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On the new convergence criteria for Fourier series of Hardy and Littlewood

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ON THE NEW CONVERGENCE CRITERIA FOR FOURIER SERIES OF HARDY AND LITTLEWOOD

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1. 1. - In a recent paper, HARDY and LITTLEWOOD (¹) have developed a new type of convergence criteria for FOURIER series. Their conclusion is that the series is convergent if the function satisfies a certain « continuity » condition and the FOURIER coefficients satisfy a certain order condition; neither condition alone, of course, is enough to secure convergence. Certain formal simplifications are made which do not impair the generality of the problem, and they will be retained throughout this paper. Thus it is supposed that $f(t)$ is even and integrable in $(-\pi, \pi)$, and periodic with period 2π; that the special point to be considered is the origin; and that $f(0)=0$. In these circumstances

$$f(t) \sim \sum a_n \cos nt$$

and our conclusion is to be

$$\sum a_n = 0.$$

1. 2. - The principal results of HARDY and LITTLEWOOD are these:

THEOREM 1. - *It is sufficient that*

$$f(t) = o(1)$$

when $t \to 0$, *and*

$$a_n = O(n^{-1})$$

when $n \to \infty$.

THEOREM 2. - *It is sufficient that*

$$f(t) = o\left\{\left(\log \frac{1}{t}\right)^{-1}\right\}$$

and

$$a_n = O(n^{-\delta})$$

for some positive δ.

(¹) HARDY and LITTLEWOOD, **2**; a short account had already appeared in **1**.

THEOREM 3. - *It is sufficient that*

$$F^*(t)=\int_0^t |f(u)|\,du=o\left\{t\left(\log\frac{1}{t}\right)^{-1}\right\}$$

and

$$a_n=O(n^{-\delta}).$$

THEOREM 4. - *If $\delta(n)$ is any function decreasing steadily to zero when n tends to infinity, then there is a function $f(t)$ such that*

$$f(t)=o\left\{\left(\log\frac{1}{t}\right)^{-1}\right\},$$
$$a_n=O(n^{-\delta(n)}),$$

and $\sum a_n$ is divergent.

Theorem 1 is familiar; Theorem 3 includes Theorem 2; and Theorem 4 shows that Theorems 2 and 3 are the « best possible » of their kind.

The classical test of DINI is that if $t^{-1}f(t)$ is integrable, then $\sum a_n$ is convergent and its sum is zero. The object of this paper is to find a continuous chain of theorems, of the type of Theorem 2, to link up DINI's test on the one hand and Theorem 1 on the other.

1. 3. - I shall assume throughout this paper that $\varphi(x)$ is a positive function which satisfies the following conditions, for large values of x:

(1.31) $$\Phi(x)=\int^x \frac{du}{u\varphi(u)}$$

(an increasing function) is unbounded,

(1.32) $\varphi'(x)$ exists and is positive,

(1.33) $$x\varphi'(x)\log x=O(\varphi(x)).$$

Writing Φ^{-1} for the inverse function of Φ, we define $\mu(x)$ by the equation

(1.34) $$\mu(x)=\mu(x, A)=\frac{1}{\Phi^{-1}(\Phi(x)-A)}.$$

Thus, when φ has the form in the first line below, Φ and μ have (approximately, at any rate) those in the second and third:

φ;	$\log\log x$,	$\log x$,	$\log x\cdot\log\log x$.
Φ;	$\log x(\log\log x)^{-1}$,	$\log\log x$,	$\log\log\log x$.
μ;	$x^{-1}(\log x)^A$,	$x^{-e^{-A}}$,	$\exp\{-(\log x)^{e^{-A}}\}$.

The main result of this paper is

THEOREM 5. - *If φ satisfies* (1.31), (1.32) *and* (1.33),

$$f(t)=o\left\{\left(\varphi\left(\frac{1}{t}\right)\right)^{-1}\right\},$$

and

$$a_n = O(\mu(n, A))$$

for some positive A, then $\sum a_n$ is convergent and its sum is zero.

1. 4. - It is possible to prove Theorem 5, in part, by analysis based on the argument used by HARDY and LITTLEWOOD in proving Theorem 2. This analysis however is by no means brief, and succeeds only when

$$\log\log x = O(\varphi(x)). \tag{1.41}$$

That its success is only partial is not altogether surprising since no direct proof is known of Theorem 1, which has been proved by a TAUBERIAN argument only. I therefore prove Theorem 5 by a TAUBERIAN argument.

The appropriate TAUBERIAN machinery is found in VALIRON's H-summation ([2]). We suppose that ([3])

$$x^{-2} \prec H(x) < 1, \tag{1.42}$$

and say that $\sum a_n$ is summable (H) to s if

$$T(x) = \sqrt{\left(\frac{H(x)}{2\pi}\right)} \sum_{-x}^{\infty} e^{-\frac{1}{2}m^2 H(x)} s_{m+x} \to s$$

when x tends to infinity through integral values. VALIRON has shown that if

(i) $\sum a_n$ is summable (H) to s

(ii) $a_n = O(\sqrt{H(n)})$

and $H(n)$ satisfies certain conditions as to regularity and growth (more restrictive than 1.42), then $\sum a_n$ is convergent and its sum is s. The theorem which I require is less difficult than VALIRON's, since I can assume that

$$a_n = o(\sqrt{H(n)})\,;$$

but my conditions on $H(n)$ are less restrictive, and a generalisation of VALIRON's theorem in this direction is essential for the application which I have to make. The summability result, Theorem 6, is modelled closely on the corresponding result (Theorem 8) in the paper of HARDY and LITTLEWOOD.

2. 1. - We shall investigate certain properties of the function

$$\eta(x) = \eta(x, K) = \Phi^{-1}(\Phi(x) - K) = \{\mu(x, K)\}^{-1} \tag{2.11}$$

([2]) VALIRON, 4. The consistency theorem for this method of summation, to which we appeal below, is quite trivial.

([3]) If g is a positive funcfion, we shall write $f \prec g$ when $f = o(g)$.

which are true for all large values of x:

(i) $\eta(x)$ is differentiable, increasing and unbounded.

(ii) $x^{-1}\eta(x)$ is decreasing, for we have

$$\Phi'(x)=\frac{1}{x\varphi(x)},\qquad \Phi'(\eta(x))\cdot\eta'(x)=\Phi'(x),$$

and therefore

$$\eta'(x)=\frac{\eta(x)\varphi(\eta(x))}{x\varphi(x)}<\frac{\eta(x)}{x}$$

for positive K, and

$$\frac{d}{dx}\left(\frac{\eta(x)}{x}\right)=-\frac{\eta(x)}{x^2}+\frac{\eta'(x)}{x}<0.$$

(iii) $$\eta(x)=o(x).$$

For if not, there is a positive δ, such that $\eta(x)>\delta x$; and

$$K=\int\limits_{\eta(x)}^{x}\frac{du}{u\varphi(u)}<\int\limits_{\delta x}^{x}\frac{du}{u\varphi(u)}<\frac{|\log\delta|}{\varphi(\delta x)}.$$

This, however, is false for large enough x.

(iv) More generally, if $A<K$, then

$$\eta(x)\mu(x)=\eta(x,K)\mu(x,A)=o(1).$$

For if not, there is a positive δ such that, for arbitrarily large x,

$$\eta(x)\mu(x)>\delta,$$

and therefore

$$K-A=\int\limits_{\eta}^{\mu^{-1}}\frac{du}{u\varphi(u)}<\int\limits_{\delta\mu^{-1}}^{\mu^{-1}}\frac{du}{u\varphi(u)}<\frac{|\log\delta|}{\varphi(\delta\mu^{-1})}.$$

This, however, is false for all large x.

(v) If a is positive, then the ratio $\frac{\eta(ax)}{\eta(x)}$ lies between a and 1. For if $f(u)$ is any differentiable function then

$$\frac{f(u+\alpha)}{f(u)}=\exp\left(\log f(u+\alpha)-\log f(u)\right)=\exp\left(\alpha\,\frac{f'(u+\theta\alpha)}{f(u+\theta\alpha)}\right)$$

where $0<\theta<1$. Now let us write

$$x=e^{u},\qquad a=e^{\alpha},\qquad \eta(x)=f(u).$$

It follows that

$$\frac{\eta(ax)}{\eta(x)}=a^{\xi\eta'(\xi),\eta(\xi)},\qquad \xi=a^{\theta}x.$$

Since

$$\frac{u\eta'(u)}{\eta(u)}=\frac{\varphi(\eta(u))}{\varphi(u)}<1,$$

our assertion is proved.

(vi) $$\log\Phi(x)=o(\eta^2(x)).$$

Since $\Phi(x)$ and $\Phi(\eta(x))$ are unbounded, and differ by a constant, we have

$$\Phi(x) \sim \Phi(\eta(x));$$

further $\Phi(x) = O(\log x)$. Thus

$$\log \Phi(x) \sim \log \Phi(\eta(x)) = O(\log\log \eta(x)) = o(\eta^2(x)).$$

3. - THEOREM 6. - *If*

(i) $f(t)$ *satisfies the conditions of* (1.1),

(ii) $\varphi(x)$ *satisfies the conditions of* (1.31-1.33),

(iii) $$f(t) = o\left\{\left(\varphi\left(\frac{1}{t}\right)\right)^{-1}\right\},$$

(iv) $$H(x)\eta^2(x, K) = O(1)$$

for some K; *then* $\sum a_n$ *is summable* (H) *to* 0.

3. 1. - We have to show that

$$T(x) = \sqrt{(H(x))} \sum_{-x}^{\infty} e^{-\frac{1}{2} m^2 H(x)} s_{m+x} \to 0$$

when x tends to infinity through integral values. Now for any positive c, and positive integral n,

$$s_n = \int_0^c f(t) \cdot \frac{\sin nt}{t} \cdot dt + o(1);$$

and, in virtue of the consistency theorem for H-summation, it will be enough to show that

$$\sqrt{(H(x))} \sum_{-x}^{\infty} e^{-\frac{1}{2} m^2 H(x)} \int_0^c f(t) \cdot \frac{\sin (m+x)t}{t} \cdot dt \to 0.$$

We show next that the lower limit of summation may be extended to $-\infty$, i. e. that

$$R(x) = \sqrt{(H(x))} \sum_{-\infty}^{-x-1} e^{-\frac{1}{2} m^2 H(x)} \int_0^c f(t) \cdot \frac{\sin (m+x)t}{t} \cdot dt \to 0.$$

To prove this we use the fact that a FOURIER series is bounded, $(C, 1)$ (4), at a point of continuity; that is to say that

$$\sum_1^n s_m = O(n)$$

or

$$S(n) = \sum_1^n \int_0^c f(t) \cdot \frac{\sin mt}{t} \cdot dt = O(n).$$

(4) By the FEJÉR-LEBESGUE Theorem.

Thus

$$R(x) = -\sqrt{(H(x))} \sum_{x+1}^{\infty} e^{-\frac{1}{2} m^2 H(x)} \int_0^c f(t) \frac{\sin (m-x)t}{t} dt$$

$$= -\sqrt{(H(x))} \sum_{x+1}^{\infty} e^{-\frac{1}{2} m^2 H(x)} (S(m-x) - S(m-x-1))$$

$$= -\sqrt{(H(x))} \sum_{x+1}^{\infty} S(m-x) \left(e^{-\frac{1}{2} m^2 H(x)} - e^{-\frac{1}{2} (m+1)^2 H(x)}\right)$$

$$= -\sqrt{(H(x))} \sum_{x+1}^{\infty} S(m-x) \int_m^{m+1} \frac{d}{dt} \left(-e^{-\frac{1}{2} t^2 H(x)}\right) dt$$

$$= -(H(x))^{\frac{3}{2}} \sum_{x+1}^{\infty} S(m-x) \int_m^{m+1} t e^{-\frac{1}{2} t^2 H(x)} dt$$

$$= O\left\{ (H(x))^{\frac{3}{2}} \int_x^{\infty} t^2 e^{-\frac{1}{2} t^2 H(x)} dt \right\}$$

$$= O\left\{ \sqrt{(\psi(x)} e^{-\psi(x)}) \right\}$$

where we write $x^2 H(x) = \psi(x)$; and since, by 1.42, $\psi(x)$ tends to infinity, it follows that $R(x) = o(1)$.

3. 2. - We have therefore

$$T(x) = \sqrt{(H(x))} \int_0^c t^{-1} f(t) Q(t, x) dt + o(1)$$

where

$$Q(t, x) = \sum_{-\infty}^{\infty} e^{-\frac{1}{2} m^2 H(x)} \sin (m+x)t = \sin xt \cdot \sum e^{-\frac{1}{2} m^2 H(x)} \cos mt$$

$$= \sqrt{\left(\frac{2\pi}{H(x)}\right)} \cdot \sin xt \cdot \sum \exp \left\{ -\frac{2\pi^2}{H(x)} \left(m - \frac{t}{2\pi}\right)^2 \right\} \quad (^5)$$

$$= \sqrt{\left(\frac{2\pi}{H(x)}\right)} \cdot \sin xt \cdot S,$$

say; and we have to prove that

$$\int_0^c t^{-1} f(t) \cdot \sin xt \cdot S \cdot dt = o(1).$$

Let us write

$$S = e^{-\frac{1}{2} H^{-1} t^2} + \sum_{m \neq 0} = S_1 + S_2.$$

(5) See, for example, TANNERY and MOLK, 3, p. 47.

If, as we may suppose $c<\pi$, then

$$\left(m-\frac{t}{2\pi}\right)^2>\left(m-\frac{1}{2}\right)^2\geqslant\frac{1}{4}\,m^2 \qquad (m\neq 0)$$

and

$$S_2<2\sum_1^\infty e^{-\frac{1}{2}m^2\pi^2H^{-1}}<4e^{-\frac{1}{2}\pi^2H^{-1}}$$

for large x. It follows that

$$\int_0^c t^{-1}f(t)\cdot\sin xt\cdot S_2\cdot dt=O\left\{e^{-\frac{1}{2}\pi^2H^{-1}}\left(\int_0^{x^{-1}}xdt+\int_{x^{-1}}^c t^{-1}\,|\,f(t)\,|\,dt\right)\right\}$$
$$=O\left(\Phi(x)\cdot e^{-\frac{1}{2}\pi^2H^{-1}}\right)$$

in virtue of condition (iii); and then, from condition (iv) and 2.1 (vi), it follows tha

$$\Phi(x)e^{\frac{1}{2}\pi^2H^{-1}}=O(\Phi(x)e^{-B\eta^2})=o(1).$$

3. 3. - The proof of the theorem is now reduced to showing that

$$U(x)=\int_0^c t^{-1}f(t)\,\sin xt\cdot\exp\left(-\frac{1}{2}H^{-1}t^2\right)dt=o(1).$$

Let us write

$$U(x)=\int_0^{x^{-1}}+\int_{x^{-1}}^{\eta^{-1}}+\int_{\eta^{-1}}^{\zeta^{-1}}+\int_{\zeta^{-1}}^{c}=U_1+U_2+U_3+U_4,$$

say, where $\eta(x)$ is defined by (2.11), and $\zeta(x)$ is at our disposal, subject only to the condition $1<\zeta(x)<\eta(x)$.

In the first place

$$U_1=o\left(\int_0^{x^{-1}}xdt\right)=o(1)$$

and

$$U_2=o\left(\int_{x^{-1}}^{\eta^{-1}}\frac{dt}{t\varphi\left(\frac{1}{t}\right)}\right)=o(1)$$

by the definition of $\eta(x)$. Next

$$U_3=O\left\{\int_{\eta^{-1}}^{\zeta^{-1}}t^{-1}\,|\,f(t)\,|\exp\left(-\frac{1}{2}H^{-1}t^2\right)dt\right\}=O\left\{\frac{1}{\varphi(\zeta)}\int_{(2H\eta^2)^{-1}}^{(2H\zeta^2)^{-1}}\frac{e^{-u}}{u}\,du\right\}=o(1)$$

since $\varphi(\zeta)\to\infty$ and $H\eta^2=O(1)$. Finally

$$U_4=O\left\{e^{-\frac{1}{2}(H\zeta^2)^{-1}}\int_{\zeta^{-1}}^{c}t^{-1}\,|\,f(t)\,|\,dt\right\}=O\left\{\Phi(\zeta)\,\exp\left(-\frac{1}{2}H^{-1}\zeta^{-2}\right)\right\}=o(1),$$

in virtue of condition (iv) and 2.1 (vi), since

$$1 < \zeta < \eta < x.$$

This completes the proof of Theorem 6.

4. - THEOREM 7. - *If*

(i) $H(x)$ *is a decreasing function, such that*

$$x^{-2} < H(x) < 1,$$

(ii) $x^2 H(x)$ *is an increasing function,*

(iii) *the ratio* $\frac{H(ax)}{H(x)}$, *for positive* a, *lies between positive bounds,*

(iv) $\sum a_n$ *is summable* (H) *to* s,

(v) $$a_n = o\{\sqrt{(H(n))}\};$$

then $\sum a_n$ *is convergent, and its sum is* s.

4. 1. - Let us write $\psi(x) = x^2 H(x)$. It follows from (v) that

$$s_m - s_n = O\left\{\sqrt{(\psi(m))} \cdot \log \frac{m}{n}\right\} \tag{4.11}$$

if $m \geqslant n$. In virtue of the consistency theorem for H-summation, Theorem 7 is proved if we show that

$$S(x) = \sqrt{(H(x))} \sum_{-x}^{\infty} e^{-\frac{1}{2} m^2 H(x)} (s_{m+x} - s_x) \to 0$$

when $x \to \infty$. We shall write

$$S(x) = \sqrt{(H(x))} \left\{ \sum_{-x}^{-\frac{1}{2}x} + \sum_{-\frac{1}{2}x}^{\frac{1}{2}x} + \sum_{\frac{1}{2}x}^{\infty} \right\} e^{-\frac{1}{2} m^2 H(x)} (s_{m+x} - s_x) = S_1 + S_2 + S_3,$$

say, and consider each sum separately.

4. 2. - In the first place

$$S_1 = \sqrt{(H(x))} \sum_{\nu=0}^{\frac{1}{2}x} e^{-\frac{1}{2}(x-\nu)^2 H(x)} (s_\nu - s_x)$$

$$= O\left\{\sqrt{(H(x))} \cdot e^{-\frac{1}{8}\psi(x)} \cdot \sqrt{(\psi(x))} \cdot \sum_{1}^{\frac{1}{2}x} \log \frac{x}{\nu}\right\} = O\left(\psi(x) e^{-\frac{1}{8}\psi(x)}\right),$$

since $\sum \log \frac{x}{\nu} < Ax$; thus $S_1 = o(1)$. Secondly

$$S_2 = \sqrt{(H(x))} \cdot \sum_{-\frac{1}{2}x}^{\frac{1}{2}x} e^{-\frac{1}{2} m^2 H(x)} (s_{m+x} - s_x)$$

$$= o\left\{ \sqrt{(H(x))} \cdot \sum_{-\frac{1}{2}x}^{\frac{1}{2}x} e^{-\frac{1}{2} m^2 H(x)} |m| \sqrt{(H(x))} \right\}$$

in virtue of conditions (v) and (iii); thus $S_2 = o(1)$. Finally

$$S_3 = \sqrt{(H(x))} \sum_{\frac{1}{2}x}^{\infty} e^{-\frac{1}{2} m^2 H(x)} (s_{m+x} - s_x)$$

$$= O\left\{ \sqrt{(H(x))} \int_{\frac{3}{2}x}^{\infty} \exp\left(-\frac{1}{2}(t-x)^2 H(x) + \frac{1}{2}\log \psi(t) + \log_2 \frac{t}{x}\right) dt \right\}$$

in virtue of (4.11). Now I say that for large x

$$\frac{1}{2} \log \psi(t) + \log_2 (tx^{-1}) \leq \frac{1}{4}(t-x)^2 H(x)$$

in the range $\frac{3}{2}x \leq t$ ([6]). For we have

$$\psi(t) = t^2 H(t) \leq 9(t-x)^2 H(t) \leq 9(t-x)^2 H(x)$$

and therefore

$$\frac{1}{2} \log \psi(t) \leq \frac{1}{8}(t-x)^2 H(x)$$

for large x; and secondly, for large x,

$$\log_2 (tx^{-1}) \leq \frac{1}{72}(tx^{-1})^2 \psi(x) \leq \frac{1}{8}(t-x)^2 H(x).$$

Thus

$$S_3 = O\left\{ \sqrt{(H(x))} \int_{\frac{3}{2}x}^{\infty} e^{-\frac{1}{4}(t-x)^2 H(x)} dt \right\} = o(1).$$

This completes the proof of Theorem 7.

5. - We prove Theorem 5 by combining Theorems 6 and 7. For if we take $K > A$, and write $H(x) = \{\eta(x, K)\}^{-2}$, it follows from Theorem 6 that $\sum a_n$ is summable (H) to zero. We see then that $H(x)$ decreases, and $x^2 H(x)$ increases, when $x \to \infty$, in virtue of 2.1 (i) and (ii); and that

$$x^{-2} < H(x) = \{\eta(x)\}^{-2} < 1,$$

([6]) Indeed, much more is true.

in virtue of 2.1 (i) and (iii). From 2.1 (v) it follows that condition (iii) of Theorem 7 is satisfied; and condition (v) is satisfied because

$$a_n = O\{\mu(n, A)\} = o\left\{\frac{1}{\eta(n, K)}\right\} = o\{\sqrt{(H(n))}\}$$

in virtue of 2.1 (iv). Thus all the conditions of Theorem 7 are satisfied, and we conclude that $\sum a_n$, summable to zero by Theorem 6, is in fact convergent to zero.

6. - It is worth adding that Theorems 3 and 4 have their general analogues; but, since their proof does not involve any idea not already present in the paper of HARDY and LITTLEWOOD or in the present paper, I content myself with stating them.

THEOREM 8. - *It is sufficient for the convergence of* $\sum a_n$ *that*

$$F^*(t) = \int_0^t |f(u)|\, du = o\left\{t\left(\varphi\left(\frac{1}{t}\right)\right)^{-1}\right\}$$

and

$$a_n = O(\mu(n)).$$

THEOREM 9. - *If* $M(x)$ *is any decreasing function, which tends to zero more slowly than* $\mu(x, A)$ *for all values of* A, *then there is a function* $f(t)$ *such that*:

(i) *the conditions of* 1.1 *are satisfied,*

(ii) $$f(t) = o\left\{\left(\varphi\left(\frac{1}{t}\right)\right)^{-1}\right\},$$

(iii) $$a_n = O\{M(n)\},$$

and

(iv) $\sum a_n$ *is divergent.*

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