dot{y}-\dot{\gamma}| \\ &\leqq \delta_1(1+\varepsilon_1)\,|\dot{\gamma}|+\varepsilon_1\,|\dot{\gamma}| \\ &\leqq [(\varepsilon-\varepsilon_1)/2+\varepsilon_1]\,|\dot{\gamma}|=|\dot{\gamma}|\,(\varepsilon+\varepsilon_1)/2, \end{aligned}$$

and $(\varepsilon+\varepsilon_1)/2<\varepsilon$, so C_1 is in $U_\varepsilon^1(\Gamma)$.

The above proof, with (2.13), shows that if C is in both $U_\delta^1(\Gamma_1)$ and $U_\delta^1(\Gamma_2)$, there exists a neighborhood $U_\varepsilon^1(C)$ which is contained in both these neighborhoods.

The neighborhood definition most evidently related to the usual definition of weak relative minimum is U_ε^1. In fact, in the usual discussion of weak relative minima the definition in § 1 could be replaced by the definition in terms of $U_\varepsilon^1(\Gamma)$ with nothing more than trifling changes. Or, directly, we notice that the family of curves satisfying the conditions in § 1 for a fixed representation of Γ contains a neighborhood $U_\varepsilon^1(\Gamma)$, while conversely if we use arc length as parameter on Γ the neighborhood $U_\delta(\Gamma)$ contains the curves of § 1 if $\varepsilon<\delta$.

4. - **Relationships.** — The relations between the topologies defined in § 2 are summed up in the following theorems.

THEOREM I. - *Let Γ be a rectifiable curve and ε a positive number. Then there is a positive δ such that each neighborhood $U_\varepsilon^i(\Gamma)$ $(i=1, 2, 3)$ contains both $U_\delta^1(\Gamma)$ and $U_\delta^3(\Gamma)$, and each neighborhood $U_\varepsilon^j(\Gamma)$ $(j=4, 5, 6)$ contains $U_\delta^i(\Gamma)$ $(i=1, 3, 4, 5, 6)$.*

THEOREM II. - *If in addition to the hypotheses of Theorem I we assume that Γ is of class D', then there is a positive $\delta<0$ such that each of the neighborhoods $U_\varepsilon^i(\Gamma)$ $(i=1,...., 6)$ contains $U_\delta^2(\Gamma)$.*

THEOREM III. - *If Γ is a non-degenerate rectifiable curve and $0<\varepsilon\leqq 1$, neither $U_\varepsilon^1(\Gamma)$, $U_\varepsilon^2(\Gamma)$ nor $U_\varepsilon^3(\Gamma)$ contain any neighborhood $U_\delta^i(\Gamma)$, $(i=4, 5, 6)$, $\delta>0$.*

THEOREM IV. - *If Γ is rectifiable but not of class D', it is possible that neighborhoods $U_\varepsilon^i(\Gamma)$ $(i=1, 3, 4, 5, 6)$ may exist which do not contain any neighborhood $U_\delta^2(\Gamma)$, $\delta>0$.*

The proof of the first theorem requires a number of separate investigations.
(A) *If* $\delta=\min\,(1/2,\ \varepsilon/2)$, *then* $U_\varepsilon^3(\Gamma)\supset U_\delta^1(\Gamma)$.

Let C be in the latter set. Then (2.7) holds with $\delta_1<\delta$ in place of ε_1. For almost all t we have $|\dot{\gamma}|>0$, so

$$\left|\frac{|\dot{y}|}{|\dot{\gamma}|}-1\right|=\frac{1}{|\dot{\gamma}|}\,\big||\dot{y}|-|\dot{\gamma}|\big|\leqq\frac{|\dot{y}-\dot{\gamma}|}{|\dot{\gamma}|}\leqq\delta_1.$$

But $\delta_1<1/2$, so

$$|\log\,(|\dot{y}|/|\dot{\gamma}|)|<2\left|1-\frac{|\dot{y}|}{|\dot{\gamma}|}\right|\leqq 2\delta_1<\varepsilon.$$

This verifies the second of inequalities (2.9). For the first,

$$\begin{aligned}\left|\frac{\dot y}{|\dot y|}-\frac{\dot\gamma}{|\dot\gamma|}\right| &\leqq \left|\frac{\dot y}{|\dot y|}-\frac{\dot y}{|\dot\gamma|}\right|+\left|\frac{\dot y}{|\dot\gamma|}-\frac{\dot\gamma}{|\dot\gamma|}\right| \\ &=\left|\dot y\left(\frac{1}{|\dot y|}-\frac{1}{|\dot\gamma|}\right)\right|+\frac{1}{|\dot\gamma|}\,|\dot y-\dot\gamma| \\ &\leqq \left|1-\frac{|\dot y|}{|\dot\gamma|}\right|+\delta_1\leqq 2\delta_1<\varepsilon.\end{aligned}$$

This verifies the first of inequalities (2.9), so C is in $U_\varepsilon^3(\Gamma)$.

(B) *If* $e^\delta<1+\varepsilon/2$, *then* $U_\varepsilon^1(\Gamma)\supset U_\delta^3(\Gamma)$.

Let C be in $U_\delta^3(\Gamma)$. Then inequalities (2.9) hold with $\delta_1<\delta$ in place of ε_1, and (since $\delta_1<e^{\delta_1}-1$)

$$\begin{aligned}\frac{|\dot y-\dot\gamma|}{|\dot\gamma|} &\leqq \left|\frac{\dot y}{|\dot y|}-\frac{\dot\gamma}{|\dot\gamma|}\right|+\left|\frac{\dot y}{|\dot\gamma|}-\frac{\dot y}{|\dot y|}\right| \\ &\leqq \delta_1+\left|\dot y\left(\frac{1}{|\dot\gamma|}-\frac{1}{|\dot y|}\right)\right| \\ &=\delta_1+\left|\frac{|\dot y|}{|\dot\gamma|}-1\right|\leqq\delta_1+(e^{\delta_1}-1)<\varepsilon.\end{aligned}$$

So C is in $U_\varepsilon^1(\Gamma)$.

(C) $U_\varepsilon^2(\Gamma)\supset U_\varepsilon^3(\Gamma)$.

Obvious.

(D) $U_\varepsilon^4(\Gamma)=U_\varepsilon^5(\Gamma)$.

If C is in $U_\varepsilon^4(\Gamma)$, inequality (2.10) holds, and on integrating we see that (2.11) holds. Therefore $U_\varepsilon^4(\Gamma)\subset U_\varepsilon^5(\Gamma)$. Conversely, suppose that C is in $U_\varepsilon^5(\Gamma)$. Since (2.11) holds, we can choose a positive number η so small that

$$\delta\equiv\eta(b-a)+\int_a^b|\dot y-\dot\gamma|\,dt<\varepsilon. \tag{4.1}$$

Define

$$\tau(t)=\delta^{-1}\int_a^t(|y-\dot\gamma|+\eta)dt. \tag{4.2}$$

Then for almost all t we have

$$\tau'(t)=(|\dot y-\dot\gamma|+\eta)/\delta>0,$$

so $\tau(t)$ has an absolutely continuous inverse $t(\tau)$, and for almost all τ its derivative is

$$t'(\tau)=\delta/(|\dot y-\dot\gamma|+\eta). \tag{4.3}$$

From (4.1) and (4.2),

$$\tau(a)=0,\qquad \tau(b)=\delta^{-1}\left[\int_a^b|\dot y-\dot\gamma|\,dt+\eta(b-a)\right]=1. \tag{4.4}$$

Introducing τ as parameter on C and Γ, we obtain the representations

$$C: \quad y=y_1(\tau)\equiv y(t(\tau)), \qquad 0\leqq\tau\leqq 1,$$
$$\Gamma: \quad y=\gamma_1(\tau)\equiv \gamma(t(\tau)), \qquad 0\leqq\tau\leqq 1.$$

Using (4.3), (4.4) and (4.1), for almost all τ

$$|\dot{y}_1(\tau)-\dot{\gamma}_1(\tau)|=|\dot{y}(t(\tau))-\dot{\gamma}(t(\tau))|\cdot t'(\tau)$$
$$=\frac{\delta\,|\dot{y}-\dot{\gamma}|}{|\dot{y}-\dot{\gamma}|+\eta}\leqq\delta<\frac{\varepsilon}{1-0},$$

so C is in $U^4_\varepsilon(\Gamma)$.

(E) $U^6_\varepsilon(\Gamma)\supset U^5_\varepsilon(\Gamma)$.

For if C is in $U^5_\varepsilon(\Gamma)$

$$|\mathfrak{L}(C)-\mathfrak{L}(\Gamma)|=\left|\int_a^b(|\dot{y}|-|\dot{\gamma}|)dt\right|\leqq\int_a^b|\dot{y}-\dot{\gamma}|\,dt<\varepsilon.$$

(F) *If δ is small enough, $U^5_\varepsilon(\Gamma)$ contains $U^1_\delta(\Gamma)$ and $U^3_\delta(\Gamma)$.*

Let $\delta=\varepsilon/(\mathfrak{L}(\Gamma)+1)$, and let C be in $U^1_\delta(\Gamma)$. Then $|\dot{y}-\dot{\gamma}|<\delta|\dot{\gamma}|$, and

$$\int_a^b|\dot{y}-\dot{\gamma}|\,dt\leqq\int_a^b\delta|\dot{\gamma}|\,dt=\delta\mathfrak{L}(\Gamma)<\varepsilon,$$

so C is in $U^5_\varepsilon(\Gamma)$. The other statement follows from this together with (B).

(G) *If δ is small enough, $U^6_\varepsilon(\Gamma)$ contains $U^1_\delta(\Gamma)$ and $U^3_\delta(\Gamma)$.* This follows from (E) and (F).

(H) *If δ is small enough, $U^5_\varepsilon(\Gamma)\supset U^6_\delta(\Gamma)$.*

Suppose that this is false. Then for every positive integer n the set $U^6_{1/n}(\Gamma)$ contains a curve C_n not in $U^5_\varepsilon(\Gamma)$.

Let $y=\gamma(t)$, $0\leqq t\leqq 1$ be the representation of Γ in terms of the parameter $t=s/\mathfrak{L}(\Gamma)$, and let $y=y_n(t)$, $0\leqq t\leqq 1$ be the analogous representation of C_n. By hypothesis, the C_n have representations $y=\eta_n(t)$, $0\leqq t\leqq 1$ such that

$$\lim_{n\to\infty}\eta_n(t)=\gamma(t) \tag{4.5}$$

uniformly on [0, 1], and

$$\lim_{n\to\infty}\mathfrak{L}(C_n)=\mathfrak{L}(\Gamma). \tag{4.6}$$

By the lower semi-continuity of length,

$$\int_0^t|\dot{\gamma}(t)|\,dt\leqq\liminf_{n\to\infty}\int_0^t|\dot{\eta}_n|\,dt\leqq\limsup_{n\to\infty}\int_0^t|\dot{\eta}_n|\,dt$$
$$=\limsup_{n\to\infty}\left[\mathfrak{L}(C_n)-\int_t^1|\dot{\eta}|\,dt\right]$$

$$= \mathfrak{L}(\Gamma) - \lim_{n\to\infty} \inf \int_t^1 |\dot{\eta}_n|\, dt$$

$$\leqq \mathfrak{L}(\Gamma) - \int_t^1 |\dot{\gamma}|\, dt$$

$$= \int_0^t |\dot{\gamma}|\, dt.$$

Hence equality holds in all these inequalities, and

$$t\mathfrak{L}(\Gamma) = \int_0^t |\dot{\gamma}|\, dt = \lim_{n\to\infty} \int_0^t |\dot{\eta}_n|\, dt. \tag{4.7}$$

The integrals in equation (4.7) are continuous monotonic increasing functions of t, and converge to a continuous limit; so (5) the convergence is uniform. The length of arc on C_n between $y_n(t)$ and $\eta_n(t)$ is

$$\left|\int_0^t |\dot{\eta}_n|\, dt - t\mathfrak{L}(C_n)\right| \leqq \left|\int_0^t |\dot{\eta}_n|\, dt - t\mathfrak{L}(\Gamma)\right| + t\,|\mathfrak{L}(C_n) - \mathfrak{L}(\Gamma)|,$$

which tends uniformly to zero as n tends to infinity, by (4.7) and (4.6). Hence $|y_n(t) - \eta_n(t)|$ tends uniformly to zero. This, with (4.5), proves

$$\lim y_n(t) = \gamma(t) \tag{4.8}$$

uniformly for $0 \leqq t \leqq 1$.

Over every interval (α, β) contained in $[0, 1]$ we have

$$\int_\alpha^\beta (\dot{y}_n^i(t) - \dot{\gamma}^i(t))\,dt = [y_n^i(\beta) - \gamma^i(\beta)] - [y_n^i(\alpha) - \gamma^i(\alpha)], \tag{4.9}$$

which tends to zero with $1/n$. The integrands in (4.9) are uniformly bounded, so by a theorem of Lebesgue (6) the integrals in (4.9) continue to approach zero if any summable function is introduced as a factor in the integrand. In particular,

$$-\int_0^1 2(\dot{y}_n^i(t) - \dot{\gamma}^i(t))\,\dot{\gamma}^i(t)\,dt \to 0. \tag{4.10}$$

Because of the choice of parameter, for almost all t

$$|\dot{y}_n(t)| = \mathfrak{L}(C_n) \to \mathfrak{L}(\Gamma) = |\dot{\gamma}(t)|.$$

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(5) H. E. Buchanan and T. H. Hildebrandt: *Note on the convergence of a sequence of functions of a certain type*. Annals of Math., vol. 9 (1908), p. 123.

(6) Hobson: *The Theory of Functions of a Real Variable*, vol. II, p. 422 or p. 464.

Hence

$$\int_0^1 \{|\dot{y}_n|^2 - |\dot{\gamma}|^2\} dt \to 0. \tag{4.11}$$

Adding (4.10) and (4.11) member by member, we obtain

$$\lim_{n\to\infty} \int_0^1 |\dot{y}_n - \dot{\gamma}|^2 dt = 0.$$

By SCHWARZ's inequality,

$$\int_0^1 |\dot{y}_n - \dot{\gamma}|\, dt \leqq \left[\int_0^1 |\dot{y}_n - \dot{\gamma}|^2 dt\right]^{\frac{1}{2}},$$

so the left member of this inequality tends to zero. In particular, it is less than ε for all large n. By equation (4.8), the inequality $|y_n(t) - \gamma(t)| < \varepsilon$ holds for all large n. Hence C_n is in $U_\varepsilon^5(\Gamma)$ for all large n. This contradicts the assumption that no C_n is in $U_\varepsilon^5(\Gamma)$, and statement (H) is established.

Statements (A) to (H) constitute the proof of Theorem I.

To prove Theorem II, we suppose first that Γ is of class C' and that $\{C_n\}$ is a sequence of curves tending to Γ in the U^2-topology; that is, for every positive ε the neighborhood $U_\varepsilon^2(\Gamma)$ contains all but a finite number of the C_n. If we can show that C_n also tends to Γ in the U^3-topology, then as in (H) we shall have shown that each $U_\varepsilon^3(\Gamma)$ contains a neighborhood $U_\delta^2(\Gamma)$.

By hypothesis, the curves Γ and C_n have representations $y = \gamma(t)$, $y = y_n(t)$ for which $|y_n(t) - \gamma(t)|$ and

$$\left| \frac{\dot{y}_n}{|\dot{y}_n|} - \frac{\dot{\gamma}}{|\dot{\gamma}|} \right| \tag{4.12}$$

tend uniformly to zero. Without loss of generality we may suppose that t represents arc length on Γ, ranging over the interval $0 \leqq t \leqq L = \mathfrak{L}(\Gamma)$. The functions $\gamma^i(t)$ can be extended so as to remain of class C' on the interval $[0, L+1]$.

The uniform convergence of expression (4.12) to zero implies, because of (2.6), that the least upper bound ε_n of the angle between $\dot{y}_n(t)$ and $\dot{\gamma}(t)$ tends to zero with $1/n$. Define (7)

$$d(t, h) = [\gamma(t+h) - \gamma(t)]/h, \qquad 0 \leqq t \leqq L, \qquad 0 < h \leqq 1.$$

Since $\dot{\gamma}$ is continuous, as h tends to zero $d(t, h)$ tends uniformly to $\dot{\gamma}(t)$. Therefore $|d(t, h)|$ tends uniformly to $|\dot{\gamma}| = 1$, and the unit vector

$$D(t, h) \equiv d(t, h)/|d(t, h)|$$

(7) If Γ is of class C'', we can replace $d(t, h)$ by $\dot{\gamma}(t)$ itself and omit the limiting process $h \to 0$.

also tends uniformly to $\dot{\gamma}(t)$. From the definition, $D(t, h)$ is a C' function of t for each h. From the uniform convergence of $D(t, h)$ to $\dot{\gamma}(t)$, we see that if $\varepsilon(h)$ is the least upper bound of the angle between these vectors, then $\varepsilon(h)$ tends to zero with h.

The angle $\vartheta(n, h, t)$ between $\dot{\gamma}_n(t)$ and $D(t, h)$ cannot exceed $\varepsilon_n + \varepsilon(h)$, so

$$\begin{aligned} \dot{y}^i_n(t) D^i(t, h) &= |\dot{y}_n| \cdot 1 \cdot \cos \vartheta(n, h, t) \\ &\geqq \cos(\varepsilon_n + \varepsilon(h)) \cdot |\dot{y}_n|. \end{aligned}$$

This, with an integration by parts, yields

$$\begin{aligned} \int_0^t |\dot{y}_n| \, dt &\leqq \sec(\varepsilon_n + \varepsilon(h)) \int_0^t \dot{y}^i_n(t) D^i(t, h) dt \\ &= \sec(\varepsilon_n + \varepsilon(h)) \left\{ y^i_n(t) D^i(t, h) \Big|_0^t - \int_0^t y^i_n(t) D^{i'}(t, h) dt \right\}. \end{aligned}$$

If we hold h fixed and let n tend to ∞, then on recalling that y_n tends uniformly to γ we obtain (with another integration by parts)

$$\begin{aligned} \limsup_{n \to \infty} \int_0^t |\dot{y}_n| \, dt &\leqq \sec \varepsilon(h) \left\{ \gamma^i(t) D^i(t, h) \Big|_0^t - \int_0^t \gamma^i(t) D^{i'}(t, h) dt \right\} = \\ &= \sec \varepsilon(h) \int_0^t \dot{\gamma}^i(t) D^i(t, h) dt. \end{aligned}$$

This holds for all h between 0 and 1. Letting h tend to 0, and recalling that $D(t, h)$ tends uniformly to $\dot{\gamma}(t)$ and $|\dot{\gamma}| = 1$

$$\limsup_{n \to \infty} \int_0^t |\dot{y}_n| \, dt \leqq \int_0^t |\dot{\gamma}|^2 dt = t.$$

But by the lower semi-continuity of length,

$$\liminf_{n \to \infty} \int_0^t |\dot{y}_n| \, dt \geqq \int_0^t |\dot{\gamma}| \, dt = t,$$

so

$$\lim_{n \to \infty} \int_0^t |\dot{y}_n| \, dt = t. \tag{4.13}$$

The integrals are continuous monotonic increasing functions of t, and converge to the continuous limit function t, so [8] the convergence in (4.13) is uniform.

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[8] H. E. BUCHANAN and T. H. HILDEBRANDT, loc. cit. [5].

As the particular case of (4.13) with $t=L$,

$$\lim \mathfrak{L}(C_n)=L=\mathfrak{L}(\Gamma). \tag{4.14}$$

The functions

$$\tau_n(t)=(L/\mathfrak{L}(C_n))\int_0^t |\dot{y}_n|\,dt$$

converge uniformly to t, by (4.13) and (4.14), and $\tau_n(0)=0$, $\tau_n(L)=L$. They have positive derivatives for almost all t, so they possess absolutely continuous inverses $t_n(\tau)$, $0\leqq\tau\leqq L$. It is easily seen that $t_n(\tau)$ converges uniformly to τ. If we introduce τ as parameter on C_n by the equation

$$y=\eta_n(\tau)\equiv y_n(t_n(\tau)), \qquad 0\leqq\tau\leqq L,$$

we compute

$$|\dot{\eta}_n(\tau)|=\mathfrak{L}(C_n)/L$$

for almost all τ. Hence

$$\left|1-\frac{|\dot{\eta}_n|}{|\dot{\gamma}|}\right|=|1-\mathfrak{L}(C_n)/L|\to 0,$$

whence $\log(|\dot{\eta}_n|/|\dot{\gamma}|)\to 0$ uniformly for $0\leqq t\leqq L$. Also,

$$|\eta_n(\tau)-\gamma(\tau)|\leqq|y_n(t_n(\tau))-\gamma(t_n(\tau))|+|\gamma(t_n(\tau))-\gamma(\tau)|,$$

which converges uniformly to zero since y_n and t_n converge uniformly to the respective limits γ and τ. Finally, recalling $|\dot{\gamma}|=1$,

$$\left|\frac{\dot{\eta}_n}{|\dot{\eta}_n|}-\frac{\dot{\gamma}}{|\dot{\gamma}|}\right|\leqq\left|\frac{\dot{y}_n(t_n(\tau))}{|\dot{y}_n(t_n(\tau))|}-\dot{\gamma}(t_n(\tau))\right|+|\dot{\gamma}(t_n(\tau))-\dot{\gamma}(\tau)|,$$

and this tends uniformly to zero because of the uniform convergence of (4.12) to 0 and of t_n to τ.

Hence C_n converges to Γ in the U^3-topology, and we have shown that if Γ is of class C', then every neighborhood $U^3_\varepsilon(\Gamma)$ contains a neighborhood $U^2_\delta(\Gamma)$.

If Γ is of class D', we have only to apply the above reasoning to each arc of Γ between corners. When we observe that each neighborhood $U^i_\varepsilon(\Gamma)$ $(i=1, 4, 5, 6)$ contains a neighborhood $U^3_\beta(\Gamma)$, and by the preceding proof this contains a neighborhood $U^2_\delta(\Gamma)$, we have completed the proof of Theorem II.

Let $y=\gamma(t)$, $a\leqq t\leqq b$ be any non-degenerate rectifiable curve. Let $v=(v^1,\dots, v^\nu)$ be a fixed vector. We define the curve C_n by the equation $y=y_n(t)$, $a\leqq t\leqq b+2$, where

$$y_n(t)\equiv\gamma(t), \qquad a\leqq t\leqq b$$
$$y_n(b+1)=\gamma(b)+(-1)^n v/n, \qquad y_n(b+2)=\gamma(b),$$

the functions $y_n(t)$ being linear on the intervals $[b, b+1]$ and $[b+1, b+2]$.

Using the metrics $d_j(C, \Gamma)$ defined (as in § 3) by the topologies U^j $(j=2,\dots, 6)$, we easily find

$$\lim d_6(C_n, \Gamma)=0,$$

whence by Theorem I we also have $d_5(C_n, \Gamma)\to 0$ and $d_4(C_n, \Gamma)\to 0$. So every neighborhood $U_\delta^6(\Gamma)$ contains all but a finite number of the C_n, and likewise every $U_\delta^5(\Gamma)$ and every $U_\delta^4(\Gamma)$. However, no matter how the C_n are represented on the interval $[a, b]$ we find that for almost all values of t near b the vectors $\dot{y}_n(t)$ and $\dot{y}_{n+1}(t)$ are oppositely directed $(n=1, 2,\dots)$. Hence $d_3(C_n, C_{n+1})\geqq 2$ and $d_2(C_n, C_{n+1})\geqq 2$.

Consequently for all n either $d_3(C_n, \Gamma)\geqq 1$ or else $d_3(C_{n+1}, \Gamma)\geqq 1$, and infinitely many curves C_n fail to lie in $U_1^3(\Gamma)$. Likewise for $U_1^2(\Gamma)$. It follows that neither $U_1^2(\Gamma)$ nor $U_1^3(\Gamma)$ can contain any $U_\delta^4(\Gamma)$, $U_\delta^5(\Gamma)$ or $U_\delta^6(\Gamma)$. Also, for almost all t near b either $|\dot{\gamma}-\dot{y}_n|\geqq|\dot{\gamma}|$ or $|\dot{\gamma}-\dot{y}_{n+1}|\geqq|\dot{\gamma}|$, so that infinitely many C_n fail to lie in $U_1^1(\Gamma)$. Hence $U_1^1(\Gamma)$ contains no neighborhod $U_\delta^i(\Gamma)$, $(i=4, 5, 6)$. This completes the proof of Theorem III.

In the interval $[0, L]$ we choose a sequence of points

$$0=q_1<p_1<q_2<p_2<\dots\to 1.$$

Let $\{a_n\}$ be a sequence of numbers tending to zero. If $\varphi(t, \{a_n\})$ is the function such that

$$\begin{aligned}
&\varphi(q_i, \{a_n\})=0, \qquad (i=1, 2,\dots),\\
&\varphi(p_i, \{a_n\})=a_i, \qquad (i=1, 2,\dots),\\
&\varphi(1, \{a_n\})=0,\\
&\varphi(t, \{a_n\}) \textit{ is linear on each of the intervals } [q, p_i] \textit{ and } [p_i, q_{i+1}],
\end{aligned}$$

then φ is continuous, and the curve C defined by the equations

$$(4.15) \qquad y^1=\varphi(t, \{a_n\}), \qquad y^2=\dots=y^n=0, \qquad 0\leqq t\leqq 1$$

is a continuous curve. In particular, let $a_i=2^{-i}$. The curve Γ thus defined is rectifiable; its length is 2. We define other curves $C_{n,m}$ by choosing

$$(4.16) \qquad a_i=2^{-i}, \quad (i=1,\dots, n, m+1,\dots); \qquad a_i=1/i, \quad (i=n+1,\dots, m).$$

If we write the equations (4.15) in the form $y=y_{n,m}(t)$, $0\leqq t\leqq 1$ for $C_{n,m}$ and $y=\gamma(t)$, $0\leqq t\leqq 1$ for Γ, we readily verify that $|y_{n,m}-\gamma|<1/n$. Also,

$$\left|\frac{\dot{y}_{n,m}}{|\dot{y}_{n,m}|}-\frac{\dot{\gamma}}{|\dot{\gamma}|}\right|=0$$

except at the values q_i, p_i of t. Hence for any sequence $m(n)>n$ of values of n the sequence $C_{n,m(n)}$ tends to Γ in the U^2-topology. But from equations (4.15) we find that for each fixed n,

$$\lim_{m\to\infty} \mathfrak{L}(C_{n,m})=\infty.$$

Hence we may choose $m(n)$ large enough so that

$$\mathfrak{L}(C_{n,\,m(n)}) > \mathfrak{L}(\Gamma) + n.$$

Then the curves $C_{n,\,m(n)}$ will not tend to Γ in the U^6-topology, nor even have uniformly bounded lengths; and by part 1 they will not tend to Γ in the U^1, U^4 or U^5 topologies. This establishes Theorem IV.

5. - **Remarks.** — As commented in § 2, the neighborhood-system $U_\varepsilon^2(C)$ defines a metric $d_2(C, \Gamma)$ in the space of non-degenerate rectifiable curves. However, with this metric the space is not complete. For example, consider the sequence of curves C_n defined by the equation $y = y_n(t)$, $0 \leqq t \leqq 1$, where $\overset{1}{y}_n(t) \equiv t/n$, $\overset{2}{y}_n(t) \equiv .... \equiv \overset{\nu}{y}_n \equiv 0$, $0 \leqq t \leqq 1$.

We readily find $|y_n - y_m| \leqq |m-n|/mn$, and

$$\left| \frac{\dot{y}_n}{|\dot{y}_n|} - \frac{\dot{y}_m}{|\dot{y}_m|} \right| \equiv 0.$$

Hence $d_2(C_n, C_m) \leqq |m-n|/mn$, and the curves C_n form a CAUCHY sequence. But $\mathfrak{L}(C_n) \to 0$, so they cannot converge to any non-degenerate curve.

The same example shows that with the metric $d_6(C, \Gamma)$ based on the U^6 topology the space of non-degenerate rectifiable curves is not complete. It remains incomplete even if the degenerate curves are added; for the curves C_n: $y = y_n(t)$, $0 \leqq t \leqq 2\pi$ defined by $\overset{1}{y}_n = n^{-1} \sin nt$, $\overset{2}{y}_n = ... = \overset{\nu}{y}_n = 0$ form a CAUCHY sequence in the d_6-metric (they all have length 4); but by (2.3) the only limit curve they can have is the degenerate curve Γ consisting of the origin alone, while $d_6(C_n, \Gamma) = 4$ for all n.

On the other hand, with the d_3-metric the space of non-degenerate rectifiable curves forms a complete space. For let $\{C_n\}$ be a CAUCHY sequence in this metric. We can select a subsequence $\{C_n^*\}$ such that $d_3(C_n^*, C_m^*) < 2^{-n}$ if $m > n$. Let C_1^* be represented by equations $y = y_1(t)$, $0 \leqq t \leqq 1$, where t is proportional to arc length. The representations of the other curves are successively determined; given the representation $y = y_n(t)$ of C_n^*, the curve C_{n+1}^* has a representation $y = y_{n+1}(t)$ such that the inequalities

$$|y_n - y_{n+1}| < 2^{-n}, \tag{5.1}$$

$$\left| \frac{\dot{y}_n}{|\dot{y}_n|} - \frac{\dot{y}_{n+1}}{|\dot{y}_{n+1}|} \right| < 2^{-n}, \tag{5.2}$$

$$|\log (|\dot{y}_{n+1}| / |\dot{y}_n|)| < 2^{-n} \tag{5.3}$$

all hold. From inequality (5.3) it is not difficult to show that $|\dot{y}_n(t)|$ tends uniformly to a bounded non-vanishing limit, except on a set of measure zero. From this, with inequality (5.2), we see that the sequence $\{\dot{y}_n(t)\}$ is uniformly convergent, if we omit a set of measure zero. Let $\dot{y}(t)$ be the limit of the functions $\dot{y}_n(t)$. If we define

$$(5.4) \qquad y(t) = \lim \left\{ y_n(0) + \int_0^t \dot{y}_n dt \right\},$$

we can verify that the curve $y = y(t)$ is the limit in the d_3-metric of C_n^*, hence of C_n.

The same proof shows that, with the d_3-metric, the space of all non-degenerate curves of class C' is complete.

The U^5-topology also gives us a metric in terms of which the space of non-degenerate rectifiable curves is complete. Let $\{C_n\}$ be a CAUCHY sequence in this metric. We can select a subsequence $\{C_n^*\}$ and successively determine representions of the C_n^* in such a way that

$$|y_n(t) - y_{n+1}(t)| < 2^{-n}$$

and

$$\int_0^1 |\dot{y}_n(t) - \dot{y}_{n+1}(t)| \, dt < 2^{-n}.$$

By the completeness of the space L_1, the functions $\dot{y}_n(t)$ tend in L_1 to a limit $\dot{y}(t)$, so that $\int_0^1 |\dot{y}_n - \dot{y}| \, dt \to 0$. If we again define $y(t)$ by (5.4), the curve $y = y(t)$ is the limit in the d_5-metric of C_n^*, hence of C_n.

We have thus shown that while the topologies U^6 and U^5 are equivalent and both lead to metrics, the metrics are not equivalent; that is, the ratio $d_5(C, \Gamma)/d_6(C, \Gamma)$ is not bounded.

In any of our six topologies the space of non-degenerate curves of class D' is a separable space. For any such curve C can be arbitrarily closely approximated in any of the six topologies by polygons whose vertices are at points y all of whose coordinates are rational. By the same approximation we can show that the space of non-degenerate rectifiable curves is separable in the U^4, U^5 and U^6 topologies. However, this space is non-separable in the U^1, U^2 and U^3 topologies. We shall show this for $\nu = 2$; it will follow at once for $\nu > 2$.

Choose points $0 = q_1 < p_1 < q_2 < p_2 < \dots \to 1$.

Corresponding to each infinite sequence $\{a_n\}$ of numbers 0 or 1 we define a curve C as follows. The functions $y(t)$ are linear on each of the intervals $[q_i, p_i]$ and $[p_i, q_{i+1}]$, and $y(1) = y(q_i) = 0$ $(i = 1, 2, \dots)$. If $a_n = 0$, we take $y(p_n) = (2^{-n}, 0)$; if $a_n = 1$, we take $y(p_n) = (0, 2^{-n})$. All these curves have length 2. They form a non-denumerable collection. We shall show that any two curves C_1, C_2 corresponding to distinct sequences a_n and b_n have distance $d_2(C_1, C_2) \geqq \sqrt{2}$, from which the non-separability of the space of rectifiable curves in the d_2-metric (hence in the U^2, U^3 and U^1 topologies) will at once follow.

Let C_1, C_2 correspond respectively to the sequences a_n, b_n and have distance $d_2(C_1, C_2)<\sqrt{2}$. Let C_1 be represented as in the paragraph. There is then a representation $y=\eta(t)$ of C_2 such that

$$\left|\frac{\dot{y}}{|\dot{y}|}-\frac{\dot{\eta}}{|\dot{\eta}|}\right|<\sqrt{2}$$

for almost all t. That is, the angle between $\dot{y}$ and $\dot{\eta}$ is less than $\pi/2$. For small t the direction of $\dot{y}(t)$ is (1, 0) or (0, 1) according as a_1 is 0 or 1, and that of $\dot{\eta}$ is (1, 0) or (0, 1) according as b_1 is 0 or 1. In order that the angle be less than $\pi/2$ we must have $b_1=a_1$. Let $q_1{}^*$ be the value of t defining the first corner of C_2. If $q_1{}^*<q_1$, then for almost all t between $q_1{}^*$ and q_1 the angle between $\dot{y}$ and $\dot{\eta}$ is π, which is impossible; so $q_1{}^*\geqq q_1$. By a similar argument, $q_1\geqq q_1{}^*$, so the two are equal. We now repeat the argument for values of t a little greater than q_1, and find that p_1 defines the second vertex of C_2 as well as of C_1. Continuing, we obtain successively the equations $b_n=a_n$, $p_n{}^*=p_n$ and $q_n{}^*=q_n$, $n=2,\dots$. Hence the sequences b_n and a_n are identical, and the proof is complete.

Theorem I and the remarks in § 3 show that topologies U^1 and U^3 are equally suited logically to define weak relative minima, the first being somewhat superior in convenience; and if Γ is of class D', topology U^2 is also logically suitable, but practically inconvenient. The other three topologies are too weak for the definition of a weak relative minimum. They have however an independent interest in that every integral

$$J(C)=\int_C F(y, \dot{y})dt$$

of the calculus of variations is continuous in these three topologies. As usual, we assume that F is of class C' for all y in a set S and all $\dot{y}\neq(0,\dots, 0)$, and is positively homogeneous of degree 1 in $\dot{y}$. Let Γ, C_1, $C_2,\dots$ be rectifiable curves such that $d_5(\Gamma, C_n)\to 0$; we shall show $\lim J(C_n)=J(\Gamma)$. For convenience we suppose S to be bounded and closed ([9]).

The curves Γ, C_n have respective representations $y=\gamma(t)$, $y=y_n(t)$, $0\leqq t\leqq 1$ such that $|y_n-\gamma|$ tends uniformly to zero and

$$\lim_{n\to\infty}\int_0^1|\dot{y}_n-\dot{\gamma}|\,dt=0. \tag{5.5}$$

([9]) This involves no loss of generality, since the set of all points y lying on one or more curves Γ, C_1, $C_2,\dots$ is such a set.

By definition,

$$J(C_n)-J(\Gamma)=\int_0^1 \{F(y_n,\dot y_n)-F(\gamma,\dot\gamma)\}\,dt \tag{5.6}$$
$$=\int_0^1 \{F(y_n,\dot y_n)-F(\gamma_n,\dot\gamma)\}\,dt$$
$$+\int_0^1 \{F(\gamma_n,\dot\gamma)-F(\gamma,\dot\gamma)\}\,dt.$$

Since F is uniformly continuous and y_n tends uniformly to γ, the second integral in the last member of the equation tends to zero. To discuss the first integral, we observe that the partial derivatives $\partial F/\partial \dot y^i$ are continuous for y in S and $|\dot y|=1$, and are positively homogeneous of degree 0, so there is an M such that

$$\Big(\sum_{i=1}^{\nu}[F_{\dot y^i}(y,\dot y)]^2\Big)^{\frac{1}{2}}\leqq M.$$

Then

$$|F(y_n,\dot y)-F(y_n,\dot\gamma)|=\Big|\int_0^1 \frac{d}{d\tau}F(y_n,\dot\gamma+\tau(\dot y_n-\dot\gamma))\,d\tau\Big|$$
$$=\Big|\int_0^1 F_{\dot y^i}\cdot(\dot y_n^i-\dot\gamma^i)\,d\tau\Big|$$
$$=\Big|(\dot y_n^i-\dot\gamma^i)\int_0^1 F_{\dot y^i}d\tau\Big|\leqq|\dot y_n-\dot\gamma|\cdot M.$$

This, with (5.5), shows that the first integral in the last member of equation (5.6) tends to zero, so that $\lim J(C_n)=J(\Gamma)$. Thus $J(C)$ is continuous in the U^5 topology, hence in all six topologies.

The U^4, U^5 and U^6 topologies are the weakest topologies in which this is true, in the following sense. Let $J(C)$ be any regular integral, and let U be a topology such that if C_n tends to Γ in the U-topology, then C_n tends to Γ in the FRÉCHET metric and $J(C_n)$ tends to $J(\Gamma)$. Then every neighborhood $U^j_\varepsilon(\Gamma)$ $(j=4, 5, 6)$ contains a neighborhood $U(\Gamma)$. If this were false, we could find a sequence C_n tending to Γ in the U-topology (hence having $J(C_n)\to J(\Gamma)$), but not tending to Γ in the U^6 topology, so that $\mathfrak{L}(C_n)$ does not tend to $\mathfrak{L}(\Gamma)$. This, however, is impossible, by a known theorem [10].

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[10] L. TONELLI: *Fondamenti di Calcolo delle Variazioni*, vol. I, p. 331; E. J. McSHANE: *Semi-continuity in the calculus of variations, and absolute minima for isoperimetric problems*, published in *Contributions to the Calculus of Variations, University of Chicago, 1930*; in particular, p. 44.