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I.E. Segal

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THE GLOBAL CAUCHY PROBLEM FOR A RELATIVISTIC SCALAR FIELD WITH POWER INTERACTION (¹);

BY

I. E. SEGAL.

The basic mathematical difficulty in dealing with the Cauchy problem for a non-linear partial differential equation of the type that occurs in relativistic quantum mechanics, i. e. that with a so-called " local " interaction, is a monotone increasing function of the order of growth of the interaction as the field intensity increases without limit. An important case, and one which is believed to be fundamentally representative, although it is the simplest non-trivial case, is that of the equation

$$\Box \varphi = F(\varphi),$$

where φ is an unknown real function of time and a space variable x ranging over euclidean n-space E_n, while F is a given function of a real variable. In most applications, $n = 3$, but higher values of n are of interest also in simulating with fewer algebraic complications the singularities which would occur in dimension three only with fields of higher spin. F is usually a polynomial in φ, for physically conceptual although empirically inconclusive reasons, and may in any event be assumed to be locally smooth. It is the order of growth of $|F(\lambda)|$ for $|\lambda| \to \infty$ which primarily determines the difficulty of solution of the Cauchy problem for this equation.

When F is uniformly Lipschitzian, the solution is straightforward, and for a significant class of applied mathematical problems this assump-

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tion may legitimately be made, since an arbitrary smooth F may be converted into this type by multiplication with a suitable factor vanishing for large arguments, which is a valid approximation if physical effects do not depend on the form of the interaction for arbitrarily large (and hence physically not realizable) field intensities. However the theoretically very interesting and traditionally central case of a power higher than the first is excluded by a uniform Lipschitz condition. Recently, K. JÖRGENS (2) has obtained, employing the energy integral method of Leray, a solution for a class of such equations in three space dimensions which includes the physically interesting one

$$\Box \varphi = m^2 \varphi + g^2 \varphi^3 ;$$

higher powers of φ are however excluded by his treatment, as well as all non-trivial powers in more than three space dimensions. Following this, it was shown (3) that the relevant results of JÖRGENS could be largely subsumed under and materially extended by results applicable to the abstract equation

$$(\bigstar) \qquad u' = Au + K(u), \qquad u(t_0) = u_0,$$

where $u = u(t)$ is an unknown function of time with values in a Hilbert space $\mathcal{H}$, A is a given skew-adjoint operator in $\mathcal{H}$, and K is an everywhere-defined locally Lipschitzian operator on $\mathcal{H}$. The difficulty which occurs in the case of higher powers or space dimensions is that the non-linear operator K which intervenes is not everywhere defined, nor continuous where defined; if the value space of u and its topology are adjusted to make K adequately regular, the energy integral method does not apply in a similarly direct fashion.

The general equation of interest in connection with relativistic physics appears in the form ($\bigstar$), but with a singular, not everywhere defined, non-linear operator K. Here it is shown, by a rather generally applicable method, that the approach employed in (3) can be combined with compactness arguments to show the existence of a global weak solution to the Cauchy problem, with rather general initial data, for the equation

$$\Box \varphi = m^2 \varphi + g^2 \varphi^p \qquad (p \text{ an odd integer}).$$

The uniqueness question for such equations appears however to be outside the scope of presently existing methods.

In the following, Δ represents the usual self-adjoint formulation of the Laplacian as an operator in the Hilbert space $L_2(E_n)$ of all square-

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(2) Das Anfangswertproblem im Grossen für eine Klasse nichtlinearer Wellengleichungen, *Math. Z.*, t. 77, 1961, p. 295-308.

(3) SEGAL (I. E.). — Non-linear semi-groups (to appear in *Annals of Math.*).

integrable functions on E_n (with the usual inner product), and B the positive square root of $m^2 - \Delta$. The domain $\mathcal{D}_B$ of B is well-defined, and consists of those elements of $L_2(E_n)$ having first partial derivatives with respect to the coordinates (as limits in mean of difference quotients) in $L_2(E_n)$. As indicated above, our procedure is to associate with the unknown function $\varphi(x, t)$ of the space variable x and time variable t the vector-valued function of time $\Phi(t)$ whose value for any time t is $\varphi(x, t)$ as a function of the space variable, and to show the existence of a vector-valued function $\Phi(t)$ satisfying the abstract differential equation corresponding to (2).

THEOREM. — *Let φ_0 and φ_1 be given functions on E_n in $\mathcal{D}_B \cap L_{p+1}(E_n)$ and in $L_2(E_n)$ respectively, and t_0 a given time. Then there exists a function $\Phi(t)$ from the reals to $L_2(E_n)$ with values in $\mathcal{D}_B \cap L_{p+1}(E_n)$, satisfying the equations*

$$-\frac{d^2}{dt^2}(\Phi(t), v) = (B\,\Phi(t), Bv) + g^2 \int (\Phi(t))^p . v;$$

for $t = t_0$, $(\Phi(t), v) = (\varphi_0, v)$ and $\left(\frac{d}{dt}\right)(\Phi(t), v) = (\varphi_1, v)$, for all functions v in $\mathcal{D}_B \cap L_{p+1}(E_n)$.

For the proof it will be assumed, to avoid somewhat lengthy although quite straightforward technical elaboration, that $m > 0$; $\mathcal{D}_B$ is then complete as a Hilbert space relative to the inner product $(x, y)_B = (Bx, By)$; as such it will be denoted $[\mathcal{D}_B]$ [while $\mathcal{D}_B$ denotes as above the same set as a subset of $L_2(E_n)$]. Putting $\mathcal{H}$ for the Hilbert space direct sum $[\mathcal{D}_B] + L_2(E_n)$, the operator A whose matrix relative to this decomposition of $\mathcal{H}$ is $\begin{pmatrix} 0 & 1 \\ -B^2 & 0 \end{pmatrix}$ is skew-adjoint. If J is any locally Lipschitzian transformation from $[\mathcal{D}_B]$ to $L_2(E_n)$ for which there exists a non-positive differentiable functional $E(\varphi)$ on $[\mathcal{D}_B]$ with differential $(J(\varphi), .)$, then as shown in ([3]), the abstract Cauchy problem in $\mathcal{H}$,

$$u' = Au + K(u), \qquad u(t_0) = u_0 \in \mathcal{H},$$

where $K(u) = 0 + J(\varphi_0)$ if $u = \varphi_0 + \varphi_1$, has for its integrated form

$$u(t) = e^{tA} u_0 + \int_{t_0}^{t} e^{A(t-s)} K(u(s))\, ds$$

a unique continuous global solution $(-\infty < t < \infty)$. Now the transformation J given by the equation $J(\varphi) = g^2 \varphi^p$ is not everywhere defined on $\mathcal{D}_B$ to $L_2(E_n)$, but if R is the operation on $L_2(E_n)$ of convolution by an integrable symmetric function whose Fourier transform has compact support, the transformation J_R given by the equation

$J_R(\varphi) = g^2 R (R\varphi)^p$ is an everywhere defined locally Lipschitzian transformation from $[\mathcal{D}_B]$ to $L_2(E_n)$, by virtue of the boundedness of $R\varphi$ in $L_\infty(E_n)$ by a multiple of the L_2-norm of $B\varphi$ which follows by Fourier transformation and the use of the Cauchy-Schwarz inequality. In addition, assuming $R = R^*$, J_R has the non-positivity property described above, with $E_R(\varphi) = g^2 \int (R\varphi)^{p+1}$. There is consequently for any such R a unique solution u_R to the integrated form of the Cauchy problem

$$u' = Au + RK(Ru), \qquad u(t_0) = u_0.$$

The energy equality, which follows also from the integrated form of the differential equation, states that the " energy "

$$(1/2)\,||\,B\Phi_R(t)\,||^2 + (1/2)\,||\,\Phi_R(t)\,||^2 + g^2 \int (\Phi_R(t))^{p+1}$$

is constant in time, and so equal to its value

$$(1/2)\,||\,B\varphi_0\,||^2 + (1/2)\,||\,\varphi_1\,||^2 + g^2 \int (B\varphi_0)^{p+1}$$

at time t_0. Restricting R henceforth to be convolution by a even function f in $L_1(E_n)$ which is non-negative and with $\int f = 1$, the norm of R as an operator on $L_2(E_n)$ is bounded by unity, and the same is true of convolution by f as an operator on any other Lebesgue space, in particular $L_{p+1}(E_n)$. It follows that $\int (R\varphi_0)^{p+1} \leqq \int (\varphi_0)^{p+1}$, which implies that each of the three terms in the expression for the energy is bounded for all t and R. The restrictions of the $\Phi_R(t)$ to an arbitrary bounded region G in E_n have accordingly uniformly bounded L_2 norms for their first partial derivatives, and so form a relatively compact subset of $L_2(G)$, by the boundedness of the first term. The $\Phi_R(.)$ form an equicontinuous collection of maps from the reals to $L_2(E_n)$, by the boundedness of the second term.

Now let R_n be convolution by the function f_n of the designated type, where the sequence $\{f_n\}$ is chosen so that the sequence of Fourier transforms converges to unity at every point, and set $\Phi_{R_n} = \Phi_R$ for convenience. Then R_n converges strongly to the identity I as a sequence of operators in L_2.

By the same argument as in the proof of Ascoli's theorem, the $\Phi_n(t)$ have relative to the sphere S_r of radius r a subsequence whose restrictions to S_r converge uniformly in $L_2(S_r)$ on each finite t-interval. By the diagonal argument, a fixed subsequence $\Phi^{(n)}(.)$ exists such that the restrictions of the $\Phi^{(n)}(t)$ to the S_r converge in $L_2(S_r)$ uniformly on each finite t-interval. Since the L_2-norms of these restrictions are bounded independently of r, t and n by a constant, the measurable limit function on E_n determined for each t is in $L_2(E_n)$, and determines a function $\Phi(t)$ on the reals to $L_2(E_n)$. In fact, $\Phi^{(n)}(t) \rightarrow \Phi(t)$ in the weak topology on $L_2(E_n)$ uniformly on each finite t-interval, by the uniform boundedness of the norms in $L_2(E_n)$ and the uniform convergence on finite t-intervals of the $F(\Phi^{(n)}(t))$ to $F(\Phi(t))$, where F is an arbitrary linear functional on $L_2(E_n)$ determined by an element vanishing outside a bounded set.

Since additionally the $\Phi^{(n)}(t)$ have the norms of their first partial derivatives in $L_2(E_n)$ bounded by a constant, $\Phi(t)$ is for each t in the domain of B, by the fact that if in a Hilbert space $\varphi_n \rightarrow \varphi$ weakly, and if each φ_n is in the domain of the self-adjoint operator B, and if $\| B\varphi_n \|$ is bounded, then φ is in the domain of B. (For the proof of this, observe that in the case of strong convergence, it follows from the spectral theorem combined with Fatou's lemma, while the case of weak convergence may be reduced to that of strong convergence with the use of the fact that a weak limit of a sequence is a strong limit of convex linear combinations of elements of the sequence.)

It remains to show that $\Phi(.)$ satisfies the stated differential equation and initial conditions. Note that by the boundedness of $\| B\Phi^{(n)}(t) \|$, the sequence $\{ B\Phi^{(n)}(t) \}$ has for any fixed t a weakly convergent subsequence. To show that

$$(B\Phi^{(n)}(t), Bv) \rightarrow (B\Phi(t), Rv),$$

it suffices to show that the limit of such a subsequence must be $B\Phi(t)$. For this it suffices in turn, by the boundedness of the $\| B\Phi_n(t) \|$, to show that

$$(B\Phi^{(n)}(t), w) \rightarrow B\Phi(t), w)$$

for a dense set of vectors w, and the domain of B provides such a dense set.

To show that the limit of $\left\{ \int (\Phi^{(n)}(t))^p\, v \right\}$ exists and equals $\int (\Phi(t))^p\, v$, it suffices, by Hölder's inequality, to show that the sequence $\{ (\Phi^{(n)}(t))^p \}$ converges weakly in $L_{(p+1)/p}(E_n)$ to $(\Phi(t))^p$. Now the norms in $L_{(p+1)/p}$ of the $(\Phi^{(n)}(t))^p$ are the same as the norms in L_{p+1} of the $\Phi^{(n)}(t)$, and so are bounded. Therefore $\{ (\Phi^{(n)}(t))^p \}$ has, for any fixed t, a subsequence which is weakly convergent in $L_{(p+1)/p}$, and it suffices to show that the limit of this subsequence coincides with $(\Phi(t))^p$. Now since $\{ \Phi^{(n)}(t) \}$ converges in L_2 on every bounded domain, there is for any fixed bounded domain a subsequence converging almost everywhere. Employing the diagonal argument again, it follows that the subsequence whose pth power is weakly convergent in $L_{(p+1)/p}$ has itself a subsequence which is convergent almost everywhere on every bounded domain. This implies convergence almost uniformly on any bounded domain, to the restriction of $\Phi(t)$ to this domain, and hence the same for every positive integral power. But this together with the boundedness of the norms implies weak convergence of the corresponding subsequence of $\{(\Phi_n(t))^p\}$ in $L_{(p+1)/p}$ to $(\Phi(t))^p$.

The function $u_R(.)$ defined above is the unique continuous solution of the equation

$$u_R(t) = e^{(t-t_0)A}u_0 + \int_{t_0}^{t} e^{(t-s)A} RK(Ru(s))\, ds.$$

Noting that e^{sA} has the matrix $\begin{pmatrix} \cos sB & B^{-1}\sin sB \\ -B\sin sB & \cos sB \end{pmatrix}$ relative to the defining decomposition of $\mathcal{H}$ (as is easily verified by checking that the given expression defines a continuous one-parameter unitary group on $\mathcal{H}$, and then differentiating with respect to s), and equating the first components in the foregoing equation, it results that

$$\Phi_R(t) = \cos[(t-t_0)B]\varphi_0 + B^{-1}\sin[(t-t_0)B]\Phi_1 + g^2\int_{t_0}^{t} B^{-1}\sin[(t-s)B[R(\Phi_R(s))^p\, ds.$$

Forming the inner product of both sides with a function v of the designated type, and choosing $R = R_n$ and taking limits of the appropriate subsequence, it results that

$$(\Phi(t), v) = (\varphi_0, \cos[(t-t_0)B]v) + (\varphi_1, B^{-1}\sin[(t-t_0)B]v) + g^2\int_{f_0}^{t} (\Phi(s))^p . B^{-1}\sin[(t-s)B]v.$$

The right side of this equation is differentiable with derivative

$$-(\varphi_0, \sin[(t-t_0)B]Bv) + (\varphi_1, \cos[t-t_0)B]v) + g^2\int_{0}^{t} (\Phi(s))^p . \cos[(t-s)B]v\, ds.$$

Substituting $t = t_0$ in the last equation shows that at time $t = t_0$, $(d/dt)(\Phi(t), v)$ agrees with (φ_1, v), and it is clear that $(\Phi(t_0), v) = (\varphi_0, v)$. Differentiating once more gives the equation

$$\left(\frac{d^2}{dt^2}\right)(\Phi(t), v) = -(B\varphi_0, \cos[(t-t_0)B]Bv) - (\varphi_1, \sin[(t-t_0)B]v) + g^2\int \Phi(s)^p . v - g^2\int_{t_0}^{t} \Phi(s)^p . \sin[(t-s)B]Bv\, ds.$$

Forming the inner product in $\mathcal{H}$ of both sides of the defining equation for u_R with $v \oplus 0$, and passing to the limit with an appropriate subsequence of the R_n, gives the equation

$$(B\Phi(t), Bv) = (B\varphi_0, \cos[(t-t_0)B]Bv) + (\varphi_1, \sin[(t-t_0)B]v)$$
$$+ g^2 \int_{t_0}^{t} (\Phi(s))^p . \sin[(t-t_0)B]Bv\, ds,$$

substitution of which in the preceding equation gives the differential equation asserted by the theorem.

Added in proof. — Since the submission of this article, J. L. Lions has obtained similar results by a different method, as yet unpublished, but referred to in (2).

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I. E. SEGAL,
Department of Mathematics, Massachusetts Institute of Technology,
Cambridge 39, Mass.
(États-Unis).