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Some new convergence criteria for Fourier series

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SOME NEW CONVERGENCE CRITERIA FOR FOURIER SERIES

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1. - In this paper (1) we are concerned with the convergence, in the classical sense, of the FOURIER series of an integrable function $\varphi(t)$. We suppose that $\varphi(t)$ is periodic, with period 2π, and even; that the fundamental interval is $(-\pi, \pi)$; that the mean value of $\varphi(t)$ over a period is 0; that the special point to be considered is the origin; and that the sum of the series is to be 0. In these circumstances

$$\varphi(t) \sim \sum_{1}^{\infty} a_n \cos nt = \sum a_n \cos nt, \tag{1.1}$$

and our conclusion is to be

$$\sum a_n = 0. \tag{1.2}$$

It is familiar that these formal simplifications do not impair the generality of the problem.

Since $\varphi(t)$ is even, any conditions which it is to satisfy may be stated for $t \geqq 0$.

Criteria containing a condition on the order of magnitude of a_n.

2. - Our main theorem (Theorem 2) involves (i) a « continuity » condition on $\varphi(t)$ and (ii) an « order » condition on a_n. One theorem of this character is known already.

THEOREM 1. - *It is sufficient that* (i)

$$\varphi(t) \to 0 \tag{2.1}$$

when $t \to 0$, *and* (ii)

$$a_n = O(n^{-1}). \tag{2.2}$$

In fact (i) implies the summability $(C, 1)$ of the series, and (ii) then implies its convergence. As is well known, the theorem covers the classical case in which $\varphi(t)$ is of bounded variation.

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(1) A short account of some of the principal results has appeared in the Journal of the London Math. Soc. (HARDY and LITTLEWOOD, 6).

It is natural to ask whether, if we strengthen the continuity condition (i), we may correspondingly relax the order condition (ii). If we replace (i) by any of

$$(2.3)\qquad \varphi(t)=O(t^{a}),\quad \varphi(t)=O\left\{\left(\log\frac{1}{t}\right)^{-1-a}\right\},\quad \varphi(t)=O\left\{\left(\log\frac{1}{t}\right)^{-1}\left(\log\log\frac{1}{t}\right)^{-1-a}\right\},\dots$$

where $a>0$, we may drop (ii) entirely, these conditions being sufficient in themselves. A natural intermediate hypothesis is

$$(2.4)\qquad \varphi(t)=o\left\{\left(\log\frac{1}{t}\right)^{-1}\right\}.$$

and it will be found that this hypothesis leads to very interesting results.

It should be observed first that (2.4) is not itself a sufficient condition for convergence (2). This is no doubt well known, though we have not met with any explicit proof; a more precise result is contained in Theorem 4 below. There is a distinction here between convergence at a point and uniform convergence, since

$$(2.5)\qquad \varphi(t+h)-\varphi(t)=o\left\{\left(\log\frac{1}{|h|}\right)^{-1}\right\},$$

uniformly in t, is a sufficient condition for the uniform convergence of the series (3).

3. - THEOREM 2. - *It is sufficient that* (i) $\varphi(t)$ *should satisfy* (2.4) *and that* (ii)

$$(3.1)\qquad a_n=O(n^{-\delta})$$

for some positive δ.

We suppose, as we may, that $\delta<1$ and

$$(3.2)\qquad |a_n|<n^{-\delta}.$$

We choose a positive c and take

$$(3.3)\qquad r=\frac{1}{2}\delta.$$

It is necessary and sufficient for convergence that

$$(3.4)\qquad S(\lambda)=\int_0^c \varphi(t)\frac{\sin\lambda t}{t}\,dt\to 0$$

when $\lambda\to\infty$.

We write

$$(3.5)\qquad S(\lambda)=\int_0^{\lambda^{-1}}+\int_{\lambda^{-1}}^{\lambda^{-r}}+\int_{\lambda^{-r}}^{c}=S_1(\lambda)+S_2(\lambda)+S_3(\lambda).$$

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(2) Indeed no condition $\varphi=o(\chi)$, with

$$\int_0 \frac{\chi(t)}{t}\,dt=\infty.$$

is sufficient (in other words, the classical test of DINI is the best possible of its kind).

(3) This is the « Dini-Lipschitz » criterion; see for example HOBSON, 7, p. 537.

Then, in the first place

$$S_1(\lambda)=o\Big(\int_0^{\lambda^{-1}}\lambda dt\Big)=o(1). \tag{3.6}$$

Here we do not require the full force of (2.4).

Next

$$|S_2(\lambda)|<\varepsilon\int_{\lambda^{-1}}^{\lambda^{-r}}\frac{dt}{t\log\frac{1}{t}}=-\varepsilon\left[\log\log\frac{1}{t}\right]_{\lambda^{-1}}^{\lambda^{-r}}=\varepsilon\log\frac{1}{r}=\varepsilon\log\frac{2}{\delta} \tag{3.7}$$

if $\lambda>\lambda_0(\varepsilon)$; and so

$$S_2(\lambda)=o(1). \tag{3.8}$$

It remain to consider $S_3(\lambda)$. Here we replace $\varphi(t)$ by its FOURIER series and integrate term by term. We thus obtain

$$S_3(\lambda)=\sum a_n\int_{\lambda^{-r}}^{c}\cos nt\frac{\sin\lambda t}{t}dt=\frac{1}{2}\sum a_n\left\{\int_{\lambda^{-r}}^{c}\frac{\sin(\lambda+n)t}{t}dt+\int_{\lambda^{-r}}^{c}\frac{\sin(\lambda-n)t}{t}dt\right\}= \tag{3.9}$$

$$=\frac{1}{2}\sum a_nu_n+\frac{1}{2}\sum a_nv_n=\frac{1}{2}U(\lambda)+\frac{1}{2}V(\lambda),$$

say; where

$$u_n=\int_{(\lambda+n)\lambda^{-r}}^{(\lambda+n)c}\frac{\sin w}{w}dw=Si\frac{\lambda+n}{\lambda^r}-Si(\lambda+n)c, \tag{3.10.1}$$

$$v_n=\operatorname{sgn}(\lambda-n)\int_{|\lambda-n|\lambda^{-r}}^{|\lambda-n|c}\frac{\sin w}{w}dw=\operatorname{sgn}(\lambda-n)\Big(Si\frac{|\lambda-n|}{\lambda^r}-Si|\lambda-n|c\Big)\quad\text{(4)}, \tag{3.10.2}$$

$$Si\,x=\int_x^\infty\frac{\sin w}{w}dw. \tag{3.10.3}$$

4. - The function $Si\,x$ satisfies the inequalities

$$|Si\,x|<A\quad(x>0),\qquad |Si\,x|<\frac{A}{x}\quad(x>1), \tag{4.1}$$

in which A is an absolute constant. Hence

$$|u_n|<A\frac{\lambda^r}{\lambda+n},$$

$$|U(\lambda)|\leqq A\lambda^r\sum\frac{|a_n|}{\lambda+n}<A\lambda^r\sum\frac{n^{-\delta}}{\lambda+n}<A\lambda^r\int_0^\infty\frac{x^{-\delta}}{\lambda+x}dx=C\lambda^{r-\delta}.$$

Here, and in the sequel, $C=C(\delta)$ denotes a number depending only on δ. It follows that

$$U=O(\lambda^{-\frac{1}{2}\delta})=o(1). \tag{4.2}$$

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(4) $v_n=0$ if $n=\lambda$.

We write V in the form

$$V=\sum v_n=\sum_{n<\lambda-\lambda^r} v_n+\sum_{|n-\lambda|\leqq\lambda^r} v_n+\sum_{n>\lambda+\lambda^r} v_n=V_1+V_2+V_3, \tag{4.3}$$

say. Here, first,

$$|V_2|<A(2\lambda^r+1)\lambda^{-\delta}=O(\lambda^{-\frac{1}{2}\delta})=o(1), \tag{4.4}$$

by (3.2), (3.3), (3.10.2) and (4.1). Next, in V_1,

$$|v_n|<A\frac{\lambda^r}{\lambda-n},$$

and so

$$|V_1|<A\lambda^r\sum_{n<\lambda-\lambda^r}\frac{n^{-\delta}}{\lambda-n}. \tag{4.5}$$

Since $x^\delta(\lambda-x)$ has one maximum, at

$$x=\frac{\delta\lambda}{\delta+1},$$

between 0 and λ, and increases to this maximum and then decreases, (4.5) gives

$$|V_1|<C\lambda^{r-\delta-1}+C\lambda^r\int_0^{\lambda-\lambda^r}\frac{x^{-\delta}}{\lambda-x}dx.$$

The first term is $o(1)$, and the second is

$$O\left\{\lambda^r\int_{\lambda^r}^{\lambda}(\lambda-w)^{-\delta}\frac{dw}{w}\right\}=O\left\{\lambda^{r-\delta}\int_{\lambda^{r-1}}^{1}(1-y)^{-\delta}\frac{dy}{y}\right\}=O(\lambda^{-\frac{1}{2}\delta}\log\lambda)=o(1).$$

Hence $V_1=o(1)$, and a similar, but rather simpler (5), argument shows that $V_3=o(1)$. Combining these results with (4.3) and (4.4), we find that $V=o(1)$. It then follows from (3.9) and (4.2) that $S_3(\lambda)=o(1)$; and this completes the proof of the theorem.

A particular case of some interest is that in which

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\varphi(t+h)-\varphi(t-h)|^p dt=O(|h|^{p\delta})$$

for some $p\geqq1$ and small $|h|$, i. e. when φ belongs to what we have called the class Lip (δ,p) (6). In this case (3.1) is certainly satisfied.

5. - There is a generalisation of Theorem 2 corresponding partly to LEBESGUE's generalisation of FEJÉR's theorem.

THEOREM 3. - *In Theorem* 2, *condition* (ii) *may be replaced by*

$$\Phi^*(t)=\int_0^t|\varphi(u)|\,du=o\left(\frac{t}{\log\frac{1}{t}}\right). \tag{5.1}$$

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(5) Because $x^\delta(x-\lambda)$ is monotonic in (λ,∞).

(6) See HARDY and LITTLEWOOD (4, 5).

In this case
$$S_1(\lambda)=o\left\{\lambda\int_0^{\lambda^{-1}}|\varphi(t)|\,dt\right\}=o\left(\frac{1}{\log\lambda}\right)=o(1);$$
while
$$|S_2(\lambda)|\leqq\int_{\lambda^{-1}}^{\lambda^{-r}}\frac{|\varphi(t)|}{t}\,dt\leqq\lambda^r\Phi^*\left(\frac{1}{\lambda^r}\right)+\int_{\lambda^{-1}}^{\lambda^{-r}}\frac{\Phi^*(t)}{t^2}\,dt,$$
which may be shown to tend to zero as in § 3. The discussion of $S_3(\lambda)$ is unaltered.

We have not been able to replace (5.1) by
$$\Phi(t)=\int_0^t\varphi(u)du=o\left(\frac{t}{\log\frac{1}{t}}\right).$$

Negative theorems.

6. - We prove next that Theorem 2 is a best possible theorem, in that the condition (3.1) cannot be replaced by any wider condition on the order of a_n.

THEOREM 4. - *Suppose that η_n decreases steadily to zero when $n\to\infty$. Then there is a function $\varphi(t)$ such that* (i) *$\varphi(t)$ satisfies* (2.4), (ii)

(6.1) $$a_n=O(n^{-\eta_n}),$$
and (iii) *$\sum a_n$ is divergent.*

We prove this by a modification of FEJÉR's well known method for the construction of divergent FOURIER series. We require

Lemma *a. - There is a constant A such that*

(6.2) $$\left|\sum_M^N\frac{\sin nt}{n\log n}\right|<\frac{A}{\log\frac{1}{t}}$$

for $1<M\leqq N$, $0<t<\frac{1}{2}$.

To prove the lemma, let $\tau=[t^{-1}]$ and write
$$S=\sum_M^N=\sum_M^\tau+\sum_{\tau+1}^N=S_1+S_2,$$
when τ falls between M and N. Then
$$|S_1|\leqq t\sum_M^\tau\frac{1}{\log n}<\frac{At\tau}{\log\tau}<\frac{A}{\log\frac{1}{t}},$$
$$|S_2|\leqq\frac{1}{\tau\log\tau}\operatorname*{Max}_{\nu>\tau}\left|\sum_{\tau+1}^{\nu}\sin nt\right|<\frac{A}{t\tau\log\tau}<\frac{A}{\log\frac{1}{t}}.$$

This proves the lemma when τ falls in (M, N), and in the contrary case the proof is simpler.

We now define $\varphi(t)$ by

(6.3) $$\varphi(t)=\sum h_r C(m_r, n_r, q_r, t)=\sum h_r C_r,$$

where

(6.4) $$C_r=2\sin q_r t\sum_{m_r}^{n_r}\frac{\sin nt}{n\log n}.$$

Here

(6.5) $$h_r>0,\qquad \sum h_r<\infty,\qquad q_r=2n_r,$$

and m_r and n_r increase rapidly with r, in a manner to be specified more precisely later.

We prove first that $\varphi(t)$ satisfies (2.4). We choose R so that

$$\sum_{R+1}^{\infty} h_r<\varepsilon.$$

Then, by (6.2),

$$|\varphi(t)|<\sum_1^R h_r|C_r|+\frac{A\varepsilon}{\log\frac{1}{t}}=\frac{A\varepsilon}{\log\frac{1}{t}}+O(t^2)<\frac{2A\varepsilon}{\log\frac{1}{t}}$$

for $0<t\leqq t_0(\varepsilon, R)=t_0(\varepsilon)$.

Next

(6.6) $$C_r=\frac{\cos(q_r-n_r)t}{n_r\log n_r}+\dots+\frac{\cos(q_r-m_r)t}{m_r\log m_r}-\frac{\cos(q_r+m_r)t}{m_r\log m_r}-\dots-\frac{\cos(q_r+n_r)t}{n_r\log n_r}.$$

If

(6.7) $$n_r>3n_{r-1}$$

then, by (6.5),

$$q_r-n_r=n_r>3n_{r-1}=q_{r-1}+n_{r-1},$$

and there is no overlapping between the cosines in different C_r, so that the FOURIER series of $\varphi(t)$ is $\sum h_r C_r$ written out at length in conformity with (6.6). When $t=0$, the series contains blocks of terms of the type

$$h_r\sum_{n_r}^{m_r}\frac{1}{n\log n},$$

and will certainly diverge if

(6.8) $$h_r s_r=h_r(\log\log n_r-\log\log m_r)\to\infty.$$

We have finally to consider the order of a_n as a function of n. The largest coefficient in C_r is $(m_r\log m_r)^{-1}$, and the highest and lowest ranks of a cosine are $q_r+n_r=3n_r$ and $q_r-n_r=n_r$. Also $h_r\to 0$. Hence condition (6.1) will certainly be satisfied if

(6.9) $$\frac{1}{m_r\log m_r}=O\{(3n_r)^{-\eta_{n_r}}\},$$

and *a fortiori* if

$$m_r^{-1}=O\{(3n_r)^{-\eta_{n_r}}\},$$

or if

$$\log m_r - \eta_{n_r} \log 3n_r \to \infty,$$

or if

(6.10)
$$-s_r - \log \eta_{n_r} = \log\log m_r - \log\log n_r - \log \eta_{n_r} \to \infty.$$

7. - A moment's reflection will show that we can always choose our sequences so as to satisfy (6.5), (6.7), (6.8) and (6.10). Suppose, for example, that

$$\eta_n = (l_3 n)^{-1}:$$

we write $l_3 n$ for $\log\log\log n$ and use a similar notation for repeated exponentials. Take

$$h_r = e_1(-r), \qquad s_r = e_1(2r).$$

Then (6.5) and (6.8) are satisfied, and (6.7) and (6.10) will be satisfied if

$$l_4 n_r - e_1(2r) \to \infty.$$

We may for example take

$$n_r = e_5(3r),$$

and then m_r is given by

$$m_r = e_2 \left\{ \frac{e_3(3r)}{e_1(2r)} \right\}.$$

8. - There are two other theorems, of the same character as Theorem 4, whose proofs we leave to the reader.

THEOREM 5. - *Suppose that χ_n tends steadily to infinity with n. Then there is a continuous function $\varphi(t)$ such that*

$$a_n = O\left(\frac{\chi_n}{n}\right)$$

and $\sum a_n$ is divergent.

THEOREM 6. - *There is a function $\varphi(t)$ such that*

(i)
$$\varphi(t) = O\left(\frac{1}{\log \frac{1}{t}}\right),$$

(ii)
$$a_n = O(n^{-\delta}) \qquad (\delta > 0),$$

(iii) *$\sum a_n$ is divergent.*

Theorem 5 shows that the condition (ii) of Theorem 1 is the best possible, while Theorem 6 shows that Theorem 2 is a best possible theorem in a second sense, viz. that condition (i) cannot be relaxed if condition (ii) is left unaltered.

An analogue of Theorem 2.

9. - It is natural to ask what happens to Theorem 2 when the o of (2.4) is replaced by O. The answer is that the order condition must then be strengthened considerably; roughly, a_n must be « very nearly $O(n^{-1})$ ».

THEOREM 7. - *It is sufficient that*

(i) $$\varphi(t)=O\left(\frac{1}{\log\frac{1}{t}}\right)$$

and

(ii) $$a_n=O(n^{-1+\delta})$$

for every positive δ.

The proof is very like that of Theorem 2, and we do not give it in full. We split up $S(\lambda)$ as in (3.5), but suppose now that $r=1-\eta$, where η is small. $S_1(\lambda)$ is $o(1)$ as before; and $S_2(\lambda)$ is bounded, and numerically less than $\varepsilon(\eta)$, tending to zero with η. Finally

$$S_3(\lambda)=O(\lambda^{r-1+\delta}\log\lambda)=O(\lambda^{\delta-\eta}\log\lambda),$$

and tends to zero if $\delta<\eta$.

The conditions (i) and (ii) are again the best possible of their kind.

Tauberian proofs and one-sided conditions.

10. - The proof of Theorem 1 is « Tauberian », and we have no direct proof corresponding to that of Theorem 2. It is natural to look for a Tauberian proof of the latter theorem, and the argument thus suggested is interesting in itself and leads to a generalisation which we cannot prove directly.

It will be convenient to introduce the notion of the « Tauberian index » of any method of summation of divergent series. Suppose that S is a method of summation, and that the hypotheses (a) $\sum a_n$ is summable (S), and (b) $a_n=O(n^{-k})$, imply the convergence of the series. Then we say that S has the Tauberian index k [7]. Thus the CESÀRO and ABEL methods have the index 1. It is plain that if we are to prove Theorem 2 by Tauberian methods, we must use some method of summation whose Tauberian index is δ.

BOREL's exponential method has the index $\frac{1}{2}$. We proved this in 1916 [8], and at the same time introduced a modification of BOREL's method. We defined the limit of a divergent sequence

$$s_n=a_0+a_1+\ldots.+a_n$$

as

$$s=\lim_{n\to\infty}\frac{1}{\sqrt{2\pi n}}\sum_{m=-\infty}^{\infty}e^{-\frac{m^2}{2n}}s_{m+n},$$

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(7) Naturally we choose k as small as possible. We need the phrase only for general explanations and it is unnecessary to be precise in our definitions.

(8) HARDY and LITTLEWOOD, 1.

where s_{m+n} is to be replaced by 0 if the suffix is negative. This definition is equivalent to BOREL's for « delicately divergent » series, and in particular for series (such as FOURIER series) whose terms tend to zero. In particular, it has the same Tauberian index $\frac{1}{2}$.

A little later VALIRON ([9]) obtained very extensive generalisations of our results. We are concerned here only with a quite special case. If we define s by

$$s = \lim_{n \to \infty} \frac{n^{\frac{1}{2}l-1}}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-\frac{1}{2}m^2 n^{l-2}} s_{m+n}, \tag{10.1}$$

where $1 \leqq l < 2$, then the Tauberian condition is

$$a_n = O(n^{\frac{1}{2}l-1}) \tag{10.2}$$

and the index is $1 - \frac{1}{2}l$. This may be made as small as we please by taking l sufficiently near to 2, and the case in which we are interested is that in which l is a little less than 2.

When $l=2$, (10.2) becomes $a_n = O(1)$, and the method cannot sum a FOURIER series unless it is convergent.

11. - We call the method of summation defined by (10.1) the method (V, l). Its use in the theory of FOURIER series depends upon the following theorem.

THEOREM 8. - *If $\varphi(t)$ satisfies (2.4), then $\sum a_n$ is summable (V, l) for $1 \leqq l < 2$, and in particular summable (B).*

We have to show that

$$T_n = \frac{n^{\frac{1}{2}l-1}}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-\frac{1}{2}m^2 n^{l-2}} t_{m+n} \to 0, \tag{11.1}$$

where

$$t_n = \int_0^c \varphi(t) \frac{\sin nt}{t} \, dt. \tag{11.2}$$

In (11.1), t_{m+n} is to be replaced by 0 if $m+n \leqq 0$. We may however drop this convention, and suppose t_{m+n} to be defined by (11.2) for all m and n. For $t_n = O(|n|)$, and the change in T_n which results from this change of convention is

$$O\left(n^{\frac{1}{2}l-1} \sum_{m=-\infty}^{-n} |m+n| e^{-\frac{1}{2}m^2 n^{l-2}}\right) = O\left(n^{\frac{1}{2}l-1} \sum_{\mu=n}^{\infty} \mu e^{-\frac{1}{2}\mu^2 n^{l-2}}\right) =$$

$$= O\left(n^{\frac{1}{2}l-1} \int_n^{\infty} x e^{-\frac{1}{2}x^2 n^{l-2}} dx\right) = O\left(n^{\frac{1}{2}l+1} \int_1^{\infty} y e^{-\frac{1}{2}y^2 n^l} dy\right),$$

which obviously tends to zero.

([9]) VALIRON, 10.

Substituting from (11.2) into (11.1), we obtain

$$T_n = \frac{n^{\frac{1}{2}l-1}}{\sqrt{2\pi}} \int_0^c \frac{\varphi(t)}{t} Q(t, n)dt,$$

where

$$Q(t, n) = Q = \sum_m e^{-\frac{1}{2}m^2 n^{l-2}} \sin(m+n)t.$$

Writing H for n^{l-2}, we have

$$Q = \mathfrak{I}\left\{\sum e^{-\frac{1}{2}m^2 H} e^{i(m+n)t}\right\} = R \sin nt,$$

where

$$R = \sum e^{-\frac{1}{2}m^2 H} \cos mt.$$

By a familiar formula in the theory of elliptic functions

$$R = \sqrt{\frac{2\pi}{H}} \sum \exp\left\{-\frac{2\pi^2}{H}\left(m - \frac{t}{2\pi}\right)^2\right\} = \sqrt{\frac{2\pi}{H}}\, S = n^{-\frac{1}{2}l+1}\sqrt{2\pi}\, S, \tag{11.3}$$

say; and what we have to prove is that

$$T_n = \int_0^c \varphi(t) \frac{\sin nt}{t} S dt \to 0. \tag{11.4}$$

12. - We write

$$S = \sum \exp\left\{-\frac{2\pi^2}{H}\left(m - \frac{t}{2\pi}\right)^2\right\} = e^{-\frac{t^2}{2H}} + S_2 = S_1 + S_2 \tag{12.1}$$

(taking in S_2 all the terms of S for which $m \neq 0$). If, as we may suppose, $c < \pi$, then

$$\left(m - \frac{t}{2\pi}\right)^2 > \left(m - \frac{1}{2}\right)^2 > \frac{1}{4}m^2 \qquad (m \neq 0)$$

and

$$S_2 < 2\sum_1^\infty e^{-\frac{m^2\pi^2}{2H}} = 2\sum_1^\infty e^{-\frac{1}{2}\pi^2 m^2 n^{2-l}} < 4e^{-\frac{1}{2}\pi^2 n^{2-l}}$$

for large n. If follows that

$$\int_0^c \varphi(t) \frac{\sin nt}{t} S_2 dt = O\left\{n e^{-\frac{1}{2}\pi^2 n^{2-l}} \int_0^c |\varphi(t)|\, dt\right\} \to 0.$$

We may therefore replace S by S_1 in (11.4), and the proof of (11.4) is reduced to a proof that

$$U_n = \int_0^c \varphi(t) \frac{\sin nt}{t} e^{-\frac{1}{2}t^2 n^{2-l}} dt \to 0. \tag{12.2}$$

It will be observed that this integral reduces to DIRICHLET's integral for $l=2$.

We now write

$$U_n = \int_0^c = \int_0^{n^{-1}} + \int_{n^{-1}}^{n^{-r}} + \int_{n^{-r}}^c = V_1 + V_2 + V_3,$$

say, choosing r small enough to make

$$h=2-l-2r>0.$$

Then

$$|V_1|\leqq n\int_0^{n^{-1}}|\varphi(t)|\,dt\to 0,$$

and

$$|V_2|\leqq\int_{n^{-1}}^{n^{-r}}\frac{|\varphi(t)|}{t}\,dt\to 0$$

as in § 3; it is here only that we use (2.4). Finally

$$|V_3|\leqq e^{-\frac{1}{2}n^h}\int_{n^{-r}}^{c}\frac{|\varphi(t)|}{t}\,dt=O(n^r e^{-\frac{1}{2}n^h})\to 0.$$

13. - We can deduce Theorem 2 (and a more general theorem) by combining Theorem 8 with appropriate Tauberian theorems, which we state as lemmas.

Lemma β. - *If* $\sum a_n$ *is summable* (V, l), *and satisfies* (10.2), *then it is convergent.*

This, as we stated in § 10, was proved by VALIRON (as a special case of a much more general theorem).

Lemma γ. - *In Lemma* β *the condition* (10.2) *may be replaced by the more general condition*

(13.1) $$a_n>-An^{\frac{1}{2}l-1}.$$

This has been proved explicitly when $k=1$ by SCHMIDT, VIJAYARAGHAVAN and WIENER. The lemma as stated requires an adaptation of their methods which has been undertaken for us by Mr. J. HYSLOP.

Taking $l=2-2\delta$, and combining Theorem 8 with Lemma β, we obtain Theorem 2. If we use Lemma γ, we obtain

THEOREM 9. - *It is sufficient that* (i) $\varphi(t)$ *should satisfy* (2.4) *and* (ii) a_n *should satisfy* (13.1).

We have no direct proof of this theorem.

Criteria of Young's type.

14. - We consider next a group of criteria suggested by the well-known criterion of YOUNG. In these there is no « order » condition, but there are two conditions on $\varphi(t)$. It is characteristic of these criteria, as of YOUNG's, that $\varphi(t)$ is assumed to be of bounded variation except at $t=0$, or at any rate for $0<\delta\leqq t\leqq c$, with some c and arbitrary δ.

YOUNG's test runs: *it is sufficient that*

$$\Phi^*(t)=\int_0^t |\varphi(u)|\,du=o(t) \quad (^{10}) \tag{14.1}$$

and

$$\int_0^t |d(u\varphi)|=O(t). \tag{14.2}$$

An interesting special case is that in which

$$\varphi(t)=o(1), \tag{14.3}$$

$\varphi(t)$ is an integral except at $t=0$, and

$$\varphi'(t)=O\left(\frac{1}{t}\right) \tag{14.4}$$

or at any rate

$$\varphi'(t)>-\frac{A}{t}. \tag{14.5}$$

In this last case, if ψ is φ' when $\varphi'<0$ and 0 otherwise, then

$$|\varphi'|=\varphi'-2\psi$$

and

$$\int_0^t |d(u\varphi)|\leqq\int_0^t u|\varphi'|\,du+\int_0^t |\varphi|\,du=\int_0^t u\varphi' du-2\int_0^t u\psi du+\int_0^t |\varphi|\,du\leqq$$

$$\leqq\left|\int_0^t u\varphi' du\right|+2At+o(t)=\left|t\varphi(t)-\int_0^t \varphi du\right|+2At+o(t)<2At+o(t),$$

so that (14.2) is certainly satisfied. A rather more detailed version of a similar argument will be given in § 16.

15. - When we modify YOUNG's criterion in the manner suggested by Theorems 2 and 3, we obtain

THEOREM 10. - *It is sufficient that* (i) $\varphi(t)$ *should satisfy* (5.1) *and* (ii) *that*

$$\Psi_\Delta^*(t)=\int_0^t |d(u^\Delta\varphi)|=O(t) \tag{15.1}$$

for some Δ.

The emphasis here is on large positive Δ, whereas in Theorems 2 and 3 it was on small positive δ. We may suppose $\Delta>1$. From (15.1) it follows that

$$\Psi_\Delta(t)=\int_0^t d(u^\Delta\varphi)=O(t) \tag{15.2}$$

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[10] This condition may be replaced by

$$\Phi(t)=\int_0^t \varphi(u)du=o(t),$$

or by still more general conditions: see POLLARD (8), HARDY and LITTLEWOOD (3). We cannot prove corresponding extensions of Theorem 10.

and so that

(15.3) $$\varphi(t)=O(t^{1-\Delta}).$$

We choose r so that

(15.4) $$r>\frac{1}{\Delta}$$

and split up $S(\lambda)$ as in § 3. Then $S_1(\lambda)\to 0$ and $S_2(\lambda)\to 0$ as in § 5. As regards $S_3(\lambda)$, we have

$$S_3(\lambda)=\int_{\lambda^{-r}}^{c}\varphi\frac{\sin\lambda t}{t}\,dt=-\frac{1}{\lambda}\int_{\lambda^{-r}}^{c}\frac{\varphi}{t}\,d(\cos\lambda t)=-\left[\varphi\frac{\cos\lambda t}{\lambda t}\right]_{\lambda^{-r}}^{c}+\frac{1}{\lambda}\int_{\lambda^{-r}}^{c}\cos\lambda t d\frac{\varphi}{t}=$$

$$=-\left[\varphi\frac{\cos\lambda t}{\lambda t}\right]_{\lambda^{-r}}^{c}-\frac{\Delta+1}{\lambda}\int_{\lambda^{-r}}^{c}\frac{\varphi}{t^2}\cos\lambda t\,dt+\frac{1}{\lambda}\int_{\lambda^{-r}}^{c}\frac{\cos\lambda t}{t^{\Delta+1}}\,d\Psi_\Delta=S_4(\lambda)+S_5(\lambda)+S_6(\lambda),$$

say. Here

$$S_4(\lambda)=O\{\lambda^{-1}(\lambda^{-r})^{-\Delta}\}=O(\lambda^{r\Delta-1})=o(1),$$

and

$$S_5(\lambda)=O\left(\frac{1}{\lambda}\int_{\lambda^{-r}}^{c}t^{-\Delta-1}dt\right)=O(\lambda^{r\Delta-1})=o(1),$$

by (15.3) and (15.4). Finally

$$|S_6(\lambda)|\leqq\frac{1}{\lambda}\int_{\lambda^{-r}}^{c}t^{-\Delta-1}|d\Psi_\Delta|=\frac{1}{\lambda}\int_{\lambda^{-r}}^{c}t^{-\Delta-1}d\Psi_\Delta^*=\frac{1}{\lambda}\left[t^{-\Delta-1}\Psi_\Delta^*\right]_{\lambda^{-r}}^{c}+\frac{\Delta+1}{\lambda}\int_{\lambda^{-r}}^{c}t^{-\Delta-2}\Psi_\Delta^*dt<$$

$$<O\left(\frac{1}{\lambda}\right)+O\left(\frac{1}{\lambda}\int_{\lambda^{-r}}^{c}\frac{dt}{t^{\Delta+1}}\right)=O(\lambda^{-1})+O(\lambda^{r\Delta-1})=o(1).$$

16. - The special case corresponding to the special case of YOUNG's theorem quoted in § 14 is

THEOREM 11. - *It is sufficient that* (i) $\varphi(t)$ *should satisfy* (2.4), (ii) $\varphi(t)$ *should be an integral except at* $t=0$, *and* (iii) *that*

(16.1) $$\varphi'(t)>-\frac{A}{t^\Delta}.$$

It is sufficient to prove that (15.1) is satisfied. In the argument which follows $0<\varepsilon<t$, and O's are uniform in t and ε (the constants which they imply are independent of both t and ε).

We have first

(16.2) $$\int_{\varepsilon}^{t}|d(u^\Delta\varphi)|\leqq\Delta\int_{\varepsilon}^{t}u^{\Delta-1}|\varphi|\,du+\int_{\varepsilon}^{t}u^\Delta|d\varphi|\leqq\Delta t^{\Delta-1}\int_{0}^{t}|\varphi|\,du+$$

$$+\int_{\varepsilon}^{t}u^\Delta|d\varphi|=O(t^\Delta)+\int_{\varepsilon}^{t}u^\Delta|d\varphi|=O(t)+\int_{\varepsilon}^{t}u^\Delta|d\varphi|.$$

Next, if we define ψ as in § 14, we have

$$(16.3)\quad \int_\varepsilon^t u^\Delta |d\varphi| = \int_\varepsilon^t u^\Delta |\varphi'|\, du = \int_\varepsilon^t u^\Delta \varphi' du - 2\int_\varepsilon^t u^\Delta \psi du < \left|\int_\varepsilon^t u^\Delta \varphi' du\right| + 2At.$$

Finally

$$(16.4)\qquad \int_\varepsilon^t u^\Delta \varphi' du = t^\Delta \varphi(t) - \varepsilon^\Delta \varphi(\varepsilon) - \Delta \int_\varepsilon^t u^{\Delta-1}\varphi(u)du = O(t^\Delta) = O(t),$$

by (2.4). From (16.2), (16.3) and (16.4) it follows that

$$\int_0^t |d(u^\Delta \varphi)| = \lim_{\varepsilon\to 0} \int_\varepsilon^t |d(u^\Delta\varphi)| = O(t),$$

which is (15.2). This proves Theorem 11.

17. - The theorem which corresponds here to Theorem 7 is

THEOREM 12. - *It is sufficient that*

(i) $$\varphi(t) = O\left(\frac{1}{\log \frac{1}{t}}\right),$$

(ii) $\varphi(t)$ *is an integral except at* $t=0$, *and*

(iii) $$\varphi'(t) = O(t^{-1-\delta})$$

for any positive δ.

We leave this theorem and its obvious generalisations to the reader.

The conjugate series.

18. - There are simular theorems concerning the convergence of the series conjugate to a FOURIER series. If we suppose (making the simplifications corresponding to those of § 1) that $\psi(t)$ is odd and

$$\psi(t) \sim \sum a_n \sin nt,$$

then the problem is that of the convergence of $\sum a_n$. We state one theorem only, which corresponds to Theorem 2.

THEOREM 13. - *If*

(i) $$\psi(t) = o\left(\frac{1}{\log\frac{1}{t}}\right),$$

(ii) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(t) \cot \tfrac{1}{2}\, t dt = \frac{1}{\pi} \lim_{\varepsilon\to 0} \int_\varepsilon^\pi \psi(t) \cot \tfrac{1}{2}\, t dt = s,$$

(iii) $a_n = O(n^{-\delta})$ *for some positive* δ, *then*

$$(18.1)\qquad \sum a_n = s.$$

The standard arguments show that (18.1) is equivalent to

$$\int_0^c \psi(t) \frac{\cos \lambda t}{t} dt \to 0;$$

and this may be proved by arguments similar to those of § 3.

Transforms of Theorems 2 and 11.

19. - There is a theorem about general trigonometrical series which is in a sense the « reciprocal » or « transform » of Theorem 2 ([11]).

THEOREM 14. - *If* (i) $a_n = O(n^{-\delta})$ *for some positive* δ, *and* (ii)

$$s_n - s = a_1 + a_2 + \dots + a_n - s = o\left(\frac{1}{\log n}\right), \tag{19.1}$$

then

$$\chi(t) = \sum_1^\infty a_n \frac{\sin nt}{nt} \to s \tag{19.2}$$

when $t \to 0$.

We may express (19.2) by saying that $\sum a_n$ *is summable* $(R, 1)$, *i. e.* by « RIEMANN's first mean », *to* s. It is familiar that $\sum a_n$ is summable $(R, 2)$ whenever it is convergent.

We may suppose $s=0$. We choose r so that

$$r\delta > 1 \tag{19.3}$$

and write

$$\mu = [t^{-1}], \qquad \nu = [t^{-r}], \tag{19.4}$$

$$\chi(t) = \sum_1^\nu + \sum_{\nu+1}^\infty = \chi_1(t) + \chi_2(t). \tag{19.5}$$

Here

$$\chi_2(t) = O\left(\frac{1}{t} \sum_{\nu+1}^\infty n^{-1-\delta}\right) = O(t^{r\delta-1}) = o(1), \tag{19.6}$$

so that it is enough to prove that

$$\chi_1(t) = o(1). \tag{19.7}$$

Summing partially, we have ([12])

$$\chi_1(t) = \sum_1^\nu a_n \frac{\sin nt}{nt} = \sum_1^{\nu-1} s_n \Delta \frac{\sin nt}{nt} + s_\nu \frac{\sin \nu t}{\nu t} = \chi_3(t) + s_\nu \frac{\sin \nu t}{\nu t} = \chi_3(t) + o(1), \tag{19.8}$$

([11]) In our note 2 we gave a general description of a heuristic process of « reciprocation » which often enables us to derive one theorem about trigonometrical series from another. Theorem 14 was derived from Theorem 2 in this way; but the process requires, as usual, a certain amount of adjustment of the data, and is difficult to describe precisely.

([12]) Here $\Delta u_n = u_n - u_{n+1}$: Δ has no connection with the Δ of § 15.

by (19.1). Also

$$(19.9)\qquad \chi_3(t)=\sum_1^{\nu-1} s_n \Delta \frac{\sin nt}{nt}=\sum_1^{\mu}+\sum_{\mu+1}^{\nu-1}=\chi_4(t)+\chi_5(t),$$

say. In χ_4, $nt\leqq 1$ and

$$\frac{\sin nt}{nt}=1-\frac{1}{6}n^2t^2+\ldots,\qquad \Delta\frac{\sin nt}{nt}=O(nt^2);$$

so that

$$(19.10)\qquad \chi_4(t)=\sum_1^{\mu} o(nt^2)=o(\mu^2t^2)=o(1).$$

On the other hand, in χ_5, $nt\geqq 1$ and

$$\Delta\frac{\sin nt}{nt}=O\left(\frac{1}{n^2t}\right)+O\left(\frac{1}{n}\right)=O\left(\frac{1}{n}\right);$$

and so

$$(19.11)\qquad \chi_5(t)=o\left(\sum_{\mu+1}^{\nu-1}\frac{1}{n\log n}\right)=o(\log\log t^{-r}-\log\log t^{-1})=o(1).$$

Collecting our results from (19.8)-(19.11), we obtain (19.7).

If $\sum a_n \cos nt$ is the FOURIER series of $\varphi(t)$, we can state the conclusion in the form

$$(19.12)\qquad \frac{1}{t}\int_0^t \varphi(u)du\rightarrow s.$$

20. - We end by proving

THEOREM 15. - *If* (i) $\sum a_n \cos nt$ *is the Fourier series, or Cauchy-Fourier series, of* $\varphi(t)$, (ii) s_n *satisfies* (19.1), *and* (ii)

$$(20.1)\qquad \varphi(t)=O(t^{-\Delta})$$

for some Δ, *then*

$$(20.2)\qquad \chi(t)=\frac{1}{t}\int_0^t \varphi(u)du=\sum a_n\frac{\sin nt}{nt}\rightarrow s$$

when $t\rightarrow 0$.

This theorem is related to Theorem 11 much as Theorem 14 is related to Theorem 2. We have however replaced (16.1) by the more restrictive condition (20.1). There is no doubt a theorem with a « one-sided » condition, but we have not attempted this generalisation.

Some hypothesis is required to establish a connection between the series and $\varphi(t)$, and the most natural hypothesis for this purpose is (i). When we say that $\sum a_n \cos nt$ is the CAUCHY-FOURIER series of $\varphi(t)$, we mean that $\varphi(t)$ is LEBESGUE integrable except at 0 and that

$$a_n=\frac{2}{\pi}\int_0^{\pi}\varphi(t)\cos nt dt=\frac{2}{\pi}\lim_{\varepsilon\rightarrow 0}\int_{\varepsilon}^{\pi}\varphi(t)\cos nt dt.$$

In these circumstances $n^{-1}a_n$ is the FOURIER sine coefficient of the odd and continuous function $\chi(t)$, and

$$\chi(t)=\sum a_n \frac{\sin nt}{nt},$$

the series being summable $(C, 1)$. We shall in fact prove incidentally that the series is convergent.

We take

(20.3) $$s=0, \qquad \Delta>1, \qquad r>2\Delta>\Delta+1, \qquad \nu=[t^{-r}],$$

and (assuming provisionally the convergence of the series) write

(20.4) $$\chi(t)=\left(\sum_{1}^{\nu}+\sum_{\nu+1}^{\infty}\right) a_n \frac{\sin nt}{nt}=\chi_1(t)+\chi_2(t).$$

We show that $\chi_1(t)\to 0$ as in § 19, and it remains to prove that

(20.5) $$\chi_2(t)=\sum_{\nu+1}^{\infty} a_n \frac{\sin nt}{nt}\to 0.$$

We write ([13])

(20.6) $$\chi_2(t)=\lim_{N\to\infty}\sum_{\nu+1}^{N} a_n \frac{\sin nt}{nt}=\lim_{N\to\infty}\chi_{2,N}(t).$$

Then

(20.7) $$\chi_{2,N}(t)=\sum_{\nu+1}^{N}\frac{\sin nt}{nt}\,\frac{2}{\pi}\int_0^{\pi}\varphi(\theta)\cos n\theta d\theta=\frac{2}{\pi t}\int_0^{\pi}\varphi(\theta)\sum_{\nu+1}^{N}\frac{\sin nt\cos n\theta}{n}\,d\theta=$$
$$=\frac{1}{\pi t}\int_0^{\pi}\varphi(\theta)S_{\nu,N}(t-\theta)d\theta+\frac{1}{\pi t}\int_0^{\pi}\varphi(\theta)S_{\nu,N}(\theta+t)d\theta=\chi_{3,N}(t)+\chi_{4,N}(t),$$

say, where

(20.8) $$S_{\nu,N}(u)=\sum_{\nu+1}^{N}\frac{\sin nu}{n}.$$

We shall prove (a) that $\chi_{3,N}(t)$ and $\chi_{4,N}(t)$ tend to limits $\chi_3(t)$ and $\chi_4(t)$ when t is positive and fixed and $N\to\infty$, and (b) that

(20.9) $$\chi_{3,N}(t)\to 0, \qquad \chi_{4,N}(t)\to 0$$

when $t\to 0$, uniformly in N. It will then follow that $\chi_{2,N}(t)$ tends to a limit $\chi_2(t)$ when $N\to\infty$ (so that the series of the theorem is convergent), and that

(20.10) $$\chi_2(t)=\lim_{N\to\infty}\chi_{2,N}(t)=\lim_{N\to\infty}\chi_{3,N}(t)+\lim_{N\to\infty}\chi_{4,N}(t)=\chi_3(t)+\chi_4(t)\to 0$$

when $t\to 0$; and this will prove the theorem.

([13]) If φ is LEBESGUE integrable, so that $\sum a_n \cos nt$ is a FOURIER series, then the introduction of N is unnecessary. We may replace N at once by ∞, the term by term integration in the argument which follows being justified by « bounded convergence ».

21. - We require the following properties of $S_{\nu,N}(u)$. In the first place

$$(21.1)\qquad |S_{\nu,N}(u)|<A$$

for all u, ν, N. Next

$$(21.2)\qquad |S_{\nu,N}(u)|<\frac{A}{\nu|u|}$$

for $|u|\leqq\frac{3}{2}\pi$ and all N. Thirdly

$$(21.3)\qquad |S'_{\nu,N}(u)|=\left|\sum_{\nu}^{N}\cos nu\right|<\frac{A}{|u|}$$

for $|u|\leqq\frac{3}{2}\pi$ and all ν, N. Finally

$$(21.4)\qquad S_{\nu,N}(u)\rightarrow S_{\nu}(u)=\sum_{\nu+1}^{\infty}\frac{\sin nu}{n},$$

for every u and ν, when $N\rightarrow\infty$, and $S_{\nu}(u)$ has the properties expressed by putting $N=\infty$ in (21.1) and (21.2).

22. - We may confine our attention to $\chi_{3,N}(t)$, the corresponding discussion for $\chi_{4,N}(t)$ being similar but a little simpler. We write

$$(22.1)\qquad \chi_{3,N}(t)=\frac{1}{\pi t}\int_{0}^{\pi}\varphi(\theta)S_{\nu,N}(t-\theta)d\theta=\frac{1}{\pi t}\left(\int_{0}^{t^2}+\int_{t^2}^{t-t^r}+\int_{t-t^r}^{t+t^r}+\int_{t+t^r}^{\pi}\right)=$$

$$=\omega_{1,N}(t)+\omega_{2,N}(t)+\omega_{3,N}(t)+\omega_{4,N}(t)=\omega_1+\omega_2+\omega_3+\omega_4,$$

and consider ω_3, ω_2, ω_4 and ω_1 in turn.

First

$$(22.2)\qquad \omega_3=\frac{1}{\pi t}\int_{t-t^r}^{t+t^r}\varphi(\theta)S_{\nu,N}(t-\theta)d\theta\rightarrow\frac{1}{\pi t}\int_{t-t^r}^{t+t^r}\varphi(\theta)S_{\nu}(t-\theta)d\theta$$

when $N\rightarrow\infty$, since φ is LEBESGUE integrable in the range and $S_{\nu,N}\rightarrow S_{\nu}$ boundedly. Also

$$(22.3)\qquad \omega_3=O\left(t^{-1}\int_{t-t^r}^{t+t^r}\theta^{-\Delta}d\theta\right)=O(t^{r-1-\Delta})=o(1),$$

uniformly in N, by (20.1), (20.3) and (21.1).

Secondly

$$(22.4)\qquad \omega_2=\frac{1}{\pi t}\int_{t^2}^{t-t^r}\varphi(\theta)S_{\nu,N}(t-\theta)d\theta\rightarrow\frac{1}{\pi t}\int_{t^2}^{t-t^r}\varphi(\theta)S_{\nu}(t-\theta)d\theta,$$

for the same reasons as in (22.2). Also

$$\omega_2=O\left(t^{-1}\int_{t^2}^{t-t^r}\frac{\theta^{-\Delta}}{\nu(t-\theta)}d\theta\right)=O\left(t^{r-1}\int_{t^2}^{t-t^r}\frac{\theta^{-\Delta}}{t-\theta}d\theta\right)=O\left(t^{r-\Delta-1}\int_{t}^{1-t^{r-1}}\frac{u^{-\Delta}}{1-u}du\right),$$

by (20.1), (20.3) and (21.2). The integral here is

$$\int_t^{\frac{1}{2}} + \int_{\frac{1}{2}}^{1-t^{r-\Delta}} = O(t^{1-\Delta}) + O(\log t) = O(t^{1-\Delta}),$$

and so, by (20.3),

(22.5) $$\omega_2 = O(t^{r-2\Delta}) = o(1),$$

uniformly in N. A similar argument shows that

(22.6) $$\omega_4 \to \frac{1}{\pi t}\int_{t+t^r}^{\pi} \varphi(\theta) S_\nu(t-\theta)\,d\theta$$

and

(22.7 $$\omega_4 = o(1),$$

uniformly in N.

Finally, if

(22.8) $$\Phi(t) = \int_0^t \varphi(u)\,du = \lim_{\varepsilon\to 0}\int_\varepsilon^t \varphi(u)\,du,$$

we have

(22.9) $$\omega_1 = \frac{1}{\pi t}\int_0^{t^2} \varphi(\theta) S_{\nu,N}(t-\theta)\,d\theta = \frac{\Phi(t^2)}{\pi t} S_{\nu,N}(t-t^2) - \\ - \frac{1}{\pi t}\int_0^{t^2} \Phi(\theta) S'_{\nu,N}(t-\theta)\,d\theta.$$

The first term tends to

$$\frac{\Phi(t^2)}{\pi t} S_\nu(t-t^2).$$

The second is

$$-\frac{1}{\pi t}\int_0^{t^2} \Phi(\theta)\sum_{\nu+1}^{N} \cos n(t-\theta)\,d\theta = -\frac{1}{\pi t}\int_0^{t^2} \Phi(\theta)\,\frac{\sin\left(N+\frac{1}{2}\right)(t-\theta) - \sin\left(\nu+\frac{1}{2}\right)(t-\theta)}{2\sin\frac{1}{2}(t-\theta)}\,d\theta$$

and (by the RIEMANN-LEBESGUE theorem) tends to

$$\frac{1}{\pi t}\int_0^{t^2} \Phi(\theta)\,\frac{\sin\left(\nu+\frac{1}{2}\right)(t-\theta)}{2\sin\frac{1}{2}(t-\theta)}\,d\theta = -\frac{1}{\pi t}\int_0^{t^2} \Phi(\theta) S'_\nu(t-\theta)\,d\theta.$$

Hence

(22.10) $$\omega_1 \to \frac{\Phi(t^2)}{\pi t} S_\nu(t-t^2) - \frac{1}{\pi t}\int_0^{t^2} \Phi(\theta) S'_\nu(t-\theta)\,d\theta = \frac{1}{\pi t}\int_0^{t^2} \varphi(\theta) S_\nu(t-\theta)\,d\theta.$$

Again, the first term in (22.9) is

$$o\left\{\frac{1}{\nu t(t-t^2)}\right\} = o\left(\frac{1}{\nu t^2}\right) = o(t^{r-2}) = o(1),$$

uniformly in N, by (21.2) and (20.3); and the second is

$$O\left(\frac{1}{t}\right)\int_0^{t^2} o(1)\,O\left(\frac{1}{t}\right)d\theta=o(1),$$

by (21.3). Hence

(22.11) $$\omega_1=o(1)$$

uniformly in N.

It follows from (22.2), (22.4), (22.6) and (22.10) that

$$\chi_{3,N}(t)\to\frac{1}{\pi t}\int_0^{\pi}\varphi(\theta)S_\nu(t-\theta)\,d\theta=\chi_3(t)$$

when $N\to\infty$, for any fixed $t>0$; and from (22.3), (22.5), (22.7) and (22.11) that

$$\chi_{3,N}(t)=o(1),$$

uniformly in N. There are similar results for $\chi_{4,N}(t)$; and the theorem follows as was explained at the end of § 20.

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