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Theory and applications of abstract normal coordinates in a general differential geometry

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THEORY AND APPLICATIONS OF ABSTRACT NORMAL COORDINATES IN A GENERAL DIFFERENTIAL GEOMETRY [1].

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Introduction.

The set of coordinates of a point in traditional differential geometry is a point of an n-dimensional arithmetic space, while in a MICHAL functional geometry [2] the coordinate space is an infinite dimensional function space. Recently the study of general differential geometries and their tensor calculi, where BANACH spaces [3] are the coordinate spaces, has been initiated by one of us [4].

The object of the present paper is to define and develop a theory of normal coordinates in a general differential geometry with a linear connection. The interest and importance of Riemannian normal coordinates in Riemannian geometry is well known [5]. Unlike VEBLEN's older treatment of normal coordinates, our general theory does not depend on power series expansions. In fact our definition and existence proofs for normal coordinates are more closely related to those given recently by WHITEHEAD and THOMAS [6]. Throughout the paper we make extensive use of results in the abstract differential calculus given in numerous papers by FRÉCHET, HILDEBRANDT, GRAVES, KERNER, MICHAL, MARTIN, ELCONIN, PAXSON and HYERS.

In section 1 we give a needed existence theorem for abstract second order differential equations in BANACH spaces. Coordinate systems and geometric objects of general differential geometry are described in section 2. Section 3 is devoted to the definition and proof of existence of normal coordinates, while in the remainder of the paper normal coordinate methods are applied to give a treatment

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(1) Presented to the Amer. Math. Society, Sept., 1936.

(2) MICHAL (6)-(10), where the numbers refer to the list at the end of the paper. For further references see KAWAGUCHI (1), CONFORTO (1).

(3) BANACH (1).

(4) MICHAL, (1)-(5).

(5) RIEMANN (1). For an extension of the notion of normal coordinates see VEBLEN (2). Also VEBLEN & THOMAS (1).

(6) WHITEHEAD (1). THOMAS (2).

of normal vector forms and extensions of multilinear forms, including a replacement theorem [7] for differential invariants.

1. - An Existence Theorem for an Abstract Differential Equation.

Theorems on the differentiability of solutions of second order differential equations in BANACH spaces subject to *two-point* boundary conditions have been given by us in a recent paper [8].

We begin this section with a theorem on the differentiability of solutions of the following differential equation

$$\frac{d^2x}{ds^2}=H\left(x,\frac{dx}{ds}\right) \tag{1.1}$$

subject to *one point* initial conditions.

THEOREM 1.1. - *Let X be a bounded convex region of a Banach space* [9] *E and let the function $H(x, \xi)$ on XE to E have the following properties:*

i) *$H(x, \xi)$ is of class $C^{(n)}$ $(n\geqslant 1)$ in (x, ξ) uniformly* [10] *on $X\Xi$* where Ξ *is the set* $\|\xi\|<1$

ii) *$H(x, \xi)$ is homogeneous in ξ of degree $r>1$.*

Denote the real interval $0\leqslant s\leqslant 1$ by I. Then for any chosen point $\bar{x}$ of X there is a neighborhood $X_0\subset X$ of $\bar{x}$, a neighborhood Y_0 of 0, and a unique function $\varphi(x_0, \xi_0, s)$ on $X_0 Y_0 I$ with the following properties:

i) *If $x_0 \varepsilon X_0$, $\xi_0 \varepsilon Y_0$, $s \varepsilon I$ then $\varphi(x_0, \xi_0, s)\,\varepsilon X$, satisfies the differential equation* (1.1) *and reduces to x_0 for $s=0$, while $\frac{\partial\varphi(x_0, \xi_0, s)}{\partial s}\,\varepsilon\,\Xi$ and reduces to ξ_0 for $s=0$*

ii) *$\varphi(x_0, \xi_0, s)$ and $\frac{\partial\varphi(x_0, \xi_0, s)}{\partial s}$ are of class $C^{(n)}$ in x_0, ξ_0 uniformly on $X_0 Y_0$.*

Proof: We may replace the differential equation (1.1) together with the initial conditions by the equivalent integral system

$$x(s)=x_0+\int_0^s \xi(\sigma)d\sigma, \qquad \xi(s)=\xi_0+\int_0^s H(x(\sigma), \xi(\sigma))d\sigma. \tag{1.2}$$

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(7) Compare with the classical replacement theorem, MICHAL and THOMAS (2).

(8) MICHAL and HYERS (1). The numbers refer to the list of references at the end of the paper.

(9) BANACH (1).

(10) For definitions of FRÉCHET differentials, regions, class $C^{(n)}$, cf. FRÉCHET (1), HILDEBRANDT and GRAVES (1), MICHAL and HYERS (1). We denote the first total FRÉCHET differential with increments $\lambda_1, \lambda_2,\ldots, \lambda_n$ of a function $f(x_1, x_2,\ldots, x_n)$ by $f(x_1, x_2,\ldots, x_n; \lambda_1, \lambda_2,\ldots, \lambda_n)$. The rth successive partial FRÉCHET differential with increments $\lambda_1,\ldots, \lambda_r$ of a function $f(x_1, x_2,\ldots, x_n)$ in the *first variable* x_1 will be denoted by $f(x_1, x_2,\ldots, x_n; \lambda_1; \lambda_2; \ldots; \lambda_r)$. Sometimes when convenient we shall write $\delta f(x)$ for $f(x; \delta x)$ and $\delta x_i f(x_1,\ldots, x_n)$ for the partial FRÉCHET differential in x_i with increment δx_i.

Let E_1 be the BANACH space of pairs $\omega=(x_0, \xi_0)$ out of E and let E_2 be the BANACH space of function pairs $y(s)\equiv(x(s), \xi(s))$ where $x(s)$ and $\xi(s)$ are continuous functions on $(0, 1)$ to E (for the construction of these product spaces cf. MICHAL and HYERS (1)). Writing

$$F_1(\omega, y)=x_0+\int_0^s \xi(\sigma)d\sigma, \qquad F_2(\omega, y)=\xi_0+\int_0^s H(x(\sigma), \xi(\sigma))d\sigma$$
$$F(\omega, y)=(F_1(\omega, y), F_2(\omega, y))$$

we can express (1.2) in the form

$$y=F(\omega, y), \tag{1.3}$$

where $F(\omega, y)$ is on $\Omega_1 Y_2$ to E_2 and where Ω_1 is the region $X\Xi$ of E_1 and Y_2 is that subset of E_2 for which $\xi(s)\,\varepsilon\,\Xi$ when $0\leqslant s\leqslant 1$.

Let W_0 be the subset $\Omega_1 Y_2$ of the composite space E_1E_2. Since X and Ξ are bounded convex regions it can be shown that W_0 is also a bounded convex region. We shall prove that with this choice of W_0 the hypotheses (H_1), (H_2), (H_3) of a known implicit function theorem ([11]) are satisfied by the function

$$G(\omega, y)=y-F(\omega, y).$$

Take $\omega_0=(\bar{x}, 0)$, $y_0=(\bar{x}, 0)$, so that (ω_0, y_0) is in W_0, and is a solution of equation (1.3) since $H(x, \xi)$ is homogeneous of degree $r>1$ in ξ. Thus (H_1) is satisfied. To show that (H_2) and (H_3) are satisfied we proceed as follows. Since $H(x, \xi)$ is of class $C^{(n)}$ uniformly on $X\Xi$, $H(y(s))$ is of class $C^{(n)}$ uniformly on Y_2, and therefore $F(\omega, y)$ is of class $C^{(n)}$ uniformly on W_0. Now

$$\delta_y F_1(\omega, y)=\int_0^s \delta\xi(\sigma)d\sigma, \qquad \delta_y F_2(\omega, y)=\int_0^s H(y(\sigma);\ \delta y(\sigma))d\sigma. \tag{1.4}$$

By an easy calculation we find that

$$\delta_y G(\omega_0, y_0)=\left(\delta x(s)-\int_0^s \delta\xi(\sigma)d\sigma,\ \delta\xi(s)\right)$$

and hence $\delta_y G(\omega_0, y_0)$ is a solvable linear function of $\delta y=(\delta x(s), \delta\xi(s))$. The existence of a solution of system (1.2) with the required properties follows with the aid of the implicit function theorem of HILDEBRANDT and GRAVES. To prove the uniqueness of such a solution we observe that the function $H(x, \xi)$ is of class $C^{(1)}$ *uniformly* on $X\Xi$ and hence, using a known formula ([12]) relating the

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([11]) Cf. Theorem 4, p. 150 of HILDEBRANDT and GRAVES (1).

([12]) Cf. for example formula (2.4), p. 652 of MICHAL and HYERS (1).

difference and differential, $H(x, \xi)$ satisfies a LIPSCHITZ condition

$$\| H(x_1, \xi_1) - H(x_2, \xi_2) \| \leqslant M \{ \| x_1 - x_2 \| + \| \xi_1 - \xi_2 \| \}$$

throughout the convex set $X\Xi$. It now follows by a standard method that there can not be two different solution pairs for system (1.2) whose values lie in $X\Xi$.

We are interested particularly in the geometrical applications in which $H(x, \xi)$ is homogeneous of degree two. If $H(x, \xi)$ satisfies the hypotheses of theorem 1.1 where r is an integer and $n \geqslant r$, $H(x, \xi)$ is necessarily a homogeneous polynomial ([13]) of degree r in ξ. For by EULER'S theorem ([14]) for abstract homogeneous functions, the rth partial differential in ξ of $H(x, \xi)$ is a homogeneous function of degree zero in ξ continuous at $\xi = 0$ and therefore is independent of ξ.

2. - Abstract Coordinates and Geometric Objects.

Before using the existence theorem of the preceding section in the study of abstract normal coordinates we give a brief discussion of abstract coordinates in general and of geometric objects. In general differential geometry ([15]) initiated by one of us, abstract coordinate systems are homeomorphic maps of neighborhoods U (called geometrical domains) of a HAUSDORFF topological space ([16]) H onto open subsets Σ (called coordinate domains) of a BANACH space E. We shall assume that every neighborhood of H can be mapped homeomorphically on some Σ and that all of the sets Σ are contained in a fixed open set $S \subset E$, where S is the homeomorphic map of some HAUSDORFF neighborhood U_0. We also postulate that if $x(P)$ is a coordinate system with the coordinate domain Σ and if $x'(x)$ is a homeomorphism taking Σ into Σ', then the homeomorphism $x'(x(P))$ is also a coordinate system. The intersection of two HAUSDORFF neighoorhoods (with maps Σ_2, $\Sigma_2 \subset S$) leads to a homeomorphism $\bar{x} = \bar{x}(x)$, called a transformation of coordinates, taking an open subset (called domain of definition of $\bar{x}(x)$) of Σ_1 into an open subset of Σ_2. Geometrical objects such as contravariant vectors and linear connections have elements of the BANACH space E as components in each coordinate system. In the intersection of two HAUSDORFF neighborhoods, each geometrical object has a characteristic transformation law relating its components in the two coordinate systems ([17]).

([13]) A continuous function $F(x)$ on E to E is said to be a polynomial of degree r if $F(x + \lambda y) = F_0(x, y) + \lambda F_1(x, y) + ... + \lambda^r F_r(x, y)$, $F_r \not\equiv 0$. Cf. MICHAL and MARTIN (1). It can be shown that the $(r+1)$ st differential of a function vanishes identically if and only if the function is a polynomial of degree $\leqslant r$. Cf. MARTIN (1).

([14]) KERNER (1).

([15]) MICHAL (1)-(5).

([16]) HAUSDORFF (1); Compare with MICHAL and PAXSON (1). In the latter reference the term « class $C^{(n)}$ » does not have the same meaning as in HILDEBRANDT and GRAVES (1).

([17]) MICHAL (1)-(5), especially (3) and (4).

The entire class of coordinate systems defined above is too wide for the purposes of differential geometry, for in general the transformations between coordinate systems will be merely continuous and not necessarily differentiable.

Definition 2.1. - *A transformation* $\bar{x}=\bar{x}(x)$ *taking an open set of a Banach space* E *into an open set of* E *will be called a regular* [18] *transformation if* $\bar{x}(x)$ *and its inverse* $x(\bar{x})$ *are differentiable throughout their respective domains.*

In previous papers [19] on abstract differential geometry the coordinate transformations were required to be regular. The present paper on normal coordinates employs special regular coordinate transformations and makes use of the HILDEBRANDT and GRAVES differentiability class [20] $C^{(m)}$ defining the class of « allowable » coordinate transformations for the geometry.

Definition 2.2. - *A function* $F(x)$ *is said to be of class* $C^{(m)}$ *locally uniformly at* x_0 *if there exists a neighborhood of* x_0 *on which* $F(x)$ *is of class* $C^{(m)}$ *uniformly. A function* $F(x, \xi_1,..., \xi_r)$ *multilinear in* $\xi_1,..., \xi_r$ *is said to be of class* $C^{(m)}$ *locally uniformly at* x_0 *if there exists a neighborhood* X *of* x_0 *such that* $F(x, \xi_1,..., \xi_r)$ *is of class* $C^{(m)}$ *uniformly on* $X\Xi^r$, *where* Ξ *is the open set* $\|\xi\|<1$.

Definition 2.3. - *A regular transformation* $\bar{x}(x)$ *will be said to be of class* $K^{(m)}$ *if the function* $\bar{x}(x)$ *and its inverse* $x(\bar{x})$ *are of class* $C^{(m)}$ *locally uniformly at each point of their domains.*

The additional postulates for a family of coordinate systems, called *allowable* $C^{(m)}$ (*allowable* $K^{(m)}$) coordinate systems, are given below:

I). *The transformation of coordinates from one allowable coordinate system to another is regular and of class* $C^{(m)}$ (*of class* $K^{(m)}$).

II). *Any coordinate system obtained by a regular transformation of class* $C^{(m)}$ (*of class* $K^{(m)}$) *from an allowable coordinate system is allowable.*

III). *If* $x(P)$ *is any allowable coordinate system taking a Hausdorff neighborhood* U *into the set* $\Sigma\subset S$, *then the correspondence* $x(P)$ *taking a Hausdorff neighborhood* $U_1\subset U$ *into its map* $\Sigma_1\subset\Sigma$ *is an allowable coordinate system.*

IV). *The coordinate system which maps the fundamental Hausdorff neighborhood* U_0 *onto* S *is allowable.*

The first three postulates are analogous to postulates A of VEBLEN and WHITEHEAD [21].

A less general theory of allowabel coordinates, more analogous to the VEBLEN

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(18) Compare VEBLEN and WHITEHEAD (1), p. 36.

(19) MICHAL (1)-(5).

(20) Cf. § 12,14 of HILDEBRAND and GRAVES (1).

(21) VEBLEN and WHITEHEAD (1), p. 76.

and WHITEHEAD treatment for n-dimensional arithmetic coordinates, will result if the geometric domains of the coordinate systems are general open sets instead of HAUSDORFF neighborhoods. In this special theory the succession of two coordinate transformations, when it exists, defines a unique coordinate transformation, and the set of regular coordinate transformations of class $C^{(m)}$ (of class $K^{(m)}$) forms a pseudo-group.

We shall be concerned mainly with three geometrical objects: a contravariant vector, a contravarient vector field and a linear connection. With each point P of the HAUSDORFF space H we associate a linear space $T(P)$ called the *tangent space* at the point P, whose definition is obtained by replacing the n-dimensional arithmetic space of VEBLEN and WHITEHEAD [22] by the BANACH space E.

Each element Q of the tangent space $T(P)$ will be called a contravariant vector (differential) associated with a point $P \varepsilon H$. Its components (maps in E) ξ and $\bar{\xi}$ in two coordinate systems $x(P)$ and $\bar{x}(P)$ respectively are related by the transformation

$$\bar{\xi}=\bar{x}(x;\ \xi) \tag{2.1}$$

for P in the intersection of the two HAUSDORFF neighborhoods. A contravariant vector field (c. v. f. for brevity) is defined as usual.

Assume now that the transformations of coordinates are of class $C^{(2)}$. With each ordered triple consisting of a point P of H and a pair of points Q_1, Q_2 of $T(P)$ we associate a range R. For each coordinate system $x(P)$ in H we postulate a biunique correspondence between R and a subset $F_{x(P)}$ of the BANACH space E. Let $L(P, Q_1, Q_2)\,\varepsilon\, R$ be a function defined for $P \varepsilon H$, Q_1, $Q_2 \varepsilon T(P)$. Suppose that ξ_1, ξ_2 are the components of Q_1, Q_2 in $x(P)$, while $\bar{\xi}_1$, $\bar{\xi}_2$ are those in any other coordinate system $\bar{x}(P)$. Let $\Gamma(x, \xi_1, \xi_2)$ be the component of $L(P, Q_1, Q_2)$ in $x(P)$ so that $\Gamma(x, \xi_1, \xi_2)\,\varepsilon\, F_{x(P)}$. We assume that $\Gamma(x, \xi_1, \xi_2)$ is bilinear in ξ_1 and ξ_2. If for the coordinate system $\bar{x}(P)$ the component $\bar{\Gamma}(\bar{x}, \bar{\xi}_1, \bar{\xi}_2)$ of $L(P, Q_1, Q_2)$ is related to $\Gamma(x, \xi_1, \xi_2)$ by the law

$$\bar{\Gamma}(\bar{x},\ \bar{\xi}_1,\ \bar{\xi}_2)=\bar{x}(x;\ \Gamma(x,\ \xi_1,\ \xi_2))-\bar{x}(x;\ \xi_1;\ \xi_2) \tag{2.2}$$

throughout the intersection of the two HAUSDORFF neighborhoods, then $L(P, Q_1, Q_2)$ will be called a *linear connection*. We may also write the transformation law (2.2) in the equivalent form

$$\bar{\Gamma}(\bar{x},\ \bar{\xi}_1,\ \bar{\xi}_2)=\bar{x}(x;\ \Gamma(x,\ \xi_1,\ \xi_2))+\bar{x}(x;\ x(\bar{x};\ \bar{\xi}_1;\ \bar{\xi}_2)). \tag{2.3}$$

It follows readily from (2.1) and (2.2) that $\bar{\Gamma}(\bar{x}, \bar{\xi}_1, \bar{\xi}_2)$ is bilinear in $\bar{\xi}_1$ and $\bar{\xi}_2$. In other words the bilinearity of a component of the linear connection is an invariant under transformations of coordinates.

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[22] VEBLEN and WHITEHEAD (1). The notion of tangent spaces in n-dimensional differential geometry was first introduced by E. CARTAN.

To avoid long circumlocutions we shall often speak of the linear connection $\Gamma(x, \xi_1, \xi_2)$. Similar abbreviations in terminology will be made for other geometric objects.

Although we can not prove that a homeomorphism taking an arbitrary open subset of the fundamental set S into another open subset of S is a transformation of coordinates, there does exist a coordinate transformation which is induced by such a homeomorphism. In fact we have.

THEOREM 2.1. - *Let $\bar{x}=\psi(x)$ be a homeomorphism taking an open $\Sigma_1 \subset S$ into an open set $\Sigma_2 \subset S$. Then there exist two coordinate systems $x(P)$ and $\bar{x}(P)$ with coordinate domains $\Sigma_1' \subset \Sigma_1$ and $\Sigma_2' \subset \Sigma_2$ respectively such that $\bar{x}=\psi(x)$ is a transformation of coordinates from the system $x(P)$ to the system $\bar{x}(P)$.*

Proof: Let $z(P)$ be the coordinate system mapping the geometric domain U_0 onto the fixed set S. Then the inverse function $P(z)$ maps Σ_1 homeomorphically on an open set $O_1 \subset U_0$. If U_1 is any HAUSDORFF neighborhood $\subset O_1$, then $x(P)=z(P)$ is a coordinate system mapping U_1 homeomorphically on some open $\Sigma_1' \subset \Sigma_1$ while $\bar{x}(P)=\psi(x(P))$ is a coordinate system mapping U_1 onto some open $\Sigma_2' \subset \Sigma_2$. Thus the required transformation of coordinates is generated by $U_1 = U_0 \cap U_1$.

Corollary. - If in Theorem 2.1, $\bar{x}=\psi(x)$ is of class $C^{(m)}$ (of class $K^{(m)}$) then the coordinate systems $x(P)$ and $\bar{x}(P)$ are allowable $C^{(m)}$ (allowable $K^{(m)}$) coordinate systems.

Had we chosen the geometrical domains of the coordinate systems as general open sets instead of HAUSDORFF neighborhoods, the sets Σ_1' and Σ_2' in the above theorem and corollary could be taken as Σ_1 and Σ_2 respectively.

The following three theorems characterize functions of class $C^{(m)}$ locally uniformly and transformations of coordinates of class $K^{(m)}$.

THEOREM 2.2. - *A necessary and sufficient condition for a function $F(x, \xi_1,...., \xi_r)$, multilinear in $\xi_1,...., \xi_r$, to be of class $C^{(m)}$ locally uniformly at x_0 is that there exist a neighborhood X of x_0 such that $F(x, \xi_1,...., \xi_r; \delta_1 x;....; \delta_m x)$ exists continuous in x uniformly with respect to its entire set of arguments for $x \varepsilon X$, $\xi_i \varepsilon \Xi$, $\delta_j x \varepsilon \Xi$, where Ξ is the neighborhood $\|\xi\|<1$.*

Proof: If the condition of the theorem is satisfied, then $F(x, \xi_1,...., \xi_r; \delta_1 x;....; \delta_m x)$ is multilinear ([23]) in $\xi_1,...., \delta_m x$ and continuous in x. Hence ([24]) there is a constant M and a neighborhood X_0 of x_0 such that $X_0 \subset X$ and

$$\| F(x, \xi_1,...., \xi_r;\ \delta_1 x;....;\ \delta_m x) \| \leqslant M \| \xi_1 \| .... \| \delta_m x \|$$

for all $x \varepsilon X_0$. The theorem now follows from lemma 3 of MICHAL and HYERS (1).

([23]) MICHAL (1).

([24]) KERNER (2).

THEOREM 2.3. - *If in Theorem 2.2 we delete the ξ's, the resulting statement about functions $F(x)$ is a true theorem.*

THEOREM 2.4. - *A necessary and sufficient condition for a regular coordinate transformation $\bar{x}=\bar{x}(x)$ to be of class $K^{(m)}$ is that the function $\bar{x}(x)$ be of class $C^{(m)}$ locally uniformly at each point of its domain of definition.*

Proof: The necessity of the condition is obvious. Now let the sufficiency hypothesis hold, and let x_0 be any chosen point of the domain of definition of $\bar{x}(x)$. Since $\bar{x}=\bar{x}(x)$ is regular, the differential $\bar{x}(x_0\,;\,\delta x)$ is a solvable linear function [25] of δx. By hypothesis there is a neighborhood of x_0 on which $\bar{x}(x)$ is of class $C^{(m)}$ uniformly, so that the hypotheses of a known implicit function theorem [26] are satisfied by the equation $\bar{x}=\bar{x}(x)$ with x_0 as the initial point. It follows that the unique solution $x=x(\bar{x})$ is of class $C^{(m)}$ locally uniformly on its domain of definition.

The question of the invariance under transformation of coordinates of the class of a linear connection and of a contravariant vector field is answered in the next theorem.

THEOREM 2.5. - *Let $x(P)$ and $\bar{x}(Q)$ be two coordinate systems with intersecting geometrical domains, generating a regular coordinate transformation $\bar{x}=\bar{x}(x)$. Denote the domains of definition of $\bar{x}(x)$ and its inverse $x(\bar{x})$ by S_1 and S_2 respectively, and suppose that $\bar{x}=\bar{x}(x)$ is of class $C^{(m)}$ (of class $K^{(m)}$) at each point of S_1. Then if the component $\xi(x)$ of a c. v. f. is of class $C^{(n)}$ (of class $C^{(n)}$ locally uniformly) on S_1, the component $\bar{\xi}(\bar{x})$ is of the same class on S_2, providing $m>n$. Similarly the property that the linear connection $\Gamma(x, \xi_1, \xi_2)$ be of class $C^{(n)}$ (of class $C^{(n)}$ locally uniformly) is a geometrical invariant for $m \geqslant n+2$.*

Proof: A known result [27] assures us that if $F(x)$ is of class $C^{(n)}$ (of class $C^{(n)}$ uniformly) on an open set X to E and $x(y)$ is of class $C^{(n)}$ (of class $C^{(n)}$ uniformly) on an open set Y to X then $F(x(y))$ is of class $C^{(n)}$ (of class $C^{(n)}$ uniformly) on Y. The theorem follows by an evident application of this result to the transformation laws (2.1) and (2.2).

3. - Abstract Normal Coordinates.

This section is devoted to the definition of abstract normal coordinates, to the proof of their existence, and to the derivation of some of their fundamental properties. We consider the HAUSDORFF space H with allowable $K^{(m)}$ coordinate systems and with a linear connection $\Gamma(x, \xi_1, \xi_2)$. From now on we assume that the fundamental set S contains the zero element of E. Let the linear connection

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(25) MICHAL and ELCONIN (1).

(26) Theorem 4, p. 150 of HILDEBRANDT and GRAVES (1).

(27) HILDEBRANDT & GRAVES (1) p. 144.

be of class $C^{(n)}$ locally uniformly in the coordinate domain Σ of an allowable $K^{(m)}$ coordinate system $x(P)$ $(m \geqslant n+2)$. Then the differential system

$$\frac{d^2x}{ds^2} + \Gamma\left(x, \frac{dx}{ds}, \frac{dx}{ds}\right) = 0, \qquad x(0) = p, \qquad \left(\frac{dx}{ds}\right)_{s=0} = \xi \tag{3.1}$$

satisfies the hypotheses of Theorem 1.1 for some neighborhood X of any chosen point $\bar{x} \varepsilon \Sigma$. Hence (3.1) has a unique solution $x = \varphi(q, p, s)$ which is of class $C^{(n)}$ in (q, p) uniformly on a region $X_0 Y_0$, for $0 \leqslant s \leqslant 1$, where $\bar{x} \varepsilon X_0$ and $0 \varepsilon Y_0$. The image $\Phi(P, Q, s)$ of $\varphi(p, \xi, s)$ in the coordinate system $x(P)$ is a parameterized curve in the HAUSDORFF space H. It is called a *path* of the linearly connected space H, with the parameter s, and the differential equation in (3.1) is called the differential equation of paths in the coordinate system $x(P)$. The class of *affine parameters* is the set of parameters t obtained from s by applying the transformations of the group $t = as + \beta$, $a \neq 0$, for it is just this group which leaves the differential equation of the paths invariant. In particular the form of this differential equation is unchanged by the transformation $t = \left(\frac{1}{\lambda}\right) s$, while the initial conditions in (3.1) go over into $x(0) = p$, $\left(\frac{dx}{dt}\right)_{t=0} = \lambda\xi$. Thus from the existence and uniqueness Theorem 1.1

$$\varphi\left(p, \lambda\xi, \frac{1}{\lambda}s\right) = \varphi(p, \lambda\xi, t) = \varphi(p, \xi, s) \quad \text{for} \quad 0 < \lambda \leqslant 1.$$

The solution of (3.1) may be written in the form

$$x = f(p, y), \tag{3.2}$$

where $y = s\xi$ and $f(p, s\xi) = \varphi(p, s\xi, 1)$ on $0 < s \leqslant 1$ and $f(p, 0) = p$.

Definition. - A coordinate system $y(P)$ in which the equation of paths through a point P_0 (with coordinate $y = 0$) takes the form $y = s\xi$ is called a normal coordinate system with center P_0.

We do not require that a normal coordinate system be an allowable $K^{(m)}$ coordinate system. Thus in general the component $\Gamma(x, \xi_1, \xi_2)$ of the linear connection in such a coordinate system will not be of class $C^{(n)}$. The following theorem shows the existence of normal coordinate systems in which the linear connection is of class $C^{(n-2)}$ locally uniformly.

THEOREM 3.1. - *Let Σ be the coordinate domain of an allowable $K^{(m)}$ coordinate system $x(P)$ and let the linear connection $\Gamma(x, \xi_1, \xi_2)$ be of class $C^{(n)}$ locally uniformly on Σ subject to the restriction $m \geqslant n+2$. Then corresponding to each point $q \varepsilon \Sigma$ there is a constant $c > 0$ and a function $h(p, x)$ of class $C^{(n)}$ uniformly on* [28] *$E^2((q)_{2c})$ such that for any*

[28] The open sphere $(x_0)_a$ in E will also be denoted by $E((x_0)_a)$.

choice of p in $(q)_c$ the transformation $y=h(p, x)$ is of class $K^{(n)}$ for $x \varepsilon (p)_c$ and defines a normal coordinate system $y(P)$ with center $P_0(p)$.

Proof: As already remarked, $\varphi(p, \xi, s)=f(p, s\xi)$ is of class $C^{(n)}$ in (p, ξ) uniformly on $X_0 Y_0$ for $0 \leqslant s \leqslant 1$, where $0 \varepsilon Y_0$. Taking $s=1$, we have that $f(p, y)$ is of class $C^{(n)}$ uniformly on $X_0 Y_0$. For any $p \varepsilon X_0$ we compute the value of the differential $\delta_y f(p, y)$ for $y=0$. Since $f(p, s\xi)$ satisfies the initial conditions in (3.1)

$$\delta_y f(p, y)]_{y=0}=\left(\frac{\delta f(p, s\delta y)}{\delta s}\right)_{s=0}=\delta y. \tag{3.3}$$

We can now show that the hypotheses of a known implicit function theorem ([29]) are satisfied by the equation $G(\omega, y)=0$, where ω is the pair (x, p) and $G(\omega, y)=x-f(p, y)$. In fact, if we take $\omega_0=(q, q)$, $y_0=0$, and W_0 as the set of points (ω, y) for which $\|x-q\|<a$, $p \varepsilon X_0$, $y \varepsilon Y_0$ where a is any chosen positive number, then $\delta_y G(\omega_0, y)]_{y=y_0}=\delta y$ is obviously a solvable linear function of δy.

From this implicit function theorem there exist positive numbers α, β and a unique function $h(p, x)$ on $E^2((q)_\beta)$ to $(0)_\alpha$ such that

1°) For each $p \varepsilon (q)_\beta$, $y=h(p, x)$ is a solution of equation (3.2).

2°) $h(p, x)$ is of class $C^{(n)}$ uniformly on $E^2((q)_\beta)$. If we take $c={}^1/{}_2\beta$, then for each $p \varepsilon (q)_c$, $h(p, x)$ is of class $K^{(n)}$ for $x \varepsilon (p)_c$.

That the transformation $y=h(p, x)$ for each p is a transformation of coordinates follows readily from the postulates of § 2 on coordinate systems. Since $x=f(p, s\xi)$ satisfies the differential equation for the paths in the coordinate system $x(P)$ and since Γ is a linear connection, one can easily show that $y=s\xi$ satisfies the differential equation of paths in the new coordinate system $y(P)$:

$$\frac{d^2y}{ds^2}+{}^+\Gamma\left(y, \frac{dy}{ds}, \frac{dy}{ds}\right)=0. \tag{3.4}$$

This shows that $y(P)$ is a normal coordinate system, and the proof of the theorem is complete.

In (3.4) and in the remaining part of the paper, the dagger $^+$ will be used to denote the components of a geometric object in a normal coordinate system.

Corollary.

$${}^+\Gamma(y, y, y)=0 \tag{3.5}$$

for y in the coordinate domain of the normal coordinate system $y(P)$

$${}^+\Gamma(0, \lambda, \lambda)=0 \tag{3.6}$$

for all λ in the Banach space E.

THEOREM 3.2. - *Let $x(P)$ and $\bar{x}(P)$ be two allowable $K^{(n+2)}$ coordinate systems $(n \geqslant 2)$ whose geometric domains have a point P_0 in common, and*

([29]) HILDEBRANDT & GRAVES (1): theorem 4.

let the linear connection $\Gamma(x, \xi_1, \xi_2)$ be of class $C^{(n)}$ locally uniformly in the coordinate domain of $x(P)$. Suppose that $y(P)$ and $\bar{y}(P)$ are the normal coordinate systems determined by the coordinate systems $x(P)$ and $\bar{x}(P)$ respectively and with the same point P_0 of H as center. Then there exist two open subsets S_y and $S_{\bar{y}}$ of the coordinate domains of $y(P)$ and $\bar{y}(P)$ respectively such that

1°) $0 \varepsilon S_y$, $0 \varepsilon S_{\bar{y}}$

2°) The linear coordinate transformation

$$\bar{y} = \bar{x}(p;\ y), \qquad (\text{where } p = x(P_0)) \tag{3.7}$$

takes S_y into $S_{\bar{y}}$; that is normal coordinates undergo a linear transformation under a general transformation of the determining coordinates.

Proof: Since the geometrical domains of $x(P)$ and $\bar{x}(P)$ both contain P_0, the intersection of the geometrical domains of the normal coordinate systems $y(P)$ and $\bar{y}(P)$ also contains P_0. Hence there will exist a *transformation of coordinates* from $y(P)$ to $\bar{y}(P)$ taking an open subset S_y of the coordinate domain of $y(P)$ into an open subset $S_{\bar{y}}$ of the coordinate domain of $\bar{y}(P)$, and taking the point $y=0$ of S_y into the point $\bar{y}=0$ of $S_{\bar{y}}$. To find the explicit form of this transformation of coordinates observe that the equations of paths in the coordinate systems $y(P)$ and $\bar{y}(P)$ are $y = s\xi$ and $\bar{y} = s\bar{\xi}$ respectively where $\bar{\xi} = \bar{x}(p;\ \xi)$, $p = x(P_0)$. Hence (3.7) is the required transformation.

4. - The Differentials of $f(p, y)$.

We shall develop the absolute differential calculus of our linearly connected space in later sections. To do this however we need the explicit expressions for the FRÉCHET differentials at the point $y=0$ of the coordinate transformation (3.2).

For simplicity *define* $\mu(y)$ and its inverse $\nu(x)$ by

$$\mu(y) = f(p, y), \qquad \nu(x) = h(p, x). \tag{4.1}$$

We have seen in (3.3) that

$$\mu(0;\ \delta y) = \frac{d\mu(\lambda \delta y)}{d\lambda}\bigg]_{\lambda=0} = \delta y. \tag{4.2}$$

More generally since $\mu(y)$ is of class $C^{(n)}$ uniformly on Y_0, it follows from the well known theorem on FRÉCHET differentials of functions of functions that for $r \leqslant n$

$$\mu_{(r)}(\lambda y;\ \delta y; \ldots;\ \delta y) = \frac{d^r \mu(\lambda \delta y)}{d\lambda^r} \tag{4.3}$$

and

$$\mu_{(r)}(0;\ \delta y; \ldots;\ \delta y) = \frac{d^r \mu(\lambda \delta y)}{d\lambda^r}\bigg]_{\lambda=0}. \tag{4.4}$$

(The notation $\pi_{(r)}(0;\ \delta y;....;\ \delta y)$ will be used to denote the rth differential of the function $\pi(y)$ with all the increments equal). Since $\mu(s\delta y)$ satisfies the differential system (3.1) for $\xi=\delta y$

$$\mu(0;\ \delta y;\ \delta y)=-\Gamma_2(p,\ \delta y,\ \delta y), \qquad (n\geqslant 2) \tag{4.5}$$

where

$$\Gamma_2(x,\ \xi_1,\ \xi_2)=\frac{1}{2}\{\Gamma(x,\ \xi_1,\ \xi_2)+\Gamma(x,\ \xi_2,\ \xi_1)\}. \tag{4.6}$$

Similarly we obtain the result that

$$\mu(0;\ \delta y;\ \delta y;\ \delta y)=-\Gamma_3(p,\ \delta y,\ \delta y,\ \delta y), \tag{4.7}$$

where $\Gamma_3(x,\ \xi_1,\ \xi_2,\ \xi_3)$ is the polar ([30]) of the third degree homogeneous polynomial

$$H_3(\xi)=\Gamma(x,\ \xi,\ \xi;\ \xi)-2\Gamma_2(x,\ \Gamma(x,\ \xi,\ \xi),\ \xi). \tag{4.8}$$

On differentiating repeatedly the differential equation of the paths satisfied by $\mu(s\delta y)$ and making use of the relation (4.4) we obtain the general formula

$$\mu_{(r)}(0;\ \delta y;\ \delta y;....;\ \delta y)=-\Gamma_r(p,\ \delta y,....,\ \delta y), \qquad 2\leqslant r\leqslant n, \tag{4.9}$$

where $\Gamma_r(x,\ \xi_1,\ \xi_2,....,\ \xi_r)$ *is the polar of the homogeneous* polynomial $H_r(\xi)$ defined by

$$\begin{aligned} H_r(\xi)=\ &\Gamma_{r-1}(x,\ \xi,....,\ \xi;\ \xi)-\\ &-\{\Gamma_{r-1}(x,\ \Gamma(x,\ \xi,\ \xi),\ \xi,....,\ \xi)+\ ....\ +\Gamma_{r-1}(x,\ \xi,....,\ \xi,\ \Gamma(x,\ \xi,\ \xi))\} \end{aligned} \tag{4.10}$$

in terms of lower order Γ's. From the properties of polars and from (4.9) it follows immediately that

$$\mu(0;\ \delta_1 y;\ \delta_2 y;.....;\ \delta_r y)=-\Gamma_r(p,\ \delta_1 y,\ \delta_2 y,....,\ \delta_r y) \quad \text{for} \quad 2\leqslant r\leqslant n. \tag{4.11}$$

One may also calculate the differentials of orders 1, 2,...., n of the inverse transformation of coordinates $y=\nu(x)$ at $x=p$. As remarked previously, $\nu(p;\ \delta x)$ is the inverse of $\mu(0;\ \delta y)$ so that

$$\nu(p;\ \delta x)=\delta x. \tag{4.12}$$

([30]) The polar of a homogeneous polynomial $h(x)$ of the rth degree may be defined as

$$h(x_1,\ x_2,....,\ x_r)=\frac{1}{r!}\ h(x;\ x_1;\ x_2;\ ....;\ x_r)$$

which is independent of x. Clearly $h(x)=h(x,\ x,....,\ x)$, This definition is equivalent to MARTIN's purely algebraic definition in terms of the rth difference of $h(x)$ Cf. MARTIN (1), MICHAL and MARTIN (1).

The second differential of $\nu(x)$ at $x=p$ is obtained by an application of the following general formula on differentials ([31])

$$(4.13) \qquad x(\bar{x};\ \delta_1\bar{x};\ \delta_2\bar{x}) = -x(\bar{x};\ \bar{x}[x(\bar{x});\ x(\bar{x};\ \delta_1\bar{x});\ x(\bar{x};\ \delta_2\bar{x})]).$$

The result is

$$(4.14) \qquad \nu(p;\ \delta_1 x;\ \delta_2 x) = \Gamma_2(p, \delta_1 x, \delta_2 x).$$

The higher differentials of $\nu(x)$ at $x=p$ can be obtained by differentiating the relation (4.13) and employing the result (4.11). An alternative method will be given in the next section.

5. - Normal Vector Forms.

A c. v. f. $f(P, Q_1,\dots, Q_k)$ which depends on k contravariant vectors $Q_1,\dots, Q_k$ will be called a *c. v. f. form* ([32]) in the contravariant vectors $Q_1,\dots, Q_k$ if its component $F(x, \xi_1,\dots, \xi_k)$ in any coordinate system $x(P)$ is a multilinear form in $\xi_1,\dots, \xi_k$. *Throughout the remainder of the paper, $x(P)$ will denote an allowable $K^{(n+2)}$ coordinate system $(n \geqslant 2)$ in whose coordinate domain the linear connection $\Gamma(x, \xi_1, \xi_2)$ is of class $C^{(n)}$ locally uniformly.* Let $y(P)$ and $\bar{y}(P)$ be two normal coordinate systems with same center P_0 determined by the two allowable $K^{(n+2)}$ coordinate systems $x(P)$ and $\bar{x}(P)$ respectively. From Theorem 3.2 we find

$$(5.1) \qquad {}^+\bar{\Gamma}_2(\bar{y}, {}^+\bar{\xi}_1, {}^+\bar{\xi}_2) = \bar{x}[p;\ {}^+\Gamma_2(y, {}^+\xi_1, {}^+\xi_2)]$$

where $p=x(P_0)$. If we differentiate both members, evaluate at the common origin and use the fact that $\bar{y}(y)$ is linear we get for $k \leqslant n-2$

$$(5.2) \qquad {}^+\bar{\Gamma}_2(0, {}^+\bar{\xi}_1, {}^+\bar{\xi}_2;\ \delta_1\bar{y};\dots;\ \delta_k\bar{y}) = \bar{x}[p;\ {}^+\Gamma_2(0, {}^+\xi_1, {}^+\xi_2;\ \delta_1 y;\dots;\ \delta_k y)].$$

Write

$$(5.3) \qquad A_k(p, \xi_1,\dots, \xi_{k+2}) = {}^+\Gamma_2(0, {}^+\xi_1, {}^+\xi_2;\ {}^+\xi_3;\dots;\ {}^+\xi_{k+2}).$$

From (5.2) we see that $A_k(x, \xi_1,\dots, \xi_{k+2})$ is a c. v. f. for $\|q-x\|<c$ since p is any point of $(q)_c$. But q is any chosen point of the coordinate domain of $x(P)$ so that A_k is defined throughout the coordinate domain of $x(P)$.

Definition. - The c. v. f. form $A_k(x, \xi_1,\dots, \xi_{k+2})$ in the contravariant vectors $\xi_1,\dots, \xi_{k+2}$ will be called the kth normal vector form.

To obtain explicit expressions for these normal vector forms we proceed as follows. Differentiate

$$(5.4) \qquad \Gamma_2(x, \xi_1, \xi_2) = -x(y;\ {}^+\xi_1;\ {}^+\xi_2) + x(y;\ {}^+\Gamma_2(y, {}^+\xi_1, {}^+\xi_2))$$

([31]) See lemma for Theorem 5.1 in Michal and Elconin (2).

([32]) Michal (3).

and evaluate at $y=0$ treating y as the independent variable. On making use of (4.2), (4.11), (4.12) and the fact that

$$^{+}\Gamma_2(0, \lambda_1, \lambda_2)=0$$

for all $\lambda_i \varepsilon E$, we find by calculation

$$A_1(x, \xi_1, \xi_2, \xi_3)=\Gamma_2(x, \xi_1, \xi_2; \xi_3)-\Gamma_3(x, \xi_1, \xi_2, \xi_3)- -\Gamma_2(x, \Gamma_2(x, \xi_1, \xi_3), \xi_2)-\Gamma_2(x, \xi_1, \Gamma_2(x, \xi_2, \xi_3)). \tag{5.5}$$

The higher order normal vector forms may be calculated by taking higher order differentials of (5.4) and using similar methods.

The normal vector forms $A_1, \ldots, A_k$ are useful in the calculation of the differentials of orders $3, 4, \ldots, k+2$ of the coordinate transformation $y=\nu(x)$ at $x=p$ (See (4.1)). For example, to calculate $\nu(p; \delta_1 x; \delta_2 x; \delta_3 x)$ consider

$$^{+}\Gamma_2(y, \delta_1 y, \delta_2 y)=-\nu(x; \delta_1 x; \delta_2 x)+\nu(x; \Gamma_2(x, \delta_1 x, \delta_2 x)). \tag{5.6}$$

Differentiating and evaluating at $x=p$ gives

$$\nu(p; \delta_1 x; \delta_2 x; \delta_3 x)=\Gamma_2(p, \Gamma_2(p, \delta_1 x, \delta_2 x), \delta_3 x)+ +\Gamma(p, \delta_1 x, \delta_2 x; \delta_3 x)-A_1(p, \delta_1 x, \delta_2 x, \delta_3 x). \tag{5.7}$$

From its definition in (5.3), A_k is symmetric in $\xi_3, \ldots, \xi_{k+2}$. Other identities for the $A's$ may be obtained by differentiating the evident identity

$$^{+}\Gamma_2(y, y, y)=0 \tag{5.8}$$

and evaluating at $y=0$. For example, A_1 satisfies

$$A_1(x, \xi_1, \xi_2, \xi_3)+A_1(x, \xi_2, \xi_3, \xi_1)+A_1(x, \xi_3, \xi_1, \xi_2)=0. \tag{5.9}$$

As in the classical n-dimensional differential geometry [33] the normal vector forms can be expressed in terms of the curvature form [34], based on Γ_2, and its covariant differentials [35], and conversely. In fact for A_1 we have

$$A_1(x, \xi_1, \xi_2, \xi_3)=\frac{1}{3}\left[B(x, \xi_1, \xi_2, \xi_3)+B(x, \xi_2, \xi_1, \xi_3)\right] \tag{5.10}$$

$$B(x, \xi_1, \xi_2, \xi_3)=A_1(x, \xi_1, \xi_2, \xi_3)-A_1(x, \xi_1, \xi_3, \xi_2) \tag{5.11}$$

where the curvature form is defined by

$$B(x, \xi_1, \xi_2, \xi_3)=\Gamma_2(x, \xi_1, \xi_2; \xi_3)-\Gamma_2(x, \xi_1, \xi_3; \xi_2)+ +\Gamma_2(x, \Gamma_2(x, \xi_1, \xi_2), \xi_3)-\Gamma_2(x, \Gamma_2(x, \xi_1, \xi_3), \xi_2). \tag{5.12}$$

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(33) VEBLEN (1); EISENHART (1); MICHAL and THOMAS (1); THOMAS (1).

(34) MICHAL (1).

(35) MICHAL (3).

Since

$$(5.13)\qquad {}^{+}\overline{\Gamma}(\overline{y}, {}^{+}\overline{\xi}_1, {}^{+}\overline{\xi}_2;\ \delta_1\overline{y};\ \delta_2\overline{y};....;\ \delta_k\overline{y}) = \overline{x}(p;\ {}^{+}\Gamma(y, {}^{+}\xi_1, {}^{+}\xi_2;\ \delta_1 y;\ \delta_2 y;....;\ \delta_k y))$$

we can define another set of normal vector forms ${}_{\Gamma}A_k(x, \xi_1,...., \xi_{k+2})$ by means of

$$(5.14)\qquad {}_{\Gamma}A_k(p, \xi_1,...., \xi_{k+2}) = {}^{+}\Gamma(y, {}^{+}\xi_1, {}^{+}\xi_2;\ {}^{+}\xi_3;....;\ {}^{+}\xi_{k+2})]_{y=0}.$$

To express ${}_{\Gamma}A_k$ in terms of A_k we need the results of the next section.

6. - Covariant Differentials and Extensions of Multilinear Forms in Contravariant Vectors.

Let $F(x, \xi_1,...., \xi_r)$ be the component in the coordinate system $x(P)$ of a c. v. f. form in the contravariant vectors $\xi_1,...., \xi_r$, and let $y(P)$ be the normal coordinate system determined by $x(P)$.

Definition. - The kth extension $F(x, \xi_1,...., \xi_r \mid \xi_{r+1},...., \xi_{r+k})$ *of the form* $F(x, \xi_1,...., \xi_r)$ *is defined by*

$$(6.1)\qquad F(p, \xi_1,...., \xi_r \mid \xi_{r+1},...., \xi_{r+k}) = {}^{+}F(y, {}^{+}\xi_1,...., {}^{+}\xi_r;\ {}^{+}\xi_{r+1};....;\ {}^{+}\xi_{r+k})]_{y=0}$$

THEOREM 6.1. - *The kth extension of a c. v. f. form* $F(x, \xi_1,...., \xi_r)$ *is a c. v. f. form in the* $r+k$ *contravariant vectors* $\xi_1,...., \xi_{r+k}$.

The proof is similar to that given in the previous section for normal vector forms. A similar theorem holds when F is an absolute scalar form, where by an absolute scalar form we mean a geometric object whose components have the transformation law

$$\overline{F}(\overline{x}, \overline{\xi}_1,....., \overline{\xi}_r) = F(x, \xi_1,...., \xi_r),$$

the values of F being numerical or more generally elements of a BANACH space.

The kth extension of a c. v. f. form $F(x, \xi_1,...., \xi_r)$ may be calculated by the method briefly outlined in the preceding section for normal vector forms. The result for the first extension is

$$(6.2)\qquad F(x, \xi_1, \xi_2,...., \xi_r \mid \xi_{r+1}) = F(x, \xi_1,...., \xi_r;\ \xi_{r+1}) - \\ - \sum_{i=1}^{r} F(x, \xi_1,...., \xi_{i-1}, \Gamma_2(x, \xi_i, \xi_{r+1}), \xi_{i+1},...., \xi_r) + \Gamma_2(x, F(x, \xi_1,...., \xi_r), \xi_{r+1}).$$

Covariant differentials of multilinear forms have already been introduced by one of us ([36]). The expression for a covariant differential of a c. v. f. form $F(x, \xi_1,...., \xi_r)$ is made clear in the following theorem, which is an immediate result of formula (6.2) and the definition ([37]) of the covariant differential based on Γ_2.

([36]) MICHAL (1)-(5).

([37]) MICHAL (3), Theorem II for the special case in which the linear connection is symmetric.

THEOREM 6.2. - *The first covariant differential, based on Γ_2, of a c. v. f.* form $F(x, \xi_1,...., \xi_r)$ is identical with the first extension $F(x, \xi_1,...., \xi_r \mid \delta x)$.

A similar theorem holds for absolute scalar forms F.

In order to obtain expressions for covariant differentials based on the linear connection $\Gamma(x, \xi_1, \xi_2)$ note that

$$\Gamma(x, \xi_1, \xi_2) = \Gamma_2(x, \xi_1, \xi_2) + \Omega(x, \xi_1, \xi_2) \tag{6.3}$$

where

$$\Omega(x, \xi_1, \xi_2) = \frac{1}{2}\{\Gamma(x, \xi_1, \xi_2) - \Gamma(x, \xi_2, \xi_1)\} \tag{6.4}$$

is a c. v. f. form, called the *torsion form.* Hence if we replace Γ_2 by Γ in the right member of (6.2) we obtain a c. v. f. form, the covariant differential [38] of F based on Γ.

By normal coordinate methods one readily obtains an elegant proof of the following BIANCHI identity

$$B(x, \xi_1, \xi_2, \xi_3 \mid \xi_4) + B(x, \xi_1, \xi_3, \xi_4 \mid \xi_2) + B(x, \xi_1, \xi_4, \xi_2) \mid \xi_3) = 0. \tag{6.5}$$

7. - Replacement Theorem for Differential Invariants.

Consider a *functional*

$$G_{\alpha\beta\,....\,\pi}[f_2(\alpha_1, \alpha_2), f_3(\beta_1, \beta_2, \beta_3),...., f_{l+2}(\pi_1, \pi_2,...., \pi_{l+2}) \| \lambda_1, \lambda_2,...., \lambda_r] \tag{7.1}$$

whose arguments are multilinear functions $f_1, f_2,...., f_{l+2}$ and whose values are multilinear functions of $\lambda_1, \lambda_2,...., \lambda_r$.

Definition. - The function

$$\begin{aligned} F(x, \xi_1,...., \xi_r) &= \\ &= G_{\alpha\beta\,....\,\pi}[\Gamma(x, \alpha_1, \alpha_2), \Gamma(x, \beta_1, \beta_2; \beta_3),...., \Gamma(x, \pi_1, \pi_2; \pi_3;....; \pi_{l+2}) \| \xi_1,...., \xi_r], \end{aligned} \tag{7.2}$$

where the $\xi's$ are contravariant vectors, will be called the component of a differential invariant of order l and of contravariant type, if it satisfies the transformation law

$$\begin{aligned} &G_{\alpha\beta\,....\,\pi}[\overline{\Gamma}(\bar{x}, \alpha_1, \alpha_2), \overline{\Gamma}(\bar{x}, \beta_1, \beta_2; \beta_3),...., \overline{\Gamma}(\bar{x}, \pi_1, \pi_2; \pi_3;....; \pi_{l+2}) \| \bar{\xi}_1,...., \bar{\xi}_r] = \\ &= \bar{x}(x; G_{\alpha\beta\,....\,\pi}[\Gamma(x, \alpha_1, \alpha_2), \Gamma(x, \beta_1, \beta_2; \beta_3),...., \Gamma(x, \pi_1, \pi_2; \pi_3;....; \pi_{l+2}) \| \xi_1,...., \xi_r]). \end{aligned} \tag{7.3}$$

A similar definition may of course be given for differential invariants of scalar type.

THEOREM. - *Under our standing hypothesis that the linear connection $\Gamma(x, \xi_1, \xi_2)$ be of class $C^{(n)}$ locally uniformly, every differential invariant of*

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[38] MICHAL (3), Theorem II.

order $l \leqslant n-2$ of either type can by a mere replacement process de written in terms of the fundamental set of differential invariants consisting of the torsion form Ω and the normal vector forms ${}_{\Gamma}A_1, \ldots, {}_{\Gamma}A_l$:

$$G_{\alpha\beta \ldots \pi}[\Gamma(x, \alpha_1, \alpha_2), \Gamma(x, \beta_1, \beta_2;\ \beta_3), \ldots, \Gamma(x, \pi_1, \pi_2;\ \pi_3; \ldots;\ \pi_{l+2}) \| \xi_1, \ldots, \xi_r] = \\ = G_{\alpha\beta \ldots \pi}[\Omega(x, \alpha_1, \alpha_2), {}_{\Gamma}A_1(x, \beta_1, \beta_2, \beta_3), \ldots, {}_{\Gamma}A_l(x, \pi_1, \ldots, \pi_{l+2}) \| \xi_1, \ldots, \xi_r]. \tag{7.4}$$

It follows immediately from (5.13) and the results of § 6 that

$$ {}_{\Gamma}A_k(x, \xi_1, \ldots, \xi_{k+2}) = A_k(x, \xi_1, \ldots, \xi_{k+2}) + \Omega(x, \xi_1, \xi_2 \mid \xi_3, \ldots, \xi_{k+2}). \tag{7.5}$$

Hence a differential invariant of order l can also be expressed in terms of another fundamental set of differential invariants consisting of the torsion form Ω, its first l extensions, and the normal vector forms $A_1, \ldots, A_l$.

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