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ON THE POLYNOMIAL EXPRESSIONS FOR THE NUMBER OF WAYS OF COLORING A MAP

by GEORGE D. BIRKHOFF (Cambridge, Mass.).

1. - Introduction.

In the past, systematic attack upon simple unsolved mathematical problems has invariably led to the discovery of new results of importance as well as to a deeper understanding of the nature of these problems, if not to their actual solution. Perhaps the simplest of all unsolved mathematical problems is that concerned with the validity of the so-called « four-color theorem ». The present paper is devoted to the study of the polynomials $P_n(\lambda)$ expressing the number of ways in which a map M_n of n regions covering the sphere can be colored in λ (or fewer) colors; in terms of these polynomials the conjecture of the four-color theorem is that 4 is not a root of any equation $P_n(\lambda)=0$.

A map M_n is said to be (properly) colored in the λ given colors, a, b, c,..., if any two regions which touch along a boundary line are colored in distinct colors. Two distinct colorings are regarded as « essentially » distinct only when they cannot be obtained from one another by a mere permutation of the colors. Let m_i $(i=1, 2,...., n)$ denote the number of « essentially » distinct colorings which involve precisely i colors. Then the polynomial of degree n in λ,

$$(1)\qquad P_n(\lambda)=m_1\lambda+m_2\lambda(\lambda-1)+m_3\lambda(\lambda-1)(\lambda-2)+.... \\ +m_n\lambda(\lambda-1)(\lambda-2)....(\lambda-n+1),$$

will give the number of distinct ways of coloring M_n in λ (or fewer) colors, provided it is not required that the colorings be « essentially » distinct. In fact the term $m_i\lambda(\lambda-1)....(\lambda-i+1)$ evidently represents the number of ways of coloring in precisely i of the λ colors $(i\leqq n)$. It may be observed that the leading coefficient m_n is 1 in $P_n(\lambda)$, while m_1 is 0 except for the trivial case of a map of one region for which $P_1(\lambda)=\lambda$.

I introduced these polynomials a long time ago ([1]), and developed recently ([2])

([1]) *A Determinant Formula for the Number of Ways of Colouring a Map.* Annals of Mathematics, vol. 14 (1912).

([2]) *On the Number of Ways of Colouring a Map.* Proceedings of the Edinburgh Mathematical Society, vol. 2, ser. 2 (1930). The result (2) is derived incidentally in the present paper.

a certain inequality, namely

$$P_n(\lambda) \geqq \lambda(\lambda-1)(\lambda-2)(\lambda-3)^{n-3} \qquad (n \geqq 3,\ \lambda \neq 4), \tag{2}$$

in which attention was restricted to integral values of λ. More recently still, H. WHITNEY has written an important paper ([3]) on the analogous polynomials for graphs (those for maps being a special case).

2. - Maximal Maps.

Let us define a map M_n as one of « maximal contact », or merely as « maximal », in case no map M_n^* exists with the same contacts of corresponding regions as M_n, and at least one further contact. We propose to limit attention almost exclusively to such maximal maps, inasmuch as the corresponding facts for non-maximal maps flow immediately from those for maximal maps.

Necessary and sufficient conditions that a map M_n $(n \geqq 3)$ be maximal are that all of its regions are simply connected, its vertices are triple, and no two of its regions touch more than once.

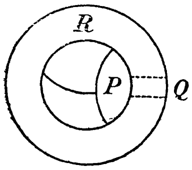

Fig. 1.

In fact if any region R in a maximal map M_n were multiply connected, for instance doubly connected as in figure 1, we could bring any two of its regions as P and Q abutting different boundaries of R into contact by means of a narrow corridor, cut across from P to Q (see the figure) and assigned arbitrarily to the region P say. The modified map M_n^* thus obtained would then possess all the contacts found in M_n, and in addition a contact of P and Q. Hence multiply connected regions cannot be present in a maximal map.

Suppose next that it were possible for a maximal map to contain a vertex V of multiplicity $\varkappa > 3$, as for instance one of multiplicity 4 (figure 2), so that the neighborhood of V is made up of four regions P, Q, R and S. It is clear that one pair of opposite regions, as P, R or Q, S, may belong to the same (simply connected) region of the map or be in contact; but evidently both pairs cannot do so. For definiteness, let us assume that P and R are distinct and do not touch one another. An opening across the vertex at V will then bring P and R into contact (see the figure) without interfering with any of the contacts in M_n. This shows that M_n is not maximal when such a vertex V of multiplicity $\varkappa > 3$ is present. Hence all of the vertices of M_n must be triple ([4]).

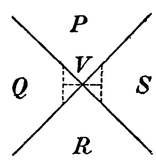

Fig. 2.

([3]) *The Coloring of Graphs*. Annals of Mathematics, vol. 33 (1932).

([4]) It is clear then that no region of M_n can touch itself at a vertex, since such a vertex

Again, let it be supposed that M_n contains two regions P and Q which touch more than once, say twice (see figure 3). In accordance with what precedes we may assume that P and Q are simply connected and all of the vertices are triple. Now one of the common sides l, which abuts on the two exterior regions R and S, may be opened up so that R and S are in contact, without destroying any existing contact, inasmuch as P and Q touch along at least one other side. Hence M_n would not be maximal. It follows then that no two of the regions in a maximal M_n touch more than once.

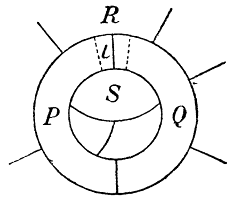

Fig. 3.

Thus the necessity of the stated conditions that M_n be maximal has been established. To prove them sufficient we need only show that there is essentially only one map having all of the contacts of a map M_n meeting these requirements.

To this end consider any region R in such a map M_n, with k regions abutting on it in cyclic order, say $R_1, R_2, ..., R_k$. We may exclude the case $k=1$ when n is of course 2 since the outer region is simply connected; likewise the case $k=2$ need not be considered since the two outer regions possess only one boundary line in common, so that n is 3 and the corresponding M_3 is obviously maximal. Hence we may assume $k \geqq 3$ for every region of M_n.

Now it is geometrically evident that, from the standpoint of *analysis situs*, the nature of the « neighborhood of R » formed by $R_1 + ... + R_k$ is completely determined by the further single contacts of the regions R_i and R_j where R_i and R_j are not adjacent along R; for instance, if there are no further contacts, this neighborhood together with R forms a simply connected region on which $R_1, ..., R_k$ abut in cyclic order. This means that we can continuously deform any other map M_n^* with contacts corresponding to those in M_n so that the corresponding regions R^*, $R_1^*, ..., R_k^*$, coincide with R, $R_1, ..., R_k$ respectively. If we proceed likewise with the adjoining regions, we can successively deform all the regions of M_n^* into the corresponding regions of M_n in a continuous manner. Thus M_n^* is seen to be essentially the same map as M_n, and in consequence not to admit of further contacts.

The number of contacts in a maximal map M_n $(n \geqq 3)$ *is* $3n-6$.

This follows at once from an application of EULER's polyhedral formula ([5]). Let f_k denote the number of k sided regions in any map M_n $(n \geqq 3)$ with simply connected regions and triple vertices, so that the number n of regions is $f_2 + f_3 + ...$

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would be of multiplicity exceeding three. Hence the regions of M_n, *considered as closed regions*, are simply connected.

([5]) For this formula, used as below, see, for instance, my paper, *On the Reducibility of Maps*. American Journal of Mathematics, vol. 35 (1915).

The number s of sides is then $(2f_2+3f_3+....)/2$, since each side is counted twice; similarly the number v of vertices is $(2f_2+3f_3+....)/3$. Thus after simple reductions, EULER's formula yields

$$4f_2+3f_3+2f_4+f_5=f_7+2f_8+ .... +12. \tag{3}$$

But f_2 is clearly 0 in the case of a maximal map $(n>3)$, while the number of contacts is s. By use of (3) the expression for s reduces to $3n-6$, as stated.

A necessary and sufficient condition that a polynomial $P_n(\lambda)$ $(n\geqq 3)$ *belong to a maximal map* M_n *is that the coefficient of* λ^{n-1} *in* $P_n(\lambda)$ *is* $-(3n-6)$.

Let us first prove the condition to be necessary. In a maximal map the number of pairs of regions in contact has just been seen to be $3n-6$; for any non-maximal map the number of contacts is less, namely $3n-6-\delta$, $\delta>0$. Now in the maximal case the number C of pairs of regions not in contact is clearly

$$C=n(n-1)/2-(3n-6).$$

Further, from equation (1), it appears that the coefficient of λ^{n-1} in $P_n(\lambda)$ is

$$m_{n-1}-(1+2+ .... +(n-1))=C-n(n-1)/2.$$

In fact we have $m_n=1$ and m_{n-1} (the number of essentially distinct ways of coloring M_n in precisely $n-1$ different colors) is obviously the number of ways of picking out a pair of regions not in contact to have the same color. Hence we infer that the coefficient in question is $-(3n-6)$.

The sufficiency of the condition is immediately obvious since this coefficient is $-(3n-6)+\delta$ in the non-maximal case.

We can prove at once the following simple facts concerning the roots of $P_n(\lambda)$ in the maximal case:

In the case of any maximal map M_n $(n\geqq 3)$ *not all of whose regions are even-sided,* 0, 1, 2 *and* 3 *are roots of* $P_n(\lambda)=0$. *If all of the regions are even-sided,* 0, 1 *and* 2 *are roots but* 3 *is not* $(P_n(3)=24)$. *In either case the center of gravity of the remaining roots lies at the point* $\lambda=3$ *of the complex plane.*

It is clear in the first place that any maximal map M_n requires at least 3 colors in the neighborhood of any (triple) vertex. Hence 0, 1 and 2 are always roots of $P_n(\lambda)=0$. Now if any region of the map is odd-sided, three colors will not suffice; for the successive abutting regions would necessarily be colored alternately in two colors if only three colors are allowed, and this is not possible if the region be odd-sided. Hence we have $P_n(3)=0$ except perhaps for the case of all even-sided regions. Let us prove that $P_n(3)=24$ not only for such maximal maps but for all other maps with only simply connected even-sided regions and triple vertices (6). Obviously $P_n(3)=0$ or 24 in such a map.

Let us eliminate the first possibility $P_n(3)=0$. By EULER's formula (3) we have, for such a map,

$$2f_2+f_4=f_8+2f_{10}+3f_{12}+\dots+6.$$

Now if the possibility $P_n(3)=0$ in question could be realized for some such map, it would be realized for some map M_n of a least number of regions. But this M_n cannot then contain a two-sided region R; for obliteration of this region R would leave a map M_{n-1} of even-sided regions, colorable in three colors; and M_n could then also be colored in three colors a, b, c by assigning to R the third color different from that of its two adjoining regions. Hence we infer that $f_2=0$. Consequently, by the above formula, there are at least six 4-sided regions in M_n.

Now let R be one of the 4-sided regions of M_n with four cyclically abutting regions A_1, A_2, A_3, A_4. Clearly either A_1 and A_3 or A_2 and A_4 are distinct and without contact. Suppose that A_1 and A_3 are. Unite A_1, R and A_3 in a single region (see figure 4). The new map M_{n-2} will then also be made up of only even-sided simply connected regions, since either A_2 and A_4 each lose two sides or A_2+A_4 loses four sides, while A_1+R+A_3 has four fewer sides than A_1 and A_3 together — an even number. Hence by hypothesis M_{n-2} $(n\geqq 6)$ can be colored in three colors with the colors around A_1+R+A_3 in alternation. But, since A_1 and A_3 have an even number of sides, A_2 and A_4 receive the same color, and we may insert R in the third color, different from that of A_1, A_3 and A_2, A_4. Thus a contradiction is reached in all cases.

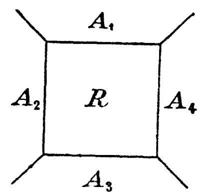


Fig. 4.

To establish the last part of the italicized statement we need only observe that $3n-6$ is the sum of all of the roots of $P_n(\lambda)=0$.

It is interesting to observe further that in the case of any M_n $(n>2)$ of all even-sided simply connected regions with simple vertices, $P_n(4)$ is always at least $12\cdot 2^{(n-1)/2}$. This is true for $n=3$ since $P_3(4)=24$. If not true for $n\geqq 3$, it fails for some M_n with least $n>3$. Now M_n contains no 2-sided region since otherwise $P_n(4)=2P_{n-1}(4)$ where M_{n-1} is obtained from M_n by shrinking the region to a point; this leads to a contradiction of course. Hence there are at least six 4-sided regions R in M_n. Distinct opposite regions abutting such a 4-sided region, R, cannot abut each other; otherwise three colors would not suffice to color M_n. Also an R must exist for which both pairs of opposite regions are

(6) Proved by P. J. HEAWOOD: *The Map-Colour Theorem.* Quarterly Journal of Mathematics, vol. 20 (1890).

distinct. In fact otherwise every R lies on a ring $R+S$ of two regions. These rings cannot cross and hence there would exist a ring $R+S$ without a 4-sided region on one side of it, I. But S has an even number of vertices in I since three colors suffice to color M_n. Hence if the exterior regions be replaced by one region T, the map $I+R+S+T$ is made up of even-sided regions, with R and T 4-sided and 2-sided respectively, and S at least 4-sided. It follows from the relation of Euler above that I must contain at least two 4-sided regions, wich is absurd. However, for an R with distinct opposite regions not in contact, we find $P_n(4)=P'_{n-2}(4)+P''_{n-2}(4)$, where M'_{n-2} and M''_{n-2} are obtained respectively by coalescence of one pair of opposite regions with R, and are composed of even-sided, simply connected regions with triple vertices. Hence a contradiction results in this case also.

The relation of the polynomials $P_n(\lambda)$ for non-maximal maps M_n to those for maximal maps is sufficiently illustrated by the following result:

The polynomial $P_n(\lambda)$ for any non-maximal map can be expressed as a sum

$$P_n(\lambda)=P_n^*(\lambda)+\sum_i P_{n_i}(\lambda) \qquad (n_i<n), \tag{4}$$

where the polynomials $P_n^*(\lambda)$, $P_{n_i}(\lambda)$ belong to maximal maps (7).

A simple illustration is that indicated in the figure 5 below in which $n=4$, $n_1=3$, $P_4=\lambda(\lambda-1)(\lambda-2)^2$, $P_4^*=\lambda(\lambda-1)(\lambda-2)(\lambda-3)$, $P_3=\lambda(\lambda-1)(\lambda-2)$.

To begin with, we observe that in all of the modifications by which a non-maximal map M_n was given further contacts (figures 1, 2, 3) a new map M_n' was obtained with the contacts of M_n and an additional one. Now if instead of an additional contact we had effected a *union*, there would have been constructed an M'_{n-1}. Moreover, since, in the original M_n the regions P and Q thus brought into contact or union must be given either different colors or the same color, we have

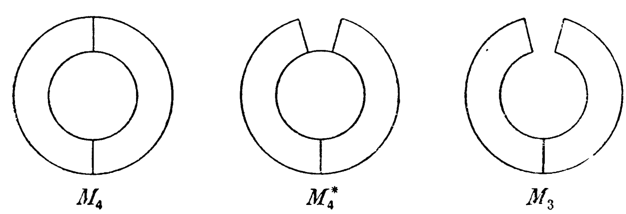


Fig. 5.

$$P_n(\lambda)=P_n'(\lambda)+P'_{n-1}(\lambda).$$

Continuing in the same manner with $P_n'(\lambda)$, $P'_{n-1}(\lambda)$, etc., we must finally express P_n as the sum of the polynomials for maximal maps as stated.

(7) A slightly imperfect notation for maps and polynomials is admitted here and later for the sake of brevity. Evidently for completeness one should write $P_{n_i}^{(i)}(\lambda)$ instead of $P_{n_i}(\lambda)$.

3. - Irreducible Maps.

We shall term any map M_n $(n>3)$ «irreducible» in case (1) all of the regions P of M_n are simply connected (i. e. are homeomorphic with a circle), (2) any two regions P, Q of M_n which touch along a side form such a simply connected region, and (3) any three regions P, Q, R of M_n of which all three pairs touch along a side, form a simply connected region about a triple vertex. In addition we shall consider a map of two or three simply connected regions to be «irreducible». All other maps M_n will be termed «reducible».

Evidently any maximal map will be irreducible provided that no doubly connected rings of three regions P, Q, R exist in it.

The polynomial $P_n(\lambda)$ for any reducible map M_n can be expressed in the form of a product,

$$P_n(\lambda)=\prod_i P_{n_i}(\lambda)/\lambda^{\alpha+\beta+\gamma}(\lambda-1)^{\beta+\gamma}(\lambda-2)^{\gamma} \qquad (n_i<n), \tag{5}$$

where the polynomials $P_{n_i}(\lambda)$ belong to irreducible maps and are $\alpha+\beta+\gamma+1$ in number, and where $n=\sum n_i-\alpha-2\beta-3\gamma$.

Suppose first that we have a r-fold $(r>1)$ connected region R in a reducible M_n. Obliterate all but one of the sets of regions into which R divides M_n, thus obtaining r (reducible or irreducible) submaps M_{n_i} with $n_i<n$ and with corresponding polynomials $P_{n_i}(\lambda)$. We have then

$$P(\lambda)=\prod_i P_{n_i}(\lambda)/\lambda^r, \tag{6}$$

since any colorings of the r maps M_{n_i} which assigns the same one of the λ colors to R combine to give a coloring of M_n, and conversely.

Evidently we may continue to apply this first process of decomposition to the maps M_{n_i} etc., until finally only maps M_{n_j} with simply connected regions remain, say $\alpha+1$, in number. In this manner $P_n(\lambda)$ is expressed in the form

$$P_n(\lambda)=\prod_j P_{n_j}(\lambda)/\lambda^{\alpha}. \tag{7}$$

Now among these maps M_{n_j} made up of simply connected regions, some may be reducible and contain two regions which touch to form a r-fold connected region R. The same process of obliteration as before may be applied to form r submaps. Colorings of these may again be combined to give a coloring of the corresponding M_{n_j}, provided that the two regions of R be given the same two of the $\lambda(\lambda-1)$ distinct pairs of colors. Here then the divisor $\lambda(\lambda-1)$ replaces λ. This second process of decomposition may also be continued until no two regions in contact along a side in the submaps form a multiply connected region, and we have then an expression of $P_n(\lambda)$ in the form

$$P_n(\lambda)=\prod_k P_{n_k}/\lambda^{\alpha+\beta}(\lambda-1)^{\beta}, \tag{8}$$

where $P_{n_k}(\lambda)$ belong to maps M_{n_k} in which all of the regions are simply connected, and in which all pairs of the regions which touch along a side form simply connected regions.

But among these maps M_{n_k} some may contain three regions P, Q, R forming a ring of three regions, dividing the rest of the map in two parts. Here again we may obliterate each of these parts in turn and so form two maps M_{n_l} and M_{n_p} with corresponding polynomials $P_{n_l}(\lambda)$, $P_{n_p}(\lambda)$. But we have

$$P_{n_k}(\lambda) = P_{n_l}(\lambda) P_{n_p}(\lambda) / \lambda(\lambda-1)(\lambda-2),$$

since any coloring of M_{n_l} and M_{n_p} which assigns the same three (necessarily distinct) colors to the three regions of the ring leads to a coloring for M_{n_k}. This third process of decomposition may also be continued until no such rings remain. Hence if three regions P, Q, R in any of these maps are in contact by pairs, they can only unite to form a simply connected region about a triple vertex, or to cover the sphere. The maps M_{n_l} so obtained are clearly irreducible, and the final expression of $P_n(\lambda)$ is evidently of the form specified in (5).

A necessary condition that a map M_n with only triple vertices be reducible is that 1, 2 *or* 3 *be a multiple root of* $P_n(\lambda) = 0$.

In fact if any region of M_n is not simply connected, the first step taken above expresses $P_n(\lambda)$ as a product in which $\lambda - 1$ is a root of multiplicity at least two. Hence the statement certainly holds unless all of the regions of M_n are simply connected. But in this case ($\alpha = 0$), $\lambda = 2$ is clearly a root of multiplicity $\beta + 1$ in the product if $\beta > 0$. Hence the statement made is certainly true unless, in addition, all pairs which touch along a side form a single simply connected region. Thus we need only to consider the case in which at least one ring of three regions is present. But in this case neither of the two partial maps can be made up of even-sided regions exclusively since each contains a three-sided region along the ring. Consequently 3 is a root of both equations $P_{n_l}(\lambda) = 0$ and $P_{n_p}(\lambda) = 0$ for the two submaps. Hence 3 is a multiple root of $P_n(\lambda) = 0$. Thus the statement made is established.

It is interesting to inquire whether the above obvious necessary condition for reducibility is sufficient. The later work of this paper throws some light on this question. In fact it will appear that in the case of irreducible maps with triple vertices, 1 and 2 cannot be multiple roots of $P_n(\lambda) = 0$. It seems to me to be probable also that 3 cannot be a multiple root, so that the stated condition is sufficient as well as necessary.

A map which is both maximal and irreducible will be called a « regular » map; accordingly, a regular map is one with simply connected regions and triple vertices, no two of whose regions touch more than once and no three of whose regions form a ring.

4. - On the Polynomials $P_n(\lambda)$ for $\lambda \leqq 0$.

The following important property of the polynomials $P_n(\lambda)$ was established by WHITNEY, and communicated orally to me nearly two years ago.

For any map M_n whatsoever, if $P_n(\lambda)$ is written in descending integral powers of λ,

(9) $$P_n(\lambda) = \lambda^n + \mu_1\lambda^{n-1} + \mu_2\lambda^{n-2} + ... + \mu_{n-1}\lambda,$$

then the successive coefficients $1, \mu_1, ..., \mu_{n-1}$ *alternate in sign.*

A simpler and basically different proof than that of WHITNEY (loc. cit., p. 690) may be given as follows: If the statement is not true, it fails for some M_n of a least number n of regions. We must have $n>3$, since for $n=2$ we have $P_2(\lambda)=\lambda(\lambda-1)$, an for $n=3$, $P_3(\lambda)=\lambda(\lambda-1)(\lambda-2)$ or $\lambda(\lambda-1)^2$.

Now there can exist no multiply connected regions R in this map M_n. Otherwise, as in the preceding section, we could write

$$P_n(\lambda) = \prod_i P_{n_i}(\lambda)/\lambda^a \qquad (n_i < n).$$

Hence, since all of the polynomials $P_{n_i}(\lambda)$ possess the stated property according to our hypothesis, $P_n(\lambda)$ would also do so. Thus M_n can contain only simply connected regions.


Fig. 6.

Consequently at any vertex V in the map M_n no region R can abut more than once, for otherwise R would be multiply connected. If then we take any boundary line which issues from V it must continue into another vertex W distinct from V (figure 6). But if we draw W down to V a map M_n^* is formed, having n regions and one less vertex. Furthermore we have either the relation

(10) $$P_n(\lambda) = P_n^*(\lambda) - P_{n-1}(\lambda),$$

where $P_{n-1}(\lambda)$ corresponds to the map obtained by the union of the two regions which abut on WV in M_n, or else

$$P_n(\lambda) = P_n^*(\lambda)$$

if these abutting regions touch elsewhere and cannot be so united in consequence. However, if $P_n^*(\lambda)$ and $P_{n-1}(\lambda)$ both possess the stated property so will $P_n(\lambda)$. We infer that the property cannot hold for $P_n^*(\lambda)$ where M_n^* has the same number of regions as M_n but one less vertex.

Thus, by successive use of (1), we are lead to maps M_n of fewer and fewer vertices, and thus finally to a contradiction of course.

The above result shows that $P_n(\lambda)$ is *completely monotonic* for $\lambda \leqq 0$ (8).

(8) More precisely, the quantities $(-1)^{n-k} d^k P_n(\lambda)/d\lambda^k$ are all positive at $\lambda = 0$ for $k = 0, 1, ..., n-1$.

We shall extend this result in the following section.

As an immediate consequence it follows that $P_n(\lambda)=0$ *has no negative roots and that* $\lambda=0$ *is a simple root of* $P_n(\lambda)$.

It may be remarked in passing that our results here and later concerning $P_n(\lambda)$ can always, if we wish, be expressed in terms of the characteristic constants m_i or the related constants $\omega_i=i!\,m_i$; these constants ω_i represent the number of ways in which the given map can be colored in precisely i colors. For example, we have from the above results the following equivalent set of inequalities,

$$P^{(n)}(0)>0, \quad (-1)P^{(n-1)}(0)>0, \ldots, \quad (-1)^{n-1}P^{(1)}(0)>0.$$

The last of these inequalities, by means of the explicit formula (1) for $P_n(\lambda)$, takes immediately the equivalent form,

$$\frac{\omega_n}{n}-\frac{\omega_{n-1}}{n-1}+\ldots\pm\frac{\omega_1}{1}>0,$$

and the other inequalities may be similarly expressed.

5. - On the Polynomials $P_n(\lambda)$ for $\lambda\leqq 2$.

If we confine attention to maximal maps or even to a somewhat more extensive set of maps we can show that $P(\lambda)/\lambda(\lambda-1)(\lambda-2)$ is completely monotonic for $\lambda\leqq 2$. For such maps this result implies of course the result of the preceding section. More precisely, we shall prove the following:

For any map M_n $(n\geqq 3)$, *all of whose vertices excepting at most one are triple, and all of whose regions are simply connected, if we write* $P_n(\lambda)=\lambda(\lambda-1)(\lambda-2)\,Q_{n-3}(\lambda)$ *we have*

$$Q_{n-3}(\lambda)=(\lambda-2)^{n-3}+\nu_{n-4}(\lambda-2)^{n-4}+\ldots+\nu_0 \tag{11}$$

where the successive coefficients $1,\ \nu_{n-4},\ \nu_{n-5},\ldots,\ \nu_0$ *alternate in sign up to the point where all the rest (if any) vanish.*

The proof will involve a method similar to that employed in the preceding section.

At the outset we may note that for $n=3$, when there are three regions in contact, we have $Q_0(\lambda)=1$, so that the stated result holds for $n=3$. Moreover it holds for $n=4$. In fact if there is no exceptional vertex we see that no region can have as many as five sides for $n=4$ since such a region would possess at least four abutting regions. Thus EULER's formula (3) reduces to

$$4f_2+3f_3+2f_4=12, \qquad (f_2+f_3+f_4=4),$$

of which the three possible solutions are

$$f_2=2,\ f_3=0,\ f_4=2; \qquad f_2=1,\ f_3=2,\ f_4=1; \qquad f_2=0,\ f_3=4,\ f_4=0.$$

It is readily seen that only the first and last of these correspond to actual maps ($n=4$); these are of the respective types of *a*) and *b*) in figure 7, with $Q_1(\lambda)=(\lambda-2)+0$ or $(\lambda-2)-1$ respectively, so that the stated result obtains if there is no exceptional vertex.

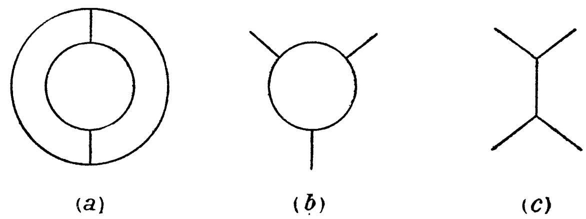


Fig. 7.

In the case of an exceptional vertex ($n=4$), its multiplicity can only be $\varkappa=4$ since the (simply connected) regions abutting at the exceptional vertex must be distinct; thus all of the four regions abut at the vertex. Since these four regions fill the sphere and all of the other vertices are triple, the only possibility is that indicated in figure 8 *c*) with $Q_1(\lambda)=(\lambda-2)+0$, when the stated result also obtains.

We can therefore restrict attention to the case $n\geqq 5$.

Suppose now that the stated result fails for some map M_n of a least number of $n\geqq 5$ regions. In particular we may choose an M_n for which the number $\varkappa\geqq 3$ of regions meeting at the exceptional vertex V, is as large as possible. This number is the same as that of the index of the vertex of course.

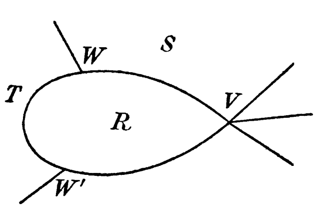


Fig. 8.

Consider now any line which issues from V. In following it along the boundary of some region R which abuts (once) at V, at least one other vertex W must be encountered (figure 8) since no region is doubly connected. But if only one vertex were encountered there would be two regions S and T abutting on R, and we would have

$$P_n(\lambda)=(\lambda-2)P_{n-1}(\lambda),$$

where M_{n-1} is formed from M_n by letting R shrink to a point. In fact any coloring of M_{n-1} would give a proper coloring for M_n, if R were inserted in one of the $\lambda-2$ colors different from those of S and T. But M_{n-1} evidently is a map of the stated type and fewer regions, so that $P_{n-1}(\lambda)$ has the specified form. Obviously $P_n(\lambda)$ would have this form also, which is not possible.

Hence there are further vertices W',.... on R, in addition to W. Furthermore none of the regions which abut at V and which are not adjacent to R at V can touch R along a side. For if a region P did, then R and P would divide M_n into two parts. By shrinking the map on one or the other side of $R+P$ to a point we would obtain submaps M_α and M_β of fewer regions with $\alpha+\beta=n+2$; and these maps would be of the stated type with $\alpha\geqq 3$ and $\beta\geqq 3$, and with V

as (possible) exceptional vertex. We would then clearly have

$$Q_{n-3}(\lambda) = (\lambda-2)\, Q_{\alpha-3}(\lambda)\, Q_{\beta-3}(\lambda) \tag{12}$$

where $Q_{\alpha-3}(\lambda)$ and $Q_{\beta-3}(\lambda)$ are of the type specified, so that $Q_{n-3}(\lambda)$ would also be, contrary to hypothesis. Therefore we may assume that none of the regions which abut at V excepting the two adjacent to R at V, touch R. In precisely the same way we may rule out the possibility that either region which abuts on R at V touches R elsewhere.

Now draw W up to V as was done in the preceding section (figure 6). A map M_n^* is thus obtained which will be of the allowed type, because of the fact that the region T adjoining S along R (figure 8) nowhere abuts at V in M_n; here it is to be recalled that there is at least one further vertex W' on R besides W and V. Similarly if we unite R and S by obliterating the boundary WV, we obtain an allowable M_{n-1}, since R and S have no further points in common. Here it is to be observed that if V is of multiplicity $\varkappa=3$ in M_n, V is no longer a vertex in M_{n-1}.

We have the obvious relation

$$Q_{n-3}(\lambda) = Q^*_{n-3}(\lambda) - Q_{n-4}(\lambda) \tag{13}$$

where $Q_{n-3}(\lambda)$, $Q^*_{n-3}(\lambda)$ and $Q_{n-4}(\lambda)$ belong to M_n, M_n^* and M_{n-1} respectively. However the two polynomials on the right have the stated property, since M_n^* has an exceptional vertex of one higher order than M_n while M_{n-1} is a map of one less region. It is seen then from the equation just written that $Q_{n-3}(\lambda)$ would have the stated property, contrary to hypothesis.

We conclude that the stated result must always hold.

It is apparent from the result just established that 0 and 1 are simple roots of $P_n(\lambda)=0$ for any map of the type considered. But the root 2 may be of maximum multiplicity $n-3$, so that $Q_{n-3}(\lambda)=(\lambda-2)^{n-3}$. This actually happens, for example, if the n regions $R_1,\ldots, R_n$ abut at the exceptional vertex V with the following pairs of regions in contact: R_2, R_1; R_3, R_2 and R_3, R_1; R_4, R_3 and R_4, R_2; etc. For then R_1 can be first colored in any of the λ colors, R_2 in the $\lambda-1$ remaining colors, R_3 in the $\lambda-2$ colors different from those of R_2 and R_1, etc.

However we proceed to prove that if, further, no two regions of M_n form a multiply connected region, then 2 is also a simple root of $P_n(\lambda)=0$, i. e. the constant term ν_0 in $Q_{n-3}(\lambda)$ is not 0; in particular this further condition is satisfied, of course, for maximal maps.

Our proof will more or less parallel the argument made above. In the first place the fact that $\nu_0 \neq 0$ in the cases $n=3$ and $n=4$ may be established by direct inspection of the various cases considered above (see, in particular, figure 7), where the two possible cases give $Q_0(\lambda)=1$ and $Q_1(\lambda)=(\lambda-2)-1$. Hence we may assume $n \geqq 5$.

Suppose now that M_n is the map of this type with least n ($n \geqq 5$) and greatest multiplicity at the exceptional vertex V, for which the stated property does not hold, i. e. for which $\nu_0=0$. We note first that no three regions P, Q, R in this map can form a ring of three regions. For, by shrinking the regions on of one side or the other of the ring to a point, we obtain two maps M_α and M_β ($\alpha+\beta=n-3$) such that

$$Q_{n-3}(\lambda)=Q_{\alpha-3}(\lambda)\,Q_{\beta-3}(\lambda). \tag{14}$$

These maps are of the stated type with $3<\alpha$, $\beta<n$ so that clearly 2 is not a root of $Q_{\alpha-3}(\lambda)=0$ or of $Q_{\beta-3}(\lambda)=0$, and so not of $Q_{n-3}(\lambda)=0$, contrary to hypothesis.

Arguing as before we deduce the situation of figure 8. If now we draw W up to V, there is obtained an M_n^*, which will clearly be of the stated type unless T in M_n touches some region P which abuts at V but not adjacent to R, so that R and the region $T+P$ in the modified map form a doubly connected region. In this case the map M_n^* clearly has a $Q_{n-3}^*(\lambda)$ with factor $\lambda-2$, because of the presence of this region. Hence the relation (13) shows that we need only show that M_{n-1}, formed by the obliteration of the side WV, is of the stated type. But there are no rings of three regions in M_n as has been seen, and hence no ring of two regions can be introduced by this obliteration. Thus M_{n-1} will be of the stated type and 2 will not be a root of $Q_{n-4}(\lambda)=0$ in this case. This completes the desired proof.

We state these results only in so far as they apply to maximal maps.

If a map M_n is maximal (so that the same restrictions are satisfied), we have in addition $\nu_0 \neq 0$. Hence for maximal maps $P_n(\lambda)=0$ has no real roots $\lambda \leqq 2$ except for simple roots 0, 1, *and* 2.

6. - On the Polynomials $P_n(\lambda)$ for $\lambda \geqq 5$.

We propose next to show that $P_n(\lambda)$ is completely monotonic for $\lambda \geqq 5$ for all maps M_n whatsoever. More precisely, we shall prove:

For all maps M_n not colorable in two colors ([9]), *so that we may write $P_n(\lambda)=\lambda(\lambda-1)(\lambda-2)\,Q_{n-3}(\lambda)$ as before, we have*

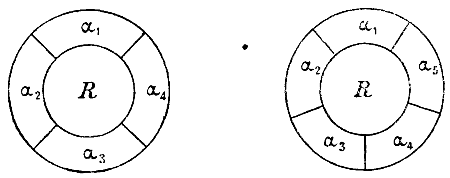
$$Q_{n-3}(\lambda)=(\lambda-5)^{n-3}+\sigma_{n-4}(\lambda-5)^{n-4}+\dots+\sigma_0, \tag{15}$$

where the successive coefficients $1, \sigma_{n-3}, \sigma_{n-4}, \dots, \sigma_0$ *are positive.*

([9]) The only maps ($n>1$) colorable in only two colors have regions bounded by a set of simple non-intersecting closed curves with at most vertices in common. In this trivial excepted case it may be readily established that $P_n(\lambda)/\lambda(\lambda-1)$ is completely monotonic for $\lambda \geqq 5$.

Let us observe first that the italicized statement holds for $n=3$ when $Q_0=1$. Hence if it does not hold for all n, it fails for some M_n with least $n \geqq 4$.

Now if this M_n is not a maximal map, we may express the corresponding polynomial $P_n(\lambda)$ as a sum of polynomials $P_n^*(\lambda)$ and $P_{n_i}(\lambda)$ $(n_i<n)$, according to the results of section 1. Here no M_{n_i} can be of the excluded type since if any M_{n_i} were colorable in two colors, then M_n would also be so colorable. We infer then that $P_n^*(\lambda)$ and $P_{n_i}(\lambda)$ possess the stated form. Consequently M_n^*, with one more contact than M_n, has a polynomial $P_n^*(\lambda)$ for which the italicized statement fails to hold. By repetition of this process we are led finally to an M_n of maximal type for which the statement fails to hold. Hence we may assume that M_n is maximal.



(a) (b)

Fig. 9.

Furthermore such a maximal M_n must necessarily be irreducible. Otherwise it must contain a ring of three regions. Hence we are led to an equation (14), in which the partial maps M_α and M_β are clearly maximal with $\alpha<n$ and $\beta<n$. Thus $Q_{\alpha-3}(\lambda)$, $Q_{\beta-3}(\lambda)$, and consequently $Q_{n-3}(\lambda)$, would satisfy the condition of the italicized statement contrary to hypothesis.

Now in this maximal, irreducible M_n there may exist a four-sided region R. If a_1, a_2, a_3, a_4 are the four abutting regions in cyclical order, they form a ring without further contacts (see a), figure 9). Furthermore if there exist no such four-sided regions, there must exist at least 12 five-sided regions R, by EULER's formula (3); and each of these will necessarily be surrounded by a similar ring of 5 regions (see b), figure 9).

At this stage I propose to make use of the first of the two following lemmas ([10]):

Lemma 1. - With the configuration of figure 9 a) in a map M_n we have the following identity:

$$P_n(\lambda)=\frac{\lambda-2}{2}\left[P_{n-2}^{(1)}(\lambda)+P_{n-2}^{(2)}(\lambda)\right]+\frac{\lambda-4}{2}\left[P_{n-1}^{(1)}(\lambda)+P_{n-1}^{(2)}(\lambda)\right]. \tag{16}$$

Here $M_{n-1}^{(i)}$ are the maps of $n-1$ regions obtained by coalescence of a_i and R, $(i=1, 2)$, respectively; and $M_{n-2}^{(i)}$ are the maps of $n-2$ regions obtained by coalescence of a_i, R, and a_{i+2} $(i=1, 2)$ respectively.

Lemma 2. - With the configuration of figure 9 b) in a map M_n we have the following identity:

$$P_n(\lambda)=\frac{2\lambda-5}{5}\sum_{i=1}^{5}P_{n-2}^{(i)}(\lambda)+\frac{\lambda-5}{5}\sum_{i=1}^{5}P_{n-1}^{(i)}(\lambda). \tag{17}$$

([10]) See my article in the Proceedings of the Edinburgh Mathematical Society cited above.

Here $M_{n-1}^{(i)}$ are the maps of $n-1$ regions obtained by the coalescence of a_i and R, $(i=1, 2,...., 5)$ respectively; and $M_{n-2}^{(i)}$ are the maps of $n-2$ regions obtained by coalescence of a_i, R, and a_{i+2}, $(i=1, 2,...., 5)$ respectively ([11]).

We shall give here the simple proof of Lemma 1; that of Lemma 2 is entirely similar. It may be stated without proof that a like formula holds for a region R surrounded by a ring of 6 regions: probably such a formula holds for a region R surrounded by a ring of any number of regions.

To prove Lemma 1, let $p_i(\lambda)$ be defined for $i=1, 2, 3, 4$ as the respective numbers of ways of coloring the map M_n in λ (or fewer) colors, *with R deleted*, according to the type of coloring on the ring as indicated in the table opposite.

	a_1	a_2	a_3	a_4
$p_1(\lambda)$	a	b	a	b
$p_2(\lambda)$	a	b	a	c
$p_3(\lambda)$	c	a	b	a
$p_4(\lambda)$	a	b	c	d

Then evidently we have the following relations:

$$(18)\qquad \begin{cases} P_n(\lambda)=(\lambda-2)p_1(\lambda)+(\lambda-3)[p_2(\lambda)+p_3(\lambda)]+(\lambda-4)p_4(\lambda), \\ P_{n-2}^{(1)}(\lambda)=p_1(\lambda)+p_2(\lambda); \qquad P_{n-2}^{(2)}(\lambda)=p_1(\lambda)+p_3(\lambda), \\ P_{n-1}^{(1)}(\lambda)=p_3(\lambda)+p_4(\lambda); \qquad P_{n-1}^{(2)}(\lambda)=p_2(\lambda)+p_4(\lambda). \end{cases}$$

From these relations we obtain the stated equation (16) by elimination of $p_i(\lambda)$, $(i=1, 2, 3, 4)$; and in addition we obtain the further relation:

$$P_{n-2}^{(1)}(\lambda)+P_{n-1}^{(1)}(\lambda)=P_{n-2}^{(2)}(\lambda)+P_{n-1}^{(1)}(\lambda).$$

Now if the M_n under consideration contains the configuration of figure 9 *a*), the polynomials on the right in (16) correspond to maps $M_{n-2}^{(i)}$, $M_{n-1}^{(i)}$ which satisfy the condition imposed in the italicized statement. Furthermore (16) may be written

$$(19)\qquad P_n(\mu+5)=\frac{\mu+3}{2}[P_{n-2}^{(1)}(\mu+5)+P_{n-2}^{(2)}(\mu+5)]+ \\ +\frac{\mu+1}{2}[P_{n-1}^{(1)}(\mu+5)+P_{n-1}^{(2)}(\mu+5)]$$

if we put $\lambda-5=\mu$. It is seen then that $Q_{n-3}(\lambda)$ must have the stated form inasmuch as $Q_{n-5}^{(i)}(\lambda)$ and $Q_{n-4}^{(i)}(\lambda)$ have this form. This is contrary to hypothesis. We infer that this M_n contains no four-sided region.

Consequently, as was remarked above, there must exist in M_n the configuration of figure 9 *b*), which leads similarly to

$$(20)\qquad P_n(\mu+5)=\frac{2\mu+5}{2}\sum_i P_{n-2}^{(i)}(\mu+5)+\frac{\mu}{5}\sum_i P_{n-1}^{(i)}(\mu+5),$$

whence we are led to a contradiction as before.

([11]) Here a_6 and a_7 are identified with a_1 and a_2 respectively.

Thus we conclude that the statement under consideration is true in all cases.

On the basis of this statement it is clear that in no case whatsoever has $P_n(\lambda)=0$ a real root for $\lambda \geqq 5$.

In this manner as far as the integral roots of $P_n(\lambda)=0$ are concerned, the only remaining question is that of a root 4; the so-called four-color theorem affirms that 4 is never a root of $P_n(\lambda)=0$. It may be remarked in passing that 1°) $P_n(4)\geqq 0$, 2°) if $P_n(4)>0$ for $n>3$, then $P_n(\lambda)$ is at least 24, and 3°) $P_n(4)=24$ holds for special maps for any n. In fact 1°) holds by definition of $P_n(\lambda)$; 2°) holds since a permutation of the colors yields a further coloring; 3°) holds, for instance, if R_2 touches R_1, if R_3 touches R_2 and R_1, if R_4 touches R_3, R_2 and R_1, if R_5 touches R_4, R_3 and R_2 etc. We see then how slight a margin of uncertainty is involved.

7. - Some Further Results for $\lambda \leqq 2$ and $\lambda \geqq 5$.

In the results of sections 5, 6, there are no limitations set to the increase of $P_n(\lambda)$ for $\lambda \leqq 2$ or for $\lambda \geqq 5$. Simple limitations of this kind are contained in the following result:

If M_n is a map of simply connected regions and triple vertices, then $Q_{n-3}(\lambda)$ is dominated by $(\lambda-2)^{n-3}$ for $\lambda \geqq 5$ and by $(\lambda-5)^{n-3}$ for $\lambda \leqq 2$ ([12]).

To establish the result stated in so far as it refers to the range $\lambda \geqq 5$, we assume it not to be true for some M_n with a least n. This n exceeds 3 since for $n=3$ we have $Q_a(\lambda)=1$.

Now M_n contains no rings of two regions. In fact otherwise we recall the relation (12) which leads at once to a contradiction since $Q_{\alpha-3}(\lambda)$ and $Q_{\beta-3}(\lambda)$ are dominated for $\lambda \geqq 5$ by $(\lambda-2)^{\alpha-3}$ and $(\lambda-2)^{\beta-3}$ respectively. Similarly it is seen that on account of relation (14) no ring of three regions exists in M_n.

Hence there must be a configuration in M_n as in figure 9 *a*) or 9 *b*). In the first case, we see, by use of (16) that $Q_{n-3}(\lambda)$ is dominated by

$$(\lambda-2)(\lambda-2)^{n-5}+(\lambda-4)(\lambda-2)^{n-4}$$

since $Q^{(i)}_{n-2}(\lambda)$ and $Q^{(i)}_{n-1}(\lambda)$ are dominated by $(\lambda-2)^{n-5}$ and $(\lambda-2)^{n-4}$ respectively, according to hypothesis. Here it is to be observed that $\lambda-2$ and $\lambda-4$ are completely monotonic for $\lambda \geqq 5$. But from this we infer at once that $Q_{n-3}(\lambda)$ is dominated by $(\lambda-3)(\lambda-2)^{n-4}$ and so by $(\lambda-2)^{n-3}$ for $\lambda \geqq 5$.

But the case of a configuration as in figure 9 *b*), we conclude by a similar use of the equation (17) that $Q_{n-3}(\lambda)$ is dominated by

$$(2\lambda-5)(\lambda-2)^{n-5}+(\lambda-5)(\lambda-2)^{n-4}=(\lambda^2-5\lambda+5)(\lambda-2)^{n-5},$$

([12]) The expression « A dominates B » means of course that A and $A-B$ are completely monotonic over the range in question and possess the same sign.

and so by $(\lambda-2)^{n-3}$ ([13]). Thus in this remaining case we are also led to a contradiction. Hence the stated result holds for $\lambda \geqq 5$.

In order to obtain the result stated for $\lambda \leqq 2$, a similar method is available. If M_n is a hypothetical map of the stated type with least n for which $Q_{n-3}(\lambda)$ is not dominated by $(\lambda-5)^{n-3}$ the possibility of rings of two or three regions in M_n can be excluded just as before. Now if the configuration of figure 9 *a*) is present we turn again to relation (16) and infer that $Q_{n-3}(\lambda)$ is dominated by $(\lambda-4)(\lambda-5)^{n-4}$, inasmuch as the first term on the right when expanded in a series in μ $(\lambda=\mu+5)$ has coefficients of opposite sign to those of the second term while those in the second term are of the sign known to hold in the expansion of $Q_{n-4}(\lambda)$ in a similar series; hence in this case $Q_{n-3}(\lambda)$ would be dominated all the more by $(\lambda-5)^{n-3}$. Again, if the configuration of figure 9 *b*) is present we find for the same reasons that $Q_{n-3}(\lambda)$ would be dominated by $(\lambda-5)(\lambda-5)^{n-4}$, a contradiction. Hence such a hypothetical map M_n cannot exist.

In connection with the above results it may be noted that for $\lambda \geqq 5$, the inequalities involved cannot be made more restrictive, for we may have $Q_{n-3}(\lambda)=(\lambda-2)^{n-3}$. On the other hand for $\lambda \leqq 2$, better results than those stated can certainly be obtained.

8. - On the Polynomials $P_n(\lambda)$ for $2 \leqq \lambda \leqq 5$.

Turning our attention to the remaining interval of the λ-axis, we shall first prove the following facts concerning $Q_{n-3}(\lambda)=P_n(\lambda)/\lambda(\lambda-1)(\lambda-2)$:

If M_n is a map of triple vertices and simply connected regions, then we have

$$|Q_{n-3}(\lambda)| \leqq 4{,}5^{n-3} \qquad (2 \leqq \lambda \leqq 5). \tag{20}$$

In fact the results of sections 5 and 7 show that we may write with $0 \leqq \theta_i \leqq 1$,

$$Q_{n-3}(\lambda)=(\lambda-5)^{n-3}+3\theta_1\binom{n-3}{1}(\lambda-5)^{n-4}+3^2\theta_2\binom{n-3}{2}(\lambda-5)^{n-5}+\dots, \tag{21}$$

since $Q_{n-3}(\lambda)$ dominates $(\lambda-5)^{n-3}$ but is dominated by $(\lambda-2)^{n-3}$ for $\lambda \geqq 5$. Hence for $\lambda \leqq 5$ we see that

$$|Q_{n-3}(\lambda)| \leqq |\lambda-8|^{n-3}.$$

Likewise the results of sections 6 and 7 show that we may write, with $0 \leqq \varphi_i \leqq 1$,

$$Q_{n-3}(\lambda)=(\lambda-2)^{n-3}-3\varphi_1\binom{n-3}{1}(\lambda-2)^{n-4}+3^2\varphi_2\binom{n-3}{2}(\lambda-2)^{n-5}+\dots$$

([13]) Note that if $\lambda=\mu+5$ we have

$$\lambda^2-5\lambda+5=5+5\mu+\mu^2, \qquad (\lambda-2)^2=9+6\mu+\mu^2.$$

Hence for $\lambda \geqq 2$ we see that

$$|Q_{n-3}(\lambda)| \leqq (\lambda+1)^{n-3}.$$

Applying the first and second of these two inequalities to the respective intervals $\left(3\frac{1}{2}, 5\right)$ and $\left(2, 3\frac{1}{2}\right)$ we infer that $|Q_{n-3}(\lambda)|$ is restricted as stated throughout the interval (2,5).

Throughout this same range $2 \leqq \lambda \leqq 5$ the integral values of λ are of most interest:

The following inequalities obtain for any map M_n of triple vertices and simply connected regions:

$$\text{(22)} \qquad \begin{cases} 0 \leqq (-1)^n Q_{n-3}(2) \leqq 3^{n-3}, & 0 \leqq Q_{n-3}(3) \leqq 1, \\ 0 \leqq Q_{n-3}(4) \leqq 2^{n-3}, & 2^{n-3} \leqq Q_{u-3}(5) \leqq 3^{n-3}. \end{cases}$$

Of these inequalities all those involving $Q_{n-3}(\lambda)$ for $\lambda = 2$, 3 and 5 are obvious consequences of what precedes. Furthermore since $P_n(4) = 24\,Q_{n-3}(4) \geqq 0$, we need only establish that $Q_{n-3}(4)$ has an upper limit as stated.

The justification for this upper limit of $Q_{n-3}(4)$ may be made as follows: Any M_n of triple vertices and simply connected regions can obviously be constructed by choosing a first region R_1, then a second region R_2 which touches R_1, then a third region R_3 which touches R_1 and R_2 (the three meeting at a triple vertex on the boundary of $R_1 + R_2$), then a fourth region which touches an adjacent pair of the regions R_1, R_2, R_3, etc. In fact at the k^{th} stage it is only necessary to select a vertex on the boundary of $R_1 + R_2 + \ldots + R_k$, and add the outer region R_{k+1} in contact with this vertex. But this construction shows that R_1 can be given one of four colors; R_2 one of three colors; R_3 one of two colors; R_4 one of at most two colors; etc. Hence $P_n(4) \leqq 4 \cdot 3 \cdot 2^{n-2}$, so that $Q_{n-3}(4) \leqq 2^{n-3}$, as was stated.

Evidently this inequality cannot be improved for the given class of maps, since we can construct such a map with regions R_1, $R_2, \ldots, R_n$ having only the contacts just indicated in which case the equality sign actually holds.

If, however, attention is restricted to the class of regular (i. e. maximal, irreducible) maps, sharper inequalities of the type

$$Q_{n-3}(4) \leqq C^{n-3} \qquad (C<2)$$

can be established. It would be of interest to determine the lower limit of C more closely. It is natural to conjecture also that, for such maps,

$$Q_{n-3}(4) \geqq D^{n-3} \qquad (D>1).$$

In other words there is probably an absolute constant $D > 1$ such that any regular map can be colored in four colors in at least $24 \cdot D^{n-3}$ different ways.

9. - Some Additional Remarks on the Roots of $P_n(\lambda)=0$.

By way of conclusion I propose to make three simple remarks concerning the roots of $P_n(\lambda)\equiv\lambda(\lambda-1)(\lambda-2)Q_{n-3}(\lambda)=0$.

a) For a regular map M_n, with n even, there is at least one real root of $Q_{n-3}(\lambda)=0$ for $2<\lambda<5$ besides a single root $\lambda=3$ and $\lambda=4$ [14]. In fact there must be an odd number of real roots of $Q_{n-3}(\lambda)=0$.

b) For a regular map M_n, not all of the roots of $Q_{n-3}(\lambda)=0$ can be real. Here we make use of the two leading terms in the formula for $P_n(\lambda)$ in its leading terms:

$$P_n(\lambda)=\lambda^n-3(n-2)\lambda^{n-1}+\frac{(n-2)(9n-25)}{2}\lambda^{n-2}+\dots$$

We have already previously computed the coefficient of λ^{n-1} in $P_n(\lambda)$; the coefficient of λ^{n-2} can either be similarly computed, or obtained by use of the formulas of WHITNEY (loc. cit.). From these terms we obtain immediately

$$P_n(\lambda)\equiv(\lambda-3)^n+6(\lambda-3)^{n-1}+(n+7)(\lambda-3)^{n-2}+\dots$$

Hence if $0, 1, 2, 3, \lambda_1,\dots, \lambda_{n-4}$ are the roots of this equation in $\lambda-3$, we obtain at once

$$\sum_i \lambda_i^2=2(4-n).$$

Since the quantity on the right is negative ($n\geqq 6$), it is clear that not all of the roots can be real.

It was observed earlier that the center of gravity of the $n-4$ roots for a maximal map other than 0, 1, 2, 3, falls at $\lambda=3$ in the complex plane. The equation written above shows that the average of their squared deviations from $\lambda=3$ is precisely -2, a negative quantity.

c) For a regular map M_n all the roots of $Q_{n-3}(\lambda)=0$ lie within a circle of radius $3/(2^{1/(n-3)}-1)$ with center at $\lambda=5$.

In fact the expression (21) for $Q_{n-3}(\lambda)$ shows that for any real or complex root of $Q_{n-3}(\lambda)=0$ we have

$$|\lambda-5|^{n-3}\leqq(|\lambda-5|+3)^{n-3}-|\lambda-5|^{n-3},$$

whence we obtain the stated conclusion.

It is obvious also that the results of sections 6, 7 permit us to name an interval $a_n<\lambda\leqq 5$ in which the equation $Q_{n-3}(\lambda)=0$ has no root. However it is natural to conjecture that there is no real root exceeding 4 in any map whatsoever. In the simpler cases, inclusive of the dodecahedral case $n=12$, this conjecture is valid; but *I* have not been able to establish its general validity even under the hypothesis that the four-color theorem is true, i. e. $P_n(4)>0$.

(14) Provided the map can be colored in four colors.

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